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REGULARITY OF CERTAIN WEAKLY P-HARMONIC MAPS INTO PSEUDOSPHERES

Mr. Wasanont Pongsawat

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By	Mr. Wasanont Pongsawat
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

. . . . Chairman
(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

. . . . Thesis Advisor
(Sujin Khomrutai, Ph.D.)

. . . . Examiner
(Assistant Professor Nataphan Kitisin, Ph.D.)

. . . . External Examiner
(Assistant Professor Tawikan Treeyaprasert, Ph.D.)

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เราได้นำเสนอแนวคิดของพลังงาน-พี ของการส่ง $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$ จากจากวงกลมหนึ่งหน่วยในปริภูมิยูคลิด $\mathbb{R}^m$ B ไปยังทรงกลมเทียม $\mathbb{S}^n_\nu$ การส่งวิกฤตสำหรับพลังงาน-พีจะนิยามเป็นการส่งพี-ฮาร์มอนิกอย่างอ่อนไปยังทรงกลมเทียม ซึ่งในกรณี $p = 2$ นั้นได้มีการศึกษาเมื่อเร็วๆนี้ โดยเริ่มต้นเราจะสร้างเอกลักษณ์ที่สำคัญซึ่งคล้ายคลึงกับเอกลักษณ์ของเวนเต้ จากนั้นจะใช้เอกลักษณ์ดังกล่าวร่วมกับเทคนิคทางด้านทฤษฎีระบบสมการอิลลิปติก เราจะสามารถพิสูจน์ ε-สม่ำเสมอบางประการของการส่งพี-ฮาร์มอนิกอย่างอ่อนได้

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We propose the notion of p-energy for maps $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$ from a Euclidean disk B into pseudospheres $\mathbb{S}^n_\nu$. Critical maps for this functional is defined to be weakly p-harmonic maps into pseudospheres generalizing the weakly harmonic maps, i.e. $p = 2$, into pseudospheres studied recently by some authors. We first derive an important identity, analogous to the Wente identity. Then, using this identity together with techniques from theory of elliptic systems, we can prove some ε-regularity results for the weakly p-harmonic maps.

Department	Mathematics and Computer Science	Student's Signature	....
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Contents

CHAPTER I

INTRODUCTION

In the recent years, there are numerous studies of p-harmonic maps motivated by their importance in physics and geometry. A p-harmonic map is a generalization of harmonic functions in classical analysis. From the point of view of the Calculus of Variation, p-harmonic maps, unlike harmonic functions which are real- (or complex-) functions, are maps that take values in a differential geometric space or even a metric space and are stationary points of some Dirichlet-like energy integral. Thus p-harmonic maps are solutions of certain nonlinear elliptic PDE systems (the Euler-Lagrange equation) of p-Laplace type. From classical complex analysis, it is well-known that if a function is harmonic except at some singular points and it satisfies a pointwise bound, then it can be extended to a harmonic function across the singularities. This fact does not carry over to p-harmonic maps however. For harmonic maps (i.e. when $p = 2$), it was proved that a certain ϵ-regularity condition has to be satisfied so that such an extension will be true.

Many previous studies until recently of p-harmonic maps concentrated on maps into a Riemannian manifold. The results in this direction can be seen from an inspiring paper by F. Duzaar and M. Fuchs[1] where the authors proved a removable singularity theorem for p-harmonic maps into a Riemannian Manifold. In this work, we consider p-harmonic maps into a pseudosphere which is the standard sphere endowed with the pseudo-Riemannian metric from its ambient Lorentz manifold. Motivated by the study of Riemannian-valued p-harmonic maps, we are expecting some similarities in the properties of p-harmonic maps into Riemannian Manifolds and pseudospheres. Because of the distinct geometric structures of pseudospheres and Riemannian manifolds, we are also expect some dissimilarities as well.

In F. Duzaar and M. Fuchs[1], they investigated the regularity problem of p-harmonic maps in higher dimensions. More precisely, they consider the following situation; the parameter domain is the unit ball $B_1(0)$ in $\mathbb{R}^m$, $m \geq 2$ (equipped with the flat metric) and the target space M is a Riemannian manifold of dimension $n \geq 1$ which is isometrically embedded in some Euclidean space $\mathbb{R}^k$, $k \geq n$. We are then interested in mappings $u{:}B_1(0) \to M$ of Soblolev class $W^{1,p}(B_1(0), M)$ being defined as the set of function u from the linear Sobolev space $W^{1,p}(B_1(0)$, $\mathbb{R}^k)$ such that $u(x) \in M$ a.e. on $B_1(0)$. The p-energy of u is defined as

$$\mathbb{E}_p(u) := \int_{B_1(0)} |Du|^p dx,$$

and u is said to be a weakly p-harmonic map if u is a weak solution of the Euler-Lagrange equations associated with the p-energy, i.e. u satisfies for all $\varphi \in C_0^1(B_1, \mathbb{R}^k)$ the equation

$$\int_{B_1(0)} |Du|^{p-2}(D_\alpha u \cdot D_\varphi u + \varphi \cdot A(u)(D_\alpha u, D_\alpha u))dx = 0,$$

where $A(q)(.,.)$ is the second fundamental form of M at q. The purpose of this paper is to prove some ε-regularity theorem for p-harmonic maps. they prove that the point singularity at the origin is removable provided the p-energy $\mathbb{E}_1(u)$ is sufficiently small. There are no a priori assumption on the image of u in M.

In N. Hungerbühler[3], the author defined a mapping $f : M \to N \subset \mathbb{R}^k$ to be weakly p-harmonic if it satisfies the Euler-Lagrange equations for p-harmonic mappings in the sense of distributions, i.e. for each coordinate chart Ω on M there holds

$$\int_\Omega \left(\gamma^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}\cdot\frac{\partial f}{\partial x^\beta}\right)^{\frac{p}{2}-1}\frac{\partial f}{\partial x^\alpha}\cdot\frac{\partial \varphi}{\partial x^\beta}\sqrt{\gamma}\,dx = 0$$

for all smooth mapping φ with compact support in Ω and satisfying $\varphi(x) \in T_{f(x)}N$ for all $x \in M$ where M is a smooth Riemannian manifolds equipped with metric $(\gamma^{\alpha\beta})$. This equation is linked to the p-energy of f

$$E(f) := \int_M e(f)$$

such that the p-energy density of f is

$$e(f)(x) := \frac{1}{p}|df_x|^p.$$

But the main topic of this paper is p-harmonic flow. He consider the p-harmonic flow into spheres and a priori estimate. He slightly attend to the weakly p-harmonic maps.

