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LINEAR TRANSFORMATION SEMIGROUPS ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO

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สุรีย์พร ชาวแพรกน้อย : กึ่งกรุปการแปลงเชิงเส้นซึ่งให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์ (LINEAR TRANSFORMATION SEMIGROUPS ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO)
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กึ่งริง คือระบบ $(A, +, \bullet)$ ซึ่ง $(A, +)$ และ $(A, \bullet)$ เป็นกึ่งกรุป และการดำเนินการ $\bullet$ กระจายบนการดำเนินการ $+$ เรากล่าวว่ากึ่งริง $(A, +, \bullet)$ *สลับที่ภายใต้การบวก* (AC) ถ้า $x + y = y + x$ สำหรับทุก $x, y \in A$ *ศูนย์*ของกึ่งริง $(A, +, \bullet)$ คือสมาชิก $0 \in A$ ซึ่ง $x + 0 = 0 + x = x$ และ $x \bullet 0 = 0 \bullet x = 0$ สำหรับทุก $x \in A$ สำหรับกึ่งกรุป S ให้ S^0 คือ S ถ้า S มีศูนย์และ S มีสมาชิกมากกว่าหนึ่งตัว นอกนั้นให้ S^0 คือกึ่งกรุป S ที่ผนวกด้วยศูนย์ 0 เรากล่าวว่ากึ่งกรุป S *ให้โครงสร้างของกึ่งริง [AC] ที่มีศูนย์* ถ้ามีการดำเนินการ $+$ บน S^0 ที่ทำให้ $(S^0, +, \bullet)$ เป็นกึ่งริง [AC] ที่มีศูนย์ โดยที่ $\bullet$ เป็นการดำเนินการบน S^0

*การดำเนินการไฮเพอร์*บนเซตไม่ว่าง H คือฟังก์ชัน $\circ : H \times H \to P^*(H)$ โดย $P(H)$ แทนเซตกำลังของ H และ $P^*(H) = P(H)\backslash\{\varnothing\}$ ในกรณีนี้ เราเรียก $(H, \circ)$ ว่า*ไฮเพอร์กรุปพอยด์* สำหรับไฮเพอร์กรุปพอยด์ $(H, \circ)$ และเซตย่อยไม่ว่าง X และ Y ของ H เราให้ $X \circ Y$ แทนส่วนร่วมของเซต $x \circ y$ ทั้งหมด โดยที่ $x \in X$ และ $y \in Y$ *กึ่งไฮเพอร์กรุป* คือ ไฮเพอร์กรุปพอยด์ $(H, \circ)$ ซึ่ง $(x \circ y) \circ z = x \circ (y \circ z)$ สำหรับทุก $x, y, z \in H$ *กึ่งไฮเพอร์ริง* คือระบบ $(A, +, \bullet)$ ซึ่งสอดคล้องสมบัติต่อไปนี้ $(A, +)$ เป็นกึ่งไฮเพอร์กรุป $(A, \bullet)$ เป็นกึ่งกรุป และการดำเนินการ $\bullet$ กระจายบนการดำเนินการไฮเพอร์ $+$ *ศูนย์*ของกึ่งไฮเพอร์ริง $(A, +, \bullet)$ คือสมาชิก $0 \in A$ ซึ่ง $x + 0 = 0 + x = \{x\}$ และ $x \bullet 0 = 0 \bullet x = 0$ สำหรับทุก $x \in A$ เช่นเดียวกัน เรากล่าวว่ากึ่งไฮเพอร์ริง*สลับที่ภายใต้การบวก* (AC) ถ้า $x + y = y + x$ สำหรับทุก $x, y \in A$ เรานิยาม *กึ่งกรุปที่ให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์* ในทำนองเดียวกัน

ให้ V เป็นปริภูมิเวกเตอร์บนริงการหาร R และ $L_R(V)$ เป็นกึ่งกรุปภายใต้การประกอบที่ประกอบด้วยการแปลงเชิงเส้น $\alpha : V \to V$ ทั้งหมด *กึ่งกรุปการแปลงเชิงเส้น* บน V หมายถึงกึ่งกรุปย่อยของ $L_R(V)$ *การแปลงเชิงเส้นบางส่วน*ของ V คือการแปลงเชิงเส้นจากปริภูมิย่อยของ V ไปยัง V เราศึกษากึ่งกรุปการแปลงเชิงเส้นหลากหลายชนิด เราศึกษาว่าเมื่อใดกึ่งกรุปเหล่านี้ให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์ เราแสดงว่ากึ่งกรุปใดๆ ที่ไม่มีศูนย์จะให้โครงสร้างของกึ่งไฮเพอร์ริง AC ที่มีศูนย์และโครงสร้างของกึ่งริงที่มีศูนย์เสมอ อย่างไรก็ตามเราให้ลักษณะว่าเมื่อใดกึ่งกรุปการแปลงเชิงเส้นเป้าหมายที่ไม่มีศูนย์จะให้โครงสร้างของกึ่งริง AC ที่มีศูนย์ ยิ่งไปกว่านั้น เรายังศึกษากึ่งกรุปของการแปลงเชิงเส้นบางส่วนบน V ทั้งหมดด้วย เราให้เงื่อนไขที่จำเป็นที่ทำให้กึ่งกรุปนี้ให้โครงสร้างของกึ่งริง AC ที่มีศูนย์

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต......................................
สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา......................................
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SUREEPORN CHAOPRAKNOI: LINEAR TRANSFORMATION SEMIGROUPS ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO. THESIS ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph. D., 60 pp. ISBN 974-17-4573-7.

A *semiring* is a symtem $(A, +, \bullet)$ such that $(A, +)$ and $(A, \bullet)$ are semigroups and the operation $\bullet$ distributes over the operation $+$. A semiring $(A, +, \bullet)$ is *additively commutative* (AC) if $x + y = y + x$ for all $x, y \in A$. The *zero* of a semiring $(A, +, \bullet)$ is an element $0 \in A$ such that $x + 0 = 0 + x = x$ and $x \bullet 0 = 0 \bullet x = 0$ for all $x \in A$. For a semigroup S, let S^0 be S if S has a zero and S contains more than one element, otherwise, let S^0 be the semigroup S with a zero 0 adjoined. We say that a semigroup S *admits the structure of a* [*AC*] *semiring with zero* if there exists an operation $+$ on S^0 such that $(S^0, +, \bullet)$ is a [AC] semiring with zero where $\bullet$ is the operation on S^0.

A *hyperoperation* on a nonempty set H is a function $\circ : H \times H \to P^*(H)$ where $P(H)$ is the power set of H and $P^*(H) = P(H)\setminus\{\varnothing\}$. For this case, $(H, \circ)$ is called a *hypergroupoid.* For a hypergroupoid $(H, \circ)$ and nonempty subsets X and Y of H, we let $X \circ Y$ denote the union of all sets $x \circ y$ where x and y run over X and Y, respectively. A *semihypergroup* is a hypergroupoid $(H, \circ)$ with $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$. A *semihyperring* is a system $(A, +, \bullet)$ satisfying the following properties: $(A, +)$ is a semihypergroup, $(A, \bullet)$ is a semigroup and the operation $\bullet$ is distributive over the hyperoperation $+$. The *zero* of a semihyperring $(A, +, \bullet)$ is an element $0 \in A$ such that $x + 0 = 0 + x = \{x\}$ and $x \bullet 0 = 0 \bullet x = 0$ for all $x \in A$. Also, a semihyperring $(A, +, \bullet)$ is *additively commutative* (AC) if $x + y = y + x$ for all $x, y \in A$. *Semigroups admitting the structure of a semihyperring with zero* are defined analogously.

Let V be a vector space over a division ring R and $L_R(V)$ the semigroup under composition of all linear transformations $\alpha : V \to V$. By a *linear transformation semigroup* on V we mean a subsemigroup of $L_R(V)$. A *partial linear transformation* of V is a linear transformation from a subspace of V into V. Various types of linear transformation semigroups are studied. We determine when they admit the structure of a semihyperring with zero. It is shown that semigroups without zero always admit the structure of an AC semihyperring with zero and the structure of a semiring with zero. However, we characterize when our target linear transformation semigroups without zero admit the structure of an AC semiring with zero. Moreover, the partial linear transformation semigroup on V is studied. Necessary conditions for this semigroup to admit the structure of an AC semiring with zero are given.

Department **Mathematics** Student's signature.......................................

Field of study **Mathematics** Advisor's signature.......................................

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

The multiplicative structure of a ring is by definition a semigroup with zero. However, ring theory is a classical subject in mathematics and had been widely studied before semigroup theory was considered important and of interest by its own. Because the multiplicative structure of a ring is a semigroup with zero, it is valid to ask which semigroups joining with zero if necessary are isomorphic to the multiplicative structure of some ring. Such semigroups are usually called *R-semigroups* or *semigroups admitting ring structure.* Equivalently, an R-semigroup or a semigroup admitting ring structure is a semigroup S having the property that there is an operation $+$ on S^0 such that $(S^0, +, \cdot)$ is a ring where $\cdot$ is the operation on S^0. Many well-known theorems in ring theory are useful to study whether a semigroup is an R-semigroup. For examples, Wedderburn's theorem tells us any finite nonabelian group is not an R-semigroup. Because every Boolean ring is a commutative ring, we conclude that any left [right] zero semigroup, that is, a semigroup S in which $xy = x$ $[xy = y]$ for all $x, y \in S$, containing more than one element is not an R-semigroup. It is interesting to know that S. R. Kogalovski [6] announced in 1961 that an axiomatic characterization of R-semigroups is impossible.

In fact, R-semigroups or semigroups admitting ring structure have long been studied. In 1970, R. E. Peinado [8] gave a brief survey of semigroups admitting ring structure. D. D. Chu and H. I. Shyr [1] proved a nice result that the multiplicative semigroup $\mathbb{N}$ of natural numbers is an R-semigroup by showing that $(\mathbb{N}^0, \cdot) \cong (\mathbb{Z}_2[X], \cdot)$. M. Satyanarayana has paid much attention to study R-semigroups. See [11], [12] and [13] for examples. Semigroups of our interest are linear transformation semigroups. There was some study of linear transformation semigroups admitting ring structure provided by M. Siripitukdet and Y. Kemprasit [14].

Krasner hyperrings are a nice generalization of rings. This is the first notion of hyperrings introduced by M. Krasner [7] himself in 1944. By the definition of Krasner hyperrings, their multiplicative structures are also semigroups with zero. Y. Kemprasit and Y. Punkla [5] have defined semigroups admitting (Krasner) hyperring structure in the same way. As mentioned above, every finite nonabelian group does not admit a ring structure. A nice result of hyperring structures is that every group admits a hyperring structure. This can be seen in an example of [2], page 170. In fact, this hyperoperation was given for any abelian group to obtain what is called a *hyperfield.* The same hyperoperation could be given for any group in order to obtain what may be called a *hyperdivision ring.* The detail of the proof for the later result can be seen in [9]. Besides [5], Y. Kemprasit has continued studying semigroups admitting hyperring structure. It has been characterized in [3] when both multiplicative interval semigroups and additive interval semigroups of real numbers admit a hyperring structure. Also, in [4], hyperring structures of some linear transformation semigroups have been investigated and a result in [4] will be referred in this research. Many results of generalized semigroups of linear transformation semigroups which admit a hyperring structure, in particular, linear transformation semigroups admitting hyperring structure, have been provided in [10].

As mentioned above, various types of linear transformation semigroups have been studied in the matters of both admitting ring structure and admitting hyperring structure. This motivates our interest to study whether or when linear transformation semigroups of various types admit the structure of a semihyperring with zero. Hyperrings generalize rings while semihyperrings with zero are a generalization of hyperrings. *Semigroups admitting the structure a semihyperring* [*semiring*] *with zero* are defined analogously. The first main purpose is to study many types of linear transformation semigroups with zero. We investigate whether or when they admit the structure of a semihypering with zero. We find out that every semigroup without zero always admits both the structure of an additively commutative (AC) semihypering with zero and the structure of a semiring with zero. However, they need not admit the structure of an additively commutative (AC) semiring with zero. For our second main purpose, we characterize when various kinds of linear transformation semigroups without zero admit the structure of an AC semiring with zero.

In the remainder of this chapter, we shall give precise definitions, notations, and basic results which will be used in Chapter II and Chapter III. Moreover, some examples are provided for better understanding.

For any set X, the cardinality of X will be denoted by $|X|$. For a semigroup S, the semigroup S^0 is defined to be S if S has a zero and S contains more than one element, otherwise, let S^0 be the semigroup S with a zero 0 adjoined, that is, $S^0 = (S \cup \{0\}, \circ)$ where $0 \notin S, 0 \circ x = x \circ 0 = 0$ for all $x \in S \cup \{0\}$ and $x \circ y = xy$ for all $x, y \in S$. Note that if $|S| = 1$, then S^0 is a semigroup of two elements and $S^0 \cong (\mathbb{Z}_2, \cdot)$. Also, if G is a group, then $G^0 = (G \cup \{0\}, \circ)$ defined as above.

For a set X, let $P(X)$ denote the power set of X and let $P^*(X) = P(X) \setminus \{\varnothing\}$.

A *hyperoperation* on a nonempty set H is a mapping of $H \times H$ into $P^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets A and B of H, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} (a \circ b),$$

$A \circ x = A \circ \{x\}$ and $x \circ A = \{x\} \circ A$ for all $x \in H$. We call $(H, \circ)$ a *commutative hypergroupoid* if and only if $x \circ y = y \circ x$ for all $x, y \in H$. An element e of H is called an *identity* of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$. An element e of H is called a *scalar identity* of $(H, \circ)$ if $(x \circ e) \cap (e \circ x) = \{x\}$ for all $x \in H$. Then H has at most one scalar identity.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$, that is,

$$\bigcup_{t \in x \circ y} t \circ z = \bigcup_{t \in y \circ z} x \circ t \quad \text{for all } \; x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ such that $H \circ x = x \circ H = H$ for all $x \in H$. For x, y in a hypergroup $(H, \circ)$, x is called an *inverse* of y if there exists an identity e of $(H, \circ)$ such that $e \in (x \circ y) \cap (y \circ x)$. A hypergroup H is called *regular* if every element of H has an inverse in H. A regular hypergroup $(H, \circ)$ is said to be *reversible* if for $x, y, z \in H, x \in y \circ z$ implies $z \in u \circ x$ and $y \in x \circ v$ for some inverse u of y and some inverse v of z.

A *canonical hypergroup* is a hypergroup $(H, \circ)$ such that

(i) $(H, \circ)$ is commutative,

(ii) $(H, \circ)$ has a scalar identity,

(iii) every element of H has a unique inverse in H and

(iv) $(H, \circ)$ is reversible.

A triple $(A, +, \cdot)$ is called a *semiring* [*semihyperring*] if

(i) $(A, +)$ is a semigroup [semihypergroup],

(ii) $(A, \cdot)$ is a semigroup and

(iii) the operation $\cdot$ is distributive over the operation [hyperoperation] $+$.

A semiring [semihyperring] $(A, +, \cdot)$ is said to be *additively commutative* if $x + y = y + x$ for all $x, y \in A$. For this case, we call $(A, +, \cdot)$ an *AC semiring* [*AC semihyperring*]. An element 0 of a semiring [semihyperring] $(A, +, \cdot)$ is called a *zero* of $(A, +, \cdot)$ if $x + 0 = 0 + x = x$ [$x + 0 = 0 + x = \{x\}$] and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in A$. By the definitions, every semiring with zero is a semihyperring with zero.

A *Krasner hyperring* is a system $(A, +, \cdot)$ where

(i) $(A, +)$ is a canonical hypergroup,

(ii) $(A, \cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A, +)$ and

(iii) the operation $\cdot$ is distributive over the hyperoperation $+$.

In this research, by a *hyperring* we mean a Krasner hyperring.

Example 1.1. ([2], page 170 and [9]) Let G be a group. Define a hyperoperation $+$ on G^0 by

$$\begin{aligned} x + 0 = 0 + x &= \{x\} && \text{for all } x \in G^0, \\ x + x &= G^0 \smallsetminus \{x\} && \text{for all } x \in G, \\ x + y &= \{x, y\} && \text{for all distinct } x, y \in G. \end{aligned}$$

Then $(G^0, +, \cdot)$ is a hyperring where $\cdot$ is the operation on G^0. Note that the zero of the hyperring $(G^0, +, \cdot)$ is 0 and the inverse of $x \in G$ in $(G^0, +)$ is x itself. Also, $(G^0, +, \cdot)$ is not a ring if $|G| > 1$.

