CHAPTER II

CONSTRUCTION AND PROOF BY THE CAUCHY - LIPSCHITZ METHOD

In this chapter, we shall apply the Cauchy - Lipschitz method to the system of two ordinary differential equations

$$\frac{du}{dx} = f(x, u, v)$$

$$\frac{dv}{dx} = g(x, u, v)$$

whose solution has the initial point (x_0, u_0, v_0).

Let $f(x, u, v)$ and $g(x, u, v)$ be two continuous functions defined in a closed sphere S with center at $P_0(x_0, u_0, v_0)$. Suppose that in S both $f(x, u, v)$ and $g(x, u, v)$ satisfy the following Lipschitz condition : there exist two positive numbers A and B such that

$$\left| f(x,u,v) - f(x,u^*,v^*) \right| < A \left| u - u^* \right| + B \left| v - v^* \right| \quad \text{and}$$

$$\left| g(x,u,v) - g(x,u^*,v^*) \right| < A \left| u - u^* \right| + B \left| v - v^* \right| \; . \quad ^{(a)}$$

Without loss of generality we may assume that P_0 is the origin of the coordinate system, so that $x_0 = u_0 = v_0 = 0$.

Let $0 = x_0, x_1, x_2, \ldots, x_n$ be successive values of x proceeding in increasing order from x_0 to x_n and such that $x_i - x_{i-1} = h$ for all $i = 1,2,\ldots, n$, and all x_i's lie in the sphere S.(b)

On the xu - plane, draw lines from the origin with slopes M_f and $- M_f$ respectively. On the xv - plane, draw lines from the origin with slopes M_g and $- M_g$ respectively, (see figure I).

(a) Let $|f(x,u,v)| \leqslant M_f$ and $|g(x,u,v)| \leqslant M_g$.

(b) and the rectangle $x = x_n$, $|u| \leqslant M_f x_n$, $|v| \leqslant M_g x_n$ lies in S.

Construct a plane passing through x_1 parallel to the uv - plane. Let the lines with slopes M_f, $-M_f$, M_g and $-M_g$ intersect the plane $x = x_1$ at the points $P(1)$, $P'(1)$, $Q(1)$ and $Q'(1)$ repectively. On the plane $x = x_1$ draw two lines passing through $P(1)$ and $P'(1)$ respectively both parallel to the v - axis and draw two lines passing through $Q(1)$ and $Q'(1)$ respectively both parallel to the u - axis intersecting the first two lines at the points A_1, B_1, C_1 and D_1 as shown in figure I. Draw OA_1, OB_1, OC_1 and OD_1, then we obtain a pyramid, namely $OA_1B_1C_1D_1$.

Let $\bar{m}_{f_1}$ and $\underline{m}_{f_1}$ be the maximum and minimum of $f(x,u,v)$ respectively, and $\bar{m}_{g_1}$ and $\underline{m}_{g_1}$ be the maximum and minimum of $g(x,u,v)$ respectively in the pyramid $OA_1B_1C_1D_1$.

On the xu - plane, draw two lines from O with slopes $\bar{m}_{f_1}$ and $\underline{m}_{f_1}$ intersecting the base of the pyramid at the points $\bar{P}_1$ and $\underline{P}_1$ respectively. Draw two lines passing through $\bar{P}_1$ and $\underline{P}_1$ respectively both parallel to the v - axis.

On the xv - plane, draw two lines from O with slopes $\bar{m}_{g_1}$ and $\underline{m}_{g_1}$ intersecting the base of the pyramid at the points $\bar{Q}_1$ and $\underline{Q}_1$ respectively. Draw two lines passing through $\bar{Q}_1$ and $\underline{Q}_1$ respectively both parallel to the u - axis intersecting the two parallel lines that pass through $\bar{P}_1$ and $\underline{P}_1$ at the points $A^{(1)}$, $B^{(1)}$, $C^{(1)}$ and $D^{(1)}$ as shown in figure I.

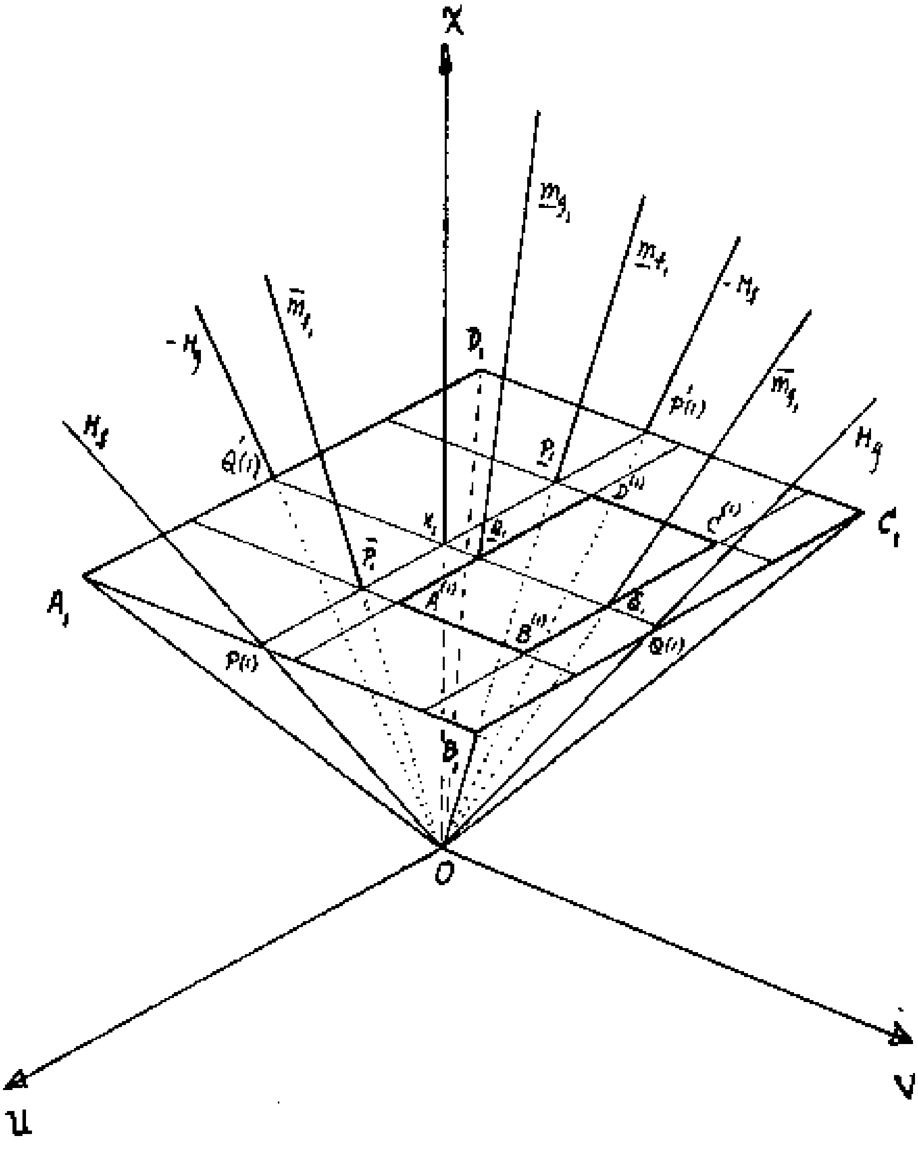

x
m_{g_1}
m_{f_1}
-M_f
m̄_{f_1}
-M_g
m̄_{g_1}
M_f
M_g
D_1
p'(1)
P_1
Q'(1)
D^{(1)}
C^{(1)}
C_1
x_1
Q_1
P̄_1
A^{(1)}
B^{(1)}
Q̄_1
A_1
p(1)
Q(1)
B_1
O
u
v

