การระเบิดของสมการความร้อน p-ลาปลาซบนแมนิโฟลด์แบบรีมันน์

นายสมเกียรติ สุนทรสวัสดิ์

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BLOW-UP OF p-LAPLACE HEAT EQUATION ON RIEMANNIAN MANIFOLDS

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ให้ M เป็นแมนิโฟลด์เรียบมีขอบมิติ n, $p \geq 2$, $p-1 < q < \dfrac{np-n+p}{n-p}$ พิจารณาปัญหาค่าเริ่มต้น-ค่าขอบในรูป

$$\begin{cases} u_t = \Delta_p u + u^q, & (x,t) \in M \times (0,\infty) \\ u(x,0) = u_0(x), & x \in M \\ u(x,t) = 0, & (x,t) \in \partial M \times [0,\infty). \end{cases}$$

โดยที่ Δ_p คือ ตัวดำเนินการ p-ลาปลาซ-เบลทรามี เราได้รับเงื่อนไขของฟังก์ชันค่าเริ่มต้น u_0 ซึ่งทำให้ผลเฉลยของสมการระเบิดในเวลาจำกัด

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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Let M be an n-dimensional smooth manifold with boundary, $p \geq 2$, $p - 1 < q < \dfrac{np - n + p}{n - p}$. Consider the initial-boundary value problems of the form

$$\begin{cases} u_t = \Delta_p u + u^q, & (x,t) \in M \times (0, \infty) \\ u(x,0) = u_0(x), & x \in M \\ u(x,t) = 0, & (x,t) \in \partial M \times [0, \infty), \end{cases}$$

where Δ_p is the p-Laplace-Beltrami operator. We obtain conditions of the initial function u_0 in which the classical solution of the problem blows up in a finite time.

Department : ...Mathematics and Computer Science... Student's Signature.........................

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CONTENTS

CHAPTER I

INTRODUCTION

The theories of blow-up of partial differential equations have been studied extensively in the past few decades inspired by Fujita's work "On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\sigma}$" in 1966. Fujita considered the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where u_0 is a nonnegative initial data, $p > 1$ and Δ is the Laplacian on $\mathbb{R}^n$. Fujita's result, which have been affected the direction of blow-up study since then, suggests that if $1 < p < 1 + 2/N$, then the solution to any nontrivial initial values blows up in a finite time and if $p > 1 + 2/N$, then the global solution to the equation exists for sufficiently small initial data and the solution blows up in a finite time for sufficiently large initial data. Such $p_c = 1 + 2/N$ is called a critical exponent and plays an important role in studying blow-up theory.

Since then, there have been several kinds of extensions to Fujita's blow-up result concerning variously on domains and initial sources. In 2007, Yang dealt with the problem involving the p-Laplace operator

$$\begin{cases} \frac{u_t}{|x|^s} = \Delta_p u + u^q, & (x,t) \in \Omega \times (0,T) \\ u(x,0) = u_0(x) \geq 0, & u_0 \neq 0,\ x \in \Omega \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > p$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $0 \le s \le 2$, $p \ge 2$, $p - 1 < q < \dfrac{np - n + p}{n - p}$. Yang's result indicates that the blow-up behavior of the solution is concerned with both p and q.

Lately, there has been some attention turning from equations on Euclidean domain $\mathbb{R}^n$ to manifolds. In 1999, Zhang considered the blow-up properties of the Cauchy problem

$$\begin{cases} u_t = \Delta u + V(x)u^q, & x \in M, t > 0 \\ u(x,0) = u_0(x), & x \in M, \end{cases}$$

where (M, g) is an n-dimensional noncompact complete Riemannian manifold with $n \ge 3$, $V(x) \in L^\infty_{loc}$, $Cr^{-m} \le V(x) \le Cr^m$ for some $C \ge 0$, $m > -2$ and Δ is the Laplace-Beltrami operator defined by $\Delta u = \dfrac{1}{\sqrt{\det g(x)}}\dfrac{\partial}{\partial x^i}(\sqrt{\det g(x)}g^{ij}(x)\dfrac{\partial u}{\partial x^j})$. Zhang further assumed that there exists a constant $\alpha > 2$ such that $|B(x, r)| \le Cr^\alpha$ when r is large and for all $x \in M$ and obtained the result suggesting if $1 < q \le 1 + (2 + m)/\alpha$ and $u_0 \ge 0$, then the problem possesses no global positive solution.