Example 1.2. ([3]) Let I be a multiplicative interval semigroup on $\mathbb{R}$, the set of real numbers, such that for every $x \in I, -x \in I$. Define a hyperoperation $\oplus$ on I by

$$\begin{aligned} \text{for } \ x, y \in I, \quad x \oplus 0 \ &= \ 0 \oplus x \ = \ \{x\}, \\ x \oplus x \ &= \ \{x\}, \\ x \oplus (-x) \ &= \ [-|x|, |x|\,], \end{aligned}$$

$$x \oplus y \;=\; y \oplus x \;= \{y\} \text{ if } |x| < |y|.$$

If $\cdot$ is the multiplication on I, then $(I, \oplus, \cdot)$ is a hyperring. Observe that 0 is the zero of $(I, \oplus, \cdot)$ and for $x \in I$, $-x$ is the inverse of x in the canonical hypergroup $(I, \oplus)$. If $|I| > 1$, then $(I, \oplus, \cdot)$ is not a ring.

In passing, we give a remark here that the following fact was proved in [3]. For a multiplicative interval semigroup I on $\mathbb{R}$ containing some positive numbers and some negative numbers, there is a hyperoperation $\oplus$ on I such that $(I, \oplus, \cdot)$ is a hyperring if and only if I has the property that $x \in I$ implies $-x \in I$.

Example 1.3. If $(S, \cdot)$ is a zero semigroup with zero 0 (that is, $x \cdot y = 0$ for all $x, y \in S$) containing more than two elements and define a hyperoperation $+$ on S by

$$x + 0 = 0 + x = \{x\} \quad \text{for all } x \in S,$$

$$x + y = S \quad \text{for all } x, y \in S \smallsetminus \{0\},$$

then $(S, +, \cdot)$ is clearly an AC semihyperring with zero which is neither a semiring with zero nor a hyperring.

Example 1.4. Let $\oplus$ be a hyperoperation defined on $[0, \infty)$ by

$$x \oplus 0 = 0 \oplus x = \{x\} \quad \text{for all } x \in [0, \infty),$$

$$x \oplus y = [\max\{x, y\}, \infty) \quad \text{for all } x, y \in (0, \infty).$$

Then $([0, \infty), \oplus, \cdot)$ is clearly an AC semihyperring with zero which is neither a semiring with zero nor a hyperring.

By the definitions, every ring is a hyperring and every hyperring and every AC semiring with zero is an AC semihyperring with zero, but the converse is not true. These can be seen from the above examples. Therefore hyperrings are a generalization of rings. Similarly, AC semihyperrings with zero generalize both AC semirings with zero and hyperrings.

A semigroup S is said to *admit a ring* [*hyperring*] *structure* if $(S^0, +, \cdot)$ is a ring [hyperring] for some operation [hyperoperation] $+$ on S^0 where $\cdot$ is the operation on S^0. *Semigroups admitting the structure of a* (*AC*) *semihyperring* [*semiring*] *with zero* are defined analogously. As mentioned previously, every finite nonabelian group does not admit a ring structure. However, Example 1.1 shows that every group admits a hyperring structure. Then every group admits the structure of a semihyperring with zero. Observe that if S is a trivial semigroup, then $S^0 \cong (\mathbb{Z}_2, \cdot)$ where $\cdot$ is the multiplication on $\mathbb{Z}_2$, so S admits a ring structure.

The following example shows that every semigroup without zero admits the structure of an AC semihyperring with zero.

Example 1.5. Let S be a semigroup without zero. Define a hyperoperation $+$ on S^0 by

$$x + 0 = 0 + x = \{x\} \qquad \text{if } x \in S^0,$$

$$x + y = \{x, y\} \qquad \text{if } x, y \in S.$$

Then $(S^0, +)$ is clearly a commutative semihypergroup with a scalar identity 0. Since 0 is not an element of $S, xy \neq 0$ for all $x, y \in S$, it follows that the multiplication $\cdot$ of S^0 distributes over the hyperoperation $+$ defined above, and thus $(S^0, +, \cdot)$ is an AC semihyperring with zero. But every $x \in S$ has no inverse in the semihypergroup $(S^0, +)$, so $(S^0, +, \cdot)$ is not a hyperring.

Also, every semigroup without zero admits the structure of a semiring with zero as shown by the next example.

Example 1.6. Let S be a semigroup without zero. Define an operation $+$ on S^0 by

$$x + 0 = 0 + x = x \qquad \text{if } x \in S^0,$$

$$x + y = x \qquad \text{if } x, y \in S.$$

Then $(S^0, +)$ is obviously a semigroup having 0 as its identity, Since $xy \neq 0$ for all $x, y \in S$, we deduce that the multiplication $\cdot$ of S^0 distributes over the operation $+$. Hence $(S^0, +, \cdot)$ is a semiring with zero, but it is not additively commutative if $|S| > 1$.

Next, let V be a vector space over a division ring R and $L_R(V)$ the semigroup under composition of all linear transformations $\alpha : V \to V$. Then $L_R(V)$ admits a ring structure under the usual addition of transformations. The image of v under $\alpha \in L_R(V)$ is written by $v\alpha$. For $\alpha \in L_R(V)$, let $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ denote the kernel and the image of α, respectively. For $A \subseteq V$, let $\langle A \rangle$ stand for the subspace of V spanned by A. The following three propositions are simple facts of vector spaces and linear transformations which will be used. The proofs are routine and elementary and they will be omitted.

Proposition 1.7. *Let B be a basis of V. If u and w are distinct elements of B, then $\{u, u + w\} \cup (B \smallsetminus \{u, w\})$ is also a basis of V.*

Proposition 1.8. *Let B be a basis of $V, A \subseteq B$ and $\varphi : B \smallsetminus A \to V$ a one-to-one map such that $(B \smallsetminus A)\varphi$ is a linearly independent subset of V. If $\alpha \in L_R(V)$ is defined by*

$$v\alpha = \begin{cases} 0 & \text{if } v \in A, \\ v\varphi & \text{if } v \in B \smallsetminus A, \end{cases}$$

then $Ker\,\alpha = \langle A \rangle$ and $Im\,\alpha = \langle B \smallsetminus A \rangle\varphi$.

Proposition 1.9. *Let B be a basis of V and $A \subseteq B$. Then*

(i) $\{v + \langle A \rangle \mid v \in B \smallsetminus A\}$ *is a basis of the quotient space* $V/\langle A \rangle$ *and*

(ii) $dim_R\,(V/\langle A \rangle) = |B \smallsetminus A|$.

Let

$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\}.$$

Then $G_R(V)$ is the unit group of the semigroup $L_R(V)$ or the group of all units of $L_R(V)$. The following known result will be referred.

Proposition 1.10.([4]) *If $\alpha \in L_R(V)$ is such that $\alpha\beta = \beta\alpha$ for all $\beta \in G_R(V)$, then $\alpha = a1_V$ for some $a \in C(R)$ where $C(R)$ is the center of R and 1_V is the identity map on V.*

Example 1.1 shows that $G_R(V)$ admits a hyperring structure. We know from the next proposition that $G_R(V)$ does not admit a ring structure if $\dim_R V > 1$.

Proposition 1.11.([14]) $G_R(V)$ *admits a ring structure if and only if* $dim_R V \leq 1$.

Next, let $M_R(V)$ and $E_R(V)$ be the set of all one-to-one linear transformations (monomorphisms) of V and the set of all onto linear transformations (epimorphisms) of V, respectively. Then

$$M_R(V) = \{\alpha \in L_R(V) \mid \operatorname{Ker}\alpha = \{0\}\},$$

$$E_R(V) = \{\alpha \in L_R(V) \mid \operatorname{Im}\alpha = V\}$$

which are subsemigroups of $L_R(V)$ containing $G_R(V)$. Moreover, it is well-known that if $\dim_R V < \infty$, then $M_R(V) = E_R(V) = G_R(V)$. In fact, if $M_R(V)$ $[E_R(V)]$ $= G_R(V)$, then $\dim_R V < \infty$. To see this, let $\dim_R V$ be infinite, B a basis of

V and $u \in B$. Since B is infinite, $|B \smallsetminus \{u\}| = |B|$. Let $\varphi : B \to B \smallsetminus \{u\}$ be a bijection. Define $\alpha \in L_R(V)$ by

$$v\alpha = v\varphi \text{ for every } v \in B.$$

By Proposition 1.8, $\operatorname{Ker}\alpha = \{0\}$ and $\operatorname{Im}\alpha = \langle B \smallsetminus \{u\} \rangle \subsetneq V$. Hence $\alpha \in M_R(V) \smallsetminus G_R(V)$. Also, if $\beta \in L_R(V)$ is defined by

$$v\beta = \begin{cases} v\varphi^{-1} & \text{if } v \in B \smallsetminus \{u\}, \\ 0 & \text{if } v = u, \end{cases}$$

then $\operatorname{Ker}\beta = \langle u \rangle$ and $\operatorname{Im}\beta = \langle B \rangle = V$ by Proposition 1.8, so $\beta \in E_R(V) \smallsetminus G_R(V)$. Consequently, $M_R(V)$ $[E_R(V)] = G_R(V)$ if and only if $\dim_R V < \infty$. Observe that if $\dim_R V \geq 1$, then none of $G_R(V)$, $M_R(V)$ and $E_R(V)$ contains 0, the zero mapping on V. Proposition 1.11 shows that $M_R(V)$ and $E_R(V)$ admits a ring structure if $\dim_R V \leq 1$. In fact, it was shown in [14] that if $\dim_R V \leq 1$ is also necessary for each of $M_R(V)$ and $E_R(V)$ to admit a ring structure. Next, let

$$OM_R(V) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha \text{ is infinite} \,\},$$

$$OE_R(V) = \{\, \alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) \text{ is infinite} \,\}.$$

If $\dim_R V$ is infinite, then 0 belongs to both $OM_R(V)$ and $OE_R(V)$. Since $\operatorname{Ker}\alpha\beta \supseteq \operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$, for all $\alpha, \beta \in L_R(V)$, it follows that $OM_R(V)$ and $OE_R(V)$ are both subsemigroups of $L_R(V)$ containing 0 if $\dim_R V$ is infinite. For this case, the semigroups $OM_R(V)$ and $OE_R(V)$ may referred to respectively as the *opposite semigroup* of $M_R(V)$ and the *opposite semigroup of* $E_R(V)$.

For $\alpha \in L_R(V), \alpha$ is said to be *almost one-to-one* if $\dim_R \operatorname{Ker}\alpha < \infty$, and α is said to be *almost onto* if $\dim_R (V/\operatorname{Im}\alpha) < \infty$. Let

$$AM_R(V) = \{\, \alpha \in L_R(V) \mid \alpha \text{ is almost one-to-one} \,\},$$

$$AE_R(V) = \{\, \alpha \in L_R(V) \mid \alpha \text{ is almost onto} \,\}.$$

Then $M_R(V) \subseteq AM_R(V)$ and $E_R(V) \subseteq AE_R(V)$. It was proved in [14] that for all $\alpha, \beta \in L_R(V)$,

$$\dim_R \operatorname{Ker} \alpha\beta \leq \dim_R \operatorname{Ker} \alpha + \dim_R \operatorname{Ker} \beta,$$

$$\dim_R (V/\operatorname{Im} \alpha\beta) \leq \dim_R (V/\operatorname{Im} \alpha) + \dim_R (V/\operatorname{Im} \beta).$$

Therefore both $AM_R(V)$ and $AE_R(V)$ are subsemigroups of the semigroup $L_R(V)$, and they do not contain 0 if $\dim_R V$ is infinite. Clearly, $AM_R(V) = AE_R(V) = L_R(V)$ if $\dim_R V < \infty$. Hence if $\dim_R V < \infty$, then both $AM_R(V)$ and $AE_R(V)$ admit a ring structure. Let $\dim_R V$ be infinite and B a basis of V. Since B is infinite, there are $B_1, B_2 \subseteq B$ such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let $\varphi : B_1 \to B_2$ be a bijection and define $\alpha \in L_R(V)$ by

$$v\alpha = \begin{cases} v\varphi & \text{if } v \in B_1, \\ 0 & \text{if } v \in B_2. \end{cases}$$

By Proposition 1.8, $\operatorname{Ker} \alpha = \langle B_2 \rangle$ and $\operatorname{Im} \alpha = \langle B_2 \rangle$, so by Proposition 1.9 (ii), $\dim_R (V/\operatorname{Im} \alpha) = |B \smallsetminus B_2| = |B_1|$. Therefore both $\dim_R \operatorname{Ker} \alpha$ and $\dim_R (V/\operatorname{Im} \alpha)$ are $|B|$. Thus $\alpha \notin AM_R(V)$ and $\alpha \notin AE_R(V)$. Therefore we conclude that $AM_R(V)$ $[AE_R(V)] = L_R(V)$ if and only if $\dim_R V < \infty$.

For any cardinal number k with $k \leq \dim_R V$, let

$$K_R(V, k) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha \geq k \,\},$$

$$I_R(V, k) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha \leq k \,\},$$

$$CI_R(V, k) = \{\, \alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im} \alpha) \geq k \,\}.$$

Then the zero map 0 on V belongs to all of the above three subsets of $L_R(V)$. Since for $\alpha, \beta \in L_R(V), \operatorname{Ker} \alpha\beta \supseteq \operatorname{Ker} \alpha$ and $\operatorname{Im} \alpha\beta \subseteq \operatorname{Im} \beta$, we conclude that all of $K_R(V, k), I_R(V, k)$ and $CI_R(V, k)$ are subsemigroups of $L_R(V)$. Observe that if $\dim_R V$ is infinite, the notations $OM_R(V)$ and $OE_R(V)$ defined previously denote

$K_R(V, \aleph_0)$ and $CI_R(V, \aleph_0)$, respectively, that is,

$$OM_R(V) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha \geq \aleph_0 \,\},$$

$$OE_R(V) = \{\, \alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im} \alpha) \geq \aleph_0 \,\}.$$

We know that if $\dim_R V$ is finite, then for $\alpha \in L_R(V)$, $\dim_R \operatorname{Ker}\alpha = \dim_R (V/\operatorname{Im}\alpha) = \dim_R V - \dim_R \operatorname{Im} \alpha$ since $\dim_R V = \dim_R \operatorname{Ker} \alpha + \dim_R \operatorname{Im} \alpha$ and $\dim_R V = \dim_R (V/\operatorname{Im} \alpha) + \dim_R \operatorname{Im} \alpha$. Hence we have

Proposition 1.12. *If $dim_R V < \infty$, then $K_R(V, k) = CI_R(V, k) = I_R(dim_R V - k)$ for every a cardinal number $k \leq dim_R V$.*

However, these are not generally true if $\dim_R V$ is infinite. This is shown by the following proposition. This proposition also shows that the semigroups $K_R(V, k), CI_R(V, k)$ and $I_R(V, k)$ should be considered independently if $\dim_R V$ is infinite.

Proposition 1.13. *Let V be an infinite dimensional vector space and a nonzero cardinal number $k \leq dim_R V$. Then the following statements hold.*

(i) $CI_R(V, k) \neq K_R(V, l)$ for every cardinal number $l \leq dim_R V$.

(ii) If $k < dim_R V$, then $I_R(V, k) \neq K_R(V, l)$ and $I_R(V, k) \neq CI_R(V, l)$ for every cardinal number $l \leq dim_R V$.

Proof. Let B be a basis of V. Since B is infinite, there are subsets B_1 and B_2 of B such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let $\varphi : B_1 \to B$ be a bijection. Define $\alpha, \beta \in L_R(V)$ by

$$v\alpha = \begin{cases} v\varphi & \text{if } v \in B_1, \\ 0 & \text{if } v \in B_2, \end{cases} \qquad v\beta = v\varphi^{-1} \text{ for all } v \in B.$$

Then by Proposition 1.8, $\operatorname{Ker}\alpha = \langle B_2\rangle, \operatorname{Im}\alpha = \langle B\rangle = V, \operatorname{Ker}\beta = \{0\}$ and $\operatorname{Im}\beta = \langle B_1\rangle$, so $\dim_R \operatorname{Ker}\alpha = |B_2| = |B| = \dim_R V, \dim_R \operatorname{Im}\alpha = |B| = \dim_R V, \dim_R (V/\operatorname{Im}\alpha) = 0, \dim_R \operatorname{Ker}\beta = 0$ and $\dim_R \operatorname{Im}\beta = |B_1| = \dim_R V$. We have from Proposition 1.9 (ii) that $\dim_R (V/\operatorname{Im}\beta) = |B_2| = \dim_R V$. Hence $\alpha \in K_R(V,l) \smallsetminus CI_R(V,k)$ for every cardinal number $l \leq \dim_R V$, so (i) is proved.

