CHAPTER IV

STOKES' THEOREM

The primary goal of this chapter is to prove a version of Stokes' Theorem (Theorem 4.4). Before we can begin its proof, however, we need the following lemmas.

Lemma 4.1 For all $n \in Z^{+}$ and all $r \in \bar{n}$,

$$\int_{\Delta(n)} x^{(r)} \, dx = \frac{1}{(n+1)!}.$$

Proof. Let $n \in Z^{+}$, and let $r \in \bar{n}$. It follows from Proposition 3.2.9 that

$$\int_{\Delta(n)} x^{(r)} \, dx = (\mathrm{R})\int_{\Delta(n)} x^{(r)} \, dx.$$

Thus, it remains to show that

$$(\mathrm{R})\int_{\Delta(n)} x^{(r)} \, dx = \frac{1}{(n+1)!}.$$

If $n = 1$, then $(\mathrm{R})\int_{\Delta(1)} x \, dx = (\mathrm{R})_0\!\int^1 x \, dx = \frac{1}{2!}$, as required. Now suppose that $n \geq 2$. Let $i_1, \ldots, i_{n-1}$ be the distinct elements of $\bar{n} \setminus \{r\}$. Note that

$(\mathrm{R})\int_{\Delta(n)} x^{(r)} \, dx$

$$= (\mathrm{R})_0\!\int^1 (\mathrm{R})_0\!\int^{1-x^{(i_{n-1})}} \cdots (\mathrm{R})_0\!\int^{1-\sum_{j=2}^{n-1} x^{(i_j)}} (\mathrm{R})_0\!\int^{1-\sum_{j=1}^{n-1} x^{(i_j)}} x^{(r)} \, dx^{(r)} dx^{(i_1)} \cdots dx^{(i_{n-1})}.$$

This implies that

$(\mathrm{R})\int_{\Delta(n)} x^{(r)} \, dx$

$$= \frac{1}{2} (\mathrm{R})_0\!\int^1 (\mathrm{R})_0\!\int^{1-x^{(i_{n-1})}} \cdots (\mathrm{R})_0\!\int^{1-\sum_{j=3}^{n-1} x^{(i_j)}} (\mathrm{R})_0\!\int^{1-\sum_{j=2}^{n-1} x^{(i_j)}} \Big(1 - \sum_{j=1}^{n-1} x^{(i_j)}\Big)^2 \, dx^{(i_1)} \cdots dx^{(i_{n-1})}.$$

To investigate $(R)\int_0^{1-\sum_{j=2}^{n-1} x^{(i_j)}} (1-\sum_{j=1}^{n-1} x^{(i_j)})^2 \, dx^{(i_1)}$, proceed as follows. Let $t = 1-\sum_{j=1}^{n-1} x^{(i_j)}$; then $dx^{(i_1)} = -dt$, $t = 1-\sum_{j=2}^{n-1} x^{(i_j)}$ if $x^{(i_1)} = 0$, and $t = 0$ if $x^{(i_1)} = 1-\sum_{j=2}^{n-1} x^{(i_j)}$. Hence

$$(R)\int_0^{1-\sum_{j=2}^{n-1} x^{(i_j)}} (1-\sum_{j=1}^{n-1} x^{(i_j)})^2 \, dx^{(i_1)} = (R)\int_0^{1-\sum_{j=2}^{n-1} x^{(i_j)}} t^2 \, dt = \frac{1}{3}(1-\sum_{j=2}^{n-1} x^{(i_j)})^3.$$

By replacing $(R)\int_0^{1-\sum_{j=2}^{n-1} x^{(i_j)}} (1-\sum_{j=1}^{n-1} x^{(i_j)})^2 \, dx^{(i_1)}$ with $\frac{1}{3}(1-\sum_{j=2}^{n-1} x^{(i_j)})^3$, we get

$(R)\int_{\Delta(n)} x^{(r)} \, dx$

$$= \frac{1}{3!}(R)\int_0^1 (R)\int_0^{1-x^{(i_{n-1})}} \cdots (R)\int_0^{1-\sum_{j=4}^{n-1} x^{(i_j)}} (R)\int_0^{1-\sum_{j=3}^{n-1} x^{(i_j)}} (1-\sum_{j=2}^{n-1} x^{(i_j)})^3 \, dx^{(i_2)} \cdots dx^{(i_{n-1})}.$$

Continuing this process inductively, we eventually obtain

$$(R)\int_{\Delta(n)} x^{(r)} \, dx = \frac{1}{n!}(R)\int_0^1 (1-x^{(i_{n-1})})^n \, dx^{(i_{n-1})}$$

$$= -\frac{1}{(n+1)!}(1-x^{(i_{n-1})})^{n+1}\Big|_0^1$$

$$= \frac{1}{(n+1)!}.$$

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Lemma 4.2 Let $k \in Z^+$ be such that $k \geq 2$. Suppose $q_0, q_1, \ldots, q_k \in R^k$ are such that $q_1 - q_0, \ldots, q_k - q_0$ are linearly independent. For each $j \in \overline{k}$, let

$$J_{j,0} = \det\begin{bmatrix}(q_2-q_1)^{(1)} & \cdots & (q_k-q_1)^{(1)} \\ \vdots & & \vdots \\ (q_2-q_1)^{(j-1)} & \cdots & (q_k-q_1)^{(j-1)} \\ (q_2-q_1)^{(j+1)} & \cdots & (q_k-q_1)^{(j+1)} \\ \vdots & & \vdots \\ (q_2-q_1)^{(k)} & \cdots & (q_k-q_1)^{(k)}\end{bmatrix},$$

and for each $a \in \bar{k}$, let

$$J_{ja} = \det\begin{bmatrix}(q_1-q_0)^{(1)} & \cdots & (q_{a-1}-q_0)^{(1)} & (q_{a+1}-q_0)^{(1)} & \cdots & (q_k-q_0)^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ (q_1-q_0)^{(j-1)} & \cdots & (q_{a-1}-q_0)^{(j-1)} & (q_{a+1}-q_0)^{(j-1)} & \cdots & (q_k-q_0)^{(j-1)} \\ (q_1-q_0)^{(j+1)} & \cdots & (q_{a-1}-q_0)^{(j+1)} & (q_{a+1}-q_0)^{(j+1)} & \cdots & (q_k-q_0)^{(j+1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ (q_1-q_0)^{(k)} & \cdots & (q_{a-1}-q_0)^{(k)} & (q_{a+1}-q_0)^{(k)} & \cdots & (q_k-q_0)^{(k)}\end{bmatrix}.$$

Then for every $j \in \bar{k}$,

(i) $\sum_{a=0}^{k}(-1)^a J_{ja} = 0,$

(ii) $\sum_{a=0}^{k}(-1)^{a+1} q_a^{(s)} J_{ja} = 0$ for all $s \in \bar{k}\backslash\{j\}$, and

(iii) if $\det\begin{bmatrix}1 & 1 & \cdots & 1 \\ q_0^{(1)} & q_1^{(1)} & \cdots & q_k^{(1)} \\ \vdots & \vdots & & \vdots \\ q_0^{(k)} & q_1^{(k)} & \cdots & q_k^{(k)}\end{bmatrix} > 0$, then

$$\mathrm{Vol}(\Delta(q_0, q_1, \ldots, q_k)) = \frac{(-1)^{j+1}}{k!}\sum_{a=0}^{k}(-1)^{a+1}q_a^{(j)}\,\mathrm{J}_{ja}.$$

Proof. Let $j \in \bar{k}$ be fixed.