FIGURE I

Then we obtain the rectangle $A^{(1)} B^{(1)} C^{(1)} D^{(1)}$ which lies on the base of the pyramid $OA_1 B_1 C_1 D_1$, where

$$A^{(1)} D^{(1)} = B^{(1)} C^{(1)} = \overline{P}_1 \underline{P}_1$$

and $$A^{(1)} B^{(1)} = C^{(1)} D^{(1)} = \overline{Q}_1 \underline{Q}_1$$

Next, construct the plane passing through x_2 parallel to the uv - plane.

Construct new coordinate axes $O'u'$, $O'v'$ and $O'x'$ by using $D^{(1)}$ as the origin O', $O'u'$ in the line $D^{(1)} A^{(1)}$ and $O'v'$ in the line $D^{(1)} C^{(1)}$. Then $O'u'$ is parallel to Ou, $O'v'$ is parallel to ov and $O'x'$ is parallel to ox (see figure II (a)).

On the $x'u'$ - plane, draw a line from $A^{(1)}$ with slope M_f intersecting the plane $x = x_2$ at the point P(2), and draw a line from O' with slope $- M_f$ intersecting the plane $x = x_2$ at the point $P'(2)$.

On the $x'v'$ - plane, draw a line from $C^{(1)}$ with slope M_g intersecting the plane $x = x_2$ at the point Q(2) and from O' draw a line with slope $- M_g$ intersecting the plane $x = x_2$ at the point $Q'(2)$.

Draw two lines passing through P(2) and $P'(2)$ respectively both parallel to $Q(2)Q'(2)$ and draw another two lines passing through Q(2) and $Q'(2)$ respectively both parallel to $P(2)P'(2)$ intersecting the two parallel lines that pass through P(2) and $P'(2)$ at the points A_2, B_2, C_2 and D_2 as shown in figure II (b). Draw $A^{(1)} A_2$, $B^{(1)} B_2$, $C^{(1)} C_2$ and $O' D_2$. Then we obtain a frustrum of a pyramid, namely $A^{(1)} B^{(1)} C^{(1)} D^{(1)} A_2 B_2 C_2 D_2$.

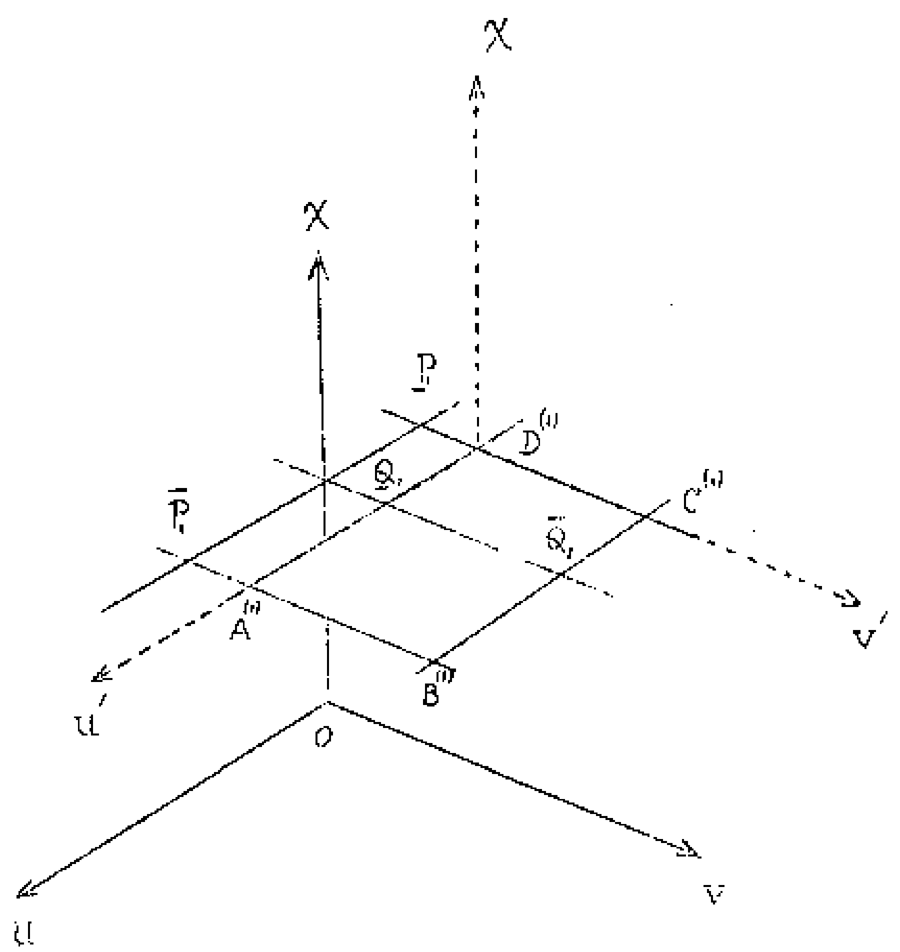
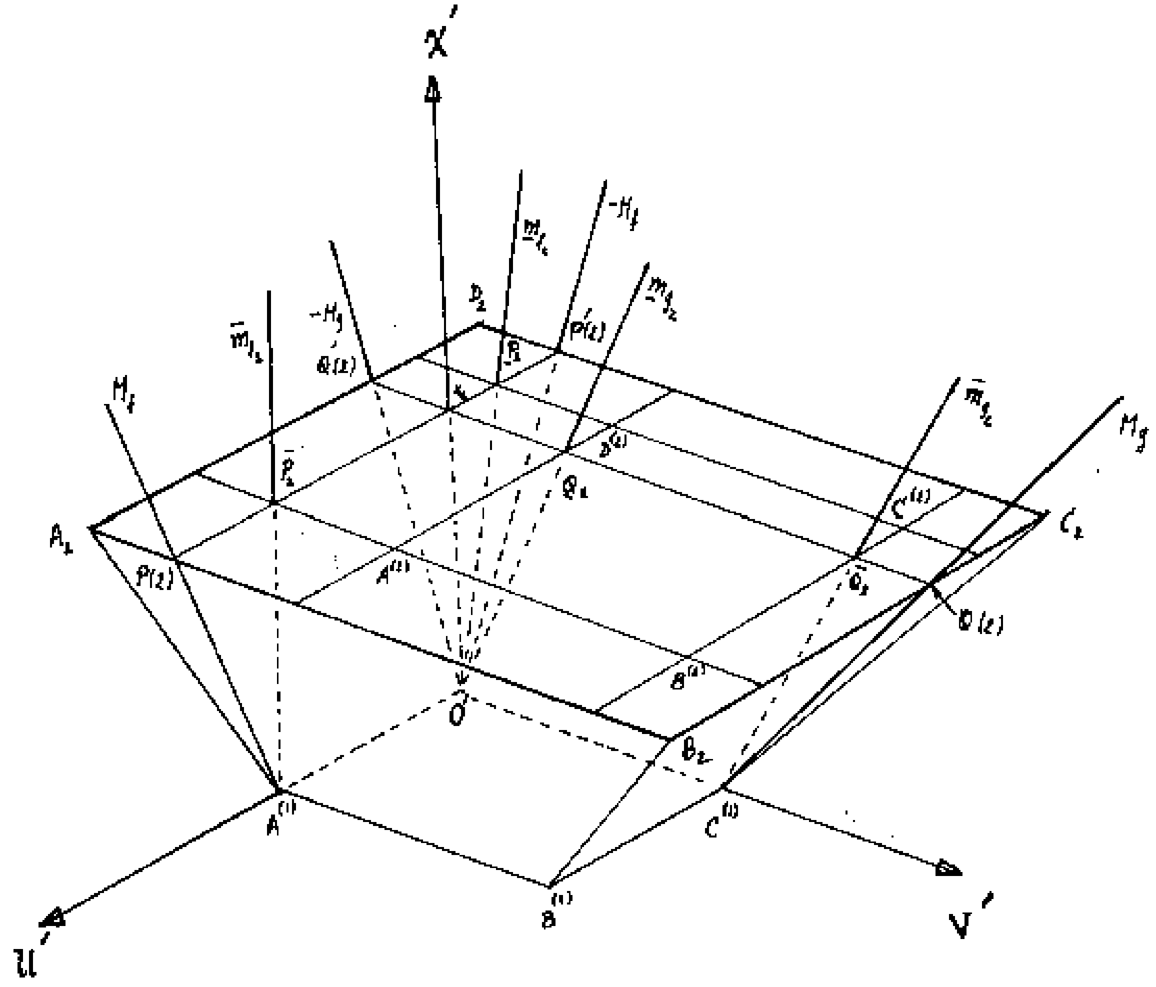