The definition of weakly p-harmonic map into pseudospheres for case $p = 2$ is defined in the recent paper of M. Zhu[8] by the energy of u given by

$$\mathbb{E}(u) := \int_{B_1(0)} (\nabla u)^T \mathcal{E} \nabla u,$$

where $\mathcal{E}$ is index matrix and u is said to be a weakly harmonic map if it is a critical point of the energy. In this paper, they prove regularity for weakly harmonic maps from the unit ball into certain pseudo-Riemannian manifolds from different points of view. Analytically, it is interesting to know how the structure of the harmonic map system is affected when the target manifolds become pseudo-Riemannian. The target pseudo-Riemannian in this paper are pseudospheres and standard stationary Lorentzian manifolds. We focus on the pseudospheres

In this study, we consider weakly p-harmonic map into pseudospheres for all $p \geq 2$ by extending idea from Zhu[8]. Theorems and Lemmas in this work are similar to the result from that paper. In Chapter II, all basic knowledges are given. In Chapter III, we prove the main result. In Chapter IV, we present conclusion and discussion.

CHAPTER II

PRELIMINARIES

We collect, in this chapter, some basic definitions from both geometry and analysis. The notion of p-energy for maps from a ball into pseudospheres is also introduced. Basic known results such as the Hodge decomposition theorem and the existence theorem of uniformly elliptic systems are also provided.

2.1 Geometry and Analysis

Throughout this work, we shall fix positive integers m, n and an integer $0 \leq \nu \leq n$. Also, we denote $B = B_1(0) \subset \mathbb{R}^m$ to be the unit ball.

Notation.

1. Let $\mathcal{E}$ be the $(n+1) \times (n+1)$-matrix given by

$$\mathcal{E} = (\varepsilon_{ij}) := \begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n+1-\nu} \end{pmatrix},$$

   where I_k, for each k, denotes the identity $k \times k$ matrix.

2. The *pseudo-Euclidean space of signature* (n, ν), denoted $\mathbb{R}^{n+1}_\nu$, is the set $\mathbb{R}^{n+1}$ equipped with the *pseudo-Riemannian metric*

$$\langle x, y \rangle_{\mathbb{R}^{n+1}_\nu} := x^T \mathcal{E} y = -\sum_{j=1}^{\nu} x^j y^j + \sum_{\alpha=\nu+1}^{n+1} x^\alpha y^\alpha$$

   for all $x = (x^1, ..., x^{n+1})^T, y = (y^1, ..., y^{n+1})^T \in \mathbb{R}^{n+1}$.

   In this work, unless specified otherwise, $\mathbb{R}^{n+1}_\nu$ always denotes the pseudo-Euclidean space and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n+1}_\nu}$ is the above pseudo-Riemannian metric.

3. The n-dimensional *pseudosphere* is the subset $\mathbb{S}^n_\nu \subset \mathbb{R}^{n+1}_\nu$ given by

$$\mathbb{S}^n_\nu := \left\{ x \in \mathbb{R}^{n+1}_\nu \mid \langle x, x \rangle_{\mathbb{R}^{n+1}_\nu} = x^T \mathcal{E} x = 1 \right\}.$$

4. Let $\Omega \subset \mathbb{R}^m$ be an open set and (X, d) a metric space. A map $f : \Omega \to X$ is said to be Hölder continuous with exponent $0 < \gamma \leq 1$ provided there exist a constant $C > 0$ such that

$$d\left(f(x), f(y)\right) \leq C|x - y|^\gamma$$

   for all $x, y \in \Omega$. We denote by $C^{0,\gamma}(\Omega, X)$ the space of all Hölder continuous maps.

5. Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The usual *Sobolev space* is denoted by $W^{k,p}(\Omega)$.

6. The vector-valued Sobolev space $W^{k,p}(\Omega, \mathbb{R}^N)$ is defined by $u \in W^{k,p}(\Omega, \mathbb{R}^N)$ if and only if $u^j \in W^{k,p}(\Omega)$ for all $j = 1, \ldots, N$ where $u = (u^1, \ldots, u^N)$.

7. Let M be a compact smooth manifold and $i : M \hookrightarrow \mathbb{R}^k$ is a smooth embedding (Whitney's theorem). The Sobolev space $W^{1,p}(\Omega, M)$ is defined by $u \in W^{1,p}(\Omega, M)$ if and only if $\tilde{u} = i \circ u \in W^{1,p}(\Omega, \mathbb{R}^k)$. □

Next, we introduce some matrix groups.

Definition 2.1. For each $k \in \mathbb{N}$, $GL(k)$ is the group of all invertible $k \times k$ matrices with real entries. We also consider the subgroups

$$O(\nu, n+1-\nu) = \left\{A \in GL(n+1) \mid A^T = \mathcal{E} A^{-1} \mathcal{E}\right\}$$
$$SO(\nu, n+1-\nu) = \left\{A \in O(\nu, n+1-\nu) \mid \det A = 1\right\}.$$

Let us recall the exponential of matrices. This will be used in our study of Wente type identity. For each $n \times n$ matrix A, the exponential of A is the $n \times n$ matrix defined by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Proposition 2.1. *Let $X, Y \in M_n(\mathbb{R})$ and let $a, b \in \mathbb{R}$. Then the matrix exponential satisfies the following properties:*

(a) $e^0 = I$, $e^{aX} e^{bX} = e^{(a+b)X}$, $e^X e^{-X} = I$.

(b) If $XY = YX$ then $e^X e^Y = e^Y e^X = e^{(X+Y)}$.

(c) If Y is invertible then $e^{YXY^{-1}} = Y e^X Y^{-1}$.

(d) $e^{(X^T)} = (e^X)^T$.

We close this section with some elementary inequalities which are used in this work.

Lemma 2.2. *Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^m$ be an open set.*

(i) (Minkowski Inequality). If $f, g \in L^p(\Omega)$, then

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

(ii) (Friedrichs Inequality). If $u \in W^{1,p}(\Omega)$ and $u|_{\partial\Omega} = 0$ in the sense of trace, then

$$\|u\|_{L^p(\Omega)} \leq \operatorname{diam}(\Omega) \|\nabla u\|_{L^p(\Omega)}.$$

2.2 Energy of Maps and Harmonic Maps

First we recall the classical Dirichlet principle: On an open subset Ω of a Euclidean space, every solution $u : \Omega \to \mathbb{R}$ to the Laplace equation

$$\triangle u = 0 \qquad \text{in } \Omega$$

is a critical point of the (Dirichlet) energy

$$E_2[w] = \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx.$$

From the point of view of the Calculus of Variations, the Laplace equation is the Euler-Lagrange equation of the functional E_2. This consideration has the following generalization. Let $1 < p < \infty$. We introduce the p-energy to be

$$E_p[w] = \frac{1}{p} \int_\Omega |\nabla w|^p \, dx,$$

where $w : \Omega \to \mathbb{R}$. The Euler-Lagrange equation of E_p is the so-called p-Laplace equation:

$$\triangle_p u = \text{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0,$$

whose solutions are called p-harmonic functions. We remark that this informal discussion should be made formal by specifying the space of functions forming the domain of E_p. Sobolev spaces are often the suitable choice in the preliminary study when there is no a priori regularity.