To prove (i), note that (using expansion by minors)

$$\sum_{a=0}^{k}(-1)^{a}\mathrm{J}_{ja} = \mathrm{J}_{j,0} - \sum_{a=1}^{k}(-1)^{a+1}\,\mathrm{J}_{ja}$$

$$= \mathrm{J}_{j,0} - \det\begin{bmatrix} 1 & \cdots & 1 \\ (q_1-q_0)^{(1)} & \cdots & (q_k-q_0)^{(1)} \\ \vdots & & \vdots \\ (q_1-q_0)^{(j-1)} & \cdots & (q_k-q_0)^{(j-1)} \\ (q_1-q_0)^{(j+1)} & \cdots & (q_k-q_0)^{(j+1)} \\ \vdots & & \vdots \\ (q_1-q_0)^{(k)} & \cdots & (q_k-q_0)^{(k)} \end{bmatrix}.$$

Thus, it is enough to show that $\mathrm{J}_{j,0}$ equals the determinant of the last matrix. By subtracting the first column from each of the other columns in the last matrix and then expanding by minors along the first row, we get

$$\det\begin{bmatrix} 1 & \cdots & 1 \\ (q_1-q_0)^{(1)} & \cdots & (q_k-q_0)^{(1)} \\ \vdots & & \vdots \\ (q_1-q_0)^{(j-1)} & \cdots & (q_k-q_0)^{(j-1)} \\ (q_1-q_0)^{(j+1)} & \cdots & (q_k-q_0)^{(j+1)} \\ \vdots & & \vdots \\ (q_1-q_0)^{(k)} & \cdots & (q_k-q_0)^{(k)} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ (q_1-q_0)^{(1)} & (q_2-q_1)^{(1)} & \cdots & (q_k-q_1)^{(1)} \\ \vdots & \vdots & & \vdots \\ (q_1-q_0)^{(j-1)} & (q_2-q_1)^{(j-1)} & \cdots & (q_k-q_1)^{(j-1)} \\ (q_1-q_0)^{(j+1)} & (q_2-q_1)^{(j+1)} & \cdots & (q_k-q_1)^{(j+1)} \\ \vdots & \vdots & & \vdots \\ (q_1-q_0)^{(k)} & (q_2-q_1)^{(k)} & \cdots & (q_k-q_1)^{(k)} \end{bmatrix}$$

$$= \mathrm{J}_{j,0}.$$

To prove (ii), suppose $s \in \overline{k}\backslash\{j\}$. We will only prove the case $s < j$; the case $s > j$ is similar. Thus, if we expand the first matrix below along its $(j+1)^{st}$ row, we see

$$0 = \det\begin{bmatrix} 1 & \cdots & 1 \\ q_0^{(1)} & \cdots & q_k^{(1)} \\ \vdots & & \vdots \\ q_0^{(s-1)} & \cdots & q_k^{(s-1)} \\ q_0^{(s)} & \cdots & q_k^{(s)} \\ q_0^{(s+1)} & \cdots & q_k^{(s+1)} \\ \vdots & & \vdots \\ q_0^{(j-1)} & \cdots & q_k^{(j-1)} \\ q_0^{(s)} & \cdots & q_k^{(s)} \\ q_0^{(j+1)} & \cdots & q_k^{(j+1)} \\ \vdots & & \vdots \\ q_0^{(k)} & \cdots & q_k^{(k)} \end{bmatrix}$$

$$= \sum_{a=0}^{k}(-1)^{(j+1)+(a+1)}\, q_a^{(s)} \det\begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ q_0^{(1)} & \cdots & q_{a-1}^{(1)} & q_{a+1}^{(1)} & \cdots & q_k^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_0^{(j-1)} & \cdots & q_{a-1}^{(j-1)} & q_{a+1}^{(j-1)} & \cdots & q_k^{(j-1)} \\ q_0^{(j+1)} & \cdots & q_{a-1}^{(j+1)} & q_{a+1}^{(j+1)} & \cdots & q_k^{(j+1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_0^{(k)} & \cdots & q_{a-1}^{(k)} & q_{a+1}^{(k)} & \cdots & q_k^{(k)} \end{bmatrix}.$$

By subtracting the first column from each of the other columns in each of the matrices in the sum above, we see

$$0 = (-1)^{j+1}(-1)q_0^{(s)} \det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ q_1^{(1)} & (q_2-q_1)^{(1)} & \cdots & (q_k-q_1)^{(1)} \\ \vdots & \vdots & & \vdots \\ q_1^{(j-1)} & (q_2-q_1)^{(j-1)} & \cdots & (q_k-q_1)^{(j-1)} \\ q_1^{(j+1)} & (q_2-q_1)^{(j+1)} & \cdots & (q_k-q_1)^{(j+1)} \\ \vdots & \vdots & & \vdots \\ q_1^{(k)} & (q_2-q_1)^{(k)} & \cdots & (q_k-q_1)^{(k)} \end{bmatrix}$$

$$+(-1)^{j+1}\sum_{a=1}^{k}(-1)^{a+1}q_a^{(s)}Q_a,$$

where for each $a \in \bar{k}$, Q_a is equal to

$$\det\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ q_0^{(1)} & (q_1-q_0)^{(1)} & \cdots & (q_{a-1}-q_0)^{(1)} & (q_{a+1}-q_0)^{(1)} & \cdots & (q_k-q_0)^{(1)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_0^{(j-1)} & (q_1-q_0)^{(j-1)} & \cdots & (q_{a-1}-q_0)^{(j-1)} & (q_{a+1}-q_0)^{(j-1)} & \cdots & (q_k-q_0)^{(j-1)} \\ q_0^{(j+1)} & (q_1-q_0)^{(j+1)} & \cdots & (q_{a-1}-q_0)^{(j+1)} & (q_{a+1}-q_0)^{(j+1)} & \cdots & (q_k-q_0)^{(j+1)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_0^{(k)} & (q_1-q_0)^{(k)} & \cdots & (q_{a-1}-q_0)^{(k)} & (q_{a+1}-q_0)^{(k)} & \cdots & (q_k-q_0)^{(k)} \end{bmatrix}.$$

