CHAPTER III

SUPER VERTEX-MAGIC GRAPHS

Our purpose in this chapter is to collect and investigate some families of graphs that admit super vertex-magic total labelings.

The circulant graphs are an important class of graphs, which can be used in the design of local area networks[2].

Theorem 3.1. ([6]) *The cycle C_n is super vertex-magic iff n is odd.*

Example 3.2. The super vertex-magic graph C_5 with the magic constant 19 is shown in Figure 3.1.

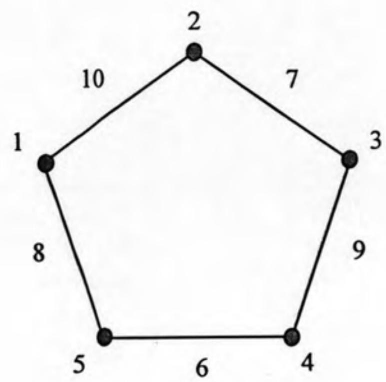

Figure 3.1 : Super vertex-magic graph C_5

Theorem 3.3. ([6]) *If $n \geq 3$ and n is odd, then the complete graph K_n is super vertex-magic.*

Example 3.4. The super vertex-magic graph K_5 with the magic constant 45 is shown in Figure 3.2.

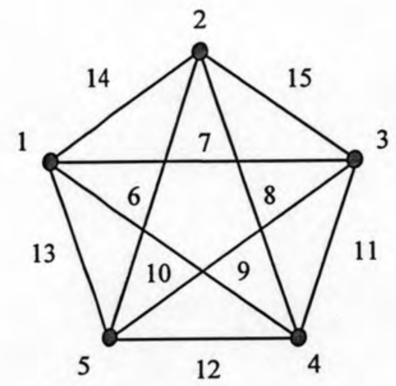

Figure 3.2 : Super vertex-magic graph K_5

Theorem 3.5. ([5]) *If* $n \equiv 0 \pmod 4$ and $n \neq 4$, *then the complete graph* K_n *is super vertex-magic.*

Example 3.6. The super vertex-magic graph K_8 with the magic constant 162 is shown in Figure 3.3.

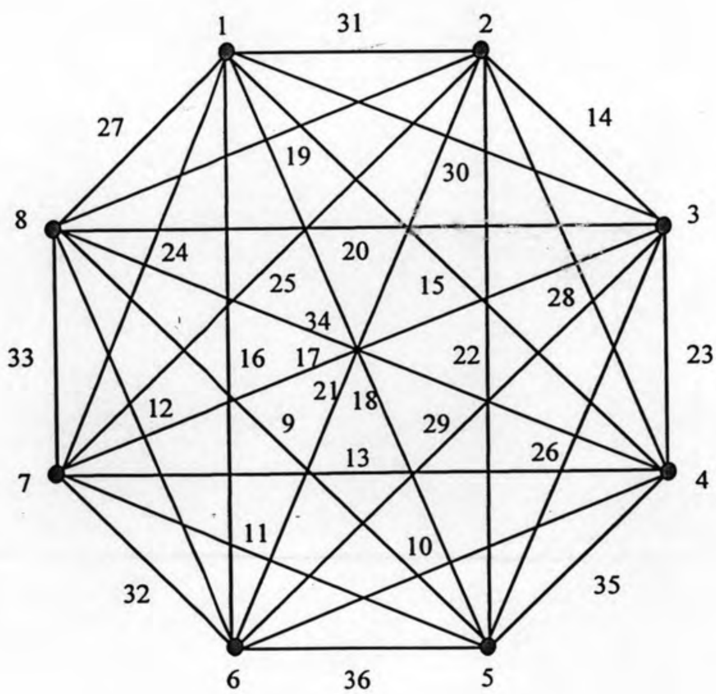

Figure 3.3 : Super vertex-magic graph K_8

Definition 3.7. Let $1 \le a_1 < a_2 < \dots < a_k \le \left\lfloor \frac{n}{2} \right\rfloor$, where n and $a_i\ (i = 1, 2, \dots, k)$ are positive integers. A *circulant graph* $C_n(a_1, a_2, \dots, a_k)$ is a regular graph whose set of vertices is $V = \{v_0, v_1, \dots, v_{n-1}\}$ and whose set of edges is

$$E = \{v_i v_{i+a_j \pmod n} : i = 0, 1, \dots, n-1,\ j = 1, 2, \dots, k\}.$$

Note that if n is odd, a circulant graph $C_n(a_1, a_2, \dots, a_k)$ is a $2k$-regular graph.

Example 3.8. The circulant graph $C_{13}(1,2,4)$ is a 6-regular graph.

