CHAPTER I

HYPERCOMPLEX NUMBER SYSTEMS

The real number system that includes the rational and irrational numbers is adequate for many problems in algebra and calculus. But in other problems an extension of the system is necessary. In 1545, Cardan used "imaginary numbers" to solve the cubic equation. (4, p. 201-206). In 1748, Euler set up the relation $e^{ix} = \cos x + i \sin x$. (3, p. 55-56).

For the first three hundred years that imaginary numbers were used, their nature was not clearly understood. Mathematicians used the relation $i^2 = -1$ even though the squares of all known numbers were positive. Finally in the nineteenth century, complex numbers in the form $x + iy$, where x and y are real numbers and i is the imaginary unit, were used as generalized forms which included both real and imaginary numbers. Today in the rigorous treatment of complex numbers, one defines them as ordered pairs (x, y) of real numbers x and y with special definitions for equality, addition and multiplication. (2, p. 101; 5, p. 678)

The complex numbers are found to have the same algebraic field properties as real numbers. The system of complex numbers is complete in that convergent sequences of complex numbers always have a limit in the system.

Mathematicians did not stop at complex numbers. They tried to set up other higher complex number systems. Finally in 1840, H. Grassman in Stettin and W.R. Hamilton in Dublin independently of each other discovered the quaternion number system (3, p. 58).

The quaternions are combinations of four real numbers with three different imaginary units. If i, j, k are three different imaginary units and x_1, x_2, x_3, x_4 are four real numbers, a quaternion is written, $Q = x_1 + ix_2 + jx_3 + kx_4$ (1, p. 58-59). This extension of the complex number system fails to be a field because the commutative law for multiplication does not hold, as we shall see later.

The most advanced system of numbers similar to those above is the system of Cayley Numbers . (10, p. 19). They are eight - term hypercomplex numbers with one real unit and seven imaginary units. A Cayley number may be written,
$C = x_0 + e_1x_1 + e_2x_2 + \cdots\cdots + e_7x_7$, where $e_1, e_2, \ldots\ldots, e_7$ are the seven imaginary units and $x_0, x_1, x_2, \ldots\ldots, x_7$ are the eight real numbers. The Cayley number system is an extension of the quaternion number system. The operations and the properties are similar but the multiplication of Cayley numbers fails to be associative.

In this thesis I have discussed the properties of these hypercomplex number systems, the relations between them and some applications of them.