By expanding by minors along the first row of each matrix, we have

$$0 = (-1)^{j+1}\sum_{a=0}^{k}(-1)^{a+1}q_a^{(s)}\mathrm{J}_{ja}.$$

This implies that

$$\sum_{a=0}^{k}(-1)^{a+1}q_a^{(s)}\mathrm{J}_{ja} = 0.$$

To prove (iii), assume that $\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_0^{(1)} & q_1^{(1)} & \cdots & q_k^{(1)} \\ \vdots & \vdots & & \vdots \\ q_0^{(k)} & q_1^{(k)} & \cdots & q_k^{(k)} \end{bmatrix} > 0$; it follows that

$\mathrm{Vol}(\Delta(q_0, q_1, \ldots, q_k))$

$$= \frac{1}{k!}\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_0^{(1)} & q_1^{(1)} & \cdots & q_k^{(1)} \\ \vdots & \vdots & & \vdots \\ q_0^{(k)} & q_1^{(k)} & \cdots & q_k^{(k)} \end{bmatrix}$$

$$= \frac{1}{k!}(-1)^{(j+1)+1} q_0^{(j)} \det \begin{bmatrix} 1 & \cdots & 1 \\ q_1^{(1)} & \cdots & q_k^{(1)} \\ \vdots & & \vdots \\ q_1^{(j-1)} & \cdots & q_k^{(j-1)} \\ q_1^{(j+1)} & \cdots & q_k^{(j+1)} \\ \vdots & & \vdots \\ q_1^{(k)} & \cdots & q_k^{(k)} \end{bmatrix}$$

$$+\frac{1}{k!}\sum_{a=1}^{k}(-1)^{(j+1)+(a+1)} q_a^{(j)} \det \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ q_0^{(1)} & \cdots & q_{a-1}^{(1)} & q_{a+1}^{(1)} & \cdots & q_k^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_0^{(j-1)} & \cdots & q_{a-1}^{(j-1)} & q_{a+1}^{(j-1)} & \cdots & q_k^{(j-1)} \\ q_0^{(j+1)} & \cdots & q_{a-1}^{(j+1)} & q_{a+1}^{(j+1)} & \cdots & q_k^{(j+1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ q_0^{(k)} & \cdots & q_{a-1}^{(k)} & q_{a+1}^{(k)} & \cdots & q_k^{(k)} \end{bmatrix}$$

$$= \frac{1}{k!}\left((-1)^{(j+1)+1} q_0^{(j)} \mathrm{J}_{j,0} + \sum_{a=1}^{k}(-1)^{(j+1)+(a+1)} q_a^{(j)} \mathrm{J}_{ja}\right)$$

$$= \frac{(-1)^{j+1}}{k!}\sum_{a=0}^{k}(-1)^{a+1} q_a^{(j)} \mathrm{J}_{ja}.$$

This completes the proof.

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Lemma 4.3 Let $k, n \in Z^+$ be such that $2 \le k \le n$. Let $e_0 = (0, \ldots, 0) \in R^k$, and let $e_1, \ldots, e_k$ be the standard basis for R^k. Let $\sigma = [p_0, p_1, \ldots, p_k]$ be an oriented affine k-simplex in R^n. For each $a \in \{0, 1, \ldots, k\}$, let $\lambda_a = [e_0, e_1, \ldots, \hat{e}_a, \ldots, e_k]$. Let $j \in \bar{k}$, and let $a \in \{0, 1, \ldots, k\}$. Then:

(i) For all $z \in \Delta(k-1)$, $\det[\pi_j \circ \lambda_a'(z)] = \begin{cases} (-1)^{j+1} & \text{if } a = 0, \\ \delta_{ja} & \text{if } a > 0. \end{cases}$

(ii) For all $i_1, \ldots, i_{k-1} \in \bar{n}$, and for all $z \in \Delta(k-1)$,

$$\frac{\partial((\sigma \circ \lambda_a)^{(i_1)}, \ldots, (\sigma \circ \lambda_a)^{(i_{k-1})})}{\partial(x^{(1)}, \ldots, x^{(k-1)})}(z) = \sum_{s=1}^{k} \mathrm{P}_s \det[\pi_s \circ \lambda_a'(z)],$$

where

$$\mathrm{P}_s = \det\begin{bmatrix} (p_1 - p_0)^{(i_1)} & \cdots & (p_{s-1} - p_0)^{(i_1)} & (p_{s+1} - p_0)^{(i_1)} & \cdots & (p_k - p_0)^{(i_1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ (p_1 - p_0)^{(i_{k-1})} & \cdots & (p_{s-1} - p_0)^{(i_{k-1})} & (p_{s+1} - p_0)^{(i_{k-1})} & \cdots & (p_k - p_0)^{(i_{k-1})} \end{bmatrix}.$$

Proof. Part (i) follows easily from the definitions of π_j and λ_a.

To prove (ii), let $i_1, \ldots, i_{k-1} \in \bar{n}$, and let $z \in \Delta(k-1)$.

Case I: $a = 0$. By part (i),

$$\sum_{s=1}^{k} \mathrm{P}_s \det[\pi_s \circ \lambda_0'(z)]$$

$$= \sum_{s=1}^{k} (-1)^{s+1} \mathrm{P}_s$$

$$= \det\begin{bmatrix} 1 & \cdots & 1 \\ (p_1 - p_0)^{(i_1)} & \cdots & (p_k - p_0)^{(i_1)} \\ \vdots & & \vdots \\ (p_1 - p_0)^{(i_{k-1})} & \cdots & (p_k - p_0)^{(i_{k-1})} \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 & \cdots & 0 \\ (p_1-p_0)^{(i_1)} & (p_2-p_1)^{(i_1)} & \cdots & (p_k-p_1)^{(i_1)} \\ \vdots & \vdots & & \vdots \\ (p_1-p_0)^{(i_{k-1})} & (p_2-p_1)^{(i_{k-1})} & \cdots & (p_k-p_1)^{(i_{k-1})} \end{bmatrix}$$

$$= \det\begin{bmatrix} (p_2-p_1)^{(i_1)} & \cdots & (p_k-p_1)^{(i_1)} \\ \vdots & & \vdots \\ (p_2-p_1)^{(i_{k-1})} & \cdots & (p_k-p_1)^{(i_{k-1})} \end{bmatrix}.$$

