# CHAPTER IV

## SOIL-PILE ELEMENT

As revealed in Chapter 3 , the proposed equivalent plane strain model is quite effective for solving three-dimensional soil-structure interaction problems when no piles are present. Comparisons were made between the result obtained by the proposed model and that of Wolf and von Arx (1978). Unfortunately, the proposed model yields unsatisfactory results. It appears that the complex soil-pile interaction behavior in three-dimensional deformation cannot be simply represented by the over-simplified equivalent two-dimensional plane strain model. Therefore, no further attempt was pursued.

### Formulation

The element stiffness of a soil-pile element can be formulated by considering the summation of the strain energy of the host soil medium and piles inserted within the soil. Piles are vertically placed, thus their neutral axes are parallel with the vertical sides of the host soil medium element as typically shown in Fig. 4.1.

By the assumption of perfect bond between the surfaces of the piles and soil, no slip occurs. Therefore, the displacement fields of the pile neutral axes can be expressed in terms of the nodal displacements of the host elements by linear interpolation.

In this study, rectangular four-node isoparametric plane strain elements are adopted for modeling the soil medium. Thus the strain energy of the soil-pile element can be splitted into two parts :

$$U_e = U_s + U_b \tag{4.1}$$

where $U_s$ is the strain energy of the host soil medium commonly described in standard textbooks on the finite element method, and $U_b$ is the strain energy of the embedded piles, considered as beam elements. For completeness, the formulation of $U_s$ is briefly described in the appendix for a four-node isoparametric plane strain element.

Since the shape function used is of $C^o$ type, the elementary beam theory based on the Euler-Bernoulli hypothesis which involves second derivatives cannot be employed. Consequently, the so-called Timoshenko's beam theory in which the transverse shear effect is included is selected.

The modulus of elasticity of the beam portion is taken as

$$E_b = E_p - E_s \tag{4.2}$$

where $E_p$ is the true value of the pile's modulus of elasticity and $E_s$ is the modulus of elasticity of the soil. The reason for reducing the modulus of elasticity of the pile is that the strain energy within the pile volume is determined twice, first for the full host soil medium and second for the pile element.

The spatial coordinate values and displacement fields of any point within the element are related through the shape functions as :

$$y(s, t) = \sum_{i=1}^{4} N_i^e(s, t) \cdot y_i \tag{4.3a}$$

$$z(s, t) = \sum_{i=1}^{4} N_i^e(s, t) \cdot z_i \tag{4.3b}$$

and

$$v\ (s, t) = \sum_{i=1}^{4} N_i^e(s, t) \cdot v_i \qquad (4.4a)$$

$$w\ (s, t) = \sum_{i=1}^{4} N_i^e(s, t) \cdot w_i \qquad (4.4b)$$

in which $y_i$ and $z_i$ are coordinates of element nodal points; $v_i$ and $w_i$ are nodal displacements in y and z directions, respectively, and $N_i^e(s, t)$ are the local shape functions of the host soil element shown in the appendix.

An elegant derivation of beam strain energy is given by Washizu (1968). The result is

$$U_b = \frac{1}{2}\int_0^{\ell} \left[E_b A_b \left(\frac{\partial w}{\partial z}\right)^2 + E_b I_b \left(\frac{\partial^2 w}{\partial z \partial y}\right)^2 + G_b \mathcal{K}_b A_b \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right] dz \qquad (4.5)$$

in which w and v are the longitudinal and transverse displacements of the pile neutral axis, respectively; $G_b$ is the shear modulus; $\mathcal{K}_b$ is the shear correction factor; $A_b$ is the pile cross-section area, and $I_b$ is the moment of inertia of the pile cross section.

