CHAPTER III

EXTENDED RANGE OF THE CAUCHY - LIPSCHITZ METHOD

We shall show the construction of the solution in every finite interval in which the solution curve is continuous.

Let us suppose that the system of two simultaneous ordinary differential equation $\frac{du}{dx} = f(x, u, v)$, $\frac{dv}{dx} = g(x, u, v)$ has a solution curve Γ which is continuous in the interval $(x_o, x_o + k)$ for any finite number k.

Without loss of generality we can take x_o at the origin, therefore the interval $(x_o, x_o + k)$ becomes $(0, k)$.

Let L be the tube whose cross section parallel to the uv-plane is a square with sides each of length 2η, where η is an arbitrary small positive number. Two parallel sides are parallel to the u - axis and the other two are parallel to the v - axis. The tube L encloses the curve Γ , and any point on the curve Γ is at distance η measured in the uv - plane, from every side of the tube L as shown in figure VI (a).

Let the interval (0, k) be divided into n equal subintervals each of length $h = \frac{k}{n}$ by the points $x_o = 0$, $x_1 = h$, $x_2 = 2h$,...., $x_n = nh = k$.

Let us suppose that the curve Γ is continuous in (0, k), and that f(x, u, v) and g(x, u, v) are continuous in the tube L, and η is so small that f(x, u, v) and g(x, u, v) satisfy the Lipschitz condition.