Note that

$[(\sigma\circ\lambda_0)'(z)]$

$$= [\sigma'(\lambda_0(z))]\times[\lambda_0'(z)]$$

$$= \begin{bmatrix} (p_1-p_0)^{(1)} & \cdots & (p_k-p_0)^{(1)} \\ \vdots & & \vdots \\ (p_1-p_0)^{(n)} & \cdots & (p_k-p_0)^{(n)} \end{bmatrix} \times \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (p_2-p_1)^{(1)} & \cdots & (p_k-p_1)^{(1)} \\ \vdots & & \vdots \\ (p_2-p_1)^{(n)} & \cdots & (p_k-p_1)^{(n)} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} D_1(\sigma\circ\lambda_0)^{(1)}(z) & D_{k-1}(\sigma\circ\lambda_0)^{(1)}(z) \\ \vdots & \vdots \\ D_1(\sigma\circ\lambda_0)^{(n)}(z) & D_{k-1}(\sigma\circ\lambda_0)^{(n)}(z) \end{bmatrix} = \begin{bmatrix} (p_2-p_1)^{(1)} & \cdots & (p_k-p_1)^{(1)} \\ \vdots & & \vdots \\ (p_2-p_1)^{(n)} & \cdots & (p_k-p_1)^{(n)} \end{bmatrix}.$$

This implies that

$$\det\begin{bmatrix} (p_2-p_1)^{(i_1)} & \cdots & (p_k-p_1)^{(i_1)} \\ \vdots & & \vdots \\ (p_2-p_1)^{(i_k)} & \cdots & (p_k-p_1)^{(i_k)} \end{bmatrix} = \frac{\partial((\sigma\circ\lambda_0)^{(i_1)},\dots,(\sigma\circ\lambda_0)^{(i_{k-1})})}{\partial(x^{(1)},\dots,x^{(k-1)})}(z).$$

Case II: $a > 0$. Then by part (i) again and a similar investigation of $[(\sigma\circ\lambda_a)'(z)]$

$$\begin{aligned} \sum_{s=1}^{k} P_s \det[\pi_s\circ\lambda_a'(z)] &= \sum_{s=1}^{k} \delta_{sa} P_s \\ &= P_a \\ &= \frac{\partial((\sigma\circ\lambda_0)^{(i_1)},\dots,(\sigma\circ\lambda_0)^{(i_{k-1})})}{\partial(x^{(1)},\dots,x^{(k-1)})}(z). \end{aligned}$$

The lemma is proved. #

We now have all tools to prove Stokes' Theorem.

Theorem 4.4 (*Stokes' Theorem*) Let $k, n \in Z^+$ be such that $k \leq n$. Let Ω be a nonempty open subset of R^n, and let $\sigma = [p_0, p_1, \ldots, p_k]$ be an oriented affine k-simplex in Ω. If ω is a differentiable $(k-1)$-form on Ω, then

$$\int_\sigma d\omega$$

exists and

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Proof. We first assume that $k = 1$. Note that $\sigma = [p_0, p_1]$, and that $\omega = g$, where $g: \Omega \to R$ is differentiable. We get

$$\begin{aligned}\int_\sigma d\omega &= \int_\sigma \sum_{i=1}^{n} \frac{\partial g}{\partial x^{(i)}}\, dx^{(i)} \\ &= (\mathrm{GR})\int_0^1 \sum_{i=1}^{n} \frac{\partial g}{\partial x^{(i)}}(\sigma(t))\, \sigma_i'(t)\, dt \\ &= (\mathrm{GR})\int_0^1 (g \circ \sigma)'(t)\, dt.\end{aligned}$$

It follows from the fundamental theorem of calculus that $\int_\sigma d\omega$ exists and

$$\begin{aligned}\int_\sigma d\omega &= g \circ \sigma(1) - g \circ \sigma(0) \\ &= g(p_1) - g(p_0) \\ &= \int_{[p_1]} \omega - \int_{[p_0]} \omega \\ &= \int_{\partial\sigma} \omega.\end{aligned}$$

From now on we assume that $k > 1$. To prove the theorem it suffices to prove that if $g: \Omega \to R$ is differentiable and if $(i_1, \ldots, i_{k-1})$ is an ascending $(k-1)$-tuple from the set $\bar{n}$, then

$$\int_\sigma d(g\ dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})})$$

exists and

(1) $$\int_\sigma d(g\ dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})}) = \int_{\partial\sigma} g\ dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})}.$$

Thus, let $g: \Omega \to R$ be differentiable, and let $(i_1, \ldots, i_{k-1})$ be an ascending $(k-1)$-tuple from the set $\bar{n}$. Let $L \in \mathrm{Lin}(R^k, R^n)$ be defined by

$$L = [0, p_1 - p_0, \ldots, p_k - p_0],$$

so $\sigma(x) = p_0 + L(x)$ for all $x \in \Delta(k)$. Put $e_0 = (0, \ldots, 0) \in R^k$, and let $e_1, \ldots, e_k$ be the standard basis for R^k.

We claim that for all $j \in \overline{k}$, $\sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma$ is integrable on $\Delta(k)$ and

$$(-1)^{j+1} \int_{\Delta(k)} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma$$

$$= \int_{\partial\Delta(k)} g \circ \sigma \, dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)}.$$

(Here we are viewing $\Delta(k)$ as the oriented affine k-simplex $[e_0, e_1, \ldots, e_k]$ in R^k.) Let $j \in \overline{k}$ be fixed. Let $\varepsilon > 0$ be given, and let $M \geq \mathrm{OV}(\Delta(k))$. Define a gauge δ on $\Delta(k)$ as follows: Let $p \in \Delta(k)$; then $\sigma(p) \in \Omega$. Since g is differentiable at $\sigma(p)$, there exists a number $r_p > 0$ such that for all $h \in R^n$, if $\sigma(p) + h \in \Omega$ and $|h| < r_p$, then

$$(2) \qquad |g(\sigma(p)+h) - g(\sigma(p)) - g'(\sigma(p))(h)| \leq \frac{\varepsilon |h|}{(\|L\|+1)M}.$$

Since σ is uniformly continuous on $\Delta(k)$, there is a number $\delta_p > 0$ such that for all a, $b \in \Delta(k)$, $|a-b| < \delta_p$ implies $|\sigma(a) - \sigma(b)| < r_p$. Define $\delta(p) = \delta_p$. We now have our $\Delta(k)$-gauge δ.

