CHAPTER II

ROTATION IN TWO , THREE AND FOUR DIMENSIONS

Let OX , OY be axes in two dimensions. Let axes OX$'$, OY$'$ rotate from axes OX , OY through an angle θ . Then, the point (x, y) is also represented by (x', y') according to the equation :-

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The matrix of the transformation is an orthogonal matrix. That is, rotation in two dimensions is an orthogonal transformation.

Next, we shall show that rotation in three dimensions is an orthogonal transformation. Let the axes OX$'$, OY$'$, OZ$'$ rotate from the axes OX, OY, OZ by the following steps : -

1) Rotate the XOY plane about the Z axis through an angle θ_{12} . The transformation is given by the equation :-

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2) Rotate the YOZ plane about the X-axis through an angle θ_{23} . The matrix of the transformation is :-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{bmatrix}$$

3) Rotate the XOZ plane about the Y-axes through an angle θ_{13}. The matrix of the transformation is

$$\begin{pmatrix} \cos\theta_{13} & 0 & -\sin\theta_{13} \\ 0 & 1 & 0 \\ \sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix}$$

After the three rotations, the point (x, y, z) is represented by (x′, y′, z′) according to the equation :-

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_{13} & 0 & -\sin\theta_{13} \\ 0 & 1 & 0 \\ \sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_{13}\cos\theta_{12} - \sin\theta_{13}\sin\theta_{23}\sin\theta_{12} & \cos\theta_{13}\sin\theta_{12} + \sin\theta_{13}\sin\theta_{23}\sin\theta_{12} & -\sin\theta_{13}\cos\theta_{13} \\ -\cos\theta_{23}\sin\theta_{12} & \cos\theta_{23}\cos\theta_{12} & \sin\theta_{23} \\ \sin\theta_{13}\cos\theta_{12} + \cos\theta_{13}\sin\theta_{23}\sin\theta_{13} & \sin\theta_{13}\sin\theta_{12} - \cos\theta_{13}\sin\theta_{23}\cos\theta_{12} & \cos\theta_{13}\cos\theta_{23} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We can check that the product of the matrix of the transformation and its transpose is equal to the identity matrix. That is, rotation in three dimensions is an orthogonal transformation.

Similarly, we shall show that the rotation in four dimensions is an orthogonal transformation.

Let OW, OX, OY and OZ be the four axes in four dimensional space. O is their common point, the origin. We define the positive rotations as in the diagram of figure 4,

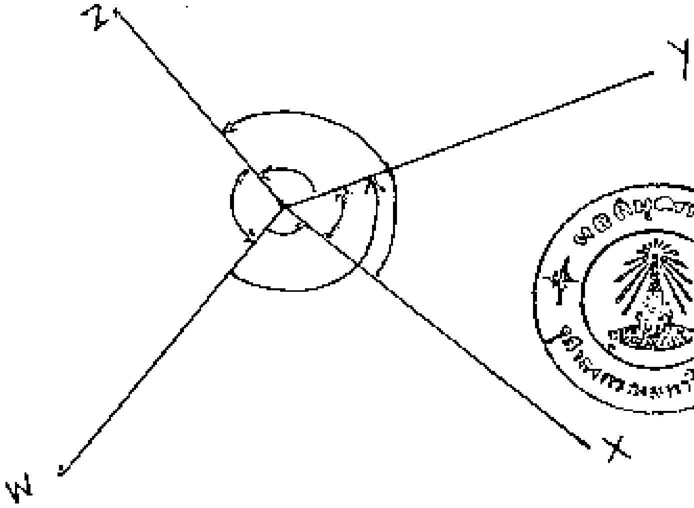

Figure 4

Consider the geometrical rotation by the following steps :-

<u>Rotation 1</u> rotates the WOX plane about the YOZ plane through an angle θ_{12} . The point (w , x, y, z) is represented by (w', x', y', z') according to the equation :-

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

or in brief , it is represented by $P' = R_1 P$

<u>Rotation 2</u> rotates the XOY plane about the WOZ plane through an angle θ_{23} . The transformation is given by the matrix:-

$$R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} & 0 \\ 0 & -\sin\theta_{23} & \cos\theta_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation 3 rotates the YOZ plane about the WOX plane through an angle θ_{34} according to the matrix

$$R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_{34} & \sin\theta_{34} \\ 0 & 0 & -\sin\theta_{34} & \cos\theta_{34} \end{bmatrix}$$

Rotation 4 rotates the WOY plane about the XOZ plane through an angle θ_{13} according to the matrix :-

$$R_4 = \begin{bmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation 5 rotates the XOZ plane about the WOY plane through an angle θ_{24} according to the matrix

$$R_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{24} & 0 & \sin\theta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_{24} & 0 & \cos\theta_{24} \end{bmatrix}$$

Rotation 6 rotates the WOZ plane about the XOY plane through an angle θ_{14} according to the matrix :-

$$R_6 = \begin{bmatrix} \cos\theta_{14} & 0 & 0 & -\sin\theta_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta_{14} & 0 & 0 & \cos\theta_{14} \end{bmatrix}$$

After the six rotations, the matrix of the transformation is the product of the six matrices. Each matrix is orthogonal, so their product is orthogonal (3, p.103) Hence rotation in four dimensions is an orthogonal transformation.

It has been shown (2, p.162) that the orthogonal matrix of order n must have $\frac{n(n-1)}{2}$ independent elements. Hence rotation in two dimensions must have $\frac{2.1}{2} = 1$ specific parameters, in three dimensions it must have $\frac{3.2}{2} = 3$ specific parameters, and in four dimensions rotation must have $\frac{4.3}{2} = 6$ specific parameters.

We have shown in chapter I that the quaternion equation (4) in chapter I cannot represent a rotation in three dimensions. We can see that it cannot represent rotation in four dimensions either, since p contains, only four arbitrary constants a, b, c, d whereas the general rotation in four dimensions contains six arbitrary constants as shown above. But the general rotation in four dimensions can be obtained by the quaternion equation

$$q' = p \cdot q \cdot \pi \qquad \ldots\ldots\ldots\ldots (1)$$

where $\pi = \delta + \alpha i + \beta j + \gamma k$

is another constant quaternion (1, p. 68).