FIGURE II (a)



X'
$\underline{m}_{f_1}$
$-H_f$
$\underline{m}_{f_2}$
$\bar{m}_{f_2}$
$-H_g$
$Q(2)$
D_2
$\underline{P}_1$
$P'(2)$
H_f
$\bar{P}_2$
$D^{(2)}$
Q_1
$\bar{m}_{f_2}$
M_g
$C^{(2)}$
A_2
$P(2)$
$A^{(2)}$
$\bar{Q}_2$
C_2
$Q(2)$
$B^{(2)}$
O
B_2
$A^{(1)}$
$C^{(1)}$
$B^{(1)}$
U'
V'

FIGURE II (b)

Let $\overline{m}_{f_2}$ and $\underline{m}_{f_2}$ be the maximum and minimum of f respectively in this frustrum, and let $\overline{m}_{g_2}$ and $\underline{m}_{g_2}$ be the maximum and minimum of g respectively in this frustrum.

On the $x'u'$ - plane, from $A^{(1)}$ and O' draw two lines with slopes $\overline{m}_{f_2}$ and $\underline{m}_{f_2}$ respectively intersecting the plane $x = x_2$ at the points $\overline{P}_2$ and $\underline{P}_2$ respectively.

On the $x'v'$ - plane, from $C^{(1)}$ and O' draw two lines with slopes $\overline{m}_{g_2}$ and $\underline{m}_{g_2}$ respectively intersecting the plane $x = x_2$ at the points $\overline{Q}_2$ and $\underline{Q}_2$ respectively.

Draw two lines passing through $\overline{P}_2$ and $\underline{P}_2$ respectively both parallel to the v' - axis, and draw another two lines passing through $\overline{Q}_2$ and $\underline{Q}_2$ respectively both parallel to the u' - axis intersecting the two parallel lines that pass through $\overline{P}_2$ and $\underline{P}_2$ at the points $A^{(2)}$, $B^{(2)}$, $C^{(2)}$ and $D^{(2)}$ as shown in figure II (b).

Then we obtain the rectangle $A^{(2)} B^{(2)} C^{(2)} D^{(2)}$ which lies on the upper base of the frustrum of a pyramid $O' A^{(1)} B^{(1)} C^{(1)} C_2 D_2 A_2 B_2$, where $A^{(2)} B^{(2)} = D^{(2)} C^{(2)} = \overline{Q}_2 \underline{Q}_2$ and $A^{(2)} D^{(2)} = B^{(2)} C^{(2)} = \overline{P}_2 \underline{P}_2$.

Continue this process until the $(n - 1)^{th}$ frustrum of a pyramid is reached, namely $A^{(n-1)} B^{(n-1)} C^{(n-1)} D^{(n-1)} A_n B_n C_n D_n$ which has the bases on the planes $x = x_{n-1}$ and $x = x_n$.