Let $\tau = \{(t_l, T_l) \mid l \in \overline{m}\}$ be any δ-fine M-bounded k-partition of $\Delta(k)$. Each k-simplex T_l can be regarded as a positively oriented affine k-simplex. For each $a \in \{0, 1, \ldots, k\}$, let

$$\lambda_a = [e_0, e_1, \ldots, \widehat{e_a}, \ldots, e_k].$$

Observe that $\partial\Delta(k) = \sum_{a=0}^{k} (-1)^a \lambda_a$, and $\partial\sigma = \sum_{a=0}^{k} (-1)^a \sigma \circ \lambda_a$. Let $l \in \overline{m}$, and let $a \in \{0, 1, \ldots, k\}$. Let $T_l = [q_0, q_1, \ldots, q_k]$ for some $q_0, q_1, \ldots, q_k \in \Delta(k)$, and let $J_{ja}^{(l)}$ be defined as J_{ja} is defined in Lemma 4.2. Define $E_{la} : \Delta(k-1) \to R$ by

$$E_{la}(x) = g(\sigma(T_l \circ \lambda_a(x))) - g(\sigma(t_l)) - g'(\sigma(t_l))(\sigma(T_l \circ \lambda_a(x)) - \sigma(t_l))$$

for all $x \in \Delta(k-1)$. Since σ, T_l, and λ_a are all affine maps, the map $g \circ \sigma \circ T_l \circ \lambda_a$ is continuous on $\Delta(k-1)$. Note that

(i) E_{la} is continuous on $\Delta(k-1)$,

(ii) $T_l \circ \lambda_a = [q_0, q_1, \ldots, \hat{q_a}, \ldots, q_k]$,

(iii) $\dfrac{\partial((T_l \circ \lambda_a)^{(1)}, \ldots, \widehat{(T_l \circ \lambda_a)^{(J)}}, \ldots, (T_l \circ \lambda_a)^{(k)})}{\partial(x^{(1)}, \ldots, x^{(k-1)})}(z) = \mathrm{J}_{ja}^{(l)}$ for all $z \in \Delta(k-1)$,

and

(iv) the integral

$$\int_{\Delta(k-1)} g \circ \sigma \circ T_l \circ \lambda_a(x)\ \mathrm{J}_{ja}^{(l)}\ dx$$

exists. We obtain

$$\int_{T_l \circ \lambda_a} g \circ \sigma\ dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(J)}} \wedge \ldots \wedge dx^{(k)}$$

$$= \int_{\Delta(k-1)} g \circ \sigma \circ T_l \circ \lambda_a(x)\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= \int_{\Delta(k-1)} [g(\sigma(t_l)) + g'(\sigma(t_l))(\sigma(T_l \circ \lambda_a(x)) - \sigma(t_l)) + E_{la}(x)]\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= \int_{\Delta(k-1)} g(\sigma(t_l))\ \mathrm{J}_{ja}^{(l)}\ dx - \int_{\Delta(k-1)} g'(\sigma(t_l))(\sigma(t_l))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$+ \int_{\Delta(k-1)} g'(\sigma(t_l))(\sigma(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx + \int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= [g(\sigma(t_l)) - g'(\sigma(t_l))(\sigma(t_l))] \frac{\mathrm{J}_{ja}^{(l)}}{(k-1)!}$$

$$+ \int_{\Delta(k-1)} g'(\sigma(t_l))(p_0 + L(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$+ \int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= [g(\sigma(t_l)) - g'(\sigma(t_l))(\sigma(t_l)) + g'(\sigma(t_l))(p_0)] \frac{\mathrm{J}_{ja}^{(l)}}{(k-1)!}$$

$$+ \int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx + \int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx.$$

Let us first investigate

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx.$$

Let $\beta_1, \ldots, \beta_{k-1}$ be the standard basis for R^{k-1}. We have 2 cases:

Case 1: $a = 0$. Let $A_0 \in \mathrm{Lin}(R^{k-1}, R^k)$ be defined by $A_0 = [0, q_2 - q_1, \ldots, q_k - q_1]$, so that $T_l \circ \lambda_0(z) = [q_1, \ldots, q_k](z) = q_1 + A_0(z)$ for all $z \in \Delta(k-1)$. Thus

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_0(x)))\ \mathrm{J}_{j,0}^{(l)}\ dx$$

$$= g'(\sigma(t_l))(L(q_1))\frac{\mathrm{J}_{j,0}^{(l)}}{(k-1)!} + \int_{\Delta(k-1)} g'(\sigma(t_l))(L \circ A_0(x))\ \mathrm{J}_{j,0}^{(l)}\ dx.$$

Look at

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L \circ A_0(x))\ \mathrm{J}_{j,0}^{(l)}\ dx$$

$$= \int_{\Delta(k-1)} \sum_{r=1}^{k-1} x^{(r)} g'(\sigma(t_l))(L \circ A_0(\beta_r))\ \mathrm{J}_{j,0}^{(l)}\ dx$$

$$= \sum_{r=1}^{k-1} g'(\sigma(t_l))(L(q_{r+1} - q_1))\ \mathrm{J}_{j,0}^{(l)} \int_{\Delta(k-1)} x^{(r)}\ dx$$

$$= \sum_{r=1}^{k-1} g'(\sigma(t_l))(L(q_{r+1} - q_1))\ \frac{\mathrm{J}_{j,0}^{(l)}}{k!} \qquad \text{(by Lemma 4.1)}$$

$$= \sum_{r=2}^{k} g'(\sigma(t_l))(L(q_r - q_1))\ \frac{\mathrm{J}_{j,0}^{(l)}}{k!}.$$

Therefore,

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_0(x)))\ \mathrm{J}_{j,0}^{(l)}\ dx$$

$$= g'(\sigma(t_l))(L(q_1))\frac{\mathrm{J}_{j,0}^{(l)}}{(k-1)!} + \sum_{r=2}^{k} g'(\sigma(t_l))(L(q_r - q_1))\ \frac{\mathrm{J}_{j,0}^{(l)}}{k!}$$

$$= k g'(\sigma(t_l))(L(q_1))\frac{\mathrm{J}_{j,0}^{(l)}}{k!} + \sum_{r=2}^{k} g'(\sigma(t_l))(L(q_r))\ \frac{\mathrm{J}_{j,0}^{(l)}}{k!}$$

$$- (k-1) g'(\sigma(t_l))(L(q_1))\frac{\mathrm{J}_{j,0}^{(l)}}{k!}$$

$$= \frac{1}{k!}\sum_{r=1}^{k} g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{j,0}^{(l)}$$

$$= \frac{1}{k!}\sum_{\substack{r=0\\ r\neq a}}^{k} g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{j,0}^{(l)}.$$