Moreover, if $k < \dim_R V$, then $\alpha \in K_R(V,l) \smallsetminus I_R(V,k)$ and $\beta \in CI_R(V,l) \smallsetminus I_R(V,k)$ for every cardinal number $l \leq \dim_R V$. Hence (ii) is proved. $\square$

Next, we define $K'_R(V,k), CI'_R(V,k)$ and $I'_R(V,k)$ which are subsets of $K_R(V,k)$, $CI_R(V,k)$ and $I_R(V,k)$ respectively as follows :

$$K'_R(V,k) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha > k \,\} \text{ where } k < \dim_R V,$$

$$CI'_R(V,k) = \{\, \alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) > k \,\} \text{ where } k < \dim_R V,$$

$$I'_R(V,k) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Im}\alpha < k \,\} \text{ where } 0 < k \leq \dim_R V.$$

Then 0 belongs to all $K'_R(V,k), CI'_R(V,k)$ and $I'_R(V,k)$, so they are respectively subsemigroups of $K_R(V,k), CI_R(V,k)$ and $I_R(V,k)$. Observe that if $k < \dim_R V$, then $K'_R(V,k) = K_R(V,k')$ and $CI'_R(V,k) = CI_R(V,k')$ where k' is the successor of k. Also, if $0 < k \leq \dim_R V, k$ is a finite cardinal number and $\tilde{k}$ is the predecessor of k, then $I'_R(V,k) = I_R(V,\tilde{k})$.

For $\alpha \in L_R(V)$, let

$$F(\alpha) = \{\ v \in V \mid v\alpha = v\ \}.$$

Then for $\alpha \in L_R(V)$, $F(\alpha)$ is a subspace of V and α is called an *almost identical linear transformation* of V if $\dim_R (V/F(\alpha))$ is finite. The set of all almost identical linear transformations of V will be denoted by $AI_R(V)$, that is,

$$AI_R(V) = \{\ \alpha \in L_R(V) \mid \dim_R (V/F(\alpha)) < \infty\ \}.$$

Observe that 1_V, the identity map on V, belongs to $AI_R(V)$. We show in the next proposition that $AI_R(V)$ is a subsemigroup of $L_R(V)$.

Proposition 1.14. *$AI_R(V)$ is a subsemigroup of $L_R(V)$.*

Proof. Let $\alpha, \beta \in AI_R(V)$. Then $\dim_R (V/F(\alpha))$ and $\dim_R (V/F(\beta))$ are finite. We claim that $\dim_R (V/F(\alpha\beta))$ is finite. Since $F(\alpha) \cap F(\beta) \subseteq F(\alpha\beta)$, it suffices to show that $\dim_R (V/(F(\alpha) \cap F(\beta)))$ is finite. Let B_1 be a basis of $F(\alpha) \cap F(\beta)$ and let $B_2 \subseteq F(\alpha) \smallsetminus B_1$ and $B_3 \subseteq F(\beta) \smallsetminus B_1$ be such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $F(\alpha)$ and $F(\beta)$, respectively. We shall show that $(B_1 \cup B_2) \cup B_3$ is linearly independent over R. Let $u_1, u_2, \ldots, u_k \in B_1 \cup B_2, v_1, v_2, \ldots, v_l \in B_3$ be distinct such that

$$\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{l} b_i v_i = 0$$

for some $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in R$. Then $\sum_{i=1}^{k} a_i u_i = -\sum_{i=1}^{l} b_i v_i \in F(\alpha) \cap F(\beta) = \langle B_1 \rangle$. Since $B_1 \cup B_3$ is linearly independent, $b_i = 0$ for all $i = 1, 2, \ldots, l$, so $\sum_{i=1}^{k} a_i u_i = 0$. This implies that $a_i = 0$ for all $i = 1, 2, \ldots, k$. Hence $B_1 \cup B_2 \cup B_3$ is linearly independent over R. Let $B_4 \subseteq V \smallsetminus (B_1 \cup B_2 \cup B_3)$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of V. It follows from Proposition 1.9(i) that $\{ v + F(\alpha) \mid v \in B_3 \cup B_4 \}$ is a basis of $V/F(\alpha)$ and $\{ v + F(\beta) \mid v \in B_2 \cup B_4 \}$ is a basis of $V/F(\beta)$. But $\dim_R (V/F(\alpha))$ and $\dim_R (V/F(\beta))$ are finite, so $B_3 \cup B_4$ and $B_2 \cup B_4$ are finite. Therefore $B_2 \cup B_3 \cup B_4$ is finite. Also, we have from Proposition 1.9 (i) that $\{ v + (F(\alpha) \cap F(\beta)) \mid v \in B_2 \cup B_3 \cup B_4 \}$ is a basis of $V/(F(\alpha) \cap F(\beta))$ which implies that $\dim_R (V/(F(\alpha) \cap F(\beta)))$ is finite.

Therefore the proposition is proved, as desired. □

Notice that if $\dim_R V < \infty$, then $AI_R(V) = L_R(V)$ which admits a ring structure. Moreover, the semigroup $AI_R(V)$ does not contain 0, the zero map on

V if $\dim_R V$ is infinite.

By a *partial linear transformation* of V, we mean a linear transformation from a subspace of V into V. Let $PL_R(V)$ be the set of all partial transformations of V. For $\alpha \in PL_R(V)$, let $\operatorname{Dom}\alpha$ and $\operatorname{Im}\alpha$ denote the domain and the image of α, respectively and the image of $v \in \operatorname{Dom}\alpha$ under α is also written by $v\alpha$. Then $PL_R(V)$ is a semigroup under the composition of maps, that is, for $\alpha, \beta \in PL_R(V)$,

$$\operatorname{Dom}\alpha\beta = \{v \in \operatorname{Dom}\alpha \mid v\alpha \in \operatorname{Dom}\beta\},$$

$$v(\alpha\beta) = (v\alpha)\beta \text{ for all } v \in \operatorname{Dom}\alpha\beta.$$

This implies that for $\alpha, \beta \in PL_R(V), \operatorname{Dom}\alpha\beta \subseteq \operatorname{Dom}\alpha$ and $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$. For a subspace W of V, let 1_W and W_0 denote respectively the identity map on W and the zero map whose domain is W. Observe that

$$\{0\}_0\alpha = \{0\}_0 \text{ and } V_0\alpha = V_0 \text{ for all } \alpha \in PL_R(V),$$

$$\alpha V_0 = V_0 \text{ for all } \alpha \in L_R(V),$$

$$\alpha\{0\}_0 = \{0\}_0 \text{ for every 1-1 map } \alpha \text{ in } PL_R(V).$$

It then follows that if $\dim_R V > 0$, then $PL_R(V)$ is a semigroup without zero. The semigroup $PL_R(V)$ is called the *partial transformation semigroup* on V. Notice that

$$L_R(V) = \{\alpha \in PL_R(V) \mid \operatorname{Dom}\alpha = V\}$$

which is a subsemigroup of $PL_R(V)$ having V_0 as its zero. Hence if $\dim_R > 0$, then $L_R(V)$ is a proper subsemigroup of $PL_R(V)$.

Since every linear transformation from a subspace of V into V can be defined on a basis of its domain, for convenience, we may write $\alpha \in PL_R(V)$ by using a bracket notation. For examples,

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}$$

means that B is a basis of $\operatorname{Dom}\alpha, B_1 \subseteq B$ and

$$\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ v & \text{if } v \in B \smallsetminus B_1 \end{cases}$$

and

$$\beta = \begin{pmatrix} u & w & v \\ w & 0 & v \end{pmatrix}_{v \,\in\, B \smallsetminus \{u,w\}}$$

means that B is a basis of $\operatorname{Dom}\alpha, u, w \in B, u \neq w$ and

$$v\beta = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v = w, \\ v & \text{if } v \in B \smallsetminus \{u, w\}. \end{cases}$$

Then the linear transformations $\alpha, \beta : V \to V$ in the proof of Proposition 1.13 can be written as

$$\alpha = \begin{pmatrix} v & B_2 \\ v\varphi & 0 \end{pmatrix}_{v \,\in\, B_1} \quad \text{and} \quad \beta = \begin{pmatrix} v \\ v\varphi^{-1} \end{pmatrix}_{v \,\in\, B}.$$

As was mentioned previously, the semigroup $PL_R(V)$ has no zero if $\dim_R V > 0$. We know that 0, the zero map on V, does not belong to $G_R(V), M_R(V)$ and $E_R(V)$ if $\dim_R V > 0$ and 0 belongs to none of $AM_R(V), AE_R(V)$ and $AI_R(V)$ if $\dim_R V$ is infinite. However, this may not be true that these semigroups have no zero for such given $\dim_R V$. The following facts should be shown.

Proposition 1.15. *The following statements hold.*

(i) Let $S_R(V)$ be $G_R(V), M_R(V)$ or $E_R(V)$. Then $S_R(V)$ has no zero if and only if either $dim_R V > 1$ or $dim_R V = 1$ and $|R| > 2$.

(ii) Let $S(V)$ be $AM_R(V), AE_R(V)$ or $AI_R(V)$. Then $S(V)$ has no zero if and

only if $dim_R V$ *is infinite.*

(iii) $PL_R(V)$ *has no zero if and only if* $dim_R V > 0$.

Proof. (i) Let B be a basis of V.

Case 1 : $\dim_R V > 1$. Let $u, w \in B$ be distinct and define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} u & w & v \\ w & u & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}.$$

Then by Proposition 1.8, $\alpha \in G_R(V) \subseteq S_R(V)$. Suppose $\theta \in S_R(V)$ is the zero of $S_R(V)$. Then $\alpha\theta = \theta\alpha = \theta$, so $u\theta = u\alpha\theta = w\theta$. Thus $0 \neq u - w \in \operatorname{Ker}\theta$. It is a contradiction if $S_R(V)$ is $G_R(V)$ or $M_R(V)$. Next let $S_R(V) = E_R(V)$. Then $z\theta = u$ for some $z \in V \smallsetminus \{0\}$, and so

$$u = z\theta = z\theta\alpha = u\alpha = w,$$

a contradiction.

Case 2 : $\dim_R V = 1$ and $|R| > 2$. Since $\dim_R V = 1$, $S_R(V)$ does not contain 0, the zero map on V. Let $u \in V \smallsetminus \{0\}$ and let $a, b \in R \smallsetminus \{0\}$ be distinct. Then $\{u\}$ is a basis of V and $\begin{pmatrix} u \\ au \end{pmatrix}$ and $\begin{pmatrix} u \\ bu \end{pmatrix}$ are distinct elements of $S_R(V)$. Assume that θ is the zero of $S_R(V)$. Then

$$\begin{pmatrix} u \\ au \end{pmatrix}\theta = \theta = \begin{pmatrix} u \\ bu \end{pmatrix}\theta,$$

so

$$a(u\theta) = u\begin{pmatrix} u \\ au \end{pmatrix}\theta = u\theta = u\begin{pmatrix} u \\ bu \end{pmatrix}\theta = b(u\theta).$$

Thus either $u\theta = 0$ or $a = b$, hence either $\theta = 0$ or $a = b$. This is a contradiction.

Conversely, assume that $\dim_R V \leq 1$ and ($\dim_R V \neq 1$ or $|R| = 2$). Then $\dim_R V = 0$ or $\dim_R V = 1$ and $|R| = 2$. Consequently, $|V| \leq 2$ and hence

$$S_R(V) = G_R(V) = \begin{cases} \{0\} & \text{if } |V| = 1, \\ \{1_V\} & \text{if } |V| = 2. \end{cases}$$

Therefore $S_R(V)$ has a zero.

(ii) If $\dim_R V < \infty$, then $AM_R(V) = AE_R(V) = AI_R(V) = L_R(V)$ which has 0 as its zero.

Conversely, assume that $\dim_R V$ is infinite. Then $S(V)$ does not contain 0, the zero map on V. Suppose that $S(V)$ has a zero θ. Let B be a basis of V and for each $u \in B$, let $\alpha_u \in L_R(V)$ be defined by

$$\alpha_u = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \setminus \{u\}} .$$

Then for every $u \in B, \operatorname{Ker} \alpha_u = \langle u \rangle$ and $\operatorname{Im} \alpha_u = \langle B \setminus \{u\} \rangle$ by Proposition 1.8, so $\dim_R (V/\operatorname{Im} \alpha_u) = 1$ by Proposition 1.9 (ii). Also, $F(\alpha_u) = \langle B \setminus \{u\} \rangle$ for every $u \in B$. Consequently, $\alpha_u \in S(V)$ for every $u \in B$. Thus

$$\alpha_u \theta = \theta \alpha_u = \theta \text{ for every } u \in B.$$

This implies that

$$\text{for every } u \in B, u\theta = u\alpha_u\theta = 0\theta = 0$$

which implies that $\theta = 0$, a contradiction.

(iii) As was shown, $\dim_R V > 0$ implies that $PL_R(V)$ has no zero.

If $\dim_R V = 0$, then $PL_R(V) = \{\{0\}_0\}$, so $PL_R(V)$ has a zero. □

Chapter II deals with linear transformation semigroups on V with zero. The purpose of Chapter II is to show that if $\dim_R V$ is infinite, the semigroups

$OM_R(V), OE_R(V)$ and some linear transformation semigroups containing $OM_R(V)$ and $OE_R(V)$ do not admit the structure of a semihyperring with zero. These results indicate that if $\dim_R V$ is infinite, then there are infinitely many subsemigroups of $L_R(V)$ which do not admit the structure of a semihyperring with zero. The semigroups $OM_R(V)$ and $OE_R(V)$ are generalized to be the semigroups $K_R(V,k)$ and $CI_R(V,k)$. We also determine in this chapter when the semigroups $K_R(V,k)$ and $CI_R(V,k)$ admit such a structure. Moreover, the semigroups $I_R(V,k)$, $K'_R(V,k)$, $CI'_R(V,k)$ and $I'_R(V,k)$ are also studied in the same matter.

In Chapter III, we intend to deal with semigroups without zero. The following semigroups are considered:

$$G_R(V), M_R(V), E_R(V), AM_R(V), AE_R(V), AI_R(V) \text{ and } PL_R(V).$$

By Proposition 1.15, the semigroups $G_R(V), M_R(V)$ and $E_R(V)$ have no zero if either $\dim_R V > 1$ or $\dim_R V = 1$ and $|R| > 2$, $PL_R(V)$ have no zero if $\dim_R V > 0$, and if $\dim_R V$ is infinite, then the semigroups $AM_R(V), AE_R(V)$ and $AI_R(V)$ have no zero. Example 1.5 and Example 1.6 show that every semigroup without zero admits both the structure of an AC semihyperring with zero and the structure of a semiring with zero. However, it need not admit the structure of an AC semiring with zero. The purpose of this chapter is to provide necessary and sufficient conditions for $G_R(V), M_R(V), E_R(V), AM_R(V), AE_R(V)$ and $AI_R(V)$ to admit the structure of an AC semiring with zero. Necessary conditions for $PL_R(V)$ to admit the structure of an AC semiring with zero are provided. In addition, a partial sufficient condition for this property also given.

Throughout, let V be the vector space over a division ring R.

CHAPTER II

SEMIGROUPS ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO

In this chapter, we deal with linear transformation semigroups on V with zero. The following linear transformation semigroups on V given in Chapter I are recalled as follows:

$$L_R(V) = \{\alpha : V \to V \mid \alpha \text{ is a linear transformation}\},$$

$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\},$$

$$OM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha \text{ is infinite }\},$$

$$OE_R(V) = \{\alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) \text{ is infinite }\},$$

$$\begin{aligned} AI_R(V) &= \{\alpha \in L_R(V) \mid \alpha \text{ is almost identical }\} \\ &= \{\alpha \in L_R(V) \mid \dim_R (V/F(\alpha)) < \infty \} \\ &\quad \text{where } F(\alpha) = \{ v \in V \mid v\alpha = v \}, \end{aligned}$$

$$\begin{aligned} K_R(V,k) &= \{ \alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha \geq k \} \\ &\quad \text{where } k \leq \dim_R V, \end{aligned}$$

$$\begin{aligned} K'_R(V,k) &= \{ \alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha > k \} \\ &\quad \text{where } k < \dim_R V, \end{aligned}$$

$$\begin{aligned} CI_R(V,k) &= \{ \alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) \geq k \} \\ &\quad \text{where } k \leq \dim_R V, \end{aligned}$$

$$\begin{aligned} CI'_R(V,k) &= \{ \alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) > k \} \\ &\quad \text{where } k < \dim_R V, \end{aligned}$$

$$I_R(V,k) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Im}\alpha \le k \,\}$$

where $k \le \dim_R V$,

$$I'_R(V,k) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Im}\alpha < k \,\}$$

where $0 < k \le \dim_R V$.

2.1 Elementary Results

We show in this section that if $\dim_R V$ is infinite, then the following subsets of $L_R(V)$ are also subsemigroups of $L_R(V)$ where H and T are respectively subsemigroups of $G_R(V)$ and $AI_R(V)$.

$$OM_R(V) \cup H\ ,\quad OE_R(V) \cup H,$$

$$OM_R(V) \cup T\ ,\quad OE_R(V) \cup T.$$

Lemma 2.1.1. *Let $\dim_R V$ be infinite. The following statements hold.*
(i) $OM_R(V)$ is a right ideal of $L_R(V)$.
(ii) $OE_R(V)$ is a left ideal of $L_R(V)$.