In the most general form, mathematicians and physicists are interested in the *p-harmonic maps*. Let M, N be smooth Riemannian manifolds. Let $2 \leq p < \infty$. A map $u : M \to N$ is called a *p-harmonic map* if it is a critical point of the *p-energy* defined by

$$E_p[u] = \frac{1}{p} \int_M |\nabla u|^p \, dvol_M.$$

Here ∇u, at each point, is the dual tangent vector to the differential du and $| \cdot |$ is the norm of tangent vectors to N. Thus u is p-harmonic if it satisfies the Euler-Lagrange equation associated with $E_p[\cdot]$ which turns out to be the *p-Laplace system of equations*:

$$\text{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

We note that to be a harmonic map, a map $u : M \to N$ is required to be a smooth map. But we have learned from above that finding harmonic maps is a sub-problem of finding solutions to the p-Laplace system. So, alternatively, one is interested in finding the *weakly p-harmonic maps* which are weak solutions, of course in some Sobolev spaces, to this p-Laplace system. An immediate advantage of considering weakly p-harmonic maps is that the function space in this case is a Banach space comparing with the non-Banach space $C^\infty(M, N)$ for plain harmonic maps. Thus, there are plenty of tools from functional analysis to get weakly p-harmonic maps.

In this work, inspired partly by the work of Zhu [see[8]], we introduce the p-energy for maps into pseudospheres and then study the weakly p-harmonic maps. Since in this work we will prove some regularity results, it is no loss of generality to assume $M = B$, the unit ball in $\mathbb{R}^m$.

Definition 2.3. For a map $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$, we define the *p-energy* by

$$\mathbb{E}_p[u] := \frac{1}{p} \int_B |(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} (\nabla u)^T \mathcal{E}(\nabla u).$$

A map $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$ is called a *weakly p-harmonic map* from B into $\mathbb{S}^n_\nu$, if it is a critical point of the p-energy $\mathbb{E}_p[\cdot]$.

2.3 Elliptic PDEs

In this section, we quote some results from the theory of elliptic PDEs. For studying the regularity aspects, the Morrey norm plays a very important role. Recall $B = B_1(0)$ is the unit ball in $\mathbb{R}^m$.

Definition 2.4. Let $1 \leq q < \infty$. The *Morrey norm* of a function $f \in L^q_{loc}(B)$ is

$$\|f\|_{M^q_q(B)} = \sup_{B_R(x) \subset B} \left(R^{q-m} \int_{B_R(x_0)} |f|^q \right)^{\frac{1}{q}}.$$

Remark. $\bigwedge^2 \mathbb{R}^m$ is the vector bundle of all alternating 2-tensors in $\mathbb{R}^m$, i.e. its elements are

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j \qquad (\omega_{ij} \in \mathbb{R}).$$

Lemma 2.5. *Let $m \geq 2$, $1 \leq s < \infty$, and $1 < q < \infty$. Let $q' \in (1, \infty)$ be the Hölder conjugate of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$. Let $B_R(x) \subset \mathbb{R}^m$. Assume $f \in W^{1,q}(B_R(x))$ and $g \in W^{1,q'}\left(B_R(x), \bigwedge^2 \mathbb{R}^m\right)$ satisfy*

$$f\big|_{\partial B_R(x)} = 0 \quad \textit{or} \quad g\big|_{\partial B_R(x)} = 0$$

and $h \in W^{1,s}(B_{2R}(x))$ satisfies

$$\|\nabla h\|_{M^s_s(B_{2R}(x))} < \infty.$$

Then, there is a uniform constant $C > 0$ independent of R such that

$$\int_{B_R(x)} (\nabla f \cdot \operatorname{curl} g)\, h \leq C \|\nabla f\|_{L^q(B_R(x))} \|\operatorname{curl} g\|_{L^q(B_R(x))} \|\nabla h\|_{M^s_s(B_{2R}(x))}.$$

Theorem 2.6 (Hodge Decomposition)**.** *If $\Phi \in L^q\left(B_R(x_0)), \bigwedge^1 \mathbb{R}^m\right)$, then there exists unique $\alpha \in W^{1,q}_0(B_R(x_0))$, $\beta \in W^{1,q}_0\left(B_R(x_0), \bigwedge^2 \mathbb{R}^m\right)$, and a harmonic $h \in C^\infty\left(B_R(x_0), \bigwedge^1 \mathbb{R}^m\right)$ such that*

$$\Phi = d\alpha + \operatorname{curl} \beta + h.$$

Moreover, we have

$$\|d\alpha\|_{L^q(B_R(x_0))} + \|\operatorname{curl}\beta\|_{L^q(B_R(x_0))} \le C\|\Phi\|_{L^q(B_R(x_0))}.$$

Theorem 2.7 (Morrey's Dirichlet growth theorem)**.** *Let $u \in W^{1,q}(B)$, $1 < q < m$. Suppose that there exist constants $0 < C < \infty$ and $\gamma \in (0,1]$ such that for all balls $B_R(x_0) \subset B$*

$$\int_{B_R(x_0)} |\nabla u|^q \le C R^{m-q+\gamma q},$$

then $u \in C^{0,\gamma}(B)$.

Theorem 2.8 (Existence theorem)**.** *Consider a linear system of PDEs with variable coefficients of the form*

$$-D_\alpha\left(A_{ij}^{\alpha\beta}(x) D_\beta u^j\right) = 0. \tag{2.1}$$

Suppose the leading coefficients $A_{ij}^{\alpha\beta}$ are in $L^\infty(\Omega)$, for all i, j, α, β, and are uniformly elliptic. Then the system (2.1) has a solution.

Lemma 2.9. *Assume the linear system (2.1) has the leading coefficients $A_{ij}^{\alpha\beta}$ in $L^\infty(\Omega)$ and is uniformly elliptic. If, in addition, $A_{ij}^{\alpha\beta} \in C^0(\Omega)$, then*

$$\int_{B_r(x_0)} |\nabla u|^q \le C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |\nabla u|^q + C(\omega_q(R))^q \int_{B_R(x_0)} |\nabla u|^q,$$

where

$$\omega_q(R) = \sup_{x \in B_R(x_0)} \left(\sum \left|A_{ij}^{\alpha\beta}(x) - A_{ij}^{\alpha\beta}(x_0)\right|^q\right)^{\frac{1}{q}}$$

and $C > 0$ is a constant independent of $R > 0$.

CHAPTER III

MAIN RESULTS

3.1 Wente type identity

The following proposition is Wente type identity for p-harmonic maps into pseudospheres.

Lemma 3.1. *Let $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$ be a weakly p-harmonic map. Then u satisfies the identity*

$$\operatorname{div}\left(\left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\left(u^i \nabla u^j - u^j \nabla u^i\right)\right) = 0 \qquad \text{in } \mathcal{D}'(B)$$

for all $i, j = 1, 2, ..., n+1$.

Proof. Fix $i \neq j \in \{1, 2, ..., n+1\}$. Let E_{ij} be an $(n+1)\times(n+1)$ matrix whose all entries are zero except the (i,j)-, and (j,i)-components which are 1 and -1, respectively. Since $(E_{ij})^T = -E_{ij}$, we have $E_{ij} \in \text{so}(n+1)$. Note that $\mathcal{E}^2 = I$ and $\mathcal{E} = \mathcal{E}^T$.