Case 2: $a \neq 0$. Let $A_a \in \mathrm{Lin}(R^{k-1}, R^k)$ be defined by

$$A_a = [0, q_1 - q_0, \ldots, \widehat{q_a - q_0}, \ldots, q_k - q_0],$$

so $T_l \circ \lambda_a(z) = [q_0, q_1, \ldots, \hat{q_a}, \ldots, q_k](z) = q_0 + A_a(z)$ for all $z \in \Delta(k-1)$. Thus

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= g'(\sigma(t_l))(L(q_0))\frac{\mathrm{J}_{ja}^{(l)}}{(k-1)!} + \int_{\Delta(k-1)} g'(\sigma(t_l))(L \circ A_a(x))\ \mathrm{J}_{ja}^{(l)}\ dx.$$

Look at

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L \circ A_a(x))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= \int_{\Delta(k-1)} \sum_{r=1}^{k-1} x^{(r)}\, g'(\sigma(t_l))(L \circ A_a(\beta_r))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= \sum_{r=1}^{k-1} g'(\sigma(t_l))(L \circ A_a(\beta_r))\ \mathrm{J}_{ja}^{(l)} \int_{\Delta(k-1)} x^{(r)}\ dx$$

$$= \frac{1}{k!}\left(\sum_{r=1}^{a-1} g'(\sigma(t_l))(L(q_r - q_0))\ \mathrm{J}_{ja}^{(l)} + \sum_{r=a}^{k-1} g'(\sigma(t_l))(L(q_{r+1} - q_0))\ \mathrm{J}_{ja}^{(l)}\right)$$

(by Lemma 4.1)

$$= \frac{1}{k!}\sum_{\substack{r=1\\ r\neq a}}^{k} g'(\sigma(t_l))(L(q_r - q_0))\ \mathrm{J}_{ja}^{(l)}\ .$$

Therefore,

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= \frac{1}{(k-1)!} g'(\sigma(t_l))(L(q_0))\,\mathrm{J}_{ja}^{(l)} + \frac{1}{k!}\sum_{\substack{r=1\\ r\neq a}}^{k} g'(\sigma(t_l))(L(q_r - q_0))\ \mathrm{J}_{ja}^{(l)}$$

$$= \frac{1}{k!}\left(k g'(\sigma(t_l))(L(q_0))\,\mathrm{J}_{ja}^{(l)} + \sum_{\substack{r=1\\ r\neq a}}^{k} g'(\sigma(t_l))(L(q_r - q_0))\ \mathrm{J}_{ja}^{(l)} \right)$$

$$= \frac{1}{k!}\sum_{\substack{r=0\\ r\neq a}}^{k} g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{ja}^{(l)}.$$

This finishes the case $a > 0$.

In either case, we eventually get

$$\int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l \circ \lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx = \frac{1}{k!}\sum_{\substack{r=0\\ r\neq a}}^{k} g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{ja}^{(l)}.$$

We then get

$$\int_{\partial T_l} g\circ\sigma\ dx^{(1)}\wedge\ldots\wedge \widehat{dx^{(l)}}\wedge\ldots\wedge dx^{(k)}$$

$$= \sum_{a=0}^{k}(-1)^a \int_{T_l\circ\lambda_a} g\circ\sigma\ dx^{(1)}\wedge\ldots\wedge \widehat{dx^{(l)}}\wedge\ldots\wedge dx^{(k)}$$

$$= \frac{g(\sigma(t_l)) - g'(\sigma(t_l))(\sigma(t_l)) + g'(\sigma(t_l))(p_0)}{(k-1)!}\sum_{a=0}^{k}(-1)^a \mathrm{J}_{ja}^{(l)}$$

$$+ \sum_{a=0}^{k}(-1)^a \int_{\Delta(k-1)} g'(\sigma(t_l))(L(T_l\circ\lambda_a(x)))\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$+ \sum_{a=0}^{k}(-1)^a \int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx$$

$$= \sum_{a=0}^{k}(-1)^a\left(\frac{1}{k!}\sum_{\substack{r=0\\r\neq a}}^{k} g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{ja}^{(l)}\right) + \sum_{a=0}^{k}(-1)^a\int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx$$

(by Lemma 4.2(i))

$$= \frac{1}{k!}\sum_{a=0}^{k}\ \sum_{\substack{r=0\\r\neq a}}^{k}(-1)^a g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{ja}^{(l)} + \sum_{a=0}^{k}(-1)^a\int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx.$$

Look at

$$\frac{1}{k!}\sum_{a=0}^{k}\ \sum_{\substack{r=0\\r\neq a}}^{k}(-1)^a g'(\sigma(t_l))(L(q_r))\ \mathrm{J}_{ja}^{(l)}$$

$$= \frac{1}{k!}\sum_{a=0}^{k}\ \sum_{\substack{r=0\\r\neq a}}^{k}(-1)^a g'(\sigma(t_l))\circ L(\sum_{s=1}^{k} q_r^{(s)}e_s)\ \mathrm{J}_{ja}^{(l)}$$

$$= \frac{1}{k!}\sum_{a=0}^{k}\ \sum_{\substack{r=0\\r\neq a}}^{k}\ \sum_{s=1}^{k}(-1)^a q_r^{(s)} g'(\sigma(t_l))\circ L(e_s)\ \mathrm{J}_{ja}^{(l)}$$

$$= \frac{1}{k!}\sum_{a=0}^{k}\ \sum_{\substack{r=0\\r\neq a}}^{k}\ \sum_{s=1}^{k}\ \sum_{w=1}^{n}(-1)^a q_r^{(s)}(p_s-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}(\sigma(t_l))\ \mathrm{J}_{ja}^{(l)}$$

$$= \frac{1}{k!}\sum_{w=1}^{n}\ \sum_{s=1}^{k}(p_s-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}(\sigma(t_l))\left(\sum_{a=0}^{k}\ \sum_{\substack{r=0\\r\neq a}}^{k}(-1)^a q_r^{(s)}\ \mathrm{J}_{ja}^{(l)}\right)$$

$$= \frac{1}{k!}\sum_{w=1}^{n}\ \sum_{s=1}^{k}(p_s-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}(\sigma(t_l))\left(\sum_{r=0}^{k}\ \sum_{\substack{a=0\\a\neq r}}^{k}(-1)^a q_r^{(s)}\ \mathrm{J}_{ja}^{(l)}\right)$$