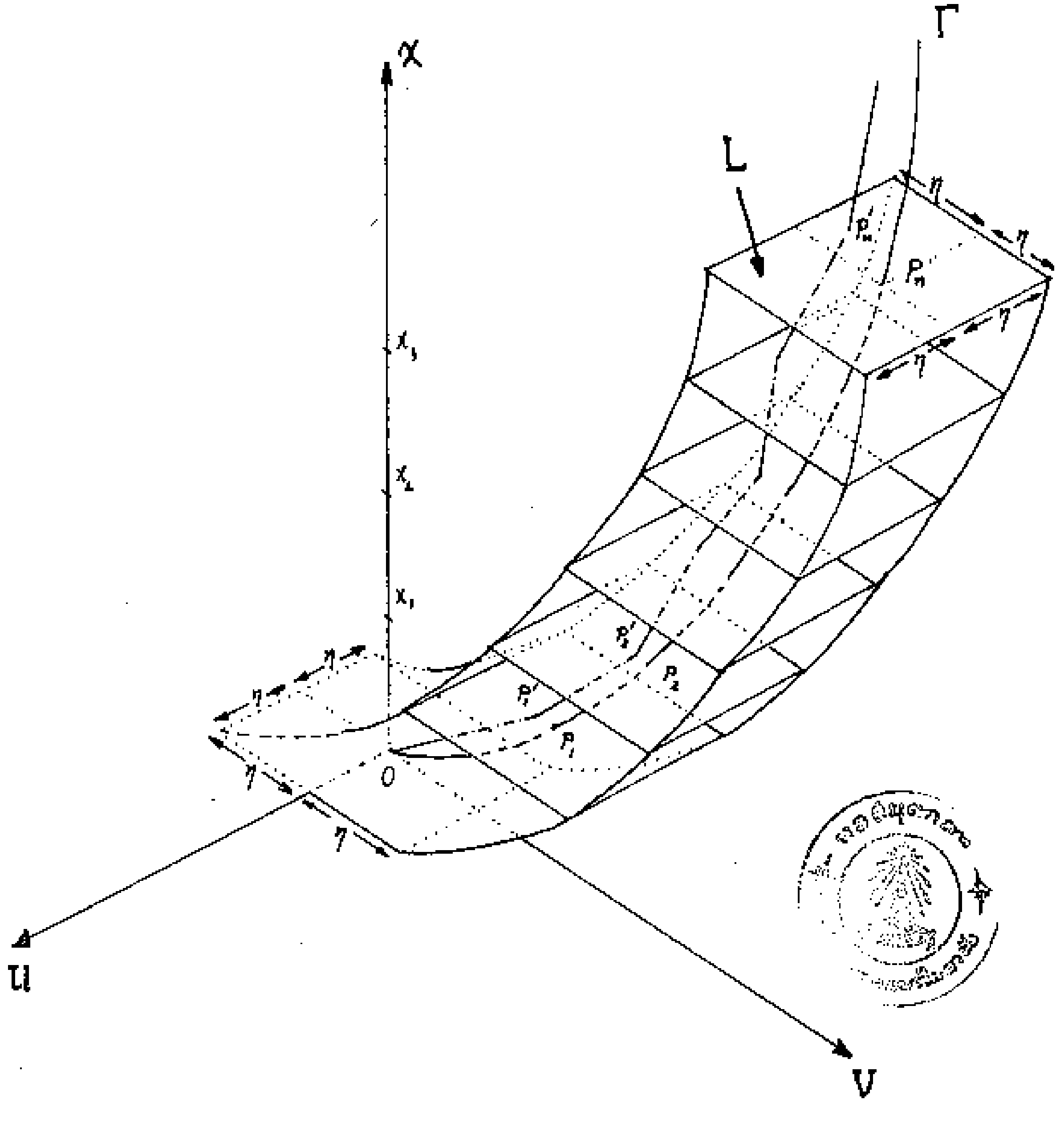

FIGURE VI(a)

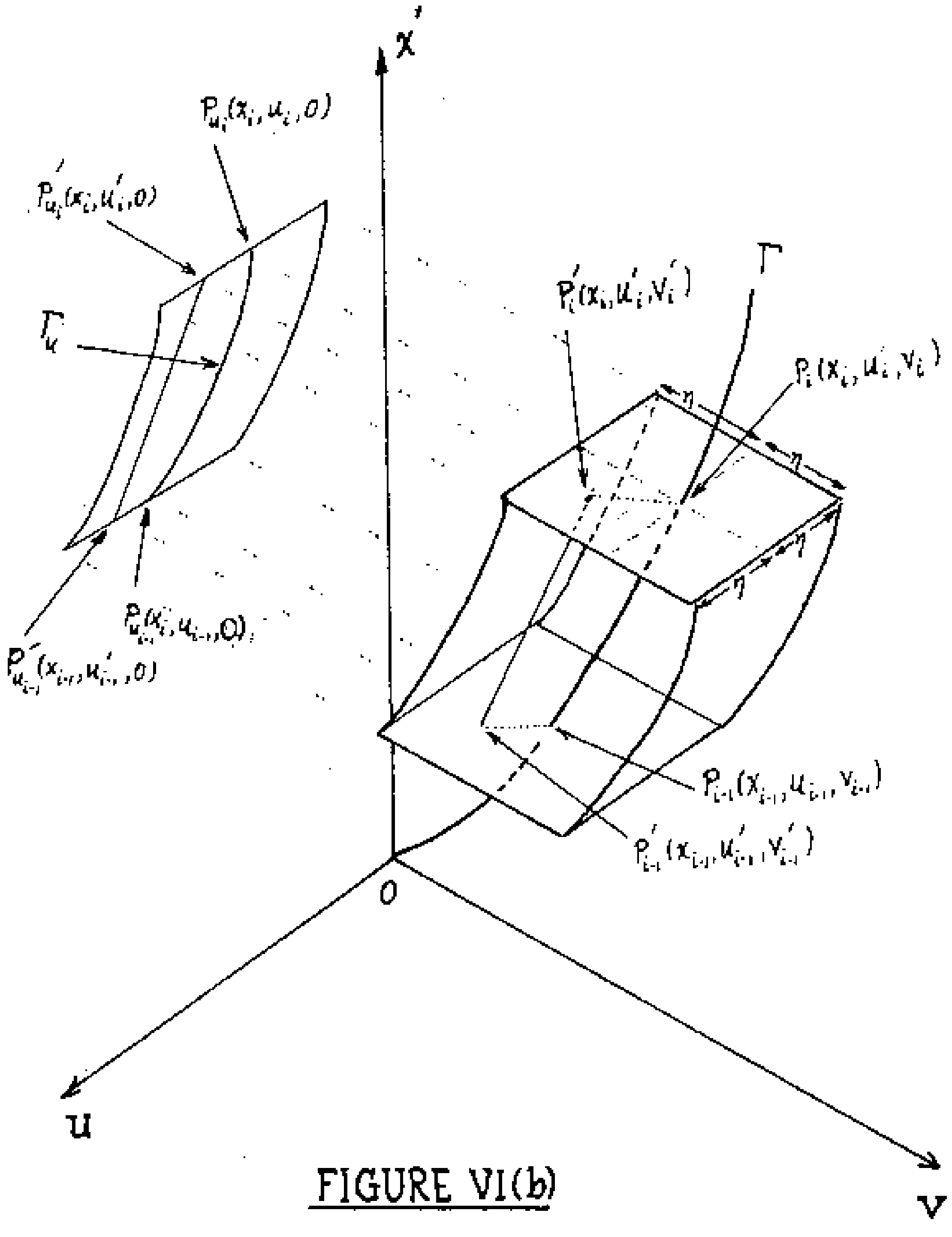
x'
u
v
O
Γ
Γ_u
$P_{u_i}(x_i, u_i, 0)$
$P'_{u_i}(x_i, u'_i, 0)$
$P_{u_{i-1}}(x'_i, u_{i-1}, 0)$
$P'_{u_{i-1}}(x_{i-1}, u'_{i-1}, 0)$
$P'_i(x_i, u'_i, v'_i)$
$P_i(x_i, u_i, v_i)$
$P_{i-1}(x_{i-1}, u_{i-1}, v_{i-1})$
$P'_{i-1}(x_{i-1}, u'_{i-1}, v'_{i-1})$
η