Since $-\mathcal{E} \cdot (E_{ij}\mathcal{E})\mathcal{E} = -\mathcal{E}E_{ij} = -\mathcal{E}E_{ij} = \mathcal{E}^T E_{ij}^T = (E_{ij}\mathcal{E})^T$, $E_{ij}\mathcal{E} \in \text{so}(\nu, n+1-\nu)$. Consider

$$\begin{aligned}\mathcal{E}\left(e^{E_{ij}\mathcal{E}}\right)^{-1}\mathcal{E} &= \mathcal{E}e^{-E_{ij}\mathcal{E}}\mathcal{E} = \mathcal{E}e^{-E_{ij}\mathcal{E}}\mathcal{E}^{-1} \\ &= e^{-\mathcal{E}(E_{ij}\mathcal{E})\mathcal{E}^{-1}} = e^{-\mathcal{E}(E_{ij}\mathcal{E})\mathcal{E}} \\ &= e^{(E_{ij}\mathcal{E})^T} = \left(e^{E_{ij}\mathcal{E}}\right)^T.\end{aligned}$$

So $e^{E_{ij}\mathcal{E}} \in O(\nu, n+1-\nu)$. For any $\varphi \in C_0^\infty(B)$, we introduce a variation

$$R_t := e^{t\varphi E_{ij}\mathcal{E}} \in C_0^\infty\left(B, O(\nu, n+1-\nu)\right).$$

Using the property of the group $O(\nu, n+1-\nu)$, we have

$$\langle R_t u, R_t u\rangle_{\mathbb{R}^{n+1}_\nu} = (R_t u)^T \mathcal{E} R_t u = u^T R_t^T \mathcal{E} R_t u = u^T \mathcal{E} u = 1,$$

almost everywhere in B. Since u is weakly p-harmonic, we have

$$0 = \frac{d}{dt}\Big|_{t=0} E_p(R_t u) = \varphi E_{ij}\mathcal{E}\frac{d}{dt}\Big|_{t=0} E_p(u).$$

Additionally, we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} E_p(R_t u) &= \frac{d}{dt}\Big|_{t=0}\frac{1}{p}\int_B \left|(\nabla R_t u)^T \mathcal{E}(\nabla R_t u)\right|^{\frac{p-2}{2}}(\nabla R_t u)^T\mathcal{E}(\nabla R_t u) \\ &= \frac{2}{p}\int_B \Bigg\{ \left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\left((\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}u\nabla\varphi + (\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}\nabla u\,\varphi\right) \\ &\quad + \left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-6}{2}}\left(\frac{p-2}{4}(\nabla u)^T\mathcal{E}(\nabla u))(2)((\nabla u)^T\mathcal{E}(\nabla u)\right) \\ &\qquad \times \left((\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}u\nabla\varphi + (\nabla u)^T\mathcal{E}E_{ij}\mathcal{E}\nabla u\,\varphi\right)\Bigg\}.\end{aligned}$$

Since $(\nabla u)^T \mathcal{E} E_{ij} \mathcal{E} u = \varepsilon_{ii}\varepsilon_{jj}(u^i \nabla u^j - u^j \nabla u^i)$ and $(\nabla u)^T \mathcal{E}(\nabla u) = 0$, we get that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} E_p(R_t u) = \frac{2}{p}\int_B \Big\{ &\left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \varepsilon_{ii}\varepsilon_{jj}(u^i \nabla u^j - u^j \nabla u^i)\cdot \nabla\varphi \\ &+ (p-2)|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \varepsilon_{ii}\varepsilon_{jj}(u^i \nabla u^j - u^j \nabla u^i)\cdot \nabla\varphi \Big\} \\ = \varepsilon_{ii}\varepsilon_{jj} \int_B &\left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \left(u^i \nabla u^j - u^j \nabla u^i\right)\cdot \nabla\varphi.\end{aligned}$$

As $\varepsilon_{ii}\varepsilon_{jj}$ is either 1 or -1, we have

$$\int_B \left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \left(u^i \nabla u^j - u^j \nabla u^i\right)\cdot \nabla\varphi = 0.$$

Since $\varphi \in C_0^\infty(B)$ is arbitrary, we obtain that

$$\operatorname{div}\left(|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} (u^i \nabla u^j - u^j \nabla u^i)\right) = 0 \qquad \text{in } \mathcal{D}'(B),$$

for all $i, j = 1, 2, ..., n+1$ with $i \neq j$. The case $i = j$ is trivial hence the proof of this lemma is complete. □

3.2 Divergence Equation

From the preceding lemma, we introduce for convenience the following matrix-valued (of rank $n+1$) vector field: Let $\Theta = (\Theta^{ij})$ where

$$\Theta^{ij} = \left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} (u^i \nabla u^j - u^j \nabla u^i)$$

for all $i, j = 1, 2, ..., n+1$.

Lemma 3.2. *Let $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$ where $1 \leq \nu \leq n$. Then u satisfies*

$$\left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \nabla u + \Theta \mathcal{E} u = 0 \qquad \textit{a.e. in } B.$$

Consequently, u satisfies

$$\operatorname{div}\left(|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \nabla u + \Theta \mathcal{E} u\right) = 0 \qquad \textit{in } \mathcal{D}'(B).$$

Proof. By the definition of $W^{1,p}(B, \mathbb{S}^n_\nu)$, we have $u^j \varepsilon_{jk} u^k = 1$ for almost every point in B. Taking ∇ both sides of the equation, we get $\nabla u^j \varepsilon_{jk} u^k = 0$ a.e. in B. So

$$\begin{aligned}&|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \nabla u^i + \Theta^{ij} \varepsilon_{jk} u^k \\ &\quad = |(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \nabla u^i + |(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} (u^i \nabla u^j - u^j \nabla u^i)\varepsilon_{jk} u^k \\ &\quad = |(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} (\nabla u^i + (u^i \nabla u^j - u^j \nabla u^i)\varepsilon_{jk} u^k) \\ &\quad = |(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} (\nabla u^i (1 - u^j \varepsilon_{jk} u^k) + u^i (\, u^j \varepsilon_{jk} u^k)) \\ &\quad = 0 \ \text{ a.e. in } \ B.\end{aligned}$$

Since $u \in W^{1,p}(B, S^n_v)$, we have $|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \nabla u^i + \Theta^{ij} \varepsilon_{jk} u^k \in L^1(B)$ for each i. Taking $-\operatorname{div}$ on both sides of the above equation we get $-\operatorname{div}(|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \nabla u + \Theta \mathcal{E} u) = 0$ in $\mathcal{D}'(B)$. □

3.3 Morrey's norm Estimate

Lemma 3.3. *Let $u \in W^{1,p}(B, \mathbb{S}^n_\nu)$, $1 \le \nu \le n$. Assume that u is a weakly p-harmonic map such that for any fixed $1 < q < \frac{m}{m-1}$ there holds*

$$\left\| \left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B)} \|\nabla u\|^2_{M^q_q(B)} < \infty.$$

Then we have the following estimate:

$$\|\Theta\|_{M^q_q(B_{1/2})} \le C \left\| \left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B)} \|\nabla u\|^2_{M^q_q(B)},$$

where $C > 0$ is a constant independent of u.