In this work, we investigate the problem

$$\begin{cases} u_t = \Delta_p u + u^q, & (x,t) \in M \times (0,T) \\ u(x,0) = u_0(x) \ge 0, & u_0 \not\equiv 0,\ x \in M \\ u(x,t) = 0, & (x,t) \in \partial M \times (0,T), \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, (M, g) is an n-dimensional smooth compact Riemannian manifold with boundary, $p \ge 2$, $p - 1 < q < \dfrac{np - n + p}{n - p}$ and Δ_p is the p-Laplace-Beltrami operator defined by $\Delta_p u = \dfrac{1}{\sqrt{\det g(x)}}\dfrac{\partial}{\partial x^i}(\sqrt{\det g(x)}g^{ij}(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial x^j})$. We apply Yang's result of blow-up criteria on $\mathbb{R}^n$ to compact manifolds. The result is obtained as follows.

Definition 1.1. Let u be a nontrivial classical solution of (1.1). We say that u *blows up in a finite time* if there exists $T_0 > 0$ such that u exists for all $x \in M$, $t \in (0, T_0)$ and $\lim_{t \to T_0^-} u(x_0, t) = \infty$ for some $x_0 \in M$.

Theorem 1.2. *Let u be a nontrivial classical solution of (1.1). Define the energy functional of (1.1) by $E(u) = \frac{1}{p} \int_M |\nabla u|^p dV - \frac{1}{q+1} \int_M |u|^{q+1} dV$. Suppose that $E(u_0) \leq 0$. Then u blows up in a finite time.*

CHAPTER II

PRELIMINARIES

In this chapter, we introduce some basic concepts in Riemannian geometry used throughout this thesis. We first start with the definition of smooth manifolds.

Definition 2.1. Let M be a topological space and $n \in \mathbb{N}$. M is called a *differentiable* (or *smooth*, or C^∞) *manifold* if the following properties hold:

(i) (Topological assumption) M is Hausdorff and second countable;

(ii) (Locally Euclidean) For each $p \in M$, there is a homeomorphism $\varphi : U \to V$ from a neighborhood U of p in M onto an open set V in $\mathbb{R}^n$, such a $\varphi : U \to V$ is called a chart of p;

(iii) (C^∞-compatibility condition) There is a collection $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in I}$ of charts with $\bigcup_{\alpha\in I} U_\alpha = M$ and for any $\alpha, \beta \in I$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\alpha \bigcap U_\beta)$$

is smooth as a map between open subsets of $\mathbb{R}^n$. Such a map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is called a chart transition map. □

Next we introduce the concept of a manifold with boundary which is of important application in PDE. Denote the upper half-space of $\mathbb{R}^n$ by

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x = (x^1, ..., x^n), x^n > 0\}.$$

Let $\overline{\mathbb{R}}^n_+$ be the closure of $\mathbb{R}^n_+$ in $\mathbb{R}^n$. The definition of a manifold with boundary is stated as follows.

Definition 2.2. Let M be a topological space which is Hausdorff and second countable. It is called a *smooth manifold with boundary* if at each $p \in M$ there exists a homeomorphism

$$\varphi : U \to \bar{\mathbb{R}}^n_+$$

from a neighborhood U of p in M onto an open subset of $\bar{\mathbb{R}}^n_+$. As before, φ is called a chart. In addition, if $p, q \in M$ and $\varphi : U \to \bar{\mathbb{R}}^n_+$, $\psi : V \to \bar{\mathbb{R}}^n_+$ are charts at p, q respectively, then $\psi \circ \varphi^{-1}$ is smooth in the sense that is can be extended to a smooth map between open subsets of $\mathbb{R}^n$. □

Now we introduce a Riemannian metric, a concept that plays an important role in Riemannian Geometry.