In matrix notation, compared with the first term of the right hand side of Eq. (2.17), Eq. (4.5) can be written as

$$U_b = \sum_{e=1}^{n} \frac{1}{2} \int_0^{e} \{q\}^T [B_b^e]^T [D_b^e][B_b^e]\{q\}\, dx \qquad (4.6)$$

in which $\{q\} = \{v_i,\ w_i\}^T$ is the vector of the nodal point displacements, and $[B_b^e] = [B_1^e, B_2^e, B_3^e, B_4^e]$ is the strain transformation matrix given by

$$[B_i^e] = \begin{bmatrix} 0 & \dfrac{\partial N_i}{\partial z} \\ 0 & \dfrac{\partial^2 N_i}{\partial z \partial y} \\ \dfrac{\partial N_i}{\partial z} & \dfrac{\partial N_i}{\partial y} \end{bmatrix} \tag{4.7}$$

and

$$[D_b^e] = \begin{bmatrix} E_b A_b & 0 & 0 \\ 0 & E_b I_b & 0 \\ 0 & 0 & G_b \mathcal{K}_b A_b \end{bmatrix} \tag{4.8}$$

Formulation of the element stiffness can be proceeded as described in Chapter 2. In view of Eq. (2.22), we have

$$[K_b^e] = \int_0^{\ell^e} [B_b^e]^T [D_b^e][B_b^e]\, dz \tag{4.9}$$

Calculation of the element stiffness matrix, Eq. (4.9), by mapping global $y-z$ domain into local $s-t$ domain can be performed using the following chain rule of derivative :

$$dz = \frac{\partial z}{\partial s} ds + \frac{\partial z}{\partial t} dt \tag{4.10}$$

For the special case of rectangular element, $\dfrac{\partial z}{\partial s} = 0$. Thus,

$$dz = \frac{\partial z}{\partial t} dt \tag{4.11}$$

in which $\dfrac{\partial z}{\partial t}$ can be easily determined from Eq. (4.3b) as

$$\frac{\partial z}{\partial t} = \sum_{i=1}^{4} \frac{\partial}{\partial t} N_i^e(s, t) \cdot z_i = \frac{\ell}{2} \tag{4.12}$$

The element stiffness matrix, Eq. (4.9), in view of Eq. (4.11) and Eq. (4.12), becomes

$$[K_b^e] = \int_{-1}^{1} [B_b^e]^T [D_b^e][B_b^e] \frac{\ell}{2} dt \tag{4.13}$$

Evaluation of the strain matrix, $[B_b^e]$, can be easily done by means of the chain rule also. The result is

$$[B_b^e] = \frac{1}{4} \left[ \begin{array}{cc|cc|cc|cc} 0 & \frac{(1+s)}{b} & 0 & \frac{(1-s)}{b} & 0 & -\frac{(1-s)}{b} & 0 & -\frac{(1+s)}{b} \\ 0 & \frac{1}{ab} & 0 & -\frac{1}{ab} & 0 & \frac{1}{ab} & 0 & -\frac{1}{ab} \\ \frac{(1+s)}{b} & \frac{(1+t)}{a} & \frac{(1-s)}{b} & -\frac{(1+t)}{a} & -\frac{(1-s)}{b} & -\frac{(1-t)}{a} & -\frac{(1+s)}{b} & \frac{(1-t)}{a} \end{array} \right] \tag{4.14}$$

## Verification of Soil-Pile Element

The proposed soil-pile element was tested for its validity in case of static loading. Since analytical solutions are readily available for a narrow two-dimensional plane stress medium reinforced with beam type elements which collectively can be treated as a composite beam, verification of the soil-pile element concept was performed for the plane stress rather than the plane strain host element. This approach is deemed sufficient in so far as the stiffness property is concerned. In this study, the limiting case of an extremely soft soil medium was considered, ie. the modulus of elasticity of the host element was almost zero. Two model shown in Fig. 4.2a and Fig. 4.2b, with concentric and eccentric piles respectively were analyzed for the effect of applied point loads at the top. The finite element solutions obtained using the soil-pile elements agree very well with the classical beam theory solutions, with only 0.03 and 3.6% discrepancies for the concentric and eccentric piles, respectively.