CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions, and results proved by other authors which will be used in this thesis.

1.1 Notations

The general notational conventions are as follows:

Z = the set of all integers;

Z^+ = the set of all positive integers;

R = the set of all real numbers;

R^+ = the set of all positive real numbers;

R^n = n-dimensional Euclidean space, where $n \in Z^+$;

$\bar{n}$ = $\{1, 2, \ldots, n\}$, where $n \in Z^+$;

$x^{(k)}$ = the k^{th} coordinate of $x \in R^n$;

$|x|$ = the Euclidean norm of $x \in R^n$;

$\mathrm{B}(a, r)$ = $\{x \in R^n \mid |x - a| < r\}$, where $a \in R^n$ and $r \in R^+$;

$\pi^{(k)}$ = the k^{th} projection map defined by

$$\pi^{(k)}(x^{(1)}, \ldots, x^{(k)}, \ldots, x^{(n)}) = x^{(k)};$$

π_k = the function mapping R^n into R^{n-1} such that

$$\pi_k(x^{(1)}, \ldots, x^{(k)}, \ldots, x^{(n)}) = (x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}, \ldots, x^{(n)});$$

$\sup(A)$ = the supremum of A, where $A \subseteq R$;

$\bar{A}$ = the closure of A, where $A \subseteq R^n$;

$\mathrm{Int}(A)$ = the set of all interior points of A, where $A \subseteq R^n$;

$\mathrm{Bd}(A)$ = the set of all boundary points of A, where $A \subseteq R^n$;

$$J_\varphi(z) = \frac{\partial(\varphi^{(1)},\ldots,\varphi^{(n)})}{\partial(x^{(1)},\ldots,x^{(n)})}(z) = \text{the Jacobian determinant of } \varphi: R^n \to R^n$$

at the point $z \in R^n$;

$\text{Lin}(R^k, R^n)$ = the set of all linear transformations of R^k into R^n ; and

$\|L\|$ = the norm of $L \in \text{Lin}(R^k, R^n)$ defined by

$$\|L\| = \sup\{|L(x)| \mid x \in R^k \text{ and } |x| = 1\}.$$

1.2 Riemann Integral

Although the purpose of this thesis is to introduce a new multi-dimensional version of the generalized Riemann integral, we still need the ordinary Riemann integral, to evaluate a certain specific integral, and the one-dimensional generalized Riemann integral, to handle the dimension-one case in Stokes' Theorem. We also want to show that the new integral reduces to the usual generalized Riemann integral in one dimension, so it is necessary to at least acquaint the reader with the latter's definition. This section and the next are devoted to sketching what we need to know about the ordinary Riemann integral and the one-dimensional generalized Riemann integral for this thesis.

For a complete account of the basics of the Riemann integral, the reader is advised to consult Bartle's book [1].

Let $n \in Z^+$. A *cell* in R is a set having one of the four forms: (a, b), $[a, b]$, $[a, b)$, or $(a, b]$, where a and b are real numbers such that $a \le b$. The numbers a and b are called the *end points* of the cell. A cell J in R^n is a Cartesian product $J = \times_{i=1}^{n} J_i$ of n cells in R. The cell is said to be *closed* (respectively, *open*) iff each of the cells J_i is closed (respectively, open) in R. If the cells J_i have end points $a_i \le b_i$ $(i = 1, \ldots, n)$, we define the *content* of $J = \times_{i=1}^{n} J_i$ to be the product $\prod_{i=1}^{n}(b_i - a_i)$, and denote it by

$c(J)$.

A subset D of R^n has *content zero* iff for each $\varepsilon > 0$ there exists a finite set $\{J_1, \dots, J_m\}$ of cells in R^n such that $D \subseteq \bigcup_{i=1}^{m} J_i$, and $\sum_{i=1}^{m} c(J_i) < \varepsilon$. A bounded subset A of R^n is said to have *content* iff its boundary Bd(A) has content zero.

Proposition 1.2.1 (see [1], p.414)

(i) Any subset of a set with content zero has content zero.

(ii) The union of a finite number of sets with content zero has content zero.