Definition 2.3. A *Riemannian metric* on M is a collection of inner products

$$g_p : T_pM \times T_pM \to \mathbb{R}, \qquad (p \in M)$$

which is symmetric and positive definite and $p \mapsto g_p$ defines a smooth map into the tensor bundle $T^{0,2}(M)$. A smooth manifold together with a Riemannian metric is called a Riemannian manifold. We sometimes denote g_p by $g_p(X, Y) = \langle X, Y \rangle_p$ for all $X, Y \in T_pM$, $p \in M$. □

Example 2.4. The *Euclidean metric* g on $\mathbb{R}^n$ is given in standard coordinates by

$$g = \delta_{ij} dx^i dx^j = (dx^1)^2 + ... + (dx^n)^2.$$

For vectors $u, v \in T_p\mathbb{R}^n$, $g_p(u, v) = \delta_{ij} u^i v^j = \sum_{i=1}^n u^i v^i = u \cdot v$. That is, g is the Euclidean dot product on $\mathbb{R}^n$. □

Having define a Riemannian metric, we can now extend the definition of some operators in PDE, such as gradient and divergence, to smooth manifolds, as presented below.

Definition 2.5. For any C^1 function f on a Riemannian manifold (M, g), define a vector field called the *gradient* of f, denoted by ∇f or grad$f$, by

$$\langle \nabla f, X \rangle = X(f)$$

for all smooth vector field X on M. In a local coordinate $\{x^i\}$, we have

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

□

To define the divergence on smooth manifolds, we begin with the definition of the Levi-Civita connection.

Definition 2.6. A *Levi-Civita connection* or *covariant derivative* on M (more precisely TM) is a map

$$\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$$

sending $(X, Y) \mapsto \nabla_X Y$ which is bilinear over $\mathbb{R}$ and the following axioms hold

(i) $\nabla_{fX} Y = f \nabla_X Y$

(ii) $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$

(iii) $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$

(iv) $\nabla_X Y - \nabla_Y X = [X, Y]$

for all $f \in C^\infty(M)$, X, Y, $Z \in \Gamma(M)$, where $\Gamma(M)$ is the space of all smooth vector fields. □

Now we can define the divergence on smooth manifolds.

Definition 2.7. For any C^1 vector field X on M, we define the *divergence* of X with respect to the Riemannian metric by

$$\mathrm{div}X = \mathrm{tr}(\xi \mapsto \nabla_\xi X).$$

□

Definition 2.8. Let f be a C^2 function on M. Define the *Laplacian* of f, Δf, by

$$\Delta f = \mathrm{div}\,\mathrm{grad}\, f = \mathrm{div}\, \nabla f.$$

In a chart $\varphi : U \to \mathbb{R}^n$,

$$\Delta f = \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x^i} (\sqrt{\det g(x)} g^{ij} \frac{\partial (f o \varphi^{-1})}{\partial x^j}).$$

□

Proposition 2.9. *Let f be a smooth function on M and X a vector field on M. Then*

$$\mathrm{div}(fX) = f\, \mathrm{div}\, X + \langle \nabla f, X \rangle.$$

In particular, if $X = \nabla h$ for some smooth function h, then

$$\mathrm{div}(f \nabla h) = f \Delta h + \langle \nabla f, \nabla h \rangle.$$

□

Now we introduce a brief concept on an integration on Riemannian manifolds.

Definition 2.10. Let $\{U_\alpha\}$ be an open cover of a smooth manifold M. A collection $\{\rho_\alpha : M \to [0, 1]\} \subset C^\infty(M)$ is called a *partition of unity* subordinate to $\{U_\alpha\}$ if the following conditions hold:

(i) $\forall p \in M$, $\rho_\alpha(p) = 0$ for all but a finite number of α's;

(ii) $\forall p \in M$, $\sum_\alpha \rho_\alpha(p) = 1$;

(iii) $\forall \alpha$, $\operatorname{supp}\rho_\alpha := \overline{\{q \in M : \rho_\alpha(q) \neq 0\}}$ is a subset of U_α. □

