CHAPTER IV

GENERALIZED REDUCED GRADIENT METHOD

4.1 The Generalized Reduced Gradient Method

In this section the main concept of the GRG method will be explained for use as reference in this work.

The general nonlinear programming problem may be stated as follows

Minimize $f(x)$

subject to $L_i \leq g_i(x) \leq U_i$ $i = 1, 2, \ldots, m$ (4.1-1)

and $L_{m+i} \leq x_i \leq U_{m+i}$ $i = 1, 2, \ldots, n$ (4.1-2)

Here $f(x)$ is the objective function

$g_i(X)$ are the constraints

X is an n - component vector whose components are X_i

L_i and U_i are the lower bounds and upper bounds, respectively.

A problem that is to be solved with Generalized Reduced Gradient (GRG) method is put in the following form :

Minimize $f(X)$

subject to $h_i(X) = 0$ $i = 1, \ldots. m$ (4.1-3)

$L_j \leq X_j \leq U_j$ $j = 1, \ldots.. m$ (4.1-4)

If there are inequality constraints $g_k(X)$, they could be written in the form

$$h_k(X) = g_k(X) - v_k^2 = 0; \ -\infty \leq v_k \leq \infty \ldots.. (4.1\text{-}5)$$

4.2 What is the Reduced Gradient ?

Consider the following two-variable problem

Min $f(X_1, X_2)$

subject to $h(X_1, X_2) = 0$

First differentiate both the objective function and constraint to obtain

$$df(X) = \frac{\partial f(x)}{\partial X_1} \cdot dX_1 + \frac{\partial f(X)}{\partial X_2} \cdot dX_2 \qquad \ldots(4.2\text{-}1)$$

$$dh(X) = \frac{\partial h(X)}{\partial X_1} \cdot dX_1 + \frac{\partial h(X)}{\partial X_2} \cdot dX_2 = 0 \qquad \ldots(4.2\text{-}2)$$

Since $dh(X) = 0$, eqn. (4.2-2) may be solved for dX_2 to get

$$dX_2 = -\frac{\partial h(X)/\partial X_1}{\partial h(X)/\partial X_2} \cdot dX_1 \qquad \ldots.(4.2\text{-}3)$$

Next substitute dX_2 in (4.2-1)

$$df(X) = \left(\frac{\partial f(X)}{\partial X_1} - \frac{\partial f(X)}{\partial X_2} \cdot \frac{\partial h(X)/\partial X_1}{\partial h(X)/\partial X_2} \right) dX_1$$

Thus we get the reduced gradient, $\frac{df(X)}{dX_1}$, as

$$\frac{df(X)}{dX_1} = \frac{\partial f(X)}{\partial X_1} - \frac{\partial f(X)}{\partial X_2} \left[\frac{\partial h(X)}{\partial X_2} \right]^{-1} \frac{\partial h(X)}{\partial X_1} \qquad \ldots.(4.2\text{-}4)$$

The necessary condition for $f(X)$ to be a minimum is $\frac{df(X)}{dX_1} = 0$

in other words, the miminizationproblem is changed to the finding of the variables X_1 and X_2 that make thereduced gradient equal zero, and the values X_1^* and X_2^* thus obtained are the optimum solution.

4.3 General Form of the Reduced Gradient

Let X_K represent the independent variables in a vector X (K = 1,2,...,)

X_I represent the dependent variables in the vector X (I = 1,2,...,)

If equation (4.2-1) is rewritten for the general case in the form of vectors and matrices, we get

$$df(X) = \nabla_{X_K} f\, dX_K + \nabla_{X_I} f\, dX_I \qquad \text{....(4.3-1)}$$

So, the reduced gradient can be obtained from

$$\frac{df(X)}{dX_K} = \nabla^T_{X_K} f + \nabla^T_{X_I} f \frac{dX_I}{dX_K} \qquad \text{..(4.3-2)}$$

The question is how to find $\frac{dX_I}{dX_K}$

From eqn. (4.2-2)

$$dh_i(X) = \nabla^T_{X_K} h_i(X)\, dX_K + \nabla^T_{X_I} h_i(X)\, dX_I = 0 \qquad \text{....(4.3-3)}$$

and eqn (4.3-3) can be rewritten as

$$\frac{dh}{dX_K} = \frac{\partial h}{\partial X_K} + \left(\frac{\partial h}{\partial X_I}\right)\left(\frac{dX_I}{dX_K}\right) = 0 \qquad \text{....(4.3-4)}$$

So

$$\frac{dX_I}{dX_K} = -\left(\frac{\partial h}{\partial X_I}\right)^{-1} \frac{\partial h}{\partial X_K} \qquad \text{....(4.3-5)}$$

Substitute eqn. (4.3-5) into eqn. (4.3-2) we obtain the generalized reduced gradient as

$$\frac{df(X)}{dX_K} = \nabla^T_{X_K} f - \nabla^T_{X_I} f \left(\frac{\partial h}{\partial X_I}\right)^{-1} \left(\frac{\partial h}{\partial X_K}\right) \qquad \text{..(4.3-6)}$$

The generalized reduced gradient $\frac{df}{dX_K}$ consists of kelements of the k independent variables at that time. This generalized reduced gradient help difine the direction that the independent variables should move in order to lower the objective function value while satisfying all the active constraints.

4.4 Algorithm of the Generalized Reduced Gradient (GRG) Method

Step 1. Find a initial feasible point, X^o, from the given starting point.

Step 2. Select X_K and X_I, Calculate

$$\frac{\partial f(X)}{\partial X_K}, \quad \frac{\partial f(X)}{\partial X_I}, \quad \frac{\partial h}{\partial X_K}, \quad \frac{\partial h}{\partial X_I}$$

step 3. Find the inverse of $\frac{\partial h}{\partial X_I}$, namely, $\left[\frac{\partial h}{\partial X_I}\right]^{-1}$

Step 4. Compute $U^T = -\nabla^T_{X_I} f\left[\frac{\partial h}{\partial X_I}\right]^{-1}$

step 5. Compute $r_g = \nabla^T_{X_K} f + U^T \cdot \left[\frac{\partial h}{\partial X_K}\right]$

step 6. Find $\Delta^{(k)}$ as

$$\Delta_j^{(k)} = \begin{cases} 0 \text{ if } r_{gi} < 0 \text{ and } X_j^{(k)} = L_j \\ 0 \text{ if } r_{gi} < 0 \text{ and } X_j^{(k)} = U_j \\ r_{gi} \text{ else} \end{cases}$$

Step 7. If $\Delta^{(k)} = 0$, then stop.

Step 8. Otherwise, chooses the value of λ that minimizes $f(X_K+\lambda\Delta^{(k)})$

Then solve a) - c) for themost appropriate values of X_K and X_I.

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\text{a)}\quad X_{Kj} = \begin{cases} L_i \text{ if } X_{Kj} + \lambda\Delta^{(k)} < L_j \\ U_j \text{ if } X_{Kj} + \lambda\Delta^{(k)} > U_j \\ X_{Kj} + \lambda\Delta^{(k)} \text{ if } L_j < X_{K_j} + \lambda\Delta^{(k)} < U_j \end{cases}
$$

b) find $X_I = g(X_K)$

c) Calculate the objective function $f(X_K, X_I)$

Step 9. Store the best solution up to now in X^o and go to step 2.