Proof. (i) and (ii) are obtained respectively from the facts that $\operatorname{Ker}\alpha\beta \supseteq \operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$ for all $\alpha, \beta \in L_R(V)$. □

Lemma 2.1.2. *If $\dim_R V$ is infinite, then $G_R(V)OM_R(V) \subseteq OM_R(V)$.*

Proof. Let $\alpha \in G_R(V)$ and $\beta \in OM_R(V)$. If $v \in \operatorname{Ker}\alpha\beta$, then $v\alpha\beta = 0$, so $v\alpha \in \operatorname{Ker}\beta$. Thus $(\operatorname{Ker}\alpha\beta)\alpha \subseteq \operatorname{Ker}\beta$. If $v \in \operatorname{Ker}\beta$, then $(v\alpha^{-1})\alpha\beta = v\beta = 0$, and hence $v = (v\alpha^{-1})\alpha \in (\operatorname{Ker}\alpha\beta)\alpha$. This proves that $(\operatorname{Ker}\alpha\beta)\alpha = \operatorname{Ker}\beta$. Since $\alpha : V \to V$ is an isomorphism, $\operatorname{Ker}\alpha\beta \cong \operatorname{Ker}\beta$. But $\dim_R \operatorname{Ker}\beta$ is infinite, so $\dim_R \operatorname{Ker}\alpha\beta$ is also infinite. Hence $\alpha\beta \in OM_R(V)$. □

The following proposition is a direct consequence of Lemma 2.1.1(i) and Lemma 2.1.2.

Proposition 2.1.3. *If $\dim_R V$ is infinite and H is a subsemigroup of $G_R(V)$, then $OM_R(V) \cup H$ is a subsemigroup of $L_R(V)$.*

Lemma 2.1.4. *If $\dim_R V$ is infinite, then $OE_R(V)G_R(V) \subseteq OE_R(V)$.*

Proof. Let $\alpha \in OE_R(V)$ and $\beta \in G_R(V)$. Define $\varphi : V/\operatorname{Im}\alpha \to V/\operatorname{Im}\alpha\beta$ by

$$(v + \operatorname{Im}\alpha)\varphi = v\beta + \operatorname{Im}\alpha\beta \quad \text{for every } v \in V.$$

Since $\beta : V \to V$ is an isomorphism, we have that φ is an isomorphism, and hence $V/\operatorname{Im}\alpha \cong V/\operatorname{Im}\alpha\beta$. But $\dim_R (V/\operatorname{Im}\alpha)$ is infinite, so $\dim_R (V/\operatorname{Im}\alpha\beta)$ is infinite. Therefore $\alpha\beta \in OE_R(V)$. This proves that $OE_R(V)G_R(V) \subseteq OE_R(V)$, as required. □

The following proposition is directly obtained from Lemma 2.1.1(ii) and Lemma 2.1.4.

Proposition 2.1.5. *If $\dim_R V$ is infinite and H is a subsemigroup of $G_R(V)$, then $OE_R(V) \cup H$ is a subsemigroup of $L_R(V)$.*

Lemma 2.1.6. *If $\dim_R V$ is infinite, then $AI_R(V)OM_R(V) \subseteq OM_R(V)$.*

Proof. Let $\alpha \in AI_R(V)$ and $\beta \in OM_R(V)$ and let B_1 be a basis of $F(\alpha) \cap \operatorname{Ker}\beta$ and $B_2 \subseteq \operatorname{Ker}\beta \smallsetminus B_1$ such that $B_1 \cup B_2$ a basis of $\operatorname{Ker}\beta$. Since $\beta \in OM_R(V)$, $B_1 \cup B_2$ is infinite. Let $v_1, v_2, \ldots, v_n$ be distinct elements of B_2 and let $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i(v_i + F(\alpha)) = F(\alpha)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\alpha)$. Since $B_2 \subseteq \operatorname{Ker}\beta$, we have that $\sum_{i=1}^{n} a_i v_i \in F(\alpha) \cap \operatorname{Ker}\beta$. But B_1 is a basis of $F(\alpha) \cap \operatorname{Ker}\beta$ and $B_1 \cup B_2$ is linearly independent over R, so we have that $a_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. This shows that $\{v + F(\alpha) \mid v \in B_2\}$ is a linearly independent subset of the quotient space $V/F(\alpha)$ and $u + F(\alpha) \neq w + F(\alpha)$ for all distinct $u, w \in B_2$. Since $\dim_R (V/F(\alpha)) < \infty$, we deduce that $\{v + F(\alpha) \mid v \in B_2\}$ is finite. But $\{v+F(\alpha) \mid v \in B_2\} = |B_2|$, thus B_2 is finite. This implies that B_1 is infinite. Since $B_1 \subseteq F(\alpha) \cap \operatorname{Ker}\beta$, we have $B_1\alpha\beta = B_1\beta = \{0\}$, so $B_1 \subseteq \operatorname{Ker}\alpha\beta$. Consequently, $\dim_R \operatorname{Ker}\alpha\beta$ is infinite. Hence $\alpha\beta \in OM_R(V)$. Therefore the lemma is proved. □

The next proposition follows directly from Lemma 2.1.1(i) and Lemma 2.1.6.

Proposition 2.1.7. *If $\dim_R V$ is infinite and T is a subsemigroup of $AI_R(V)$, then $OM_R(V) \cup T$ is a subsemigroup of $L_R(V)$.*

Lemma 2.1.8. *For every $\alpha \in AI_R(V)$, $\dim_R \operatorname{Ker}\alpha < \infty$.*

Proof. Let $\alpha \in AI_R(V)$ and B a basis of $\operatorname{Ker}\alpha$. Let $v_1, v_2, \ldots, v_n \in B$ be distinct and let $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i(v_i + F(\alpha)) = F(\alpha)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\alpha)$ which implies that $(\sum_{i=1}^{n} a_i v_i)\alpha = \sum_{i=1}^{n} a_i v_i$. But $v_1, v_2, ..., v_n \in \operatorname{Ker}\alpha$, so $(\sum_{i=1}^{n} a_i v_i)\alpha = 0$. Thus $\sum_{i=1}^{n} a_i v_i = 0$. Since $v_1, v_2, \ldots, v_n$ are linearly independent over R, it follows that $a_i = 0$ for every $i \in \{1, 2, \ldots, n\}$. This proves that $\{v + F(\alpha) \mid v \in B\}$ is a linearly independent subset of $V/F(\alpha)$ and $v + F(\alpha) \neq w + F(\alpha)$ for all distinct $v, w \in B$. Since $\dim_R (V/F(\alpha))$ is finite, $\{v + F(\alpha) \mid v \in B\}$ is finite. But $|\{v + F(\alpha) \mid v \in B\}| = |B|$, so $\dim_R \operatorname{Ker}\alpha < \infty$. □

Lemma 2.1.9. *If $\dim_R V$ is infinite, then $OE_R(V)AI_R(V) \subseteq OE_R(V)$.*

Proof. Let $\alpha \in OE_R(V)$ and $\beta \in AI_R(V)$. Define $\varphi : V/\operatorname{Im}\alpha \to \operatorname{Im}\beta/\operatorname{Im}\alpha\beta$ by

$$(v + \operatorname{Im}\alpha)\varphi = v\beta + \operatorname{Im}\alpha\beta \quad \text{for all } v \in V. \tag{1}$$

Then φ is an epimorphism from $V/\operatorname{Im}\alpha$ onto $\operatorname{Im}\beta/\operatorname{Im}\alpha\beta$. Hence

$$(V/\operatorname{Im}\alpha)/\operatorname{Ker}\varphi \cong \operatorname{Im}\beta/\operatorname{Im}\alpha\beta. \tag{2}$$

To show that $\dim_R \operatorname{Ker}\varphi < \infty$, let $B \subseteq V$ be such that

$$\begin{gathered} \{v + \operatorname{Im}\alpha \mid v \in B\} \text{ is a basis of } \operatorname{Ker}\varphi \text{ and} \\ v + \operatorname{Im}\alpha \neq w + \operatorname{Im}\alpha \text{ for all distinct } v, w \in B. \end{gathered} \tag{3}$$

Then from (1) and (3), we have that for every $v \in B$, $v\beta + \operatorname{Im}\alpha\beta = (v + \operatorname{Im}\alpha)\varphi = \operatorname{Im}\alpha\beta$. Thus $v\beta \in \operatorname{Im}\alpha\beta = (\operatorname{Im}\alpha)\beta$ for all $v \in B$, so for each $v \in B$, there exists an element $w_v \in \operatorname{Im}\alpha$ such that $v\beta = w_v\beta$. Consequently,

$$\{v - w_v \mid v \in B\} \subseteq \operatorname{Ker}\beta. \tag{4}$$

If $v_1, v_2, \ldots, v_n \in B$ are distinct and $\sum_{i=1}^{n} a_i(v_i - w_{v_i}) = 0$ where $a_1, a_2, \ldots, a_n \in R$, then $\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i w_{v_i} \in \operatorname{Im}\alpha$, and hence $\sum_{i=1}^{n} a_i(v_i + \operatorname{Im}\alpha) = \operatorname{Im}\alpha$ in $V/\operatorname{Im}\alpha$. By (3), $a_i = 0$ for every $i \in \{1, 2, \ldots, n\}$. This shows that

$$\begin{gathered} \{v - w_v \mid v \in B\} \text{ is linearly independent over } R \\ \text{and } u - w_u \neq v - w_v \text{ for all distinct } u, v \in B. \end{gathered} \tag{5}$$

We therefore deduce from (4) and (5) that $|B| \leq \dim_R \operatorname{Ker}\beta$. Since $\dim_R \operatorname{Ker}\beta < \infty$ by Lemma 2.1.8, we have that B is finite, and hence $\dim_R \operatorname{Ker}\varphi < \infty$ by (3). But $\dim_R (V/\operatorname{Im}\alpha)$ is infinite and

$$\dim_R (V/\operatorname{Im}\alpha) = \dim_R ((V/\operatorname{Im}\alpha)/\operatorname{Ker}\varphi) + \dim_R \operatorname{Ker}\varphi,$$

so we have that $\dim_R((V/\mathrm{Im}\alpha)/\mathrm{Ker}\varphi)$ is infinite. Then from (2), $\dim_R(\mathrm{Im}\beta/\mathrm{Im}\alpha\beta)$ is infinite. Consequently, $\dim_R(V/\mathrm{Im}\,\alpha\beta)$ is infinite, so $\alpha\beta \in OE_R(V)$.

Therefore the proof is complete. $\square$

The last result of this section is obtained directly from Lemma 2.1.1(ii) and Lemma 2.1.9.

Proposition 2.1.10. *If $\dim_R V$ is infinite and T is a subsemigroup of $AI_R(V)$, then $OE_R(V) \cup T$ is a subsemigroup of $L_R(V)$.*

2.2 The Semigroups $OM_R(V)$ and $OE_R(V)$

Throughout this section, $\dim_R V$ is assumed to be infinite. Recall that 0, the zero map on V belongs to both $OM_R(V)$ and $OE_R(V)$ and note that $1_V \notin OM_R(V)$ and $1_V \notin OE_R(V)$. In this section, we aim to prove the following theorem.

Theorem 2.2.1. *If $S(V)$ is $OM_R(V)$ or $OE_R(V)$, then $S(V)$ does not admit the structure of a semihyperring with zero.*

Proof. We prove the theorem by contradiction. Suppose that there is a hyperoperation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \cdot)$ is a semihyperring with zero 0 where $\cdot$ is the operation on $S(V)$. Let B be a basis of V. Then B is infinite, so there are subsets B_1, B_2 of B such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2}. \tag{1}$$

By Proposition 1.8 and Proposition 1.9 (ii), $\text{Ker}\,\alpha = \langle B_2 \rangle$, $\dim_R (V/\text{Im}\,\alpha) = \dim_R (V/\langle B_1 \rangle) = |B_2|$, $\text{Ker}\,\beta = \langle B_1 \rangle$ and $\dim_R (V/\text{Im}\,\beta) = \dim_R (V/\langle B_2 \rangle) = |B_1|$. Then α and β are elements of $OM_R(V)$ and $OE_R(V)$. Hence $\alpha \oplus \beta \subseteq S(V)$. Obviously,

$$\alpha^2 = \alpha,\ \beta^2 = \beta,\ \alpha\beta = \beta\alpha = 0. \tag{2}$$

We have from (2) that

$$\begin{aligned} \alpha(\alpha \oplus \beta) &= \alpha \oplus 0 = \{\alpha\}, \\ \beta(\alpha \oplus \beta) &= 0 \oplus \beta = \{\beta\}. \end{aligned} \tag{3}$$

Let $\lambda \in \alpha \oplus \beta$. By (3), $\alpha\lambda = \alpha$ and $\beta\lambda = \beta$. We therefore deduce from these equalities and (1) that

$$\begin{aligned} &\text{for every } v \in B_1,\ v\lambda = v\alpha\lambda = v\alpha = v, \\ &\text{for every } v \in B_2,\ v\lambda = v\beta\lambda = v\beta = v. \end{aligned} \tag{4}$$

Consequently, $v\lambda = v$ for every $v \in B$. Since B is a basis of V, $\lambda = 1_V$. This is a contradiction because $1_V \notin OM_R(V)$ and $1_V \notin OE_R(V)$.

Therefore the theorem is proved. □

Since every hyperring [ring] is an AC semihyperring with zero, we have

Corollary 2.2.2. *The following statements hold.*

(i) The semigroup $OM_R(V)$ does not admit a hyperring [ring] structure.

(ii) The semigroup $OE_R(V)$ does not admit a hyperring [ring] structure.

2.3 Semigroups Containing $OM_R(V)$ and Semigroups Containing $OE_R(V)$

Also, $\dim_R V$ is assumed to be infinite in this section. From Proposition 2.1.3, Proposition 2.1.5, Proposition 2.1.7 and Proposition 2.1.10, we know respectively that

(1) $OM_R(V) \cup H$ where H is a subsemigroup of $G_R(V)$,

(2) $OE_R(V) \cup H$ where H is a subsemigroup of $G_R(V)$,

(3) $OM_R(V) \cup T$ where T is a subsemigroup of $AI_R(V)$,

(4) $OE_R(V) \cup T$ where T is a subsemigroup of $AI_R(V)$

are also subsemigroups of $L_R(V)$. It is shown in this section that any linear transformation semigroup on V of type (1) – (4) does not admit the structure of a semihyperring with zero.

Theorem 2.3.1. *If H is a subsemigroup of $G_R(V)$ and $S(V)$ is the semigroup $OM_R(V) \cup H$ or the semigroup $OE_R(V) \cup H$, then $S(V)$ does not admit the structure of a semihyperring with zero.*

Proof. Suppose that there exists a hyperoperation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \cdot)$ is a semihyperring with zero 0 where $\cdot$ is the operation on $S(V)$. Let B be a basis of V and $u \in B$ a fixed element. Since B is infinite, $B \smallsetminus \{u\}$ has subsets B_1, B_2 such that $B \smallsetminus \{u\} = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B \smallsetminus \{u\}|(= |B|)$. Then $B = B_1 \cup B_2 \cup \{u\}$ and these three sets are pairwise disjoint. Define $\alpha, \beta, \gamma \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \cup \{u\} \\ v & 0 \end{pmatrix}_{v \in B_1}, \beta = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v \end{pmatrix}_{v \in B_2}, \gamma = \begin{pmatrix} u & B_1 \cup B_2 \\ u & 0 \end{pmatrix}. \quad (1)$$

By Proposition 1.8 and Proposition 1.9 (ii), $\operatorname{Ker}\alpha = \langle B_2 \cup \{u\}\rangle$, $\dim_R(V/\operatorname{Im}\alpha) = \dim_R(V/\langle B_1\rangle) = |B_2 \cup \{u\}|$, $\operatorname{Ker}\beta = \langle B_1 \cup \{u\}\rangle$, $\dim_R(V/\operatorname{Im}\beta) = \dim_R(V/\langle B_2\rangle) = |B_1 \cup \{u\}|$, $\operatorname{Ker}\gamma = \langle B_1 \cup B_2\rangle$, and $\dim_R(V/\operatorname{Im}\gamma) = \dim_R(V/\langle u\rangle) = |B_1 \cup B_2|$. Then $\alpha, \beta, \gamma \in S(V)$. Thus $\alpha \oplus \beta \subseteq S(V)$. Obviously,

$$\alpha^2 = \alpha,\ \beta^2 = \beta,\ \alpha\beta = \beta\alpha = 0,\ \gamma\alpha = \gamma\beta = 0. \tag{2}$$

From (2), we have

$$\begin{aligned} \alpha(\alpha \oplus \beta) &= \alpha \oplus 0 = \{\alpha\}, \\ \beta(\alpha \oplus \beta) &= 0 \oplus \beta = \{\beta\}, \\ \gamma(\alpha \oplus \beta) &= 0 \oplus 0 = \{0\}. \end{aligned} \tag{3}$$

Let $\lambda \in \alpha \oplus \beta$. We therefore have from (3) that $\alpha\lambda = \alpha, \beta\lambda = \beta$ and $\gamma\lambda = 0$. Hence from these equalities and (1), we get

$$\begin{aligned} &\text{for every } v \in B_1,\ v\lambda = v\alpha\lambda = v\alpha = v, \\ &\text{for every } v \in B_2,\ v\lambda = v\beta\lambda = v\beta = v, \\ &u\lambda = u\gamma\lambda = 0, \end{aligned} \tag{4}$$

that is,

$$\lambda = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B_1 \cup B_2}.$$

Hence $\operatorname{Ker}\lambda = \{u\}$ and $\dim_R(V/\operatorname{Im}\lambda) = \dim_R(V/\langle B \smallsetminus \{u\}\rangle) = |\{u\}| = 1$. Therefore $\lambda \notin S(V)$ which is contrary to that $\lambda \in \alpha \oplus \beta \subseteq S(V)$.