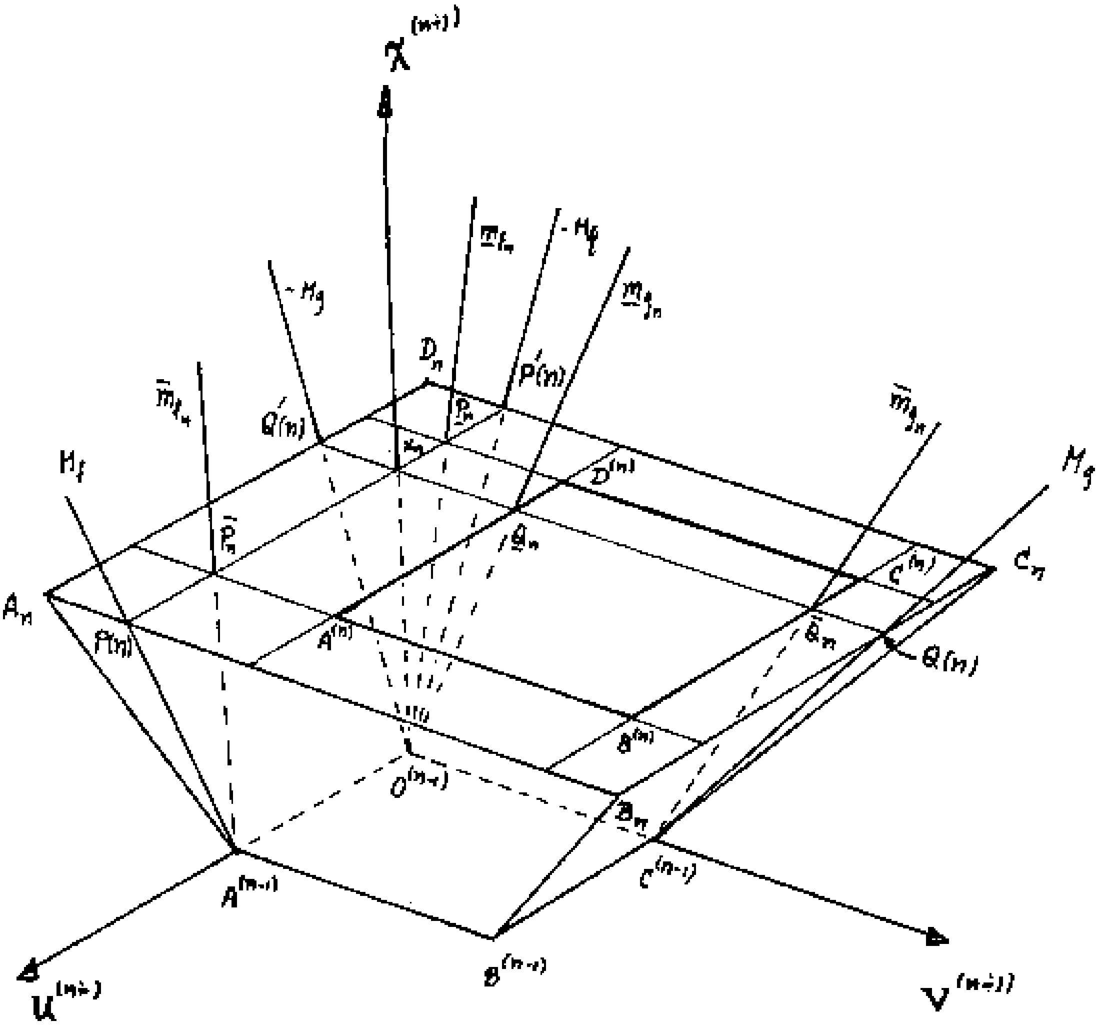

$X^{(n+1)}$
$\underline{m}_{f_n}$
$-M_f$
$\underline{m}_{g_n}$
$-M_g$
D_n
$P'(n)$
$\bar{m}_{f_n}$
$Q'(n)$
$\underline{P}_n$
x_n
$D^{(n)}$
$\bar{m}_{g_n}$
M_g
M_f
$\bar{P}_n$
$\underline{Q}_n$
$C^{(n)}$
C_n
A_n
$P(n)$
$A^{(n)}$
$\bar{Q}_n$
$Q(n)$
$B^{(n)}$
$O^{(n-1)}$
B_n
$A^{(n-1)}$
$C^{(n-1)}$
$B^{(n-1)}$
$U^{(n+1)}$
$V^{(n+1)}$

FIGURE III

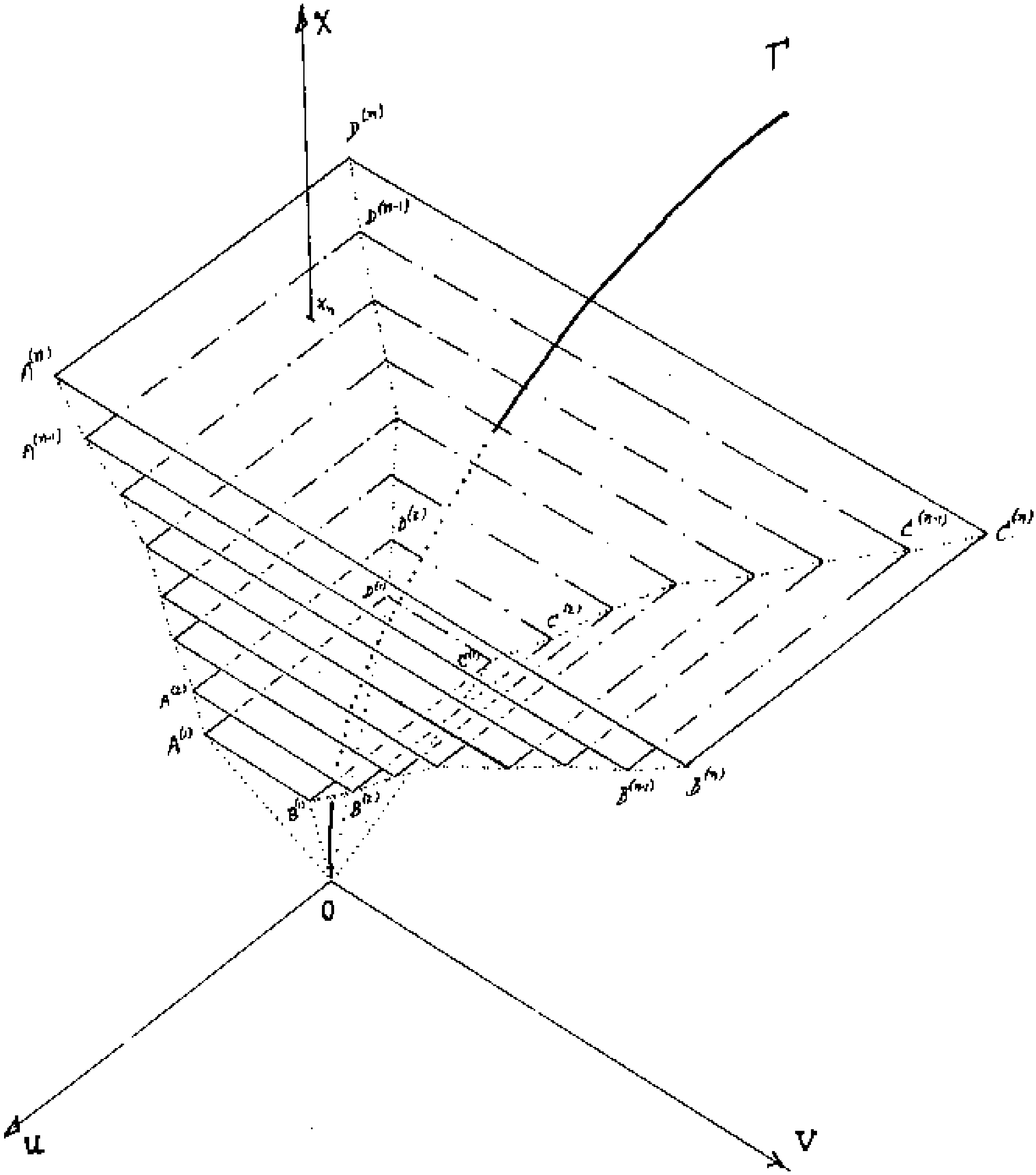


FIGURE IV

Let $\overline{m}_{f_n}$ and $\underline{m}_{f_n}$ be the maximum and the minimum of f respectively in the $(n - 1)^{th}$ frustrum of a pyramid and let $\overline{m}_{g_n}$ and $\underline{m}_{g_n}$ be the maximum and minimum of g respectively in this frustrum.