FIGURE VI(b)

Construct the polygonal line $OP_1'P_2', \ldots, P_n'$ whose the coordinates (x_i, u_i', v_i') of P_i' are given by

$$u_o' = v_o' = 0,$$

$$u_i' = u_{i-1}' + hf(x_{i-1}, u_{i-1}', v_{i-1}'),$$

and $$v_i' = v_{i-1}' + hg(x_{i-1}, u_{i-1}', v_{i-1}'), \text{ for } i = 1,2,\ldots, n.$$

Let the curve Γ intersect the plane $x = x_i$ at the point P_i; then the coordinate of the point P_i is (x_i, u_i, v_i).

Consider the i^{th} frustrum of the tube L, and suppose that all points up to and including the point P_{i-1}' are included in L.

We shall show that if the subdivision of the interval (0,k) is sufficiently fine then the points $P_1', P_2', \ldots, P_n'$ will all be within the tube L.

The distance between P_i' and P_i is

$$d(P_i', P_i) = \sqrt{(u_i' - u_i)^2 + (v_i' - v_i)^2}.$$

Project the part of Γ in this frustrum and the straight line $P_{i-1}' P_i'$ onto the xu- plane and let the projections of P_i, P_{i-1}, P_i', P_{i-1}' and Γ be P_{u_i}, $P_{u_{i-1}}$, P_{u_i}', $P_{u_{i-1}}'$ and Γ_u respectively on the xu - plane, as shown in figure VI (b).

Similarly, the projections of P_i, P_{i-1}, P_i', P_{i-1}' and Γ on the xv - plane may be written $P_{v_i}(x_i, 0, v_i)$, $P_{v_{i-1}}(x_{i-1}, 0, v_{i-1})$, $P_{v_i}'(x_i, 0, v_i')$, $P_{v_{i-1}}'(x_{i-1}, 0, v_{i-1}')$ and Γ_v respectively.

On the xu - plane

$u_i = u_{i-1} + hf(x_i^*, u_i^*, v_i^*)$ where (x_i^*, u_i^*, v_i^*) lies on Γ in the i^{th} frustrum., by the mean valued theorem.

Let $d_{u_i} = | u_i - u_i' |$, and note that

$$u_i - u_i' = \left\{ u_{i-1} + hf(x_i^*, u_i^*, v_i^*) \right\} - \left[u_{i-1}' + hf(x_{i-1}, u_{i-1}', v_{i-1}') \right]$$

$$= (u_{i-1} - u_{i-1}') + h \left\{ f(x_i^*, u_i^*, v_i^*) - f(x_{i-1}, u_{i-1}', v_{i-1}') \right\} .$$