$$= \frac{1}{k!}\sum_{w=1}^{n}\ \sum_{s=1}^{k}(p_s-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}\circ\sigma(t_l)\left(\sum_{r=0}^{k} q_r^{(s)}\left(\sum_{a=0}^{k}(-1)^a\mathrm{J}_{ja}^{(l)} - (-1)^r\mathrm{J}_{jr}^{(l)}\right)\right)$$

$$= \frac{1}{k!}\sum_{w=1}^{n}\sum_{s=1}^{k}(p_s-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}\circ\sigma(t_l)\left(\sum_{r=0}^{k}(-1)^{r+1}q_r^{(s)}\ \mathrm{J}_{jr}^{(l)}\right)$$

$$= \frac{1}{k!}\sum_{w=1}^{n}(p_j-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}\circ\sigma(t_l)\left(\sum_{r=0}^{k}(-1)^{r+1}q_r^{(j)}\ \mathrm{J}_{jr}^{(l)}\right)$$

$$= (-1)^{j+1}\sum_{w=1}^{n}(p_j-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}\circ\sigma(t_l)\ \mathrm{Vol}(T_l).$$

(Here we used Lemma 4.2 (i), (ii), and (iii) in the last four lines.)

Hence

$$\int_{\partial T_l} g\circ\sigma\ dx^{(1)}\wedge\ldots\wedge\widehat{dx}^{(j)}\wedge\ldots\wedge dx^{(k)}$$

$$= (-1)^{j+1}\sum_{w=1}^{n}(p_j-p_0)^{(w)}\frac{\partial g}{\partial x^{(w)}}\circ\sigma(t_l)\ \mathrm{Vol}(T_l)$$

$$+ \sum_{a=0}^{k}(-1)^{a}\int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx.$$

We now investigate $\left|\int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx\right|$ for all $a\in\{0,1,\ldots,k\}$. Let $a\in\{0,1,\ldots,k\}$ be fixed; then

$$\left|\int_{\Delta(k-1)} E_{la}(x)\ \mathrm{J}_{ja}^{(l)}\ dx\right|$$

$$\leq \left|\mathrm{J}_{ja}^{(l)}\right|\int_{\Delta(k-1)}\left|E_{la}(x)\right|dx$$

$$= \left|\mathrm{J}_{ja}^{(l)}\right|\int_{\Delta(k-1)}\Big|g(\sigma(T_l\circ\lambda_a(x)))-g(\sigma(t_l)) - g'(\sigma(t_l))(\sigma(T_l\circ\lambda_a(x))-\sigma(t_l))\Big|\ dx$$

$$\leq \left|\mathrm{J}_{ja}^{(l)}\right|\int_{\Delta(k-1)}\frac{\varepsilon\left|\sigma(T_l\circ\lambda_a(x))-\sigma(t_l)\right|}{(\|L\|+1)M}\ dx \qquad \text{(by inequality (2))}$$

$$\leq \frac{\varepsilon\,\mathrm{diam}(\sigma(T_l))\left|\mathrm{J}_{ja}^{(l)}\right|}{(\|L\|+1)M(k-1)!}.$$

Note that for all $y, z \in T_l$,

$$\begin{aligned} |\sigma(y) - \sigma(z)| &= |(p_0 + L(y)) - (p_0 + L(z))| \\ &= |L(y-z)| \\ &\le \|L\| |y - z| \\ &\le \|L\| \operatorname{diam}(T_l). \end{aligned}$$

Thus, $\operatorname{diam}(\sigma(T_l)) \le \|L\| \operatorname{diam}(T_l)$. Hence

$$\begin{aligned} \left| \int_{\Delta(k-1)} E_{la}(x)\ J^{(l)}_{ja}\ dx \right| &\le \frac{\varepsilon \|L\| \operatorname{diam}(T_l) \left| J^{(l)}_{ja} \right|}{(\|L\|+1) M (k-1)!} \\ &\le \frac{\varepsilon \operatorname{diam}(T_l) \left| J^{(l)}_{ja} \right|}{M (k-1)!} \\ &\le \frac{\varepsilon}{M} \cdot \frac{\operatorname{diam}(T_l)}{(k-1)!} \left(\sum_{r=1}^{k} \left| J^{(l)}_{ra} \right|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\int_{\partial\Delta(k)} g \circ \sigma\ dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)}$$

$$= \sum_{l=1}^{m} \int_{\partial T_l} g \circ \sigma\ dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)}$$

$$= \sum_{l=1}^{m} \left((-1)^{j+1} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(t_l)\ \operatorname{Vol}(T_l) \right)$$

$$+ \sum_{l=1}^{m} \sum_{a=0}^{k} (-1)^a \int_{\Delta(k-1)} E_{la}(x)\ J^{(l)}_{ja}\ dx,$$

it follows that

$$\left| \int_{\partial\Delta(k)} g \circ \sigma\ dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)} - \mathrm{R}\left((-1)^{j+1} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma, \tau \right) \right|$$

$$= \left| \int_{\partial\Delta(k)} g \circ \sigma \; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)} - \sum_{l=1}^{m} \left((-1)^{j+1} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(t_l) \; \mathrm{Vol}(T_l) \right) \right|$$

$$= \left| \sum_{l=1}^{m} \sum_{a=0}^{k} (-1)^a \int_{\Delta(k-1)} E_{la}(x) \; \mathrm{J}_{ja}^{(l)} \; dx \right|$$

$$\leq \sum_{l=1}^{m} \sum_{a=0}^{k} \left| \int_{\Delta(k-1)} E_{la}(x) \; \mathrm{J}_{ja}^{(l)} \; dx \right|$$

$$\leq \sum_{l=1}^{m} \left(\sum_{a=0}^{k} \frac{\varepsilon}{M} \cdot \frac{\mathrm{diam}(T_l)}{(k-1)!} \left(\sum_{r=1}^{k} \left| \mathrm{J}_{ra}^{(l)} \right|^2 \right)^{\frac{1}{2}} \right)$$

$$= \frac{\varepsilon}{M} \sum_{l=1}^{m} \frac{\mathrm{diam}(T_l)}{(k-1)!} \left(\sum_{a=0}^{k} \left(\sum_{r=1}^{k} \left| \mathrm{J}_{ra}^{(l)} \right|^2 \right)^{\frac{1}{2}} \right)$$

$$= \frac{\varepsilon}{M} \sum_{l=1}^{m} \frac{\mathrm{diam}(T_l)}{(k-1)!} \left(\left(\sum_{r=1}^{k} \left| \mathrm{J}_{r,0}^{(l)} \right|^2 \right)^{\frac{1}{2}} + \sum_{a=1}^{k} \left(\sum_{r=1}^{k} \left| \mathrm{J}_{ra}^{(l)} \right|^2 \right)^{\frac{1}{2}} \right)$$