Definition 2.11. Let (M^n, g) be a Riemannian manifold, $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}$ an atlas of M and $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$. Define

$$dV = \sum_{\alpha \in A} \rho_\alpha \sqrt{\det g(x)}\, dx^1 \cdots dx^n.$$

Then, we define an integration of a smooth function f on M by

$$\int_M f dV = \sum_{\alpha \in A} \int_{\varphi(U_\alpha)} f \rho_\alpha \sqrt{\det g(x)}\, dx^1 \cdots dx^n.$$

□

Theorem 2.12 (The Divergence Theorem for Compact Riemannian Manifold)**.** *Let (M, g) be a compact Riemannian manifold with boundary. For any C^1 vector field X,*

$$\int_M (\operatorname{div} X) dV = \int_{\partial M} \langle X, \nu \rangle dS,$$

where ν is the outward-pointing unit normal vector field along ∂M, dS is the corresponding volume form on ∂M. □

The next theorem is cited from real analysis used for application in differentiation under the integral sign, as demonstrated in the next chapter.

Theorem 2.13. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f : X \times [a,b] \to \mathbb{R}$ $(-\infty < a < b < \infty)$ and that $f(\cdot, t) : X \to \mathbb{R}$ is integrable for each $t \in [a,b]$. Let $F(t) = \int_X f(x,t) d\mu(x)$. If $\frac{\partial f}{\partial t}$ exists and there is a $g \in L^1(\mu)$ such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ for all x, t, then F is differentiable and $F'(t) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x)$.*
□

CHAPTER III

PROOF OF THEOREM 1.1

Proof. Suppose that u does not blow up in a finite time. We assume further that there exist $t_0 \geq 0$ such that $E(u(t_0)) \leq 0$. First we claim that $\frac{d}{dt}E(u(t)) = -\int_M |u_t|^2 dV$. To see this, compute

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \frac{d}{dt}\left(\frac{1}{p}\int_M |\nabla u|^p dV - \frac{1}{q+1}\int_M |u|^{q+1} dV\right)\\ &= \frac{d}{dt}\left(\frac{1}{p}\int_M |\nabla u|^p dV\right) - \frac{d}{dt}\left(\frac{1}{q+1}\int_M |u|^{q+1} dV\right). \qquad (3.1)\end{aligned}$$

Consider the first term of on the RHS of (3.1). We aim to use Theorem 2.13 to differentiate under the integral sign. Note that $|\nabla u|^p$ is integrable since M is compact. Note that

$$\frac{\partial}{\partial t}|\nabla u|^p = \frac{\partial}{\partial t}\langle \nabla u, \nabla u\rangle^{\frac{p}{2}} = \frac{p}{2}\langle \nabla u, \nabla u\rangle^{\frac{p-2}{2}}\frac{\partial}{\partial t}\langle \nabla u, \nabla u\rangle.$$

Since the metric tensor is smooth, $\frac{\partial}{\partial t}|\nabla u|^p$ exists. By compactness of M and continuity of the metric tensor, $\frac{\partial}{\partial t}|\nabla u|^p = p\langle \nabla u, \nabla u\rangle^{\frac{p-2}{2}}\langle \nabla u, \nabla u_t\rangle$ is bounded, i.e., $\frac{\partial}{\partial t}|\nabla u|^p \leq k$ for some $k > 0$. Clearly, $k \in L^1(M)$. Therefore, by Theorem 2.13, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{p}\int_M |\nabla u|^p dV\right) &= \frac{1}{p}\int_M \frac{\partial}{\partial t}|\nabla u|^p dV\\ &= \frac{1}{p}\int_M \frac{p}{2}\langle \nabla u, \nabla u\rangle^{\frac{p-2}{2}}\frac{\partial}{\partial t}\langle \nabla u, \nabla u\rangle dV. \qquad (3.2)\end{aligned}$$

In a local coordinate $\{x^i\}$, we have $\langle \nabla u, \nabla u \rangle = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}$ and so