Hence the theorem is completely proved. $\square$

The following corollary is a direct consequence of the above theorem.

Corollary 2.3.2. *If H is a subsemigroup of $G_R(V)$, then the semigroups $OM_R(V) \cup H$ and $OE_R(V) \cup H$ do not admit a hyperring [ring] structure.*

Remark 2.3.3. Let B be a basis of V and for distinct $u, w \in B$, let $\alpha_{u,w} \in G_R(V)$ be defined by

$$\alpha_{u,w} = \begin{pmatrix} u & w & v \\ w & u & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}.$$

Then $H_{u,w} = \{1_V, \alpha_{u,w}\}$ is a subgroup of $G_R(V)$ for all distinct $u, w \in B$, and $H_{u,w} \neq H_{u',w'}$ if u, w, u', w' are elements of B such that $(u, w) \neq (u', w')$. This fact and Theorem 2.3.1 show that if $\dim_R V$ is infinite, there are infinitely many subsemigroups of $L_R(V)$ containing $OM_R(V)$ and infinitely many subsemigroups of $L_R(V)$ containing $OE_R(V)$ which do not admit the structure of a semihyperring with zero.

Theorem 2.3.4. *If T is a subsemigroup of $AI_R(V)$ and $S(V)$ is the semigroup $OM_R(V) \cup T$ or the semigroup $OE_R(V) \cup T$, then $S(V)$ does not admit the structure of a semihyperring with zero.*

Proof. Suppose that $(S(V), \oplus, \cdot)$ is a semihyperring with zero for some hyperoperation $\oplus$ on $S(V)$ where $\cdot$ is the operation on $S(V)$. Let B be a basis of V and let $B_1, B_2 \subseteq B$ be such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Then there is a bijection $\varphi : B_1 \to B_2$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B_2 \\ v\varphi & 0 \end{pmatrix}_{v \in B_1} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v \\ 0 & v\varphi^{-1} \end{pmatrix}_{v \in B_2}. \tag{1}$$

Then from Propositon 1.8 and Propositon 1.9 (ii), $\operatorname{Ker}\alpha = \langle B_2 \rangle$, $\dim_R(V/\operatorname{Im}\alpha) = \dim_R(V/\langle B_2 \rangle) = |B_1|$, $\operatorname{Ker}\beta = \langle B_1 \rangle$ and $\dim_R(V/\operatorname{Im}\beta) = \dim_R(V/\langle B_1 \rangle) = |B_2|$. Thus $\alpha, \beta \in OM_R(V) \cap OE_R(V)$, and so $\alpha, \beta \in S(V)$. It is clear from (1) that

$$\begin{aligned} &\alpha^2 = \beta^2 = 0, \\ &\text{for every } v \in B_1,\ v\alpha\beta = v, \\ &\text{for every } v \in B_2,\ v\beta\alpha = v. \end{aligned} \tag{2}$$

It follows from (2) that

$$\alpha(\alpha \oplus \beta) = 0 \oplus \alpha\beta = \{\alpha\beta\},\ \beta(\alpha \oplus \beta) = \beta\alpha \oplus 0 = \{\beta\alpha\}. \tag{3}$$

Let $\lambda \in \alpha \oplus \beta$. We therefore have from (3) that $\alpha\lambda = \alpha\beta$ and $\beta\lambda = \beta\alpha$. Hence from these facts, (1) and (2), we have

$$\begin{aligned} &\text{for every } v \in B_1,\ (v\varphi)\lambda = v\alpha\lambda = v\alpha\beta = v = (v\varphi)\varphi^{-1}, \\ &\text{for every } v \in B_2,\ (v\varphi^{-1})\lambda = v\beta\lambda = v\beta\alpha = v = (v\varphi^{-1})\varphi. \end{aligned} \tag{4}$$

We can see from (4) that

$$\begin{aligned} \lambda_{|_{B_2}} &= \varphi^{-1} : B_2 \to B_1 \text{ is a bijection,} \\ \lambda_{|_{B_1}} &= \varphi : B_1 \to B_2 \text{ is a bijection,.} \end{aligned} \tag{5}$$

Since $B = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$, it follows from (5) that $\lambda_{|_B} : B \to B$ is a bijection, so $\lambda \in G_R(V)$. Hence $\lambda \notin OM_R(V)$ and $\lambda \notin OE_R(V)$. Claim that $\lambda \notin AI_R(V)$, let $v_1, v_2, ..., v_n \in B_1$ be distinct and let $a_1, a_2, ..., a_n \in R$ be such that $\sum\limits_{i=1}^{n} a_i(v_i + F(\lambda)) = F(\lambda)$. Then $\sum\limits_{i=1}^{n} a_i v_i \in F(\lambda)$, so $(\sum\limits_{i=1}^{n} a_i v_i)\lambda = \sum\limits_{i=1}^{n} a_i v_i$. But $(\sum\limits_{i=1}^{n} a_i v_i)\lambda = \sum\limits_{i=1}^{n} a_i(v_i\lambda) \in \langle B_2 \rangle$ by (5), so we have, $\sum\limits_{i=1}^{n} a_i v_i \in \langle B_1 \rangle \cap \langle B_2 \rangle = \{0\}$, which implies that $a_i = 0$ for all $i \in \{1, 2, \ldots, n\}$ since B_1 is linearly independent over R. This shows that $\{v + F(\lambda) \mid v \in B_1\}$ is a linearly independent subset of $V/F(\lambda)$ and $v + F(\lambda) \neq w + F(\lambda)$ for all distinct $v, w \in B_1$. Hence $\dim_R (V/F(\lambda)) \geq |B_1|$. But B_1 is infinite, so $\lambda \notin AI_R(V)$. Therefore we have $\lambda \notin OM_R(V) \cup OE_R(V) \cup AI_R(V)$. Thus $\lambda \notin S(V)$. This is a contradiction since $\lambda \in \alpha \oplus \beta \subseteq S(V)$.

This proves that there is no hyperoperation $\oplus$ on $S(V)$ such that $(S(V), \oplus, \cdot)$ is a semihyperring with zero. Hence the theorem is proved. $\square$

Also, we have a corollary of Theorem 2.3.4 as follows:

Corollary 2.3.5. *If T is a subsemigroup of $AI_R(V)$, then the semigroups $OM_R(V) \cup T$ and $OE_R(V) \cup T$ do not admit a hyperring [ring] structure.*

Remark 2.3.6. Let B be a basis of V and for each $u \in B$, define $\alpha_u \in L_R(V)$ by

$$\alpha_u = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}} .$$

Then $F(\alpha_u) = \langle B \smallsetminus \{u\} \rangle$, and hence by Proposition 1.9 (ii), $\dim_R (V/F(\alpha_u)) = |\{u\}|$ for every $u \in B$. Clearly, $\alpha_u \neq \alpha_w$ if u and w are distinct elements of B and for each $u \in B, \{\alpha_u\}$ is a subsemigroup of $AI_R(V)$ since $\alpha_u^2 = \alpha_u$. This fact and Theorem 2.3.4 show that there are infinitely many subsemigroups of $L_R(V)$ containing $OE_R(V)$ which do not admit the structure of a semihyperring with zero.

2.4 The Semigroups $K_R(V, k)$ and $K'_R(V, k)$

We shall characterize when $K_R(V, k)$ admits the structure of a semihyperring with zero. The characterization will generalize Theorem 2.2.1 for the case of $OM_R(V)$ since $OM_R(V) = K_R(V, \aleph_0)$ if $\dim_R V$ is infinite. Since $K'_R(V, k) = K_R(V, k')$ if k' is the successor of k, by the characterization of $K_R(V, k)$ admitting this structure, necessary and sufficient conditions for $K'_R(V, k)$ to admit such a structure are also obtained.

Theorem 2.4.1. *Let k be a cardinal number with $k \leq dim_R V$. Then $K_R(V,k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$.

Proof. To prove sufficiency, assume that (i) or (ii) holds. If $k = 0$, Then $K_R(V,k) = K_R(V,0) = L_R(V)$ which admits a ring structure. Assume that $\dim_R V < \infty$ and $k = \dim_R V$. If $\alpha \in K_R(V,k) = K_R(V, \dim_R V)$, then $\dim_R \operatorname{Ker}\alpha = \dim_R V < \infty$ which implies that $\operatorname{Ker}\alpha = V$, so $\alpha = 0$. Hence $K_R(V,k) = \{0\}$ which admits a ring structure.

To prove necessity, assume that there is a hyperoperation $\oplus$ on $K_R(V,k)$ such that $(K_R(V,k), \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $K_R(V,k)$. To prove that (i) or (ii) must hold, suppose on the contrary that (i) and (ii) are false. Then either (1) $0 < k < \dim_R V < \infty$ or (2) $k > 0$ and $\dim_R V$ is infinite.

Case 1 : $0 < k < \dim_R V < \infty$. Let B be a basis of V and $B_1 \subseteq B$ such that $|B_1| = k$. Since $|B_1| = k > 0$, there exists an element $u \in B_1$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \setminus B_1}, \quad \beta = \begin{pmatrix} u & B \setminus \{u\} \\ u & 0 \end{pmatrix}. \tag{1}$$

Then $\operatorname{Ker}\alpha = \langle B_1 \rangle$ and $\operatorname{Ker}\beta = \langle B \setminus \{ u \} \rangle$, so $\dim_R \operatorname{Ker}\alpha = k$ and $\dim_R \operatorname{Ker}\beta = \dim_R V - 1 \geq k$. Consequently, $\alpha, \beta \in K_R(V,k)$. By (1), we clearly have

$$\alpha^2 = \alpha, \ \beta^2 = \beta \quad \text{and} \quad \alpha\beta = \beta\alpha = 0,$$

and thus,

$$\alpha(\alpha \oplus \beta) = \{ \alpha \} \quad \text{and} \quad \beta(\alpha \oplus \beta) = \{ \beta \}. \tag{2}$$

Let $\gamma \in \alpha \oplus \beta$. It then follows from (2) that $\alpha\gamma = \alpha$ and $\beta\gamma = \beta$, and hence

$$\operatorname{Im}\alpha = \operatorname{Im}\alpha\gamma \subseteq \operatorname{Im}\gamma \quad \text{and} \quad \operatorname{Im}\beta = \operatorname{Im}\beta\gamma \subseteq \operatorname{Im}\gamma. \tag{3}$$

Thus we deduce from (1) and (3) that

$$B \setminus (B_1 \setminus \{u\}) = (B \setminus B_1) \cup \{u\} \subseteq \operatorname{Im}\alpha \cup \operatorname{Im}\beta \subseteq \operatorname{Im}\gamma.$$

This implies that

$$\dim_R \operatorname{Im}\gamma \geq |B \setminus (B_1 \setminus \{u\})| = \dim_R V - (k-1). \tag{4}$$

Since $\infty > \dim_R V = \dim_R \operatorname{Ker}\gamma + \dim_R \operatorname{Im}\gamma$, we have that

$$\begin{aligned} \dim_R \operatorname{Ker}\gamma &= \dim_R V - \dim_R \operatorname{Im}\gamma \\ &\leq \dim_R V - (\dim_R V - (k-1)) \quad \text{from (4)} \\ &= k-1 < k \end{aligned}$$

which implies that $\gamma \notin K_R(V,k)$. This yields a contradiction since $\gamma \in \alpha \oplus \beta \subseteq K_R(V,k)$.

Case 2 : $k > 0$ and $\dim_R V$ is infinite. Let B be a basis of V. Then B is infinite, so there exist subsets B_1 and B_2 of B such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let $\alpha, \beta \in L_R(V)$ be defined by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2}, \quad \beta = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1}. \tag{5}$$

Then $\operatorname{Ker}\alpha = \langle B_1 \rangle$ and $\operatorname{Ker}\beta = \langle B_2 \rangle$, so $\dim_R \operatorname{Ker}\alpha = |B_1| = |B|$ and $\dim_R \operatorname{Ker}\beta = |B_2| = |B|$. Since $k \leq \dim_R V = |B|$, we have $\alpha, \beta \in K_R(V,k)$. It is clear from (5) that

$$\alpha^2 = \alpha,\ \beta^2 = \beta \ \text{ and } \ \alpha\beta = \beta\alpha = 0,$$

and thus

$$\alpha(\alpha \oplus \beta) = \{\, \alpha \,\} \;\text{ and }\; \beta(\alpha \oplus \beta) = \{\, \beta \,\}. \tag{6}$$

Let $\gamma \in \alpha \oplus \beta$. Then from (6), $\alpha\gamma = \alpha$ and $\beta\gamma = \beta$. Consequently,

$$\text{for every } v \in B_2,\; v\gamma = v\alpha\gamma = v\alpha = v,$$

$$\text{for every } v \in B_1,\; v\gamma = v\beta\gamma = v\beta = v$$

which implies that $\gamma = 1_V$, the identity map on V. It then follows that $\dim_R \mathrm{Ker}\gamma = 0 < k$ since $k > 0$. Hence $\gamma \notin K_R(V, k)$, a contradiction.

Therefore the proof is complete. □

We give a remark here that from Theorem 2.4.1, we conclude that Theorem 2.2.1 for that case of $OM_R(V)$ is a consequence of Theorem 2.4.1.

Corollary 2.4.2. *Assume that $k < dim_R\, V$. Then $K'_R(V,k)$ admits the structure of a semihyperring with zero if and only if $dim_R\, V < \infty$ and $k = dim_R\, V - 1$.*

Proof. Let k' be the successor of k. Then $k' > 0$ and $K'_R(V,k) = K_R(V,k')$. If $K'_R(V,k)$ admits the structure of a semihyperring with zero, then by Theorem 2.4.1, $\dim_R V < \infty$ and $k' = \dim_R V$, so $k = \dim_R V - 1$.

If $\dim_R V < \infty$ and $k = \dim_R V - 1$, then $k' = \dim_R V$, thus by Theorem 2.4.1, $K_R(V,k')$ admits the structure of a semihyperring with zero, and so does $K'_R(V,k)$ since $K'_R(V,k) = K_R(V,k')$. □

We can see from the proofs of Theorem 2.4.1 and Corollary 2.4.2 that $K_R(V,k) = L_R(V)$ or $\{0\}$ and $K'_R(V,k) = \{0\}$ are necessary conditions of Theorem 2.4.1 and Corollary 2.4.2, respectively. Hence we have

Corollary 2.4.3. *For a cardinal number k with $k \leq dim_R V$, $K_R(V,k)$ admits a hyperring [ring] structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$.

Corollary 2.4.4. *For a cardinal number k with $k < dim_R V, K'_R(V,k)$ admits a hyperring [ring] structure if and only if $dim_R V < \infty$ and $k = dim_R V - 1$.*

Remark 2.4.5. If k_1 and k_2 are cardinal numbers such that $k_1 < k_2 \leq \dim_R V$, then $K_R(V,k_1) \supsetneq K_R(V,k_2)$. To see this, let B a basis of V. Then $k_1 < k_2 \leq |B|$, so there is a subset B_1 of B such that $|B_1| = k_1$. Define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\dim_R \operatorname{Ker} \alpha = |B_1| = k_1 < k_2$. Thus $\alpha \in K_R(V,k_1) \smallsetminus K_R(V,k_2)$. It then follows that if $\dim_R V$ is infinite, then

$$K_R(V,1) = K'_R(V,0) \supsetneq K_R(V,2) = K'_R(V,1) \supsetneq K_R(V,3) = K'_R(V,2) \supsetneq \dots$$

and by Theorem 2.4.1, none of these subsemigroups of $L_R(V)$ admits the structure of a semihyperring with zero.

2.5 The Semigroups $CI_R(V,k)$ and $CI'_R(V,k)$

From Proposition 1.12, if $\dim_R V < \infty$, then $K_R(V,k) = CI_R(V,k)$ for every cardinal number k with $k \leq \dim_R V$. However, it is also shown in Proposition 1.13 (i) that if $\dim_R V$ is infinite, $CI_R(V,k) \neq K_R(V,l)$ for all cardinal numbers

k, l with $0 < k \leq \dim_R V$ and $l \leq \dim_R V$. Then characterizing when $CI_R(V, k)$ admits the structure of a semihyperring with zero should be also considered.