Proof. Let $s = \frac{q}{q-1} > m$ be the conjugate exponent of q and let

$$B_R := B_R(x_0) \subset B_{1/2}.$$

For any $\Phi \in L^s\left(B_R, \bigwedge^1 \mathbb{R}^m\right)$ with $\|\Phi\|_{L^s(B_R)} \le 1$ and $0 < \rho < R$, let $\tau \in C^\infty_0(B_R, [0,1])$ be a cut-off function satisfying $\tau \equiv 1$ on $B_\rho(x_0)$. Then $\tau\Phi$ is supported in $B_R(x_0)$ and vanishes on $\partial B_R(x_0)$.

By Hodge decomposition, there exists $\alpha \in W^{1,s}_0(B_R)$, $\beta \in W^{1,s}_0\left(B_R, \bigwedge^2 \mathbb{R}^m\right)$ and a harmonic function $h \in C^\infty\left(B_R, \bigwedge^1 \mathbb{R}^m\right)$ such that $\tau\Phi = \nabla\alpha + \operatorname{curl}\beta + h$. Moreover, we have

$$\begin{aligned}\|\nabla\alpha\|_{L^s(B_R)} + \|\operatorname{curl}\beta\|_{L^s(B_R)} &\le C\|\tau\Phi\|_{L^s(B_R)} \\ &\le C_1\|\Phi\|_{L^s(B_R)} \le C_1,\end{aligned}$$

where $C_1 > 0$ is a constant independent of ρ and R. Since $\nabla\alpha\big|_{\partial B_R} = \operatorname{curl}\beta\big|_{\partial B_R} = 0$, we get $h\big|_{\partial B_R} = (\tau\Phi)\big|_{\partial B_R} = 0$. Since h is harmonic, it follows that $h \equiv 0$ in $B_R(x_0)$.

Now using that u is a weakly p-harmonic map, we obtain that

$$\Theta = \left(|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} \left(u^i \nabla u^j - u^j \nabla u^i\right)\right)$$

is divergence free. So $\int_{B_R(x_0)} \Theta^{ij} \cdot (\nabla\alpha) = 0$. Then, we estimate for fixed $i, j = 1, 2, ..., n+1$ the integral:

$$\begin{aligned}\int_{B_R(x_0)} (\tau\Theta^{ij}) \cdot \Phi &= \int_{B_R(x_0)} \Theta^{ij} \cdot (\tau\Phi) = \int_{B_R(x_0)} \Theta^{ij} \cdot (\nabla\alpha + \operatorname{curl}\beta) = \int_{B_R(x_0)} \Theta^{ij} \cdot \operatorname{curl}\beta \\ &= \int_{B_R(x_0)} |(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} (u^i \nabla u^j - u^j \nabla u^i) \cdot \operatorname{curl}\beta \\ &= \int_{B_R(x_0)} (\nabla u^j \cdot \operatorname{curl}\beta) \left(|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} u^i\right) \\ &\qquad - \int_{B_R(x_0)} \left(\nabla u^i \cdot \operatorname{curl}\beta\right) \left(|(\nabla u)^T \mathcal{E}(\nabla u)|^{\frac{p-2}{2}} u^j\right) \\ &\le C_2 \left\| \left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B)} \|\nabla u\|_{L^q(B_R)} \|\operatorname{curl}\beta\|_{L^s(B_R)} \|\nabla u\|_{M^q_q(B_{2R})} \\ &\le C_1 C_2 \left\| \left|(\nabla u)^T \mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B)} \|\nabla u\|_{L^q(B_R)} \|\nabla u\|_{M^q_q(B_{2R})},\end{aligned}$$

where $C_2 > 0$ is a constant independent of R and $B_{2R} = B_{2R}(x_0)$. By the duality characterization of L^p functions, we have

$$\begin{aligned}\left\|\tau\Theta^{ij}\right\|_{L^q(B_R)} &= \sup_{\|\Phi\|_{L^q(B_R)}\le 1}\left|\int_{B_R(x_0)}(\tau\Theta^{ij})\cdot\Phi\right| \\ &\le C_1C_2\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\|\nabla u\|_{L^q(B_R)}\|\nabla u)\|_{M_q^q(B_{2R})}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\Theta^{ij}\|_{L^q(B_\rho)} &\le \|\tau\Theta^{ij}\|_{L^q(B_R)} \\ &\le C_1C_2\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\|\nabla u\|_{L^q(B_R)}\|\nabla u\|_{M_q^q(B_{2R})}.\end{aligned}$$

Since $\rho \in (0, R)$ is arbitrary, letting $\rho \nearrow R$, we get

$$\|\Theta^{ij}\|_{L^q(B_R)} \le C_1C_2\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\|\nabla u\|_{L^q(B_R)}\|\nabla u\|_{M_q^q(B_{2R})}.$$

Then by the definition of the Morrey norm $\|\nabla u\|_{M_q^q(B_{2R})}$ and the fact that $B_{2R} \subset B$, we obtain

$$\begin{aligned}\|\nabla u\|_{L^q(B_R)} &= \left(\int_{B_R}|\nabla u|^q\right)^{\frac{1}{q}} \\ &= R^{\frac{m}{q}-1}\left(R^{q-m}\int_{B_R(x_0)}|\nabla u|^q\right)^{\frac{1}{q}} \\ &\le R^{\frac{m}{q}-1}\|\nabla u\|_{M_q^q(B)}.\end{aligned}$$

We estimate

$$\begin{aligned}\|\Theta\|_{L^q(B_R)} &= \sum_{i,j}|\Theta^{ij}\|_{L^q(B_R)} \\ &\le (n+1)^2C_1C_2\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\|\nabla u\|_{L^q(B_R)}\|\nabla u\|_{M_q^q(B_{2R})} \\ &\le (n+1)^2CR^{\frac{m}{q}-1}\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\|\nabla u\|_{M_q^q(B)}\|\nabla u\|_{M_q^q(B)}.\end{aligned}$$

Since the ball $B_R(x_0)$ is arbitrary, it follows that

$$\begin{aligned}\|\Theta\|_{M_q^q(B_{1/2})} &= \sup_{B_R(x_0)\subset B_{1/2}}\left(R^{q-m}\int_{B_R(x_0)}|\Theta|^p\right)^{\frac{1}{p}} \\ &\le (n+1)^2C_1C_2\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\|\nabla u\|^2_{M_q^q(B)}.\end{aligned}$$

Thus, we have completed the proof of the lemma. □

3.4 Regularity Results

The following Lemma is the main key to prove Theorem 3.6. The important idea of the proof of this lemma is an elliptic system estimate from M. Giaquinta[2].