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \nabla u, \nabla u \rangle &= \frac{\partial}{\partial t} \left(g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \right) \\
&= g^{ij} \left(\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x^i} \right) \frac{\partial u}{\partial x^j} + \frac{\partial u}{\partial x^i} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x^j} \right) \right) \\
&= g^{ij} \left(\frac{\partial^2 u}{\partial x^i \partial t} \frac{\partial u}{\partial x^j} + \frac{\partial u}{\partial x^i} \frac{\partial^2 u}{\partial x^j \partial t} \right) \\
&= 2g^{ij} \left(\frac{\partial u}{\partial x^i} \frac{\partial^2 u}{\partial x^j \partial t} \right) \\
&= 2 \langle \nabla u, \nabla u_t \rangle. \qquad (3.3)
\end{aligned}$$

Substituting (3.3) into (3.2), we obtain

$$\begin{aligned}
\frac{d}{dt} \left(\frac{1}{p} \int_M |\nabla u|^p dV \right) &= \frac{1}{p} \int_M \frac{p}{2} \langle \nabla u, \nabla u \rangle^{\frac{p-2}{2}} \left(2 \langle \nabla u, \nabla u_t \rangle \right) dV \\
&= \int_M \langle \nabla u, \nabla u \rangle^{\frac{p-2}{2}} \langle \nabla u, \nabla u_t \rangle dV \\
&= \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla u_t \rangle dV \\
&= \int_M \langle |\nabla u|^{p-2} \nabla u, \nabla u_t \rangle dV. \qquad (3.4)
\end{aligned}$$

By Proposition 2.9, we get

$$\langle |\nabla u|^{p-2} \nabla u, \nabla u_t \rangle = \mathrm{div}(|\nabla u|^{p-2} (\nabla u) u_t) - \mathrm{div}(|\nabla u|^{p-2} \nabla u) u_t. \qquad (3.5)$$

By the divergence theorem, Theorem 2.12, we have

$$\int_M \mathrm{div}(|\nabla u|^{p-2} (\nabla u) u_t) dV = \int_{\partial M} \langle |\nabla u|^{p-2} (\nabla u) u_t, \nu \rangle dS. \qquad (3.6)$$

Since $u(x,t) = 0$ for all $x \in \partial M, t \geq 0$, $u_t(x,t) = 0$ on ∂M. Thus $\int_{\partial M} \langle |\nabla u|^{p-2} (\nabla u) u_t, \nu \rangle dS = 0$. By (3.4), (3.5) and (3.6), we get

$$\begin{aligned}
\frac{d}{dt} \left(\frac{1}{p} \int_M |\nabla u|^p dV \right) &= \int_M \mathrm{div}(|\nabla u|^{p-2} (\nabla u) u_t) dV - \int_M \mathrm{div}(|\nabla u|^{p-2} \nabla u) u_t dV \\
&= \int_{\partial M} \langle |\nabla u|^{p-2} (\nabla u) u_t, \nu \rangle dS - \int_M \mathrm{div}(|\nabla u|^{p-2} \nabla u) u_t dV \\
&= - \int_M \mathrm{div}(|\nabla u|^{p-2} \nabla u) u_t dV = - \int_M (\Delta_p u) u_t dV. \qquad (3.7)
\end{aligned}$$

Consider the second term on the RHS of (3.1). Observe that $\dfrac{\partial}{\partial t}|u|^{q+1}$ exists since $q+1 \geq 2$. Since M is compact, $|u|^{q+1}$ is bounded. Then Theorem 2.13 gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{q+1}\int_M |u|^{q+1}dV\right) &= \frac{1}{q+1}\int_M \frac{\partial}{\partial t}(u^2)^{(q+1)/2}dV \\ &= \frac{1}{q+1}\int_M \frac{q+1}{2}(u^2)^{(q-1)/2}2uu_t dV \\ &= \int_M |u|^q u_t dV. \qquad (3.8)\end{aligned}$$

Thus, by (1.1), (3.7) and (3.8)

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= -\int_M (\Delta_p u)u_t dV - \int_M |u|^q u_t dV \\ &= -\int_M ((\Delta_p u) + |u|^q)u_t dV \\ &= -\int_M |u_t|^2 dV.\end{aligned}$$

It follows that, for all $t \geq t_0$,

$$-\int_t^{t_0}\int_M |u_\tau|^2 dV d\tau = \int_t^{t_0} E'(u(\tau))d\tau = E(u(t_0)) - E(u(t)).$$