Theorem 2.5.1. *Let k be a cardinal number with $k \leq dim_R V$. Then $CI_R(V, k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $dim_R V < \infty$ and $k = dim_R V$

Proof. By Proposition 1.12, $CI_R(V, k) = K_R(V, k)$ if $\dim_R V < \infty$. Hence by Theorem 2.4.1, $CI_R(V, k)$ admits a ring structure if (i) or (ii) holds.

Conversely, assume that there is a hyperoperation $\oplus$ on $CI_R(V, k)$ such that $(CI_R(V, k), \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $CI_R(V, k)$. To prove that (i) or (ii) holds, suppose instead that they both are false. Then either (1) $0 < k < \dim_R V < \infty$ or (2) $k > 0$ and $\dim_R V$ is infinite.

Case 1 : $0 < k < \dim_R V < \infty$. Since $\dim_R V < \infty, K_R(V, k) = CI_R(V, k)$. By Theorem 2.4.1, $CI_R(V, k)$ does not admit the structure of a semihyperring with zero. This is a contradiction.

Case 2 : $k > 0$ and $\dim_R V$ is infinite. Let B be a basis of V and let $B_1, B_2 \subseteq B$ such that $B = B_1 \cup B_2, B_1 \cap B_2 = \varnothing$ and $|B_1| = |B_2| = |B|$. Let $\alpha, \beta \in L_R(V)$ be defined by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_2} \quad , \quad \beta = \begin{pmatrix} v & B_2 \\ v & 0 \end{pmatrix}_{v \in B_1} .$$

Then $\dim_R(V/\mathrm{Im}\alpha) = \dim_R(V/\langle B_2\rangle) = |B_1| = |B| = \dim_R V$ and $\dim_R(V/\mathrm{Im}\beta) = \dim_R(V/\langle B_1\rangle) = |B_2| = |B| = \dim_R V$. Consequently, $\alpha, \beta \in CI_R(V, k)$. Hence $\alpha \oplus \beta \subseteq CI_R(V, k)$. As shown in the proof of Case 2 of Theorem 2.4.1 that if

$\gamma \in \alpha \oplus \beta$, then $\gamma = 1_V$. But $\dim_R (V/\mathrm{Im}\, 1_V) = 0 < k$, so this is contrary to that $\alpha \oplus \beta \subseteq CI_R(V,k)$.

Therefore the proof is complete. □

Corollary 2.5.2. *Assume that k is a cardinal number such that $k < \dim_R V$. Then $CI'_R(V,k)$ admits the structure of a semihyperring with zero if and only if $\dim_R V < \infty$ and $k = \dim_R V - 1$.*

Proof. Let k' be the successor of k. Then $k' > 0$ and $CI'_R(V,k) = CI_R(V,k')$. It therefore follows from Theorem 2.5.1 that $CI'_R(V,k)$ admits the structure of a semihyperring with zero if $\dim_R V < \infty$ and $k' = \dim_R V$, or equivalently, $\dim_R V < \infty$ and $k = \dim_R V - 1$.

Conversely, assume that $\dim_R V < \infty$ and $k = \dim_R V - 1$. Then $k' = \dim_R V$, and thus by Theorem 2.5.1, $CI_R(V,k')$ admits the structure of a semihyperring with zero. But $CI_R(V,k') = CI'_R(V,k)$, so $CI'_R(V,k)$ admits the structure of a semihyperring with zero. □

Notice from the proofs of Theorem 2.4.1, Theorem 2.5.1, Corollary 2.4.2 and Corollary 2.5.2 that necessary conditions of Theorem 2.5.1 and Corollary 2.5.2 are $CI_R(V,k) = L_R(V)$ or $\{0\}$ and $CI'_R(V,k) = \{0\}$, respectively. Hence the following corollaries are obtained directly.

Corollary 2.5.3. *For a cardinal number k with $k \leq \dim_R V, CI_R(V,k)$ admits a hyperring [ring] structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $\dim_R V < \infty$ and $k = \dim_R V$.

Corollary 2.5.4. *For a cardinal number k with $k \leq dim_R V, CI'_R(V,k)$ admits a hyperring [ring] structure if and only if $dim_R V < \infty$ and $k = dim_R V - 1$.*

Remark 2.5.5. Let $\dim_R V$ be infinite and B a basis of V. If $k_1 < k_2 \leq \dim_R V$, let B_1 of B such that $|B_1| = k_1$. Define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}.$$

Then $\dim_R(V/\operatorname{Im}\alpha) = \dim_R(V/\langle B \smallsetminus B_1\rangle) = |B_1| = k_1 < k_2$, so $\alpha \in CI_R(V,k_1) \smallsetminus CI_R(V,k_2)$. It then follows that if $\dim_R V$ is infinite, then

$$CI_R(V,1) = CI'_R(V,0) \supsetneq CI_R(V,2) = CI'_R(V,1) \supsetneq CI_R(V,3) = CI'_R(V,2) \supsetneq \dots$$

and by Theorem 2.5.1, none of them admits the structure of a semihyperring with zero.

2.6 The Semigroups $I_R(V,k)$ and $I'_R(V,k)$

We have shown in Proposition 1.13 (ii) if $\dim_R V$ is infinite, then for a nonzero cardinal number k with $k < \dim_R V$, $I_R(V,k)$ is not equal to $K_R(V,l)$ and $CI_R(V,l)$ for any cardinal number $l \leq \dim_R V$. This is also true for $I'_R(V,k)$, $K'_R(V,l)$ and $CI'_R(V,l)$ where $0 < k \leq \dim_R V$ and $0 \leq l < \dim_R V$. The next theorem and corollary, Theorem 2.4.1, Corollary 2.4.2, Theorem 2.5.1 and Corollary 2.5.2 also show that what we have mentioned is true.

Theorem 2.6.1. *Let k be a cardinal number such that $k \leq dim_R V$. Then $I_R(V,k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $k = dim_R V$.

(iii) *k is an infinite cardinal number.*

Proof. To prove sufficiency, assume (i), (ii) or (iii) holds. Since $I_R(V, 0) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha \leq 0 \,\} = \{\, 0 \,\}$ and $I_R(V, \dim_R V) = \{\, \alpha \in L_R(V) \mid \dim_R \operatorname{Im} \alpha \leq \dim_R V \,\} = L_R(V)$. Therefore if (i) or (ii) holds, then $I_R(V, k)$ admits a ring structure.

Next, assume that (iii) holds. Then $k + k = k$. We know that for $\alpha, \beta \in L_R(V), \operatorname{Im}(\alpha + \beta) \subseteq \operatorname{Im} \alpha + \operatorname{Im} \beta$ and $\operatorname{Im}(-\alpha) = \operatorname{Im} \alpha$ where $+$ is the usual addition on $L_R(V)$. Thus for $\alpha, \beta \in I_R(V, k)$,

$$\begin{aligned} \dim_R \operatorname{Im}(\alpha - \beta) &\leq \dim_R \operatorname{Im} \alpha + \dim_R \operatorname{Im} \beta \\ &\leq k + k = k. \end{aligned}$$

It therefore follows that $I_R(V, k)$ is a subring of $(L_R(V), +, \cdot)$, so $I_R(V, k)$ admits a ring structure.

Conversely, assume that $(I_R(V, k), \oplus, \cdot)$ is a semihyperring with zero for some hyperoperation $\oplus$ on $I_R(V, k)$ where $\cdot$ is the operation on $I_R(V, k)$. To show that one of (i), (ii) and (iii) must hold, suppose on the contrary that (i), (ii) and (iii) are all false. Then $0 < k < \dim_R V$ and k is finite. Let B be a basis of V and $B_1 \subseteq B$ such that $|B_1| = k$. Since $k < \dim_R V$, there exists an element $u \in B \smallsetminus B_1$. Define $\alpha, \beta \in L_R(V)$ by

$$\alpha = \begin{pmatrix} v & B \smallsetminus B_1 \\ v & 0 \end{pmatrix}_{v \,\in\, B_1} \quad \text{and} \quad \beta = \begin{pmatrix} u & B \smallsetminus \{u\} \\ u & 0 \end{pmatrix}. \tag{1}$$

Then $\operatorname{Im} \alpha = \langle B_1 \rangle$ and $\operatorname{Im} \beta = \langle u \rangle$, so $\dim_R \operatorname{Im} \alpha = k$ and $\dim_R \operatorname{Im} \beta = 1 \leq k$. This implies that $\alpha, \beta \in I_R(V, k)$ and so $\alpha \oplus \beta \subseteq I_R(V, k)$. Clearly,

$$\alpha^2 = \alpha, \;\; \beta^2 = \beta \quad \text{and} \quad \alpha\beta = \beta\alpha = 0. \tag{2}$$

Therefore (2) yields

$$\alpha(\alpha \oplus \beta) = \{\, \alpha \,\} \quad \text{and} \quad \beta(\alpha \oplus \beta) = \{\, \beta \,\}. \tag{3}$$

Let $\gamma \in \alpha \oplus \beta$. It thus follows from (3) that $\alpha\gamma = \alpha$ and $\beta\gamma = \beta$. We have from these equalities and (1) that

$$v\gamma = v\alpha\gamma = v\alpha = v \quad \text{for every } v \in B_1 \quad \text{and}$$
$$u\gamma = u\beta\gamma = u\beta = u,$$

so $\operatorname{Im}\gamma \supseteq \langle B_1 \cup \{\, u \,\}\rangle$ which implies that $\dim_R \operatorname{Im}\gamma \geq |B_1 \cup \{\, u \,\}| = k+1 > k$. This contradicts the fact that $\gamma \in \alpha \oplus \beta \subseteq I_R(V,k)$.

Hence the theorem is proved. □

Corollary 2.6.2. *For a cardinal number k with $0 < k \leq dim_R V$, the semigroup $I'_R(V,k)$ admits the structure of a semihyperring with zero if and only if either $k = 1$ or k is an infinite cardinal number.*

Proof. If $I'_R(V,1) = I_R(V,0) = \{0\}$, then $I'_R(V,1)$ admits a ring structure. Next, assume that k is an infinite cardinal number. Then $k + k = k$. If $\alpha, \beta \in I'_R(V,k)$, then $\dim_R \operatorname{Im}\alpha < k$ and $\dim_R \operatorname{Im}\beta < k$, and hence

$$\begin{aligned} \dim_R \operatorname{Im}(\alpha - \beta) &\leq \dim_R \operatorname{Im}\alpha + \dim_R \operatorname{Im}\beta \\ &< k + k = k. \end{aligned}$$

It follows that $(I'_R(V,k), +, \cdot)$ is a ring where $+$ is the usual addition of linear transformations. Therefore the sufficiency is proved.

To prove necessity, suppose that $1 < k$ and k is finite. Then $I'_R(V,k) = I_R(V,k-1)$, $0 < k-1 < \dim_R V$ and $k-1$ is finite. It therefore follows from Theorem 2.6.1 that $I'_R(V,k)$ does not admits the structure of a semihyperring with zero. □

Theorem 2.6.1 and Corollary 2.6.2 and their proofs yield the following results.

Corollary 2.6.3. *For a cardinal number k with $k \leq \dim_R V$, the semigroup $I_R(V, k)$ admits a hyperring [ring] structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $k = \dim_R V$.

(iii) k is an infinite cardinal number.

Corollary 2.6.4. *For a cardinal number k with $0 < k \leq \dim_R V$, the semigroup $I'_R(V, k)$ admits a hyperring [ring] structure if and only if either $k = 1$ or k is an infinite cardinal number.*

Remark 2.6.5. Assume that $\dim_R V$ is infinite and let B a basis of V. Then B contains a subset $\{ u_n \mid n \in \mathbb{N}\}$ where $u_n \neq u_m$ if $n \neq m$. For each positive integer n, let $\alpha_n \in L_R(V)$ be define by

$$\alpha_n = \begin{pmatrix} u_1 & u_2 & \dots & u_n & B \smallsetminus \{u_1, u_2, \dots, u_n\} \\ u_1 & u_2 & \dots & u_n & 0 \end{pmatrix}.$$

Then $\dim_R \operatorname{Im} \alpha_n = \dim_R \langle u_1, \dots, u_n \rangle = n$ for every $n \in \mathbb{N}$, so $\alpha_n \in I_R(V, n) \smallsetminus I_R(V, n-1)$ for every $n \geq 1$. Consequently,

$$I_R(V, 1) = I'_R(V, 2) \supsetneq I_R(V, 2) = I'_R(V, 3) \supsetneq I_R(V, 3) = I'_R(V, 4) \supsetneq \dots$$

and Theorem 2.6.1 shows that none of these semigroups admits the structure of a semihyperring with zero.

CHAPTER III

SEMIGROUPS ADMITTING THE STRUCTURE OF AN ADDITIVELY COMMUTATIVE SEMIRING WITH ZERO

Example 1.5 and Example 1.6 show that every semigroup without zero always admits both the structure of an AC semihyperring with zero and the structure of a semiring with zero. As can be seen in this chapter that a semigroup without zero need not admit the structure of an AC semiring with zero. In this chapter,we aim to study various kinds of linear transformation semigroups which need not have a zero. We shall determine when they admit the structure of an AC semiring with zero.

First, let us recall the following linear transformation semigroups on V.

$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\},$$

$$M_R(V) = \{\alpha \in L_R(V) \mid \operatorname{Ker}\alpha = \{0\}\},$$

$$E_R(V) = \{\alpha \in L_R(V) \mid \operatorname{Im}\alpha = V\},$$

$$AM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker}\alpha < \infty\},$$

$$AE_R(V) = \{\alpha \in L_R(V) \mid \dim_R (V/\operatorname{Im}\alpha) < \infty\},$$

$$AI_R(V) = \{\alpha \in L_R(V) \mid \dim_R (V/F(\alpha)) < \infty \}$$

$$\text{where } F(\alpha) = \{ v \in V \mid v\alpha = v \},$$

$$PL_R(V) = \{\alpha \colon W \to V \mid W \text{ is a subspace of } V$$

$$\text{and } \alpha \text{ is a linear transformation}\}.$$

For convenience in writing, the following notations will be used in this chapter. If B is a basis of V and $u, w \in B$ are distinct, let $(u, w)_B$ and $(u \to w)_B$ be the elements of $L_R(V)$ defined respectively by

$$(u, w)_B = \begin{pmatrix} u & w & v \\ w & u & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}} \qquad \text{and } (u \to w)_B = \begin{pmatrix} u & v \\ w & v \end{pmatrix}_{v \in B \smallsetminus \{u\}}.$$

Then for all distinct $u, w \in B$, $(u, w)_B \in G_R(V)$ and also $(u, w)_B^2 = 1_V$.

3.1 The Semigroups $G_R(V)$, $M_R(V)$ and $E_R(V)$

Note that from Proposition 1.15(i), each of $G_R(V)$, $M_R(V)$, and $E_R(V)$ has a zero if and only if either $\dim_R V = 0$ or $\dim_R V = 1$ and $|R| = 2$. The purpose is to show that $\dim_R V = 0$ or $\dim_R V = 1$ is necessary and sufficient for each of $G_R(V)$, $M_R(V)$, and $E_R(V)$ to admit the structure of an AC semiring with zero.

Lemma 3.1.1. *If $\dim_R V = 1$, then $G_R(V) \cong (R \smallsetminus \{0\}, \cdot)$, the multiplicative group of nonzero elements of the division ring R.*

Proof. Let $u \in V \smallsetminus \{0\}$. Since $\dim_R V = 1$, $\{u\}$ is a basis of V, so $V = Ru = \{au \mid a \in R\}$ and it is clear that

$$\begin{gathered}\text{for every } \alpha \in G_R(V), \text{ there is a unique } a_\alpha \in R \smallsetminus \{0\} \\ \text{such that } u\alpha = a_\alpha u.\end{gathered} \tag{1}$$

Define φ: $G_R(V) \to R \smallsetminus \{0\}$ by $\alpha\varphi = a_\alpha$ for every $\alpha \in G_R(V)$. Then φ is well-defined by (1). If α, $\beta \in G_R(V)$, then

$$u\alpha\beta = (a_\alpha u)\beta = a_\alpha(u\beta) = (a_\alpha a_\beta)u,$$

so by (1), $a_{\alpha\beta} = a_\alpha a_\beta$. Hence φ is a homomorphism from $G_R(V)$ into $(R \smallsetminus \{0\}, \cdot)$. If $\alpha \in G_R(V)$ is such that $a_\alpha = 1$, then $u\alpha = u$ which implies that $\alpha = 1_V$ since

$V = Ru$. Therefore φ is one-to-one. For $b \in R \smallsetminus \{0\}$, if define $\alpha \in L_R(V)$ by $u\alpha = bu$, then $\alpha \in G_R(V)$ since $\{u\}$ is a basis of V and hence $\alpha\varphi = b$.