Lemma 3.4. *Let $m \geq 2$, $1 < q < \frac{m}{m-1}$, and $\delta > 0$. There exists a constant $\varepsilon_{m,q,\delta} > 0$ such that if $\Omega \in L^p\left(B, M_n(\mathbb{R}) \otimes \bigwedge^1 \mathbb{R}^m\right)$ satisfies $\operatorname{div}\Omega = 0$ in $\mathcal{D}'(B)$, $u \in W^{1,p}(B,\mathbb{R}^n)$ weakly solves $\operatorname{div}\left((|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla u) + \Omega u\right) = 0$ in $\mathcal{D}'(B)$ with $\left|(\nabla u)^T\mathcal{E}(\nabla u)\right| \geq \delta$ for all $x \in B$, and*

$$\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)} + \|\nabla u\|_{M_q^q(B)} + \|\Omega\|_{M_q^q(B)} < \varepsilon_{m,q,\delta},$$

then u is Hölder continuous in B.

Proof. Fix any $1 < q < \frac{m}{m-1}$ and $\delta > 0$. Since $\operatorname{div}\Omega = 0$, by Hodge decomposition there exists $\xi \in W^{1,q}\left(B, M_n(\mathbb{R}) \otimes \bigwedge^2 \mathbb{R}^m\right)$ such that

$$\omega = \operatorname{curl}\xi.$$

Let $B_{2R} = B_{2R}(x_0) \subset B$ and as before $B_R := B_R(x_0)$. The assumption $\left|(\nabla u)^T\mathcal{E}(\nabla u)\right| \geq \delta$ on B implies that the following equation is uniformly elliptic:

$$\begin{cases} -\operatorname{div}\left(|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla w\right) = 0 \quad in \quad B_{\mathbb{R}}(x_0), \\ w = u \quad in \quad \partial B_{\mathbb{R}}(x_0). \end{cases} \tag{3.1}$$

Let w be the unique weak solution to the above problem.

Then $v = u - w \in W^{1,p}(B_R(x_0), \mathbb{R}^n)$ weakly solves

$$\begin{cases} -\operatorname{div}\left((|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla v) + \Omega u\right) = 0 \quad in \quad B_{\mathbb{R}}(x_0) \\ v = 0 \quad in \quad \partial B_{\mathbb{R}}(x_0) \end{cases}$$

Let $s = \frac{q}{q-1} > m$. For any $\varphi \in W_0^{1,s}(B_R(x_0))$ with $\|\varphi\|_{W^{1,s}(B_R(x_0))} \leq 1$, since

$$-\operatorname{div}\left((|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla v) + \Omega u\right) = 0,$$

we estimate for each i that

$$\begin{aligned}
\int_{B_R}\left(|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla v^i\right)\cdot\nabla\varphi &= -\int_{B_R(x_0)}(\Omega^{ij}u^j)\cdot\nabla\varphi \\
&= -\int_{B_R(x_0)}(\operatorname{curl}\xi^{ij}\cdot\nabla\varphi)u^j \\
&\leq C_1\|\operatorname{curl}\xi^{ij}\|_{L^q(B_R)}\|\nabla\varphi\|_{L^s(B_R)}\|\nabla u\|_{M_q^q(B_{2R})} \\
&\leq C_1\|\Omega^{ij}\|_{L^q(B_R)}\|\nabla u\|_{M_q^q(B_{2R})},
\end{aligned}$$

where $C_1 > 0$ is constant independent of R.

By the definition of Morrey norm, we have

$$\begin{aligned}
\|\Omega^{ij}\|_{L^q(B_R)} &= \left(\int_{B_R(x_0)}|\Omega^{ij}|^q\right)^{\frac{1}{q}} \\
&= R^{\frac{m}{q}-1}\left(R^{q-m}\int_{B_R(x_0)}|\Omega^{ij}|^q\right)^{\frac{1}{q}} \\
&\leq R^{\frac{m}{q}-1}\|\Omega^{ij}\|_{M_q^q(B)}.
\end{aligned}$$

So

$$\int_{B_R}\left(|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla v^i\right)\cdot\nabla\varphi \le C_1 R^{\frac{m}{q}-1}\|\Omega^{ij}\|_{M_q^q(B)}\|\nabla u\|_{M_q^q(B_{2R})}$$
$$\le C_1 R^{\frac{m}{q}-1}\varepsilon_{m,q}\|\nabla u\|_{M_q^q(B_{2R})}.$$

Since $v\big|_{\partial B_R(x_0)}=0$, we have by Friedrich's inequality and the L^p-duality that

$$\|\nabla v\|_{L^q(B_R)} \le C_2 \sup_{\|\varphi\|_{W^{1,s}(B_R)}\le 1}\int_{B_R}\nabla v\cdot\nabla\varphi,$$

where $C_2>0$ is constant independent of R. Since $|(\nabla u)^T\mathcal{E}(\nabla u)|\ge\delta$ for all $x\in B$, we have

$$C_2 \sup_{\|\varphi\|_{W^{1,s}(B_R)}\le 1}\int_{B_R}\nabla v\cdot\nabla\varphi \le \frac{C_2}{\delta^{\frac{p-2}{2}}}\sup_{\|\varphi\|_{W^{1,s}(B_R)}\le 1}\int_{B_R}|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla v\cdot\nabla\varphi.$$

So

$$\|\nabla v\|_{L^q(B_R)} \le \frac{C_1C_2(n+1)}{\delta^{\frac{p-2}{2}}}R^{\frac{m}{q}-1}\varepsilon_{m,q,\delta}\|\nabla u\|_{M_q^q(B_{2R})}.$$

Since w solves (3.1), we have that for any $r<R$ that the following estimates hold:

$$\int_{B_r}|\nabla w|^q \le C_3\left(\frac{r}{R}\right)^m\int_{B_R(x_0)}|\nabla w|^q + C_3(\omega_q(R))^q\int_{B_R(x_0)}|\nabla w|^q,$$

where

$$\omega_p(R)=\sup_{x\in B_R(x_0)}\left|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}(x)-\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}(x_0)\right|$$

and $C_3>0$ is a constant independent of R.

Since $u=v+w$, applying the Minkowski and the Power Mean inequalities, we obtain

$$\int_{B_r(x_0)}|\nabla u|^q \le \left(\left(\int_{B_r(x_0)}|\nabla w|^q\right)^{\frac{1}{q}}+\left(\int_{B_r(x_0)}|\nabla v|^q\right)^{\frac{1}{q}}\right)^q$$
$$\le 2^{q-1}\int_{B_r(x_0)}|\nabla w|^q + 2^{q-1}\int_{B_r(x_0)}|\nabla v|^q.$$

Similary, we have

$$\int_{B_R}|\nabla w|^q \le 2^{q-1}\int_{B_R}|\nabla u|^q + 2^{q-1}\int_{B_R}|\nabla v|^q.$$