$$= \frac{\varepsilon}{M} \sum_{l=1}^{m} \frac{\mathrm{diam}(T_l)}{(k-1)!} \left(S_k((q_1, \ldots, q_k)) + \sum_{a=1}^{k} S_k((q_0, q_1, \ldots, \widehat{q_a}, \ldots, q_k)) \right)$$

$$= \frac{\varepsilon}{M} \sum_{l=1}^{m} \frac{\mathrm{diam}(T_l)}{(k-1)!} \left(\sum_{a=0}^{k} S_k((q_0, q_1, \ldots, \widehat{q_a}, \ldots, q_k)) \right)$$

$$= \frac{\varepsilon}{M} \sum_{l=1}^{m} \mathrm{OV}(T_l)$$

$$\leq \varepsilon .$$

This shows that $(-1)^{j+1} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma$ is integrable on $\Delta(k)$ and

$$\int_{\Delta(k)} (-1)^{j+1} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(z)\; dz$$

$$= \int_{\partial\Delta(k)} g \circ \sigma \; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)};$$

therefore, $\sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma$ is integrable on $\Delta(k)$ and

$$(-1)^{j+1} \int_{\Delta(k)} \sum_{w=1}^{n} (p_j - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(z)\; dz$$

$$= \int_{\partial\Delta(k)} g \circ \sigma \; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(j)}} \wedge \ldots \wedge dx^{(k)}.$$

The claim is proved.

It follows from Lemma 4.3(ii) that for all $r \in \{0, 1, \ldots, k\}$, and all $z \in \Delta(k-1)$,

$$\frac{\partial((\sigma \circ \lambda_r)^{(i_1)}, \ldots, (\sigma \circ \lambda_r)^{(i_{k-1})})}{\partial(x^{(1)}, \ldots, x^{(k-1)})}(z) = \sum_{s=1}^{k} \mathrm{P}_s \det[\, \pi_s \circ \lambda_r{}'(z)],$$

where

$$\mathrm{P}_s = \det\begin{bmatrix} (p_1 - p_0)^{(i_1)} & \cdots & (p_{s-1} - p_0)^{(i_1)} & (p_{s+1} - p_0)^{(i_1)} & \cdots & (p_k - p_0)^{(i_1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ (p_1 - p_0)^{(i_{k-1})} & \cdots & (p_{s-1} - p_0)^{(i_{k-1})} & (p_{s+1} - p_0)^{(i_{k-1})} & \cdots & (p_k - p_0)^{(i_{k-1})} \end{bmatrix}.$$

Note that for all $r \in \{0, 1, \ldots, k\}$,

$$\int_{\sigma \circ \lambda_r} g \; dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})} = \int_{\Delta(k-1)} g \circ \sigma \circ \lambda_r(z) \frac{\partial((\sigma \circ \lambda_r)^{(i_1)}, \ldots, (\sigma \circ \lambda_r)^{(i_{k-1})})}{\partial(x^{(1)}, \ldots, x^{(k-1)})}(z)\; dz$$

$$= \int_{\Delta(k-1)} g \circ \sigma \circ \lambda_r(z) \left(\sum_{s=1}^{k} \mathrm{P}_s \det[\, \pi_s \circ \lambda_r{}'(z)] \right) dz$$

$$= \sum_{s=1}^{k} \mathrm{P}_s \int_{\Delta(k-1)} g \circ \sigma \circ \lambda_r(z) \det[\, \pi_s \circ \lambda_r{}'(z)]\; dz$$

$$= \sum_{s=1}^{k} P_s \int_{\lambda_r} g \circ \sigma \;\; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(s)}} \wedge \ldots \wedge dx^{(k)}.$$

We conclude that

$$\int_{\partial\sigma} g \;\; dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})}$$

$$= \sum_{r=0}^{k} (-1)^r \int_{\sigma\circ\lambda_r} g \;\; dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})}$$

$$= \sum_{r=0}^{k} (-1)^r \left(\sum_{s=1}^{k} P_s \int_{\lambda_r} g \circ \sigma \;\; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(s)}} \wedge \ldots \wedge dx^{(k)} \right)$$

$$= \sum_{s=1}^{k} P_s \left(\sum_{r=0}^{k} (-1)^r \int_{\lambda_r} g \circ \sigma \;\; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(s)}} \wedge \ldots \wedge dx^{(k)} \right)$$

$$= \sum_{s=1}^{k} P_s \int_{\partial\Delta(k)} g \circ \sigma \;\; dx^{(1)} \wedge \ldots \wedge \widehat{dx^{(s)}} \wedge \ldots \wedge dx^{(k)}$$

$$= \sum_{s=1}^{k} P_s \left((-1)^{s+1} \int_{\Delta(k)} \sum_{w=1}^{n} (p_s - p_0)^{(w)} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(z) \;\; dz \right)$$

$$= \int_{\Delta(k)} \sum_{w=1}^{n} \left(\sum_{s=1}^{k} (-1)^{s+1} (p_s - p_0)^{(w)} P_s \right) \frac{\partial g}{\partial x^{(w)}} \circ \sigma(z) \;\; dz$$

$$= \int_{\Delta(k)} \sum_{w=1}^{n} \det \begin{bmatrix} (p_1 - p_0)^{(w)} & \cdots & (p_k - p_0)^{(w)} \\ (p_1 - p_0)^{(i_1)} & \cdots & (p_k - p_0)^{(i_1)} \\ \vdots & & \vdots \\ (p_1 - p_0)^{(i_{k-1})} & \cdots & (p_k - p_0)^{(i_{k-1})} \end{bmatrix} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(z) \;\; dz$$

$$= \int_{\Delta(k)} \sum_{w=1}^{n} \frac{\partial g}{\partial x^{(w)}} \circ \sigma(z) \;\; \frac{\partial(\sigma^{(w)}, \sigma^{(i_1)}, \ldots, \sigma^{(i_{k-1})})}{\partial(x^{(i_1)}, \ldots, x^{(i_k)})}(z) \;\; dz$$

$$= \int_{\sigma} \sum_{w=1}^{n} \frac{\partial g}{\partial x^{(w)}} \;\; dx^{(w)} \wedge dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})}$$

$$= \int_{\sigma} d(g \;\; dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})}).$$

This shows that $\int_{\sigma} \quad d(g\ dx^{(i_1)} \wedge \ldots \wedge dx^{(i_{k-1})})$ exists and equation (1) holds, as required.

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