Since $f(x_i^*, u_i^*, v_i^*) - f(x_{i-1}, u_{i-1}', v_{i-1}') = \left\{ f(x_i^*, u_i^*, v_i^*) - f(x_{i-1}, u_{i-1}, v_{i-1}) \right\} + \left\{ f(x_{i-1}, u_{i-1}, v_{i-1}) - f(x_{i-1}, u_{i-1}', v_{i-1}') \right\}$,

and since f satisfies the Lipschitz condition in L so that there exist two positive numbers A and B such that

$$\left| f(x_{i-1}, u_{i-1}, v_{i-1}) - f(x_{i-1}, u_{i-1}', v_{i-1}') \right| < A \left| u_{i-1} - u_{i-1}' \right| + B \left| v_{i-1} - v_{i-1}' \right| ,$$

and since f is also continuous in L, and it is therefore a continuous function of x along Γ, so that if λ_f is arbitrarily assigned, δ_f may be chosen sufficiently small and h can be put so fine that

$$\left| x_i^* - x_{i-1} \right| \leqslant x_i - x_{i-1} = h < \delta_f \Longrightarrow \left| f(x_i^*, u_i^*, v_i^*) - f(x_{i-1}, u_{i-1}, v_{i-1}) \right| < \lambda_f ,$$

it follows that

$$\left| f(x_i^*, u_i^*, v_i^*) - f(x_{i-1}, u_{i-1}', v_{i-1}') \right| < \lambda_f + A \left| u_{i-1} - u_{i-1}' \right| + B \left| v_{i-1} - v_{i-1}' \right| ,$$

and $d_{u_i} < d_{u_{i-1}} + h\lambda_f + Ah \left| u_{i-1} - u_{i-1}' \right| + Bh \left| v_{i-1} - v_{i-1}' \right|$ ... (3)

On the xv - plane

$v_i = v_{i-1} + hg(x_i^o, u_i^o, v_i^o)$ where (x_i^o, u_i^o, v_i^o) lies on Γ in the i^{th} frustrum.

Let $d_{v_i} = |v_i - v_i'|$, and note that

$$v_i - v_i' = \left[v_{i-1} + hg(x_i^o, u_i^o, v_i^o)\right] - \left[v_{i-1}' + hg(x_{i-1}, u_{i-1}', v_{i-1}')\right]$$

$$= (v_{i-1} - v_{i-1}') + h\left[g(x_i^o, u_i^o, v_i^o) - g(x_{i-1}, u_{i-1}', v_{i-1}')\right].$$

Since $g(x_i^o, u_i^o, v_i^o) - g(x_{i-1}, u_{i-1}', v_{i-1}') = \Big[g(x_i^o, u_i^o, v_i^o) - g(x_{i-1}, u_{i-1}, v_{i-1})\Big] + \Big[g(x_{i-1}, u_{i-1}, v_{i-1}) - g(x_{i-1}, u_{i-1}', v_{i-1}')\Big]$,

and since g satisfies the Lipschitz condition in L, so that there exist two positive numbers A and B such that

$$\left|g(x_{i-1}, u_{i-1}, v_{i-1}) - g(x_{i-1}, u_{i-1}', v_{i-1}')\right| < A\left|u_{i-1} - u_{i-1}'\right| + B\left|v_{i-1} - v_{i-1}'\right|,$$

and since g is also continuous in L, and is therefore a continuous function of x along Γ, so that if λ_g is arbitrarily assigned, δ_g may be chosen sufficiently small and h can be put so fine that

$$\left|x_i^o - x_{i-1}\right| \leqslant x_i - x_{i-1} = h < \delta_g \Longrightarrow \left|g(x_i^o, u_i^o, v_i^o) - g(x_{i-1}, u_{i-1}', v_{i-1}')\right| < \lambda_g.$$

it follows that

$$\left|g(x_i^o, u_i^o, v_i^o) - g(x_{i-1}, u_{i-1}', v_{i-1}')\right| < \lambda_g + A\left|u_{i-1} - u_{i-1}'\right| + B\left|v_{i-1} - v_{i-1}'\right|,$$

and $d_{v_i} < d_{v_{i-1}} + h\lambda_g + Ah\left|u_{i-1} - u_{i-1}'\right| + Bh\left|v_{i-1} - v_{i-1}'\right|$ ..... (4)

Put $\lambda = \max(\lambda_f, \lambda_g)$ and choose $\delta = \min(\delta_f, \delta_g)$ and put $h < \delta$.