By the same argument as above we obtain the points $\overline{P}_n$, $\underline{P}_n$, $\overline{Q}_n$ and $\underline{Q}_n$, and we obtain the n^{th} rectangle $A^{(n)}B^{(n)}C^{(n)}D^{(n)}$ lying on the upper base of the $(n - 1)^{th}$ frustrum of a pyramid, where $A^{(n)}B^{(n)} = D^{(n)}C^{(n)} = \overline{Q}_n\underline{Q}_n$ and $A^{(n)}D^{(n)} = B^{(n)}C^{(n)} = \overline{P}_n\underline{P}_n$ as shown in figure III.

The complete construction is shown in figrue IV. We obtain n rectangles, namely $A^{(1)}B^{(1)}C^{(1)}D^{(1)}$, $A^{(2)}B^{(2)}C^{(2)}D^{(2)}$,..., $A^{(n)}B^{(n)}C^{(n)}D^{(n)}$.

Consider the i^{th} rectangle, where $1 \leqslant i \leqslant n$. The area of the i^{th} rectangle is $(A^{(i)}B^{(i)})(A^{(i)}D^{(i)})$.

Since $A^{(i)}D^{(i)} = \overline{P}_i\underline{P}_i$ and $A^{(i)}B^{(i)} = \overline{Q}_i\underline{Q}_i$, therefore the area of the i^{th} rectangle is $(\overline{P}_i\underline{P}_i)(\overline{Q}_i\underline{Q}_i)$

We shall show that the sides of the i^{th} rectangle are not greater than the sides of the $(i + 1)^{th}$ one, for $i = 1, 2, \ldots, n - 1$, and the sides of the n^{th} rectangle approach zero as n increases indefinitely.

Now, suppose we project all the $\overline{P}_i$'s and $\underline{P}_i$'s onto the xu - plane. $\overline{P}_i$ is projected onto $\overline{\overline{P}}_i$ and $\underline{P}_i$ is projected onto $\underline{\underline{P}}_i$ respectively, for $i = 1, 2, \ldots, n$. Since the $x'u'$ - plane, the $x''u''$ - plane, ... , and the $x^{(n)}u^{(n)}$ - plane are parallel to the xu - plane, therefore $\overline{P}_i\underline{P}_i = \overline{\overline{P}}_i\underline{\underline{P}}_i$ for all $i = 1, 2, \ldots, n$. The coordinates of $\overline{\overline{P}}_i$ and $\underline{\underline{P}}_i$ can be found as follows.

Let the coordinates of $\bar{\bar{P}}_1, \bar{\bar{P}}_2, \ldots, \bar{\bar{P}}_n$ be $(h, \bar{u}_1), (2h, \bar{u}_2), \ldots, (nh, \bar{u}_n)$ respectively and the coordinates of $\underline{\underline{P}}_1, \underline{\underline{P}}_2, \ldots, \underline{\underline{P}}_n$ be $(h, \underline{u}_1), (2h, \underline{u}_2), \ldots, (nh, \underline{u}_n)$ respectively and note that $\bar{u}_o = \underline{u}_o = 0$.

The positions of $\bar{\bar{P}}_1, \bar{\bar{P}}_2, \ldots, \bar{\bar{P}}_n$ and $\underline{\underline{P}}_1, \underline{\underline{P}}_2, \ldots, \underline{\underline{P}}_n$ are shown in figure V (a). The ordinates can be calculated as follows.

$$\bar{u}_1 = h\,\bar{m}_{f_1},$$

$$\bar{u}_2 = \bar{u}_1 + h\,\bar{m}_{f_2}$$

$$= h(\bar{m}_{f_1} + \bar{m}_{f_2}),$$

$$\bar{u}_3 = \bar{u}_2 + h\,\bar{m}_{f_3}$$

$$= h(\bar{m}_{f_1} + \bar{m}_{f_2} + \bar{m}_{f_3}),$$

$$- - - - - - - - - - - -$$

$$\bar{u}_n = h(\bar{m}_{f_1} + \bar{m}_{f_2} + \ldots + \bar{m}_{f_n}).$$

Similarly, $\underline{u}_i = h(\underline{m}_{f_1} + \underline{m}_{f_2} + \ldots + \underline{m}_{f_i})$ for $i = 1, 2, \ldots, n$.

We now have

$$\bar{\bar{P}}_i\underline{\underline{P}}_i = \bar{u}_i - \underline{u}_i = h(\bar{m}_{f_1} + \bar{m}_{f_2} + \ldots + \bar{m}_{f_i}) - h(\underline{m}_{f_1} + \underline{m}_{f_2} + \ldots + \underline{m}_{f_i})$$

$$= h\left[(\bar{m}_{f_1} - \underline{m}_{f_1}) + (\bar{m}_{f_2} - \underline{m}_{f_2}) + \ldots + (\bar{m}_{f_i} - \underline{m}_{f_i})\right].$$

Next, suppose we project all $\bar{Q}_i$'s and $\underline{Q}_i$'s onto the xv - plane, that is $\bar{Q}_i$ is projected onto $\bar{\bar{Q}}_i$ and $\underline{Q}_i$ is projected onto $\underline{\underline{Q}}_i$ respectively, for $i = 1, 2, \ldots, n$. Since the $x'v'$ - plane, the $x''v''$ - plane, ..., and the $x^{(n)}v^{(n)}$ - plane are parallel to the xv - plane, therefore, $\bar{Q}_i\underline{Q}_i = \bar{\bar{Q}}_i\underline{\underline{Q}}_i$ for all $i = 1, 2, \ldots, n$. The coordinates of $\bar{\bar{Q}}_i$ and $\underline{\underline{Q}}_i$ can be found as follows.