Let $I = \times_{i=1}^{n} [a_i, b_i]$ be a closed cell in R^n. For each $i \in \bar{n}$, let P_i be a partition of $[a_i, b_i]$ into a finite number of closed cells in R. The partition of I induced by $P_1, \dots, P_n$ is the set $\{ \times_{i=1}^{n} J_i \mid J_i \in P_i \text{ for all } i \in \bar{n}\}$. By a *partition* of I we mean a partition induced by $P_1, \dots, P_n$, where each P_i is a partition of $[a_i, b_i]$ into a finite number of closed cells in R. If P and P' are partitions of I, we say that P is a *refinement* of P' iff each cell in P is a subset of some cell in P'.

We shall now define the integral of a bounded real function f defined on a closed cell $I \subseteq R^n$.

A *Riemann sum* corresponding to a partition $P = \{J_1, \dots, J_m\}$ of I, denoted by $S(f, P)$, is given by

$$S(f, P) = \sum_{i=1}^{m} f(x_i)c(J_i),$$

where x_i is any point in J_i, $i = 1, \dots, m$. A real number L is defined to be a *Riemann integral of f over I* iff for every $\varepsilon > 0$ there is a partition P_ε of I such that if P is any refinement of P_ε and $S(f, P)$ is any Riemann sum corresponding to P, then

$$| S(f, P) - L | < \varepsilon.$$

In case this integral exists we say that f is *Riemann integrable on I* (or briefly, f is *(R)-integrable on I*).

Note if f is (R)-integrable on I then f has a unique integral L on I ; we shall write $(R)\int_I f$, or sometimes $(R)\int_I f(x)\,dx$, for L.

Let A be a bounded subset of R^n, and let $f: A \to R$ be a bounded function. Since A is bounded, there exists a closed cell I in R^n such that $A \subseteq I$. We define $f_I : I \to R$ by

$$f_I(x) = \begin{cases} f(x) & \text{for } x \in A, \\ 0 & \text{for } x \in I \setminus A. \end{cases}$$

Note that if J is another closed cell containing A, then $(R)\int_I f_I$ exists iff $(R)\int_J f_J$ exists, in which case these integrals are equal. Because of this we shall say that f *is (R)-integrable on A* iff there is some closed cell I containing A such that f_I is (R)-integrable on I, and in that case we define $(R)\int_A f = (R)\int_I f_I$, since the right hand side depends only on f and A.

Let B be a nonempty subset of A; we shall say that $f: A \to R$ is (R)-integrable on B iff the restriction $f\big|_B$ of f to B is (R)-integrable on B. We shall write $(R)\int_B f$ for the (R)-integral of f on B.

Theorem 1.2.2 **([1], Theorem 43.5)** Suppose A is a bounded subset of R^n. Let f, g : $A \to R$ be (R)-integrable on A, and let $\alpha, \beta \in R$. Then the function $\alpha f + \beta g$ is (R)-integrable on A and

$$(R)\int_A (\alpha f + \beta g) = \alpha(R)\int_A f + \beta(R)\int_A g.$$

Theorem 1.2.3 **([1], Theorem 43.6)** Suppose A is a bounded subset of R^n. If f: $A \to R$ is (R)-integrable on A and if $f(x) \geq 0$ for all $x \in A$, then

$$(\mathrm{R})\int_A f \geq 0.$$

Corollary 1.2.4 Suppose A is a bounded subset of R^n. If $f, g : A \to R$ are (R)-integrable on A, and if $f(x) \geq g(x)$ for all $x \in A$, then

$$(\mathrm{R})\int_A f \geq (\mathrm{R})\int_A g.$$

Theorem 1.2.5 (*Integrability Theorem*) ([1], Theorem 43.9) Let I be a closed cell in R^n, and let $f : I \to R$ be bounded. If there exists a subset E of I with content zero such that f is continuous on $I \setminus E$, then f is (R)-integrable on I.

Theorem 1.2.6 ([1], Theorem 44.8) Let $A \subseteq R^n$ have content, and let $f : A \to R$ be bounded and continuous on A. Then f is (R)-integrable on A.