Then, (3) and (4) can be written as

$$d_{u_i} < d_{u_{i-1}} + h\lambda + Ahd_{u_{i-1}} + Bhd_{v_{i-1}}$$

$$d_{v_i} < d_{v_{i-1}} + h\lambda + Ahd_{u_{i-1}} + Bhd_{v_{i-1}}$$

and it follows that

$$d_{u_i} + d_{v_i} < d_{u_{i-1}} + d_{v_{i-1}} + 2\lambda h + 2\,Ahd_{u_{i-1}} + 2\,Bhd_{v_{i-1}} .$$

Let $C = \max(2A, 2B)$.

Then $d_{u_i} + d_{v_i} < (d_{u_{i-1}} + d_{v_{i-1}}) + 2\lambda h + Ch\,(d_{u_{i-1}} + d_{v_{i-1}})$,

$$d_{u_i} + d_{v_i} + \frac{2\lambda}{C} < (d_{u_{i-1}} + d_{v_{i-1}})(1+Ch) + 2\lambda h + \frac{2\lambda}{C}$$

$$= (d_{u_{i-1}} + d_{v_{i-1}})(1+Ch) + \frac{2\lambda}{C}(1 + Ch)$$

$$= (d_{u_{i-1}} + d_{v_{i-1}} + \frac{2\lambda}{C})(1 + Ch)$$

$$< (d_{u_{i-1}} + d_{v_{i-1}} + \frac{2\lambda}{C})\,e^{hC} .$$

Put $i = 1, 2, \ldots, r$ for $r \leqslant n$.

For $i = 1$, $d_{u_1} + d_{v_1} + \frac{2\lambda}{C} < \frac{2\lambda}{C}\,e^{hC}$, since $d_{u_0} = d_{v_0} = 0$.

For $i = 2$, $d_{u_2} + d_{v_2} + \frac{2\lambda}{C} < (d_{u_1} + d_{v_1} + \frac{2\lambda}{C})\,e^{hC} < \frac{2\lambda}{C}\,e^{2hC}$.

For $i = 3$, $d_{u_3} + d_{v_3} + \frac{2\lambda}{C} < (d_{u_2} + d_{v_2} + \frac{2\lambda}{C})\,e^{hC}$

$$< \frac{2\lambda}{C}\,e^{3hC} .$$

In general , for $i = r$,

$$d_{u_r} + d_{v_r} + \frac{2\lambda}{C} < \frac{2\lambda}{C}\,e^{rhC} ,$$

or $d_{u_r} + d_{v_r} < \frac{2\lambda}{C}\,(e^{rhC} - 1)$.

Since rhC is a constant, and λ can be made so small that $2\lambda(e^{rhC}-1) < C\eta$, we have

$$d_{u_r} + d_{v_r} < \eta .$$

Since d_{u_r} and d_{v_r} are both positive, it follows that $d_{u_r} < \eta$ and $d_{v_r} < \eta$, for all $r = 1,2,\ldots, n$, and $d(P'_r, P_r) < \sqrt{\eta^2+\eta^2} = \sqrt{2}\,\eta$ for all $r = 1,2,\ldots, n$. Hence all points $P'_1, P'_2,\ldots, P'_n$ are within the tube L.

Since η can be made as small as we desire, therefore, all points $P'_1, P'_2,\ldots, P'_n$ have the limiting positions on the curve Γ; and since h can be made as small as we desire, therefore, all points on the polygonal line $OP'_1P'_2\ldots P'_n$ have the limiting positions on the curve Γ. That is, the polygonal line $OP'_1P'_2\ldots P'_n$ has the curve Γ as its limit. Therefore, for any finite interval $(0,k)$ we obtain an approximation to the solution curve as close as we like by constructing the polygonal line $OP'_1P'_2\ldots P'_n$ where the coordinates (x_i, u'_i, v'_i) of P'_i are given by

$$u'_0 = v'_0 = 0$$

$$u'_i = u'_{i-1} + hf(x_{i-1}, u'_{i-1}, v'_{i-1}), \text{ and}$$

$$v'_i = v'_{i-1} + hg(x_{i-1}, u'_{i-1}, v'_{i-1}), \text{ for } i = 1,2,\ldots, n.$$