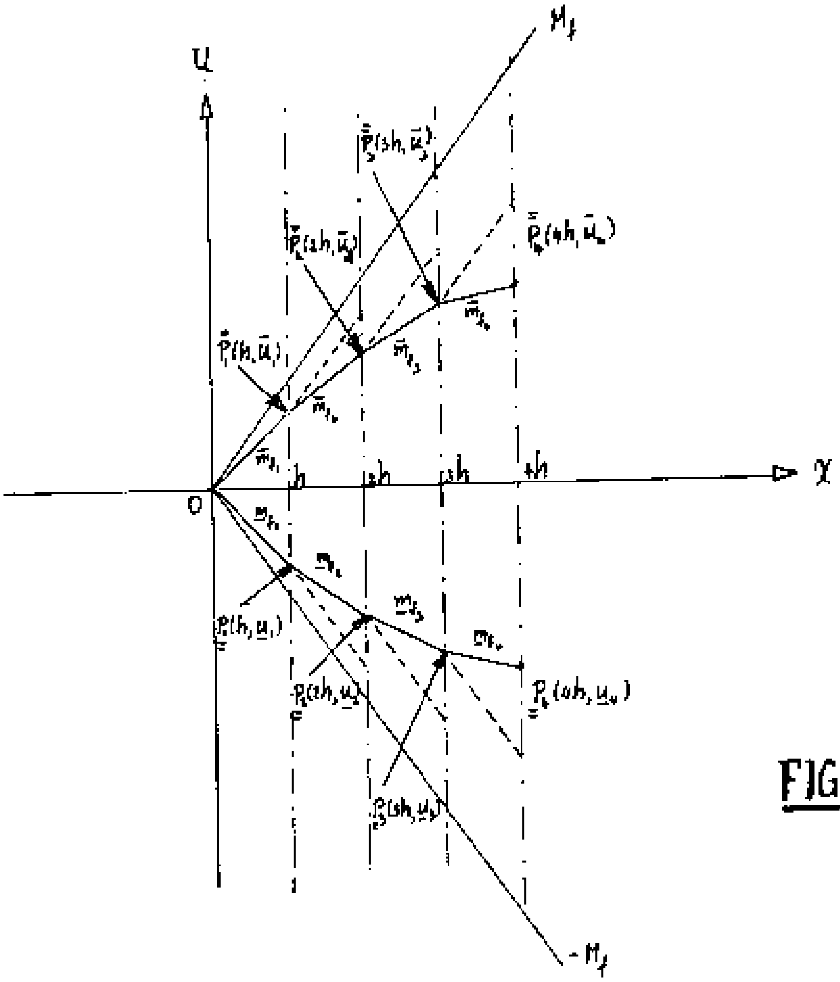


FIGURE V (a)

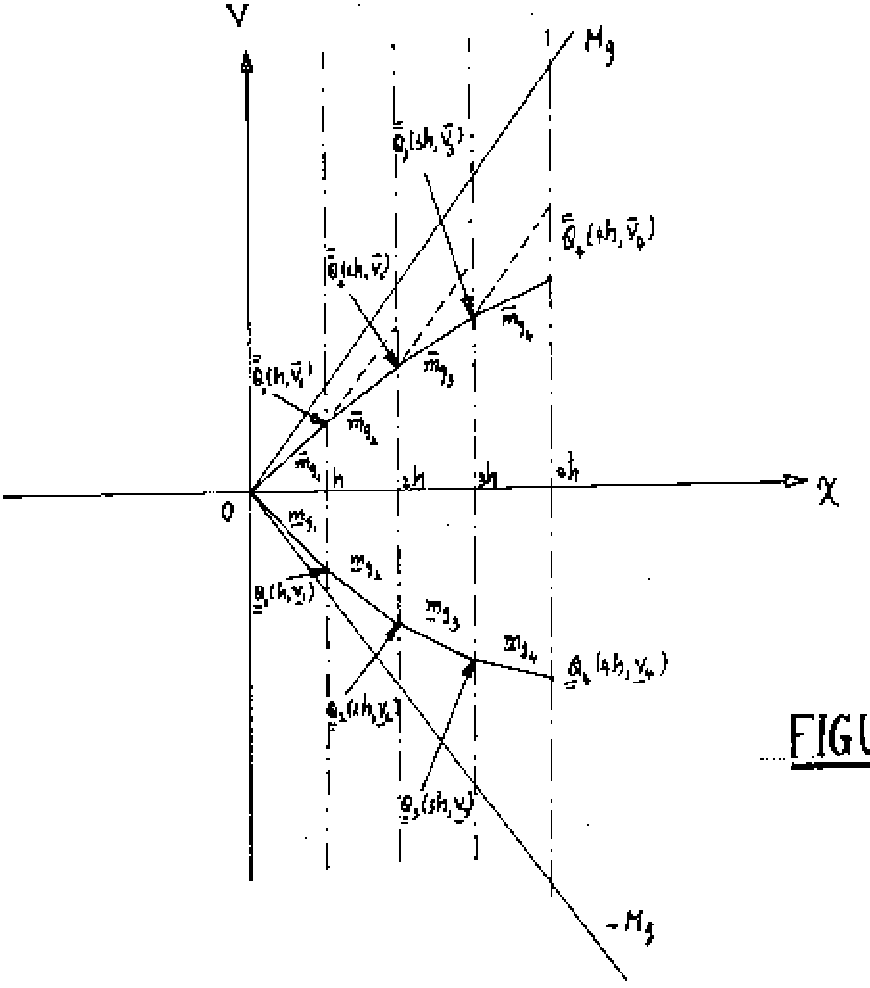


FIGURE V (b)

Let the coordinates of $\bar{\bar{Q}}_1, \bar{\bar{Q}}_2, \ldots, \bar{\bar{Q}}_n$ be $(h, \bar{v}_1)$, $(2h, \bar{v}_2)$, $\ldots$, $(nh, \bar{v}_n)$ respectively and the coordinates of $\underline{\underline{Q}}_1, \underline{\underline{Q}}_2, \ldots, \underline{\underline{Q}}_n$ be $(h, \underline{v}_1)$, $(2h, \underline{v}_2), \ldots, (nh, \underline{v}_n)$ respectively, and note that $\bar{v}_o = \underline{v}_o = 0$.

The positions of $\bar{\bar{Q}}_1, \bar{\bar{Q}}_2, \ldots, \bar{\bar{Q}}_n$ and $\underline{\underline{Q}}_1, \underline{\underline{Q}}_2, \ldots, \underline{\underline{Q}}_n$ are shown in figure V (b).

By the same arguments as above we can show that

$$\bar{v}_i = h(\bar{m}_{g_1} + \bar{m}_{g_2} + \ldots + \bar{m}_{g_i}) \quad \text{and}$$

$$\underline{v}_i = h(\underline{m}_{g_1} + \underline{m}_{g_2} + \ldots + \underline{m}_{g_i}) \quad \text{for } i = 1, 2, \ldots, n.$$

We may have

$$\bar{\bar{Q}}_i\underline{\underline{Q}}_i = \bar{v}_i - \underline{v}_i = h(\bar{m}_{g_1} + \bar{m}_{g_2} + \ldots + \bar{m}_{g_i}) - h(\underline{m}_{g_1} + \underline{m}_{g_2} + \ldots + \underline{m}_{g_i})$$
$$= h\left\{(\bar{m}_{g_1} - \underline{m}_{g_1}) + (\bar{m}_{g_2} - \underline{m}_{g_2}) + \ldots + (\bar{m}_{g_i} - \underline{m}_{g_i})\right\}.$$

The area of the i^{th} rectangle is

$$(\bar{P}_i\underline{P}_i)(\bar{Q}_i\underline{Q}_i) = h^2\left\{(\bar{m}_{f_1} - \underline{m}_{f_1}) + (\bar{m}_{f_2} - \underline{m}_{f_2}) + \ldots + (\bar{m}_{f_i} - \underline{m}_{f_i})\right\}\left\{(\bar{m}_{g_1} - \underline{m}_{g_1}) + (\bar{m}_{g_2} - \underline{m}_{g_2}) + \ldots + (\bar{m}_{g_i} - \underline{m}_{g_i})\right\}.$$

We now show that the area of the i^{th} rectangle is not greater than the area of the $(i + 1)^{th}$ one.