So, for $B_r = B_r(x_0)$ with $0 < r < R$, we obtain

$$\begin{aligned}
\int_{B_r} |\nabla u|^q &\leq 2^{q-1}\int_{B_r} |\nabla w|^q + 2^{q-1}\int_{B_r} |\nabla v|^q \\
&\leq C_3 2^{q-1}\left(\frac{r}{R}\right)^m \int_{B_R} |\nabla w|^q + C_3 2^{q-1}(\omega_p(R))^q \int_{B_R} |\nabla w|^q + 2^{q-1}\int_{B_r} |\nabla v|^q \\
&\leq C_3 2^{q-1}\left(\frac{r}{R}\right)^m \int_{B_R} |\nabla u|^q + C_3 2^{2(q-1)}(\omega_p(R))^q \int_{B_R} |\nabla u|^q + 2^{q-1}\int_{B_r} |\nabla v|^q \\
&\qquad + C_3 2^{2(q-1)}(\omega_p(R))^q \int_{B_R} |\nabla v|^q + 2^{2(q-1)}\int_{B_R} |\nabla v|^q \\
&\leq C_3 2^{q-1}\left(\frac{r}{R}\right)^m \int_{B_R} |\nabla u|^q + C_3 2^{3q-2}\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\|^q_{L^\infty(B)} \int_{B_R} |\nabla u|^q \\
&\qquad + 2^{q-1}\int_{B_r} |\nabla v|^q + C_3 2^{3q-2}\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|^q_{L^\infty(B)} \int_{B_R} |\nabla v|^q \\
&\qquad\qquad + 2^{2(q-1)}\int_{B_R} |\nabla v|^q \\
&\leq C_3 2^{q-1}\left(\frac{r}{R}\right)^m \int_{B_R} |\nabla u|^q + C_3 2^{3q-2}\varepsilon^q_{m,q,\delta}\int_{B_R} |\nabla u|^q \\
&\qquad + \left(\frac{C_1C_2(n+1)}{\delta}^{\frac{p-2}{2}}\right)^q 2^{2q-1}R^{m-q}\varepsilon^q_{m,q,\delta}\|\nabla u\|^q_{M^q_q(B_{2R})} \\
&\qquad\qquad + \left(\frac{C_1C_2(n+1)}{\delta^{\frac{p-2}{2}}}\right)^q C_3 2^{3q-2}\varepsilon^{2q}_{m,q,\delta}R^{m-q}\|\nabla u\|^q_{M^q_q(B_{2R})}.
\end{aligned}$$

Divide by r^{m-q} to get

$$\begin{aligned}
\frac{1}{r^{m-q}}\int_{B_r} |\nabla u|^q &\leq C\left(\frac{r}{R}\right)^q \frac{1}{R^{m-q}}\int_{B_R} |\nabla u|^q + C\frac{1}{R^{m-q}}\varepsilon^q_{m,q,\delta}\int_{B_R} |\nabla u|^q \\
&\qquad + C\left(\frac{R}{r}\right)^{m-q}\varepsilon^q_{m,q,\delta}\|\nabla u\|^q_{M^q_q(B_{2R})} + C\left(\frac{R}{r}\right)^{m-q}\varepsilon^{2q}_{m,q,\delta}\|\nabla u\|^q_{M^q_q(B_{2R})},
\end{aligned}$$

where $C = \max\left\{C_3 2^{3q-2}, (C_1C_2(n+1)/\delta^{\frac{p-2}{2}})^q C_3 2^{3q-2}\right\}$.

Let

$$\Psi(R) = \sup_{B_r(x_0)\subset B, 0<r\leq R}\left(r^{q-m}\int_{B_r(x_0)} |\nabla u|^q\right).$$

Choose $\gamma \in (0, \frac{1}{2})$ such that $C\gamma^{\frac{q-1}{2}} \leq \frac{1}{4}$ and set $\varepsilon^q_{m,q,\delta} := \gamma^m$. Then for $r := \gamma R$, we have

$$\begin{aligned}
\frac{1}{r^{m-q}}\int_{B_r} |\nabla u|^q &\leq C\gamma^q\|\nabla u\|^q_{M^q_q(B_{2R})} + C\gamma^m\|\nabla u\|^q_{M^q_q(B_{2R})} \\
&\qquad + C\gamma^q\|\nabla u\|^q_{M^q_q(B_{2R})} + C\gamma^{2q}\|\nabla u\|^q_{M^q_q(B_{2R})} \\
&\leq 4C\gamma^q\|\nabla u\|^q_{M^q_q(B_{2R})} \\
&\leq \gamma^{\frac{q+1}{2}}\|\nabla u)\|^q_{M^q_q(B_{2R})} \\
&\leq \gamma^{\frac{q+1}{2}}\Psi(2R)
\end{aligned}$$

so by taking the supremum, we find that

$$\Psi(\gamma R) \leq \gamma^{\frac{q+1}{2}}\Psi(2R).$$

Finally, for any $r \in [0, \frac{\gamma}{2}]$, let $k \in \mathbb{N}$ be such that $(\frac{\gamma}{2})^{k+1} < r \le (\frac{\gamma}{2})^k$ and $x_0 \in B$ be such that $B_r(x_0) \subset B$. Then

$$\begin{aligned}\Psi(r) &\le \Psi\left(\left(\frac{\gamma}{2}\right)^k\right)\\ &\le \left(\left(\frac{\gamma}{2}\right)^{\frac{q+1}{2}}\right)^k \Psi(1)\\ &\le \left(\frac{\gamma}{2}\right)^{k+1}\Psi(1)\\ &\le r\|\nabla u)\|^q_{M^q_q(B)}\\ &\le \varepsilon^q_{m,q,\delta} r.\end{aligned}$$

By Morrey's Dirichlet growth theorem, we now conclude that $u \in C^{0,1/q}(B)$. □

Similarly to lemma 3.4, we can prove the following result.

Lemma 3.5. *Let $m \ge 2$, $1 < q < \frac{m}{m-1}$, and $\delta > 0$. There exists $\varepsilon_{m,q,\delta} > 0$ such that if $\Omega \in L^p\left(B, M_n(\mathbb{R}) \otimes \bigwedge^1 \mathbb{R}^m\right)$ satisfies $\operatorname{div}\Omega = 0$ in $\mathcal{D}'(B)$, $u \in W^{1,p}(B,\mathbb{R}^n)$ weakly solves $\operatorname{div}\left((|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla u) + \Omega u\right) = 0$ in $\mathcal{D}'(B)$ with $|(\nabla u)^T\mathcal{E}(\nabla u)| \ge \delta > 0$ for all $x \in B$, and*

$$\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla u\|_{M^q_q(B_{\frac{1}{2}})} + \|\Omega\|_{M^q_q(B_{\frac{1}{2}})} < \varepsilon_{m,q,\delta},$$

then u is Hölder continuous in B.

3.5 Main Theorem

Finally, we prove the main theorem.

Theorem 3.6. *Let $m \ge 2$, $1 < q < \frac{m}{m-1}$, and $\delta > 0$. There exists a constant $\varepsilon'_{m,q,\delta} > 0$ such that if $u \in W^{1,p}(B,\mathbb{S}^n_\nu)$ is a weakly p-harmonic map satisfying $|(\nabla u)^T\mathcal{E}(\nabla u)| \ge \delta > 0$ for all $x \in B$ and*

$$\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B)} + \|\nabla u\|_{M^q_q(B)} < \varepsilon'_{m,q,\delta},$$

then u is Hölder continuous in B.