Theorem 1.2.7 ([1], Theorem 44.9)

(i) Let A_1 and A_2 be subsets of R^n having content. Suppose that $A_1 \cap A_2$ has content zero. If $A = A_1 \cup A_2$, and if $f : A \to R$ is (R)-integrable on A_1 and A_2, then f is (R)-integrable on A, and

$$(*) \qquad (\mathrm{R})\int_A f = (\mathrm{R})\int_{A_1} f + (\mathrm{R})\int_{A_2} f.$$

(ii) Let A, A_1, A_2 be subsets of R^n having content. Suppose that $A = A_1 \cup A_2$ and that $A_1 \cap A_2$ has content zero. If f is (R)-integrable on A and if the restrictions of f to A_1 and A_2 are (R)-integrable, then $(*)$ holds.

Theorem 1.2.8 ([1], Theorem 44.14) Let $A \subseteq R^2$ be given by

$$A = \{(x, y) \mid c \leq y \leq d \text{ and } \alpha(y) \leq x \leq \beta(y)\},$$

where α and β are continuous functions on $[c, d]$ with values in $[a, b]$. If $f : A \to R$ is continuous on A, then f is integrable on A and

$$\int_A f = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} f(x, y) \, dx \right) dy.$$

Note that this theorem can be generalized in the obvious way to R^n when $n > 2$.

Theorem 1.2.9 ([1], Theorem 45.2) Let Ω be an open subset of R^n, and let $\varphi:\Omega \to R^n$ be a C^1-mapping. If $A \subseteq \Omega$ has content zero and if $\overline{A} \subseteq \Omega$, then $\varphi(A)$ has content zero.

Theorem 1.2.10 ([1], Theorem 45.4) Let $\Omega \subseteq R^n$ be open, and let φ: $\Omega \to R^n$ be a C^1-mapping. Suppose that A has content, $\overline{A} \subseteq \Omega$, and $J_\varphi(x) \neq 0$ for all $x \in \text{Int}(A)$. Then $\varphi(A)$ has content.

Theorem 1.2.11 (*Change of Variables Theorem*) ([1], Theorem 45.9) Let $\Omega \subseteq R^n$ be open, and let φ: $\Omega \to R^n$ be a 1-1 C^1-mapping such that $J_\varphi(x) \neq 0$ for all $x \in \Omega$. Suppose that A has content with $\overline{A} \subseteq \Omega$ and that f : $\varphi(A) \to R$ is bounded and continuous. Then

$$(R)\int_{\varphi(A)} f = (R)\int_A (f \circ \varphi)\left|J_\varphi\right|.$$

1.3 The Generalized Riemann Integral

The generalized Riemann integral or *gauge integral* is similar to the ordinary Riemann integral, yet it integrates a much wider class of functions and has much nicer properties, as described in Chapter 13 of [2]. We shall not say much about this topic. The reader is advised to consult [2] for more details.

Let a and b be real numbers such that $a \leq b$, and let f : $[a, b] \to R$ be a function.

A function $\delta : [a, b] \to R^+$ will be called a *gauge* on $[a, b]$. A *tagged partition* τ of the closed interval $[a, b]$ is a finite collection of ordered pairs $\{(t_i , [x_{i-1}, x_i]) \mid i \in \overline{m}\}$, where $a = x_0 \leq x_1 \leq \ldots \leq x_m = b$ and each t_i is in $[x_{i-1}, x_i]$. Given a gauge δ on $[a, b]$, a tagged partition τ is said to be *δ-fine* iff $[x_{i-1}, x_i] \subseteq B(t_i , \delta(t_i))$ for all

$i \in \overline{m}$. The following lemma ensures the existence of a δ-fine tagged partition of $[a, b]$.

Lemma 1.3.1 (*Cousin's Lemma*) ([2], Theorem 13.5) For every gauge δ on $[a, b]$, there exists a δ-fine tagged partition of $[a, b]$.

Given a tagged partition $\tau = \{(t_i\ , [x_{i-1}, x_i\ ]) \mid i \in \overline{m}\}$, we call the sum

$$\sum_{i=1}^{m} f(t_i)(x_i - x_{i-1})$$

the *Riemann sum* of f given by τ, and denote it by $R(f, \tau)$. We call a real number L a *generalized Riemann integral of* f *over* $[a, b]$ iff for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$|\mathrm{R}(f, \tau) - L| < \varepsilon$$

for every δ-fine tagged partition τ of $[a, b]$.