$$\text{Since } \bar{P}_i\underline{P}_i = \bar{\bar{P}}_i\underline{P}_i = h\left\{(\bar{m}_{f_1} - \underline{m}_{f_1}) + (\bar{m}_{f_2} - \underline{m}_{f_2}) + \ldots + (\bar{m}_{f_i} - \underline{m}_{f_i})\right\}$$

and $\bar{P}_{i+1}\underline{P}_{i+1} = \bar{P}_{i+1}\underline{P}_{i+1} = h\left[(\bar{m}_{f_1} - \underline{m}_{f_1}) + (\bar{m}_{f_2} - \underline{m}_{f_2}) + \cdots + (\bar{m}_{f_i} - \underline{m}_{f_i}) + (\bar{m}_{f_{i+1}} - \underline{m}_{f_{i+1}})\right]$, and since $\bar{m}_{f_i} \geqslant \underline{m}_{f_i}$ so that $\bar{m}_{f_i} - \underline{m}_{f_i} \geqslant 0$, for all i = 1,2,...., n, it follows that

$$\bar{P}_i\underline{P}_i \leqslant \bar{P}_{i+1}\underline{P}_{i+1}$$

Similarly $\bar{Q}_i\underline{Q}_i \leqslant \bar{Q}_{i+1}\underline{Q}_{i+1}$

Therefore $(\bar{P}_i\underline{P}_i)(\bar{Q}_i\underline{Q}_i) \leqslant (\bar{P}_{i+1}\underline{P}_{i+1})(\bar{Q}_{i+1}\underline{Q}_{i+1})$

that is, the area of the ith rectangle is not greater than the area of the (i + 1)th one.

Next, we shall show that the area of the nth rectangle approaches zero as n increases indefinitely, that is $\bar{P}_i\underline{P}_i$ and $\bar{Q}_i\underline{Q}_i$ approach zero as n increases indefinitely.

We have $\bar{u}_i = h(\bar{m}_{f_1} + \bar{m}_{f_2} + \dots + \bar{m}_{f_i})$

$= h(\bar{m}_{f_1} + \bar{m}_{f_2} + \dots + \bar{m}_{f_{i-1}}) + h\,\bar{m}_{f_i}$

$= \bar{u}_{i-1} + h\,\bar{m}_{f_i}$.

Similarly, $\underline{u}_i = \underline{u}_{i-1} + h\,\underline{m}_{f_i}$.

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Let $\delta_{u_i} = \bar{u}_i - \underline{u}_i$

$= (\bar{u}_{i-1} - \underline{u}_{i-1}) + h(\bar{m}_{f_i} - \underline{m}_{f_i})$

$= \delta_{u_{i-1}} + h(\bar{m}_{f_i} - \underline{m}_{f_i})$.

Since f is continuous in S,

$\forall \lambda_f > 0$, $\exists\, \delta_f > 0$ such that

$d\left((x,u,v), (x^*,u^*,v^*)\right) < \delta_f \Longrightarrow \left|f(x,u,v) - f(x^*,u^*,v^*)\right| < \lambda_f$.

Since f satisfies the Lipschitz condition, there exist two positive numbers A and B such that

$|f(x,u,v) - f(x,u^*,v^*)| < A|u - u^*| + B|v - v^*|$ where (x,u,v) and (x,u^*,v^*) lie in S.

But

$$\bar{m}_{f_i} - \underline{m}_{f_i} = f(x_i^*, u_i^*, v_i^*) - f(x_i^{**}, u_i^{**}, v_i^{**})$$

$$= \left[f(x_i^*, u_i^*, v_i^*) - f(x_i^{**}, u_i^*, v_i^*) \right] + \left[f(x_i^{**}, u_i^*, v_i^*) - f(x_i^{**}, u_i^{**}, v_i^{**}) \right]$$

where (x_i^*, u_i^*, v_i^*), (x_i^{**}, u_i^*, v_i^*) and $(x_i^{**}, u_i^{**}, v_i^{**})$ lie in the frustrum $A^{(i-1)}B^{(i-1)}C^{(i-1)}D^{(i-1)}A_iB_iC_iD_i$, and we may assume that when λ_f has been specified, $h < \delta_f$. For if not, we can make the subdivision finer until $h < \delta_f$ holds. Therefore, since $|x_i^* - x_i^{**}| \leqslant x_i - x_{i-1} = h < \delta_f$, it follows that $\bar{m}_{f_i} - \underline{m}_{f_i} < \lambda_f + A|u_i^* - u_i^{**}| + B|v_i^* - v_i^{**}|$ .... (1)

Let $\delta_{v_i} = \bar{v}_i - \underline{v}_i$, then we can show similarly that

$$\delta_{v_i} = \delta_{v_{i-1}} + h(\bar{m}_{g_i} - \underline{m}_{g_i}).$$

Since g is continuous in S,

$\forall \lambda_g > 0, \exists \delta_g > 0$ such that

$d((x,u,v), (x^*,u^*,v^*)) < \delta_g \Longrightarrow |g(x,u,v) - g(x^*,u^*,v^*)| < \lambda_g$.

Since g satisfies the Lipschitz condition.

There exist two positive numbers A and B such that

$|g(x,u,v) - g(x,u^*,v^*)| < A|u - u^*| + B|v - v^*|$ where (x,u,v) and (x,u^*,v^*) lie in S.

But $\bar{m}_{g_i} - \underline{m}_{g_i} = g(x_i^o, u_i^o, v_i^o) - g(x_i^{oo}, u_i^{oo}, v_i^{oo})$

$$= \Big[g(x_i^o, u_i^o, v_i^o) - g(x_i^{oo}, u_i^o, v_i^o) \Big] + \Big[g(x_i^{oo}, u_i^o, v_i^o) - g(x_i^{oo}, u_i^{oo}, v_i^{oo}) \Big],^{*}$$

and we may assume that when λ_g has been specified, $h < \delta_g$. For, if not, we can make the subdivision finer until $h \leqslant \delta_g$ holds.

Therefore, since $\left| x_i^o - x_i^{oo} \right| \leqslant x_i - x_{i-1} = h < \delta_g$, it follows that $\bar{m}_{g_i} - \underline{m}_{g_i} < \lambda_g + A \left| u_i^o - u_i^{oo} \right| + B \left| v_i^o - v_i^{oo} \right|$ ...(2)

Now $\left| u_i^* - u_i^{**} \right| \leqslant \bar{u}_i - \underline{u}_i = \delta_{u_{i-1}} + h(\bar{m}_{f_i} - \underline{m}_{f_i})$

$$< \delta_{u_{i-1}} + 2hM_f ,$$

and $\left| u_i^o - u_i^{oo} \right| \leqslant \bar{u}_i - \underline{u}_i = \delta_{u_{i-1}} + h(\bar{m}_{f_i} - \underline{m}_{f_i})$

$$< \delta_{u_{i-1}} + 2hM_f ,$$

and similarly, $\left| v_i^* - v_i^{**} \right| < \delta_{v_{i-1}} + 2hM_g ,$

$$\left| v_i^o - v_i^{oo} \right| < \delta_{v_{i-1}} + 2hM_g .$$

Therefore (1) and (2) can be written as follows.