Therefore the lemma is proved, as required. □

Theorem 3.1.2. *Let $S(V)$ be $G_R(V)$, $M_R(V)$ or $E_R(V)$. Then $S(V)$ admits the structure of an AC semiring with zero if and only if $\dim_R V \leq 1$.*

Proof. If $\dim_R V = 0$, then $G_R(V) = M_R(V) = E_R(V) = \{0\}$, and if $\dim_R V = 1$, then $G_R(V) = M_R(V) = E_R(V) \cong (R \smallsetminus \{0\}, \cdot)$ by Lemma 3.1.1. Thus $S(V)$ admits a ring structure if $\dim_R V \leq 1$.

Assume that $\dim_R V > 1$. To show that $S(V)$ does not admit the structure of an AC semiring with zero, suppose on the contrary that there is an operation $\oplus$ on $S^0(V)$ such that $(S^0(V), \oplus, \cdot)$ is an AC semiring with zero 0 where $\cdot$ is the operation on $S^0(V)$. Let B be a basis of V. Then $|B| > 1$. Let u and w be distinct elements of B and let $B' = \{u, u+w\} \cup (B \smallsetminus \{u, w\})$. By Proposition 1.7, B' is also a basis of V. Then $(u, w)_B$ and $(u, u+w)_{B'}$ are elements of $G_R(V) \subseteq S(V)$, and so $1_V \oplus (u, w)_B$ and $1_V \oplus (u, u+w)_{B'}$ are elements of $S(V)$. Since $(u, w)_B^2 = 1_V = (u, u+w)_{B'}^2$, we have the following equalities.

$$(u, w)_B(1_V \oplus (u, w)_B) = 1_V(1_V \oplus (u, w)_B), \tag{1}$$

$$(1_V \oplus (u, w)_B)(u, w)_B = (1_V \oplus (u, w)_B)1_V, \tag{2}$$

$$(u, u+w)_{B'}(1_V \oplus (u, u+w)_{B'}) = 1_V(1_V \oplus (u, u+w)_{B'}), \tag{3}$$

$$(1_V \oplus (u, u+w)_{B'})(u, u+w)_{B'} = (1_V \oplus (u, u+w)_{B'})1_V. \tag{4}$$

Case 1 : $1_V \oplus (u, w)_B \neq 0$. Then $1_V \oplus (u, w)_B \in S(V)$. If $S(V)$ is $G_R(V)$ or $M_R(V)$, then $1_V \oplus (u, w)_B$ is a one-to-one map, so from (1), we have $(u, w)_B = 1_V$. If $S(V)$

is $E_R(V)$, then $\text{Im}\,(1_V \oplus (u,w)_B) = V$ which implies by (2) that $(u,w)_B = 1_V$. But $u \neq w$, so we have a contradiction.

Case 2 : $1_V \oplus (u, u+w)_{B'} \neq 0$. Then $1_V \oplus (u, u+w)_{B'} \in S(V)$. If $S(V)$ is $G_R(V)$ or $M_R(V)$, then $1_V \oplus (u, u+w)_{B'}$ is a one-to-one map and hence $(u, u+w)_{B'} = 1_V$ by (3). If $S(R) = E_R(V)$, then $\text{Im}\,(1_V \oplus (u, u+w)_{B'}) = V$, so $(u, u+w)_{B'} = 1_V$ by (4). These yield a contradiction since $u \neq u + w$.

Case 3 : $1_V \oplus (u,w)_B = 0 = 1_V \oplus (u, u+w)_{B'}$. Then $(u,w)_B = (u,w)_B \oplus 0 = (u,w)_B \oplus 1_V \oplus (u, u \oplus w)_{B'} = 0 \oplus (u, u \oplus w)_{B'} = (u, u \oplus w)_{B'}$ and hence $w = u(u,w)_B = u(u, u+w)_{B'} = u + w$ which is a contradiction.

Therefore the proof is complete. □

The following fact has been given in [14]. However, it can be considered as a consequence of Theorem 3.1.2 and its proof for the sufficiency part.

Corollary 3.1.3. *Let $S(V)$ be $G_R(V)$, $M_R(V)$ or $E_R(V)$. Then $S(V)$ admits a ring structure if and only if $dim_R\, V \leq 1$.*

Remark 3.1.4. We know from Example 1.5 and Example 1.6 that every semigroup without zero always admits both the structure of an AC semihyperring with zero and the structure of a semiring with zero. Also, if $\dim_R V > 1$, then all the semigroups $G_R(V)$, $M_R(V)$ and $E_R(V)$ have no zero. Hence, from these facts and Theorem 3.1.2, we have that all the semigroups $G_R(V)$, $M_R(V)$ and $E_R(V)$ always admit the structure of an AC semihyperring with zero and the structure of a semiring with zero.

3.2 The Semigroups $AM_R(V)$ and $AE_R(V)$

We first recall that $G_R(V) \subseteq M_R(V) \subseteq AM_R(V)$, $G_R(V) \subseteq E_R(V) \subseteq AE_R(V)$, $AM_R(V)$ and $AE_R(V)$ have no zero if $\dim_R V$ is infinite by Proposition 1.15(ii), and if $\dim_R V < \infty$, then $AM_R(V)$ and $AE_R(V)$ admit a ring structure. The purpose is to show that $\dim_R V < \infty$ is also necessary for $AM_R(V)$ and $AE_R(V)$ to admit the structure of an AC semiring with zero.

Theorem 3.2.1. *Let $S(V)$ be $AM_R(V)$ or $AE_R(V)$. Then $S(V)$ admits the structure of an AC semiring with zero if and only if $\dim_R V < \infty$.*

Proof. As was mentioned above if $\dim_R V < \infty$, then $S(V)$ admits a ring structure.

Assume that $\dim_R V$ is infinite. To prove that $S(V)$ does not admit the structure of an AC semiring with zero, suppose instead that there is an operation $\oplus$ on $S^0(V)$ such that $(S^0(V), \oplus, \cdot)$ is an AC semiring with zero 0 where $\cdot$ is the operation on $S^0(V)$. Note that $0 \notin S(V)$, so for $\alpha, \beta \in S^0(V), \alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$. Let B be a basis of V and let u, w be distinct elements of B. Define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} \{u, w\} & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}. \tag{1}$$

Then $\dim_R \mathrm{Ker}\, \alpha = \dim_R \langle u, w \rangle = 2$ and $\dim_R (V/\mathrm{Im}\, \alpha) = \dim_R (V/\langle B \smallsetminus \{u, w\}\rangle) = |\{u, w\}| = 2$. We deduce that $\alpha \in S(V)$. It is clear that $(u, w)_B\alpha = \alpha = \alpha(u, w)_B$. Hence

$$(u, w)_B(1_V \oplus (u, w)_B) = 1_V \oplus (u, w)_B, \tag{2}$$

$$\alpha \oplus \alpha = (1_V \oplus (u, w)_B)\alpha = \alpha(1_V \oplus (u, w)_B). \tag{3}$$

Since $\beta(1_V \oplus 1_V) = \beta \oplus \beta = (1_V \oplus 1_V)\beta$ for every $\beta \in S(V)$, we have by Proposition 1.10 that, $1_V \oplus 1_V = a1_V$ for some $a \in C(R)$. Then

$$\alpha \oplus \alpha = (1_V \oplus 1_V)\alpha = (a1_V)\alpha = a\alpha. \tag{4}$$

If $a = 0$, then $1_V \oplus 1_V = 0$ and from (3) and (4), $1_V \oplus (u, w)_B = 0$ since $\alpha \in S(V)$, so $1_V = (u, w)_B$, a contradiction. This shows that $a \neq 0$. From (3) and (4), we have

$$(1_V \oplus (u, w)_B)\alpha = \alpha(1_V \oplus (u, w)_B) = a\alpha \neq 0. \tag{5}$$

By (1) and (5), we have

$$v(1_V \oplus (u, w)_B)\alpha \;=\; v\alpha(1_V \oplus (u, w)_B) \;=\; v(a\alpha) \;=\; av \quad \text{for every } v \in B \smallsetminus \{u, w\},$$

$$\begin{aligned} u(1_V \oplus (u, w)_B)\alpha &= u(a\alpha) = 0, \\ w(1_V \oplus (u, w)_B)\alpha &= w(a\alpha) = 0. \end{aligned} \tag{6}$$

We deduce from (2) that

$$u(1_V \oplus (u, w)_B) = u(u, w)_B(1_V \oplus (u, w)_B) = w(1_V \oplus (u, w)_B). \tag{7}$$

Thus (6) and (7) yield the fact that

$$u(1_V \oplus (u, w)_B) = w(1_V \oplus (u, w)_B) \in \operatorname{Ker} \alpha = \langle u, w \rangle,$$

so

$$u(1_V \oplus (u, w)_B) = w(1_V \oplus (u, w)_B) = bu + cw \quad \text{for some } b, c \in R. \tag{8}$$

Define $\gamma \in L_R(V)$ by

$$\gamma = \begin{pmatrix} \{u, w\} & v \\ u + w & v \end{pmatrix}_{v \in B \smallsetminus \{u, w\}}. \tag{9}$$

Then $\operatorname{Ker}\gamma \subseteq \langle u, w\rangle = \operatorname{Ker}\alpha$ and $\operatorname{Im}\gamma \supseteq \langle B \setminus \{u, w\}\rangle = \operatorname{Im}\alpha$, so $\gamma \in S(V)$. Since $u\gamma(u, w)_B = w\gamma(u, w)_B = (u + w)(u, w)_B = u + w$, it follows that $\gamma(u, w)_B = \gamma$, and hence

$$\gamma(1_V \oplus (u, w)_B) = \gamma \oplus \gamma = (1_V \oplus 1_V)\gamma = a\gamma. \tag{10}$$

Thus

$$\begin{aligned} 2bu + 2cw &= (u + w)(1_V \oplus (u, w)_B) && \text{from (8)} \\ &= u\gamma(1_V \oplus (u, w)_B) && \text{from (9)} \\ &= u(a\gamma) && \text{from (10)} \\ &= a(u + w) && \text{from (9)} \\ &= au + aw \end{aligned}$$

which implies that $2b = 2c = a \neq 0$. This shows that $\operatorname{char}R \neq 2$. Because $-1_V \in S(V)$ and $\beta(1_V \oplus (-1_V)) = \beta \oplus (-\beta) = (1_V \oplus (-1_V))\beta$ for all $\beta \in S(V)$, by Proposition 1.10, $1_V \oplus (-1_V) = a'1_V$ for some $a' \in C(R)$. If $a' = 0$, then $1_V \oplus (-1_V) = 0$ and so

$$0 = \alpha \oplus (-\alpha) = \alpha \oplus (-\alpha(u, w)_B) = \alpha(1_V \oplus (-(u, w)_B))$$

which implies that $1_V \oplus (-(u, w)_B) = 0$, and hence $-1_V = -(u, w)_B$, a contradiction. Then $a' \neq 0$. But

$$a'1_V = 1_V \oplus (-1_V) = -1_V(1_V \oplus (-1_V)) = -a'1_V,$$

$2a'1_V = 0$, and thus $2a' = 0$ since $V \neq \{0\}$. Due to the facts that $a' \neq 0$ and $\operatorname{char}R \neq 2$, we have a contradiction.

Hence the theorem is completely proved. $\square$

The following fact has been given in [14]. It is also considered as a consequence of Theorem 3.2.1 and the first line of its proof.

Corollary 3.2.2. *Let $S(V)$ be $AM_R(V)$ or $AE_R(V)$. Then $S(V)$ admits a ring structure if and only if $dim_R V < \infty$.*

Remark 3.2.3. Since $AM_R(V) = AE_R(V) = L_R(V)$ if $\dim_R V < \infty$ and $AM_R(V)$ and $AE_R(V)$ have no zero if $\dim_R V$ is infinite, we have from Example 1.5, Example 1.6 and Theorem 3.2.1 that $AM_R(V)$ and $AE_R(V)$ always admit both the structure of an AC semihyperring with zero and the structure of a semiring with zero.

3.3 The Semigroup $AI_R(V)$

It has been proved in Section 2.3 that if $\dim_R V$ is infinite, then for every subsemigroup T of $AI_R(V), OM_R(V) \cup T$ and $OE_R(V) \cup T$ do not admit the structure of a semihyperring with zero. If $\dim_R V < \infty$, then $AI_R(V) = L_R(V)$, so $AI_R(V)$ admits a ring structure. If $\dim_R V$ is infinite, then $AI_R(V)$ has no zero by Proposition 1.15(ii). It will be shown that if $\dim_R V$ is infinite, then $AI_R(V)$ does not admit the structure of an AC semiring with zero, so it does not admit a ring structure.

The following lemma is required.

Lemma 3.3.1. *Assume that $dim_R V$ is infinite. If $(AI_R^0(V), \oplus, \cdot)$ is an AC semiring with zero. Then $1_V \oplus 1_V = 1_V$.*

Proof. Suppose on the contrary that $1_V \oplus 1_V \neq 1_V$. Then either $1_V \oplus 1_V = 0$ or $1_V \oplus 1_V = \beta$ for some $\beta \in AI_R(V) \smallsetminus \{1_V\}$.

Case 1 : $1_V \oplus 1_V = 0$. Let B be a basis of V and choose distinct elements u, w in B. Define $\alpha \in L_R(V)$ by

$$\alpha = \begin{pmatrix} \{u,w\} & v \\ 0 & v \end{pmatrix}_{v \,\in\, B \smallsetminus \{u,w\}}.$$

Then $F(\alpha) = \langle B \smallsetminus \{u,w\}\rangle$ and so $\dim_R(V/F(\alpha)) = |\{u,w\}| = 2$. Thus $\alpha \in AI_R(V)$. Also, $(u,w)_B\alpha = \alpha$ and $\alpha \oplus \alpha \in AI_R^0(V)$. Since $0 = (1_V \oplus 1_V)\alpha = \alpha \oplus \alpha = (1_V \oplus (u,w)_B)\alpha$, we have $1_V \oplus 1_V = 1_V \oplus (u,w)_B = 0$. This implies that $1_V = (u,w)_B$, a contradiction.

Case 2 : $1_V \oplus 1_V = \beta$ for some $\beta \in AI_R(V) \smallsetminus \{1_V\}$. Then $F(\beta) \subsetneq V$. Since $\dim_R(V/F(\beta)) < \infty$ and $\dim_R V$ is infinite, $F(\beta) \neq \{0\}$. Let $u \in V \smallsetminus F(\beta)$ and $w \in F(\beta) \smallsetminus \{0\}$. Then $u\beta \neq u$ and $w\beta = w$. Since $\langle w\rangle \subseteq F(\beta)$ and $u \notin F(\beta)$, we have that u and w are linearly independent. Let B be a basis of V containing u, w. Then

$$\begin{aligned} \beta = 1_V \oplus 1_V &= (u,w)_B 1_V (u,w)_B \oplus (u,w)_B 1_V (u,w)_B \\ &= (u,w)_B(1_V \oplus 1_V)(u,w)_B \\ &= (u,w)_B \beta (u,w)_B. \end{aligned}$$

Consequently, $u(u,w)_B = w = w\beta = w(u,w)_B\beta(u,w)_B = u\beta(u,w)_B$. Since $(u,w)_B$ is one-to-one, $u = u\beta$, a contradiction.

Hence the proof is complete. □

Theorem 3.3.2. *The semigroup $AI_R(V)$ admits the structure of an AC semiring with zero if and only if $\dim_R V < \infty$.*

Proof. As mentioned above, if $\dim_R V < \infty$, then $AI_R(V)$ admits a ring structure.