Proof. By lemma 3.2, we have

$$\operatorname{div}\left(|(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\nabla u + \Theta\mathcal{E}u\right) = 0 \qquad \text{in } \mathcal{D}'(B).$$

for all $u \in W^{1,p}(B,\mathbb{S}^n_\nu)$. By lemma 3.1, $\operatorname{div}\Theta = 0$. Since $\mathcal{E}$ is a constant matrix, $\operatorname{div}(\Theta\mathcal{E}) = 0$. Note that $\|\Theta\mathcal{E}\|_{M^q_q(B_{1/2})} = \|\Theta\|_{M^q_q(B_{1/2})}$. By lemma 3.5, there exists $\varepsilon_{m,q,\delta} > 0$ with for any weakly p-harmonic map $u \in W^{1,p}(B,\mathbb{S}^n_\nu)$ such that $|(\nabla u)^T\mathcal{E}(\nabla u)| \ge \delta$ for all $x \in B$ we have

$$\left\|\left|(\nabla u)^T\mathcal{E}(\nabla u)\right|^{\frac{p-2}{2}}\right\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla u\|_{M^q_q(B_{\frac{1}{2}})} + \|\Theta\|_{M^q_q(B_{\frac{1}{2}})} < \varepsilon_{m,q,\delta}.$$

By lemma 3.3, we have the estimate

$$\|\Theta\|_{M_q^q(B_{1/2})} \le C\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\|_{L^\infty(B)}\|\nabla u\|^2_{M_q^q(B)},$$

where $C>0$ is a constant. We choose $\varepsilon'_{m,q,\delta} = \min\left\{\varepsilon_{m,q,\delta}/3, \sqrt{27/4C}\right\}$. Let $u \in W^{1,p}(B,\mathbb{S}^n_\nu)$ be a weakly p-harmonic map such that $|(\nabla u)^T\mathcal{E}(\nabla u)| \ge \delta$ for all $x\in B$ and

$$\left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B)} + \|\nabla u\|_{M_q^q(B)} < \varepsilon'_{m,q,\delta}.$$

Consider

$$\begin{aligned}&\left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B_{1/2})} + \|\nabla u\|_{M_q^q(B_{1/2})} + \|\Theta\|_{M_q^q(B_{1/2})}\\ &\le \left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B_{1/2})} + \|\nabla u\|_{M_q^q(B_{1/2})} + C\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\|_{L^\infty(B)}\|\nabla u\|^2_{M_q^q(B)}.\end{aligned}$$

By the arithmetic-geometric means inequality, we get

$$\begin{aligned}&\left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B)}\left(\frac{\|\nabla u\|_{M_q^q(B)}}{2}\right)^2\\ &\quad\le \frac{1}{27}\left(\left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B)} + \|\nabla u\|_{M_q^q(B)}\right)^3.\end{aligned}$$

So

$$\begin{aligned}&\left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B_{1/2})} + \|\nabla u\|_{M_q^q(B_{1/2})} + \|\Theta\|_{M_q^q(B_{1/2})}\\ &\le \left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B_{1/2})} + \|\nabla u\|_{M_q^q(B_{1/2})}\\ &\qquad + \frac{4C}{27}\left(\left\||(\nabla u)^T\mathcal{E}(\nabla u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B)} + \|\nabla u\|_{M_q^q(B)}\right)^3\\ &\le \frac{\varepsilon_{m,q,\delta}}{3} + \frac{4C}{27}\cdot\frac{27}{4C}\frac{\varepsilon_{m,q,\delta}}{3}\\ &= \frac{2\varepsilon_{m,q,\delta}}{3} < \varepsilon_{m,q,\delta}.\end{aligned}$$

Thus u is Hölder continuous on $B_{1/2}(0)$.

Finally, we use a rescaling argument to prove that u is Hölder continuous on the whole B. Fix $x_0\in B$. Choose $0<r<1$ such that $B_r(x_0)\subset B$. Define $\tilde{u} = u|_{B_r(x_0)} : B_r(x_0)\to\mathbb{S}^n_\nu$ and define $\tilde{\tilde{u}} : B\to\mathbb{S}^n_\nu$ by $\tilde{\tilde{u}}(x) = \tilde{u}(\frac{x-x_0}{r})$. Let $y=\frac{x-x_0}{r}$. So that

$$\|\nabla_x\tilde{\tilde{u}}\|_{M_q^q(B)} = r\|\nabla_y u\|_{M_q^q(B_r(x_0))}$$

and

$$\left\||(\nabla_x\tilde{\tilde{u}})^T\varepsilon(\nabla_x\tilde{\tilde{u}})|^{\frac{p-2}{2}}\right\|_{L^\infty(B)} = r^{p-2}\left\||(\nabla_y u)^T\varepsilon(\nabla_y u)|^{\frac{p-2}{2}}\right\|_{L^\infty(B_r(x_0))}.$$

Thus we have

$$\begin{aligned}
&\|\nabla_x \tilde{\tilde{u}}\|_{M_q^q(B)} + \left\| \left|(\nabla_x \tilde{\tilde{u}})^T \varepsilon(\nabla_x \tilde{\tilde{u}})\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B)} \\
&\qquad = r\|\nabla_y u\|_{M_q^q(B_r(x_0))} + r^{p-2} \left\| \left|(\nabla_y u)^T \varepsilon(\nabla_y u)\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B_r(x_0))} \\
&\qquad \leq \|\nabla_y u\|_{M_q^q(B_r(x_0))} + \left\| \left|(\nabla_y u)^T \varepsilon(\nabla_y u)\right|^{\frac{p-2}{2}} \right\|_{L^\infty(B_r(x_0))} \\
&\qquad \leq \varepsilon'_{m,q,\delta}.
\end{aligned}$$

Thus $\tilde{\tilde{u}}$ is Hölder continuous on $B_{1/2}$. This implies that u is Hölder continuous in $B_{r/2}(x_0)$ for all $x_0 \in B$. We therefore conclude that u is locally Hölder continuous on B. □

CHAPTER IV

CONCLUSION AND DISCUSSION

The results of this work generalization some results from the paper of M. Zhu [8]. We can prove some ε-regularity for weakly p-harmonic maps into pseudospheres under assumption on a strictly positive lower bound for the term $|(\nabla u)^T \mathcal{E}(\nabla u)|$. So ε of the main result depends on p, the dimension of the domain of the Euclidean Disk m, and a lower bound δ. However, in [8], epsilon of the main result depends only on p and m. The author does not know whether the strict δ-lower bound is a necessary condition or it can be derived a priori, so that our main ε-regularity result is true. It is therefore very interesting to settle this open question.

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VITA

Name : Mr. Wasanont Pongsawat

Date of Birth : 17 September 1990.

Place of Birth : Bangkok, Thailand.

Education : B.Sc. (Mathematics), (First Class Honors),
Chulalongkorn University, 2011.