$$\bar{m}_{f_i} - \underline{m}_{f_i} < \lambda_f + A(\delta_{u_{i-1}} + 2hM_f) + B(\delta_{v_{i-1}} + 2hM_g) ,$$

$$\bar{m}_{g_i} - \underline{m}_{g_i} < \lambda_g + A(\delta_{u_{i-1}} + 2hM_f) + B(\delta_{v_{i-1}} + 2hM_g) .$$

Let $\lambda = \max(\lambda_f, \lambda_g)$, and choose $\delta = \min(\delta_f, \delta_g)$ and put $h < \delta$. Then we obtain.

$$\bar{m}_{f_i} - \underline{m}_{f_i} < \lambda + A(\delta_{u_{i-1}} + 2hM_f) + B(\delta_{v_{i-1}} + 2hM_g)$$

$$= \lambda + A\delta_{u_{i-1}} + B\delta_{v_{i-1}} + 2h(AM_f + BM_g),$$

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[*] where $(x_i^o, u_i^o, v_i^o), (x_i^{oo}, u_i^o, v_i^o)$ and $(x_i^{oo}, u_i^{oo}, v_i^{oo})$ lie in the frustrum $A^{(i-1)} B^{(i-1)} C^{(i-1)} D^{(i-1)} A_i B_i C_i D_i$.

and $\bar{m}_{g_i} - \underline{m}_{g_i} < \lambda + A(\delta_{u_{i-1}} + 2hM_f) + B(\delta_{v_{i-1}} + 2hM_g)$

$$= \lambda + A\delta_{u_{i-1}} + B\delta_{v_{i-1}} + 2h(AM_f + BM_g).$$

Since h can be made so small that

$2h(AM_f + BM_g) < \lambda$, we obtain

$$\bar{m}_{f_i} - \underline{m}_{f_i} < 2\lambda + A\delta_{u_{i-1}} + B\delta_{v_{i-1}},$$

$$\bar{m}_{g_i} - \underline{m}_{g_i} < 2\lambda + A\delta_{u_{i-1}} + B\delta_{v_{i-1}},$$

and $\delta_{u_i} < \delta_{u_{i-1}} + 2h\lambda + Ah\delta_{u_{i-1}} + Bh\delta_{v_{i-1}}$,

$$\delta_{v_i} < \delta_{v_{i-1}} + 2h\lambda + Ah\delta_{u_{i-1}} + Bh\delta_{v_{i-1}}.$$

Therefore $\delta_{u_i} + \delta_{v_i} < \delta_{u_{i-1}} + \delta_{v_{i-1}} + 4h\lambda + 2Ah\delta_{u_{i-1}} + 2Bh\delta_{v_{i-1}}.$

Let C = max (2A, 2B).

Then $\delta_{u_i} + \delta_{v_i} < (\delta_{u_{i-1}} + \delta_{v_{i-1}}) + 4\lambda h + Ch(\delta_{u_{i-1}} + \delta_{v_{i-1}})$,

and $\delta_{u_i} + \delta_{v_i} + \frac{4\lambda}{C} < (\delta_{u_{i-1}} + \delta_{v_{i-1}})(1+Ch) + 4\lambda h + \frac{4\lambda}{C}$

$$= (\delta_{u_{i-1}} + \delta_{v_{i-1}})(1+Ch) + \frac{4\lambda}{C}(1 + Ch)$$

$$= (\delta_{u_{i-1}} + \delta_{v_{i-1}} + \frac{4\lambda}{C})(1 + Ch)$$

$$< (\delta_{u_{i-1}} + \delta_{v_{i-1}} + \frac{4\lambda}{C})e^{Ch}.$$

Put i = 1, 2, ... , n successively.

For i = 1, $\delta_{u_1} + \delta_{v_1} + \frac{4\lambda}{C} < \frac{4\lambda}{C}e^{hC}$, since $\delta_{u_o} = \delta_{v_o} = 0$.

For i = 2, $\delta_{u_2} + \delta_{v_2} + \frac{4\lambda}{C} < (\delta_{u_1} + \delta_{v_1} + \frac{4\lambda}{C})e^{hC}$.

$$< \frac{4\lambda}{C}e^{2hC}.$$

For $i = 3$, $\delta_{u_3} + \delta_{v_3} + \frac{4\lambda}{C} < (\delta_{u_2} + \delta_{v_2} + \frac{4\lambda}{C})\, e^{hC}$

$$< \frac{4\lambda}{C}\, e^{3hC} .$$

Continue this process until the n^{th} step is reached.

We obtain $\delta_{u_n} + \delta_{v_n} + \frac{4\lambda}{C} < \frac{4\lambda}{C}\, e^{nhC}$,

or $\delta_{u_n} + \delta_{v_n} < \frac{4\lambda}{C}\, (e^{nhC} - 1)$.

Since λ can be made as small as we desire, and $e^{nhC} - 1$ is constant, it follows that

$\delta_{u_n} + \delta_{v_n}$ converges to zero as a limit. But, both δ_{u_n} and δ_{v_n} are positive and their sum converges to zero, therefore, both δ_{u_n} and δ_{v_n} converge to zero as limit.

Therefore $\bar{P}_nP_n = \delta_{u_n}$ and $\bar{Q}_nQ_n = \delta_{v_n}$ converge to zero as limit, and the area of the n^{th} rectangle $= (\bar{P}_nP_n)(\bar{Q}_nQ_n)$ approaches zero as a limit.

That is, the n^{th} rectangle approaches to a single point. Since the area of the n^{th} rectangle is not less than the area of the $(n - 1)^{th}$ one, it follows that as n increases indefinitely, each of the rectangles approaches to a single point. Then, as n increases indefinitely all the rectangles approach a single curve Γ as shown in the figure IV .

If we project this curve Γ onto the xu - plane and onto the xv - plane we obtain the graphs of $u = u(x)$ and $v = v(x)$ which represent the solution of the system of two simultaneous ordinary differential equations

$$\frac{du}{dx} = f(x, u, v)$$

$$\frac{dv}{dx} = g(x, u, v).$$

In the construction, all rectangles are joined one to another successively by the lines joining the corresponding vertices. That is, all rectangles are joined one to another by four polygonal lines $OA^{(1)}A^{(2)} \ldots A^{(n)}$, $OB^{(1)} B^{(2)} \ldots B^{(n)}$, $OC^{(1)} C^{(2)} \ldots C^{(n)}$ and $OD^{(1)} D^{(2)} \ldots D^{(n)}$. These four polygonal lines meet at the point O. Since the $(n - 1)^{th}$ rectangle is not greater than the n^{th} one, and as n increases indefinitely, the n^{th} rectangle converges to a single point, therefore the four polygonal lines $OA^{(1)} A^{(2)} \ldots A^{(n)}$, $OB^{(1)} B^{(2)} \ldots B^{(n)}$, $OC^{(1)} C^{(2)} \ldots C^{(n)}$ and $OD^{(1)} D^{(2)} \ldots D^{(n)}$ converge uniformly to the curve Γ . But those four polygonal lines are continuous. Therefore, Γ is continuous, and so are $u(x)$ and $v(x)$.