Assume that $\dim_R V$ is infinite and suppose that there is an operation $\oplus$ on $AI_R^0(V)$ such that $(AI_R^0(V), \oplus, \cdot)$ is an AC semiring with zero 0 where $\cdot$ is the operation on $AI_R^0(V)$. From Lemma 3.3.1, $1_V \oplus 1_V = 1_V$ and thus

$$\alpha \oplus \alpha = \alpha \quad \text{for every } \alpha \in AI_R(V). \tag{1}$$

Let B be a basis of V. Then for all distinct $u, w \in B$, $F((u,w)_B) = \langle B \smallsetminus \{u,w\} \rangle$ and $F((u \to w)_B) = \langle B \smallsetminus \{u\} \rangle$. Consequently, $(u,w)_B, (u \to w)_B \in AI_R(V)$ for all distinct $u, w \in B$. Next, let u and w be fixed distinct elements of B. We clearly have

$$\begin{aligned} (u \to w)_B^2 &= (u \to w)_B = (w \to u)_B(u \to w)_B \\ &= (w \to u)_B(u,w)_B = (u,w)_B(u \to w)_B, \\ (w \to u)_B^2 &= (w \to u)_B = (u \to w)_B(w \to u)_B \\ &= (u \to w)_B(u,w)_B = (u,w)_B(w \to u)_B. \end{aligned} \tag{2}$$

We therefore have from (2) that $(w \to u)_B[1_V \oplus (u \to w)_B] = (w \to u)_B \oplus (u \to w)_B$ and $(u \to w)_B[1_V \oplus (w \to u)_B] = (u \to w)_B \oplus (w \to u)_B$. Thus

$$(w \to u)_B[1_V \oplus (u \to w)_B] = (u \to w)_B[1_V \oplus (w \to u)_B]. \tag{3}$$

For each $v \in B \smallsetminus \{u\}$,

$$\begin{aligned} v[1_V \oplus (u \to w)_B] &= v(u \to w)_B[1_V \oplus (u \to w)_B] \\ &= v[(u \to w)_B \oplus (u \to w)_B] \quad \text{from (2)} \\ &= v(u \to w)_B = v \quad \text{from (1)}. \end{aligned}$$

Let $u[1_V \oplus (u \to w)_B] = au + bw + \sum_{i=1}^{n} c_i v_i$ for some $a, b, c_1, c_2, \ldots, c_n \in R$ and distinct $v_1, v_2, \ldots, v_n \in B \smallsetminus \{u, w\}$. We therefore have

$$\begin{aligned} u &= u(w \to u)_B \\ &= u[(w \to u)_B \oplus (w \to u)_B] \quad \text{from (1)} \\ &= u[1_V \oplus (u \to w)_B](w \to u)_B \quad \text{from (2)} \\ &= (au + bw + \sum_{i=1}^{n} c_i v_i)(w \to u)_B \end{aligned}$$

$$= \ au + bu + \sum_{i=1}^{n} c_i v_i$$

which implies that $a + b = 1$ and $c_i = 0$ for all $i = 1, 2, \ldots, n$. Consequently,

$$\begin{aligned} v[1_V \oplus (u \to w)_B] &= v && \text{if } \ v \in B \setminus \{u\}, \\ u[1_V \oplus (u \to w)_B] &= au + bw && \text{where } \ a + b = 1. \end{aligned} \tag{4}$$

Let $v_1, v_2, \ldots, v_m \in B \setminus \{u\}$ be distinct and let $d_0, d_1, \ldots, d_m \in R$ be such that $(d_0 u + \sum_{i=1}^{m} d_i v_i)[1_V \oplus (u \to w)_B] = 0$. Then from (4),

$$d_0 au + d_0 bw + \sum_{i=1}^{m} d_i v_i = 0$$

which implies that $d_0 a = d_0 b = d_1 = \ldots = d_m = 0$. But $a + b = 1$, so $d_0 = 0$. This shows that

$$1_V \oplus (u \to w)_B \text{ is a one-to-one map.} \tag{5}$$

Since

$$\begin{aligned} [1_V \oplus (u \to w)_B]^2 &= 1_V \oplus (u \to w)_B \oplus (u \to w)_B \oplus (u \to w)_B^2 \\ &= 1_V \oplus (u \to w)_B \quad \text{from (2) and (1),} \end{aligned}$$

it follows from (5) that

$$1_V \oplus (u \to w)_B = 1_V. \tag{6}$$

We therefore have

$$\begin{aligned} 1_V &= (u, w)_B^2 \\ &= [1_V (u, w)_B]^2 \\ &= [(1_V \oplus (u \to w)_B)(u, w)_B]^2 \quad \text{from (6)} \\ &= [(u, w)_B \oplus (w \to u)_B]^2 \quad \text{from (2)} \\ &= (u, w)_B^2 \oplus (u, w)_B (w \to u)_B \oplus (w \to u)_B (u, w)_B \oplus (w \to u)_B^2 \\ &= 1_V \oplus (w \to u)_B \oplus (u \to w)_B \oplus (w \to u)_B \quad \text{from (2)} \end{aligned}$$

$$= 1_V \oplus (u \to w)_B \oplus (w \to u)_B \quad \text{from (1)}$$

$$= 1_V \oplus (w \to u)_B \quad \text{from (6).} \tag{7}$$

Hence from (3), (6) and (7), we get $(w \to u)_B = (u \to w)_B$. This is a contradiction since $w(w \to u)_B = u \neq w = w(u \to w)_B$.

This proves that if $\dim_R V$ is infinite, then $AI_R(V)$ does not admit the structure of an AC semiring with zero. Therefore the theorem is proved. □

From Theorem 3.3.2 and its proof of the sufficiency part, we have

Corollary 3.3.3. *The semigroup $AI_R(V)$ admits a ring structure if and only if $dim_R V < \infty$.*

Remark 3.3.4. Since $AI_R(V) = L_R(V)$ if $\dim_R V < \infty$ and $AI_R(V)$ has no zero if $\dim_R V$ is infinite, it follows from Example 1.5 and Example 1.6 that the semigroup $AI_R(V)$ always admits both the structure of an AC semihyperring with zero and the structure of a semiring with zero.

3.4 Partial Linear Transformation Semigroups

Let us recall the following facts of $PL_R(V)$.

$\{0\}_0\alpha = \{0\}_0$ and $V_0\alpha = V_0$ for all $\alpha \in PL_R(V)$,

$\alpha V_0 = V_0$ for every $\alpha \in L_R(V)$,

$\alpha\{0\}_0 = \{0\}_0$ for every one-to-one map $\alpha \in PL_R(V)$.

If $\dim_R V > 0$, then $PL_R(V)$ has no zero.

The purpose is to show that if $PL_R(V)$ admits the structure of an AC semiring with zero, then either $\dim_R V = 0$ or $\dim_R V = 1$ and $\mathrm{char} R = 2$. Also $PL_R(V)$ admits a ring structure if and only if $\dim_R V = 0$. To obtain the main results, the following three lemmas are required.

Lemma 3.4.1. *If $\dim_R V > 0$ and $(PL_R^0(V), \oplus, \cdot)$ is an AC semiring with zero where $\cdot$ is the operation of the semigroup $PL_R^0(V)$, then the following statements hold.*

(i) $1_V \oplus (-1_V) = a1_V$ *for some $a \in C(R) \smallsetminus \{0\}$.*

(ii) $W_0 \oplus W_0 = W_0$ *for every subspace W of V.*

Proof. First we note that $PL_R(V)$ is a semigroup without zero, so for $\alpha, \beta \in PL_R^0(V), \alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$.

To show that $1_V \oplus (-1_V) \neq 0$, suppose on the contrary that $1_V \oplus (-1_V) = 0$. Then $\alpha \oplus (-\alpha) = 0$ for all $\alpha \in PL_R^0(V)$. In particular, $V_0 \oplus V_0 = V_0 \oplus (-V_0) = 0$. But $V_0 \neq 0$ and

$$V_0(V_0 \oplus \{0\}_0) = V_0 \oplus V_0 = 0,$$

thus $V_0 \oplus \{0\}_0 = 0$. Therefore $V_0 \oplus V_0 = 0 = V_0 \oplus \{0\}_0$. This implies that $V_0 = \{0\}_0$ which is contrary to that $\dim_R V > 0$. Hence $1_V \oplus (-1_V) \in PL_R(V)$. Since

$$V_0(1_V \oplus (-1_V)) = V_0 \oplus V_0 = (1_V \oplus (-1_V))V_0,$$

it follows that Dom $(1_V \oplus (-1_V)) \supseteq$ Dom $((1_V \oplus (-1_V))V_0) =$ Dom $(V_0(1_V \oplus (-1_V))) = V$. It is clear that $\alpha(1_V \oplus (-1_V)) = (1_V \oplus (-1_V))\alpha$ for all $\alpha \in PL_R(V)$. By Proposition 1.10, there is an element $a \in C(R)$ such that $1_V \oplus (-1_V) = a1_V$. If $a = 0$, then $1_V \oplus (-1_V) = V_0$ which implies that

$$\begin{aligned} V_0 = V_0\{0\}_0 &= (1_V \oplus (-1_V))\{0\}_0 \\ &= \{0\}_0(1_V \oplus (-1_V)) \end{aligned}$$

$$= \{0\}_0 V_0 \ = \ \{0\}_0$$

which is a contradiction since $\dim_R V > 0$. Therefore $a \neq 0$. If W is a subspace of V, then

$$W_0 \oplus W_0 = (1_V \oplus (-1_V))W_0 = (a1_V)W_0 = W_0.$$

Therefore (i) and (ii) are obtained, as required. □

Lemma 3.4.2. *If $\dim_R V > 0$ and the semigroup $PL_R(V)$ admits the structure of an AC semiring with zero, then char$R = 2$.*

Proof. Assume that $\oplus$ is an operation on $PL_R^0(V)$ such that $(PL_R^0(V), \oplus, \cdot)$ is an AC semiring with zero where $\cdot$ is the operation on $PL_R^0(V)$. By Lemma 3.4.1, $1_V \oplus (-1_V) = a1_V$ for some $a \in C(R) \smallsetminus \{0\}$. Then

$$\begin{aligned} a^2 1_V \ &= \ (a1_V)(a1_V) \ = \ (1_V \oplus (-1_V))(a1_V) \\ &= \ ((-1_V) \oplus 1_V)(-a1_V) \\ &= \ (1_V \oplus (-1_V))(-a1_V) \\ &= \ (a1_V)(-a1_V) \ = \ -a^2 1_V, \end{aligned}$$

so $2a^2 = 0$. But $a^2 \neq 0$, thus char$R = 2$. □

Lemma 3.4.3. *If $\dim_R V > 1$, then the semigroup $PL_R(V)$ does not admit the structure of an AC semiring with zero.*

Proof. Let $\dim_R V > 1$ and suppose that the semigroup $PL_R(V)$ admits the structure of an AC semiring with zero. Then there is an operation $\oplus$ on $PL_R^0(V)$ such that $(PL_R^0(V), \oplus, \cdot)$ is an AC semiring with zero where $\cdot$ is the operation on $PL_R^0(V)$. Since $\dim_R V > 1$, there are elements $u, w \in V$ such that u and w are linearly independent over R. Then $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}$ is an element of $PL_R^0(V)$.

Case 1 : $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = 0$. Then $0 = V_0\left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}\right) = V_0 \oplus V_0$. This contradicts Lemma 3.4.1(ii).

Case 2 : $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = \{0\}_0$. Then $\{0\}_0 = \{0\}_0 V_0 = \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}\right) V_0 = \begin{pmatrix} u \\ 0 \end{pmatrix} \oplus \begin{pmatrix} u \\ 0 \end{pmatrix} = \langle u \rangle_0 \oplus \langle u \rangle_0$ which is contrary to Lemma 3.4.1(ii).

Case 3 : $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = \alpha$ for some $\alpha \in PL_R(V)$ with $\operatorname{Dom}\alpha \neq \{\,0\,\}$. Since

$$\begin{pmatrix} u \\ u \end{pmatrix}\left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}\right) = \begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix},$$

we have $\begin{pmatrix} u \\ u \end{pmatrix}\alpha = \alpha$. Then $\operatorname{Dom}\alpha = \operatorname{Dom}\left(\begin{pmatrix} u \\ u \end{pmatrix}\alpha\right) \subseteq \operatorname{Dom}\begin{pmatrix} u \\ u \end{pmatrix} = \langle u \rangle$, and thus $\dim_R(\operatorname{Dom}\alpha) \leq \dim_R \langle u \rangle = 1$. But $\operatorname{Dom}\alpha \neq \{0\}$, so $\operatorname{Dom}\alpha = \langle u \rangle$. Also, since

$$\begin{aligned}\left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}\right)\begin{pmatrix} u & w \\ w & u \end{pmatrix} &= \begin{pmatrix} u \\ w \end{pmatrix} \oplus \begin{pmatrix} u \\ u \end{pmatrix} \\ &= \begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix},\end{aligned}$$

it follows that $\alpha\begin{pmatrix} u & w \\ w & u \end{pmatrix} = \alpha$. Consequently, $\operatorname{Im}\alpha = \operatorname{Im}\left(\alpha\begin{pmatrix} u & w \\ w & u \end{pmatrix}\right) \subseteq \operatorname{Im}\begin{pmatrix} u & w \\ w & u \end{pmatrix} = \langle u, w \rangle$. Now we have

$$\operatorname{Dom}\alpha = \langle u \rangle \text{ and } \operatorname{Im}\alpha \subseteq \langle u, w \rangle.$$

Then there are $a, b \in R$ such that $u\alpha = au+bw$. This implies that $\alpha = \begin{pmatrix} u \\ au+bw \end{pmatrix}$.

But since $\alpha \begin{pmatrix} u & w \\ w & u \end{pmatrix} = \alpha$, we have

$$\begin{pmatrix} u \\ au+bw \end{pmatrix} = \begin{pmatrix} u \\ au+bw \end{pmatrix} \begin{pmatrix} u & w \\ w & u \end{pmatrix} = \begin{pmatrix} u \\ aw+bu \end{pmatrix},$$

and hence $a = b$. Thus $\alpha = \begin{pmatrix} u \\ a(u+w) \end{pmatrix}$. Now, we have

$$\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u \\ a(u+w) \end{pmatrix}. \tag{1}$$

Consequently,

$$\begin{aligned}
\begin{pmatrix} u \\ u+w \end{pmatrix} \oplus \begin{pmatrix} u \\ u+w \end{pmatrix} &= \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}\right) \begin{pmatrix} u & w \\ u+w & u+w \end{pmatrix} \\
&= \begin{pmatrix} u \\ a(u+w) \end{pmatrix} \begin{pmatrix} u & w \\ u+w & u+w \end{pmatrix} \quad \text{from (1)} \\
&= \begin{pmatrix} u \\ a(u+w+u+w) \end{pmatrix} \\
&= \begin{pmatrix} u \\ 2a(u+w) \end{pmatrix} \\
&= \begin{pmatrix} u \\ 0 \end{pmatrix} \quad \text{by Lemma 3.4.2.} \qquad (2)
\end{aligned}$$

Since u and w are linearly independent, $u+w \neq 0$, and so $\begin{pmatrix} u \\ u+w \end{pmatrix}$ is a one-to-one partial linear transformation of V. Hence

$$\begin{aligned}
\{0\}_0 \oplus \{0\}_0 &= \begin{pmatrix} u \\ u+w \end{pmatrix} \{0\}_0 \oplus \begin{pmatrix} u \\ u+w \end{pmatrix} \{0\}_0 \\
&= \left(\begin{pmatrix} u \\ u+w \end{pmatrix} \oplus \begin{pmatrix} u \\ u+w \end{pmatrix} \right) \{0\}_0 \\
&= \begin{pmatrix} u \\ 0 \end{pmatrix} \{0\}_0 \qquad \text{from (2)} \\
&= \begin{pmatrix} u \\ 0 \end{pmatrix}
\end{aligned}$$

which is a contradiction because of Lemma 3.4.1(ii). □

Theorem 3.4.4. *If the semigroup $PL_R(V)$ admits the structure of an AC semiring with zero, then either $dim_R V = 0$ or $dim_R V = 1$ and $charR = 2$.*

Proof. Assume that $PL_R(V)$ admits the structure of an AC semiring with zero. We therefore have from Lemma 3.4.3 that $\dim_R V = 0$ or $\dim_R V = 1$. If $\dim_R V = 1$, then by Lemma 3.4.2, $\text{char}R = 2$. □

Notice that if $\dim_R V = 0$, then $PL_R(V) = \{\{0\}_0\}$, so $PL_R(V)$ admits a ring structure. If $\dim_R V > 0$, Lemma 3.4.1(ii) shows that $PL_R(V)$ does not admit a ring structure. Hence we have

Theorem 3.4.5. *The semigroup $PL_R(V)$ admits a ring structure if and only if $dim_R V = 0$.*

Remark 3.4.6. (1) Since $PL_R(V)$ has no zero if $\dim_R V > 0$, by the same reason as before, we have that $PL_R(V)$ always admits the structure of an AC semihyperring with zero and the structure of a semiring with zero.

(2) It is natural to ask whether it is possible that $\dim_R V = 1$, $\operatorname{char} R = 2$ and the semigroup $PL_R(V)$ admits the structure of an AC semiring with zero. The answer is " yes ". To see this, assume that $\dim_R V = 1$ and $|R| = 2$. Then $|V| = 2$ which implies that

$$PL_R(V) = \{\{0\}_0, V_0, 1_V\},$$

and thus

$$PL_R^0(V) = \{0, \{0\}_0, V_0, 1_V\}.$$

Define an operation $\oplus$ on $PL_R^0(V)$ as follows:

$\oplus$	0	$\{0\}_0$	V_0	1_V
0	0	$\{0\}_0$	V_0	1_V
$\{0\}_0$	$\{0\}_0$	$\{0\}_0$	V_0	1_V
V_0	V_0	V_0	V_0	V_0
1_V	1_V	1_V	V_0	1_V

It is straightforward to verify that $(PL_R^0(V), \oplus, \cdot)$ is an AC semiring with zero 0.

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