Note that a function f has at most one integral over an interval $[a, b]$. The function f is said to be *generalized Riemann integrable on* $[a, b]$ (or briefly, *(GR)-integrable on* $[a, b]$) whenever a generalized Riemann integral on $[a, b]$ of f exists. This integral will be denoted by

$$(\mathrm{GR})\int_a^b f(x)\ dx,$$

or frequently just by

$$(\mathrm{GR})\int_a^b f.$$

It is easy to see that $f: [a, b] \to R$ is (R)-integrable iff it is (GR)-integrable with respect to a constant gauge. In particular, every (R)-integrable function is (GR)-integrable and these integrals are equal.

The following example shows that there is a function that is (GR)-integrable, but not (R)-integrable.

Example 1.3.2 ([2], Example 13.8) Let $f : [a, b] \to R$ have constant value c except possibly at a countable number of points. Then f is (GR)-integrable over $[a, b]$ with integral

$$(\mathrm{GR})\int_a^b f = c(b-a).$$

The above example is applicable to the function $f : [0,1] \to R$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases}$$

and so provides an example of a function that is (GR)-integrable, but not (R)-integrable.

One very nice property of the generalized Riemann integral is the fundamental theorem of calculus. To prove this we need the following lemma:

Lemma 1.3.3 (*Straddle Lemma*) ([2], Lemma 13.9) Let $f : [a, b] \to R$ be differentiable at $z \in [a, b]$. Then for each $\varepsilon > 0$, there is $\delta_z > 0$ such that

$$|f(v) - f(u) - f'(z)(v-u)| \le \varepsilon(v-u)$$

whenever $u \le z \le v$ and $[u, v] \subseteq [a, b] \cap (z - \delta_z, z+\delta_z)$.

Proof. Since f is differentiable at z, there is $\delta_z > 0$ such that

$$\left|\frac{f(x) - f(z)}{x - z} - f'(z)\right| < \varepsilon$$

whenever $x \in [a, b]$ and $0 < |x - z| < \delta_z$. Let u, v be such that $u \le z \le v$ and $[u, v] \subseteq [a, b] \cap (z - \delta_z, z+\delta_z)$. If $z = u$ or $z = v$, the conclusion of the lemma is immediate; so suppose $u < z < v$. Then

$$\begin{aligned} |f(v) - f(u) - f'(z)(v-u)| &\le |f(v) - f(z) - f'(z)(v-z)| + |f(z) - f(u) - f'(z)(z-u)| \\ &\le \varepsilon(v-z) + \varepsilon(z-u) \\ &= \varepsilon(v-u). \end{aligned}$$

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Theorem 1.3.4 (*The Fundamental Theorem of Calculus*) ([2], Lemma 13.10) If $f : [a, b] \to R$ is differentiable on $[a, b]$, then f' is integrable on $[a, b]$ and

$$(\mathrm{GR})\int_a^b f' = f(b) - f(a).$$

Proof. Let $\varepsilon > 0$ be given. Define a gauge δ on $[a, b]$ as follows: For each $z \in [a, b]$, define $\delta(z) = \delta_z$ which is given by the Straddle Lemma. Let $\tau = \{(t_i , [x_{i-1}, x_i]) \mid i \in \overline{m}\}$ be any δ-fine tagged partition of $[a, b]$. Note that

$$f(b) - f(a) = \sum_{i=1}^{m} (f(x_i) - f(x_{i-1})),$$

so

$$\begin{aligned}
|\mathrm{R}(f', \tau) - (f(b) - f(a))| &= \left| \sum_{i=1}^{m} \{ f'(t_i)(x_i - x_{i-1}) - (f(x_i) - f(x_{i-1}))\} \right| \\
&\le \sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) - f'(t_i)(x_i - x_{i-1}) \right| \\
&\le \sum_{i=1}^{m} \varepsilon(x_i - x_{i-1}) \\
&= \varepsilon(b - a),
\end{aligned}$$

by the Straddle Lemma. This shows that f' is (GR)-integrable on $[a, b]$ and

$$(\mathrm{GR})\int_a^b f' = f(b) - f(a).$$

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