CHAPTER V

METHOD OF WAVELET IN FIELD THEORY

Contents

5.1 Definition

Following Best. C. and Schafer. A., hep-lat/9311031, hep-lat/9402012, a wavelet $\Psi \in L^2(\Re)$ is a function whose binary dilations and dyadic translations generate a Riesz basis of $L^2(\Re)$. That means any function in $L^2(\Re)$ can be expanded into a wavelet series,

$$f(x) = \sum_{n=-\infty}^{\infty} \sum_{x' \in \Gamma^n} \hat{S}^{(n)}(x') \Psi^{(n)}(x')(x) \tag{5.1}$$

where $\Psi^{(n)}(x')(x) \in L^2(\Re)$. Denotes that the dilated and translated wavelet defined by

$$\Psi^{(n)}(x')(x) = 2^{-n/2} \Psi(2^{-n}(x - x')) \tag{5.2}$$

where $n \in Z$ gives the scale of the wavelet, corresponding to a dilation by 2^n, and $x' \in \Gamma^n$ gives the position translation on the sublattice $\Gamma^n \in 2^n Z$ of scale n. The

coefficients $\hat{S}^{(n)}(x')$ characterize a feature of the wavelet at scale n and position x'. In this sense, the wavelet transform offers a mixed position-frequency representation of the signal.

Wavelet in more than one dimension can be constructed by direct product. We use an additional label $t = 1,..., 2^D-1$. That specifies their composition. It is advantageous to put both wavelet and scaling function on the same footing using an index $t = 0, 1$

$$\psi_t^{(1D,n)}(x')(x) = \begin{cases} \phi^{(n)}(x')(x) & if\ t=0 \\ \psi^{(n)}(x')(x) & if\ t=1 \end{cases} \tag{5.3}$$

Then the N dimensional scaling function ($t=0$) and wavelet ($t=1,...,2^D-1$) are given by

$$\psi_t^{(n)}(x')(x) = \prod_{k=1}^{D} \psi_{t_k}^{(1D,n)}(x'_k)(x_k) \tag{5.4}$$

$$t = \sum_{k=1}^{D} t_k 2^{k-1} \tag{5.5}$$

where t_k are the digits 0 or 1 of the binary representation of t

5.2 Method

5.2.1 Wavelet Representation (Best and Schafer hep-lat/9402012)

Best and Schafer studied statistical field theories on the lattice in wavelet representations. The lattice field $S(x)$, $x \in \Gamma$, is expanded over a wavelet basis. Since our lattice systems are finite, N, t are also finite. We get

$$S(x) = \sum_{n=1}^{N}\sum_{t}^{n_t}\sum_{x'\in\Gamma^n} \hat{S}_t^{(n)}(x')\Psi_t^{(n)}(x')(x) \tag{5.4}$$

$$\hat{S}_t^{(n)}(x') = \sum_{x'\in\Gamma^n} \Psi_t^{(n)}(x')(x)S(x) \tag{5.5}$$

on a D-dimensional lattice $\Gamma=\Gamma^0$ with 2^{DN} sites, N giving the number of scales. The sublattices Γ^n have accordingly $2^{D(N-n)}$ sites; t=0 and 1,...,$n_t = 2^D$-1 denotes the Cartesian composition of multidimensional wavelets; t= 0 is included only at the topmost level n = N, where $\Psi_0^{(N)}$ is the constant function. And, $\hat{S}_0^N$ thus gives the average of $S(x)$ over all sites.

5.2.2 Variational Principle

We employ the principle of minimal free energy to obtain an approximate description of the probability distribution on the partition sum. In a trial ensemble with probability distribution $P(\alpha;\{S(x)\})$ characterized by a set of variational parameters α, we calculate the entropy S

$$S = \int d\{S(x)\}P(\alpha,\{S(x)\})\ln P(\alpha,\{S(x)\}) \tag{5.6}$$

and internal energy

$$U = \int d\{S(x)\}P(\alpha,\{S(x)\})H(\{S(x)\}) \tag{5.7}$$

where H is Hamiltonian of the system and $U = < H >_0$

By minimizing the free energy

$$F = U - \frac{1}{\beta} S \tag{5.8}$$

with respect to the parameters α, the best fit probability distribution is obtained. We choose a Gaussian trial probability distribution characterized by the expectation values

$$< \hat{S}_{t_1}^{(n_1)}(x_1') \hat{S}_{t_2}^{(n_2)}(x_2') > = \delta_{n_1 n_2} \delta_{x_1' x_2'} A_{t_1 t_2}^{(n_1)} \tag{5.9}$$

i.e., fluctuations of wavelet coefficients are assumed to be uncorrelated between scales and between different wavelet positions. This ensemble make calculations sufficiently simple and is, at the same time, able to describe nonlocal correlations in position space. It can be considered an extension of purely local ansatz

$$< \hat{S}(x) \hat{S}(y) > = \delta_{x,y} A \tag{5.10}$$

The introduction of different fluctuation scales in the wavelet transform hence enables us to improve the description of nonlocal fluctuations.

The entropy associated with the Gaussian ensemble is the logarithm of the determinant of the correlation matrix which can be written as

$$S = \frac{1}{2} \sum_{n=1}^{N} \sum_{t}^{n_n} N_n \ln A_t^{(n)} + \text{const} \quad \text{or}$$

$$S = \frac{1}{2} \sum_{n} N_n Tr \ln A^{(n)} + \text{const} \tag{5.11}$$

where $N_n = 2^{-nD} N_0$ is the number of sites on the sublattice Γ^n and the trace is taken over Cartesian composition indices t.