Therefore

$$\int_{t_0}^t \int_M |u_\tau|^2 dV d\tau + \frac{1}{p}\int_M |\nabla u|^p dV - \frac{1}{q+1}\int_M |u|^{q+1}dV = E(u(t_0)). \qquad (3.9)$$

Define $f(t) = \dfrac{1}{2}\displaystyle\int_{t_0}^t \int_M u^2 dV d\tau$. Then, $f'(t) = \dfrac{1}{2}\displaystyle\int_M u^2 dV$ and

$$\begin{aligned}f''(t) &= \int_M uu_t dV \\ &= \int_M u(\Delta_p u + u^q)dV \\ &= \int_M u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)dV + \int_M |u|^{q+1}dV.\end{aligned}$$

Using Proposition 2.9 and Theorem 2.12, we obtain,

$$\begin{aligned}\int_M u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)dV &= \int_M \mathrm{div}(|\nabla u|^{p-2}u\nabla u)dV - \int_M |\nabla u|^{p-2}\langle \nabla u, \nabla u\rangle dV \\ &= \int_{\partial M}\langle |\nabla u|^{p-2}u\nabla u), \nu\rangle dS - \int_M |\nabla u|^p dV \\ &= -\int_M |\nabla u|^p dV.\end{aligned}$$

Thus,

$$f''(t) = -\int_M |\nabla u|^p dV + \int_M |u|^{q+1} dV.$$

Hence, by (3.9),

$$f''(t) = \frac{q+1-p}{p}\int_M |\nabla u|^p dV - (q+1)E(u(t_0)) + (q+1)\int_{t_0}^t \int_M |u_\tau|^2 dV d\tau. \tag{3.10}$$

Since $E(u(t_0)) \le 0$,

$$\frac{q+1-p}{p}\int_M |\nabla u|^p dV - (q+1)E(u(t_0)) > 0 \tag{3.11}$$

and so, for all $t \ge t_0$,

$$f''(t) > (q+1)\int_{t_0}^t \int_M |u_\tau|^2 dV d\tau.$$

The inequality (3.11) is strict because $\int_M |\nabla u|^p dx \neq 0$; if $\int_M |\nabla u|^p dx = 0$, then u is constant in x and so u is not a solution of the problem.

By choosing $c = (q+1)\int_{t_0}^{t_1} \int_M |u_\tau|^2 dV d\tau$ for some fixed $t_1 > t_0$, we have

$$f''(t) > (q+1)\int_{t_0}^t \int_M |u_\tau|^2 dV d\tau > (q+1)\int_{t_0}^{t_1} \int_M |u_\tau|^2 dV d\tau = c \ge 0$$

for all $t > t_1$.

If $c > 0$, then we have

$$f'(t) > \int_{t_0}^t c d\tau + f'(t_0) = c(t-t_0) + f'(t_0) \text{ and}$$

$$f(t) \ge \int_{t_0}^t (c(\tau - t_0) + f'(t_0)) d\tau + f(t_0) = \frac{c}{2}(t-t_0)^2 + f'(t_0)(t-t_0) + f(t_0)$$

which implies that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = \infty$.

Consider the case $c = 0$ for all $t_1 \ge t_0$. Then $\int_{t_0}^{t_1} \int_M |u_\tau|^2 dV d\tau = 0$ which implies that $u_t = 0$. By (3.10), we have

$$f''(t) = \frac{q+1-p}{p}\int_M |\nabla u|^p dV - (q+1)E(u(t_0)). \tag{3.12}$$

Since $u_t = 0$, u is constant in t and so $f''(t)$ is a constant. If $f''(t) = 0$, then, by (3.12), we have $\int_M |\nabla u|^p dV = 0$. Then u is a trivial solution which is a contradiction. On the other hand, if $f''(t) = c > 0$, then $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = \infty$ as previously done.

We note that

$$\begin{aligned} f''(t) &\geq (q+1)\int_{t_0}^{t}\int_M |u_\tau|^2 dV d\tau \\ f(t)f''(t) &\geq \frac{q+1}{2}\left(\int_{t_0}^{t}\int_M |u|^2 dV d\tau\right)\left(\int_{t_0}^{t}\int_M |u_\tau|^2 dV d\tau\right). \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} f(t)f''(t) &\geq \frac{q+1}{2}\left(\int_{t_0}^{t}\int_M u u_\tau dV d\tau\right)^2 \\ &= \frac{q+1}{2}\left(\int_{t_0}^{t} f''(\tau) d\tau\right)^2 \\ &= \frac{q+1}{2}(f'(t) - f'(t_0))^2. \end{aligned}$$

Claim that there exists a constant $\alpha > 0$ such that $f(t)f''(t) \geq (\alpha+1)(f'(t))^2$ as $t \to \infty$. Choose any $0 < \alpha < \frac{q-1}{2}$. Then $0 < \alpha + 1 < \frac{q+1}{2}$. Since $\lim_{t\to\infty}(f'(t))^2 = \infty$, we can choose $t_2 > t_0$ such that $(\sqrt{\frac{q+1}{2}} - \sqrt{\alpha+1})f'(t_2) \geq \sqrt{\frac{q+1}{2}} f'(t_0)$. Then

$$\begin{aligned} \tfrac{q+1}{2}(f'(t) - f'(t_0))^2 - (\alpha+1)(f'(t))^2 &= \left(\left(\sqrt{\tfrac{q+1}{2}} - \sqrt{\alpha+1}\right) f'(t) - \sqrt{\tfrac{q+1}{2}} f'(t_0)\right) \\ &\quad \times \left(\left(\sqrt{\tfrac{q+1}{2}} + \sqrt{\alpha+1}\right) f'(t) - \sqrt{\tfrac{q+1}{2}} f'(t_0)\right) \\ &\geq 0. \end{aligned}$$

Thus, for $t \geq t_2$,

$$f(t)f''(t) \geq \frac{q+1}{2}(f'(t) - f'(t_0))^2 \geq (\alpha+1)(f'(t))^2$$

as claim. Let $g(t) = (f(t))^{-\alpha}$. Then

$$\begin{aligned} g''(t) &= -\alpha(f^{-\alpha-1}(t)f''(t) + (-\alpha-1)f^{-\alpha-2}(t)(f'(t))^2) \\ &= -\alpha f^{-\alpha-2}(t)(f(t)f''(t) + (-\alpha-1)(f'(t))^2) \\ &\leq -\alpha f^{-\alpha-2}(t)((\alpha+1)(f'(t))^2) + (-\alpha-1)(f'(t))^2) \\ &= 0 \end{aligned}$$

for all $t \geq t_2$. Hence $g(t) = f(t)^{-\alpha}$ is concave for all $t \geq t_2$.

Since $f^{-\alpha}(t) > 0$ and $\lim_{t\to\infty} f(t) = \infty$, $\lim_{t\to\infty} f^{-\alpha}(t) = 0$.

Using the mean value theorem of concave functions, since $f^{-\alpha}(\mathrm{t})$ is concave for all $s, t \in (t_0, \infty)$,

$$f^{-\alpha}(t) \leq f^{-\alpha}(s) + (f^{-\alpha}(s))'(t-s) \text{ for all } t \geq t_0.$$

Observe that $(f^{-\alpha}(s))' = -\alpha f^{-\alpha-1}(s)f'(s) = -\alpha\left(\frac{f^{-\alpha}}{f}f'\right)(s) < 0$.

Thus, it can be concluded that

$$f^{-\alpha}(t) \leq f^{-\alpha}(s) + (f^{-\alpha}(s))'(t-s) \to -\infty$$

as $t \to \infty$ which is a contradiction.

Therefore u blows up in a finite time. $\square$

CHAPTER IV

CONCLUSION

A blow-up criterion in Yang's result was studied and adjusted in this work. We obtained a condition of the initial function suggesting that a nontrivial classical solution of the p-Laplace heat equation blows up in a finite time whenever $E(u_0) \leq 0$ where u_0 is the initial function of the problem and the energy functional $E(u)$ is defined as in Chapter I.

Another unfinished goal in this work is to extend Yang's existence of solution theorem to manifolds.

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