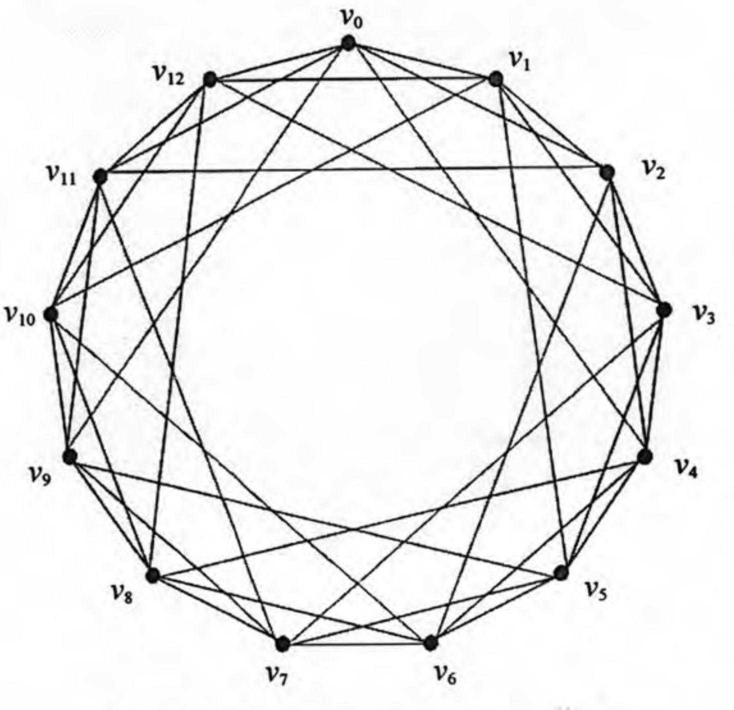

Figure 3.4 : Circulant graph $C_{13}(1,2,4)$

Remark 3.9. The circulant graph $C_n(a_1, a_2, \dots, a_k)$ where n is odd, has $v = n$ and $e = kn$.

Theorem 3.10. ([1]) *If $n \ge 5$ and n is odd, and $m \in \{2, 3, \dots, \frac{n-1}{2}\}$, then a circulant graph $C_n(1, m)$ is super vertex-magic.*

Example 3.11. Super vertex-magic graphs $C_9(1,2)$ and $C_9(1,4)$ with the magic constant 79 are shown in Figure 3.5.

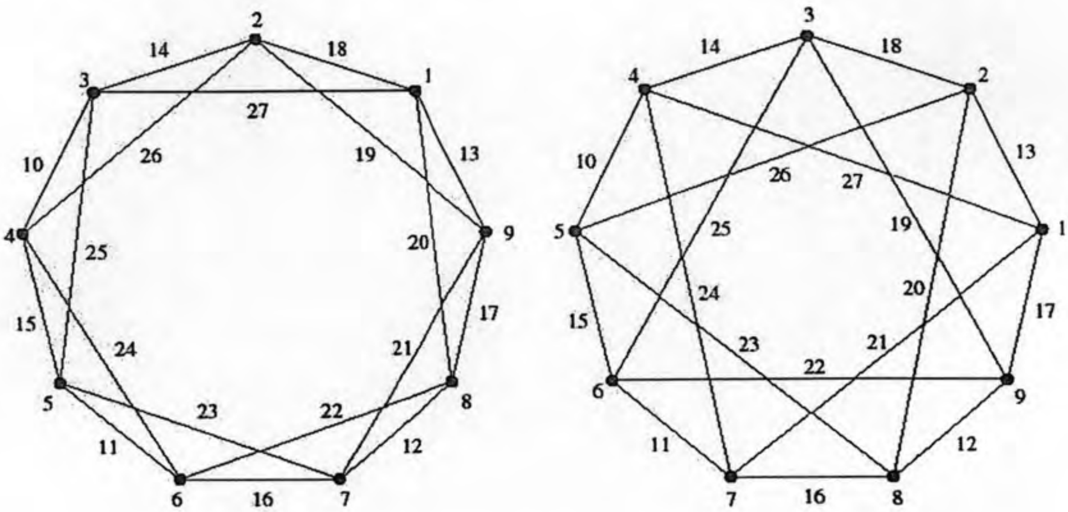

Figure 3.5 : Super vertex-magic graphs $C_9(1,2)$ and $C_9(1,4)$

Theorem 3.12. ([1]) *If* $n \geq 7$ *and n is odd, then circulant graph* $C_n(1,2,3)$ *is super vertex-magic.*

Example 3.13. Super vertex-magic graphs $C_7(1,2,3)$ and $C_9(1,2,3)$ with the magic constant 112 and 143, respectively, are shown in Figure 3.6.

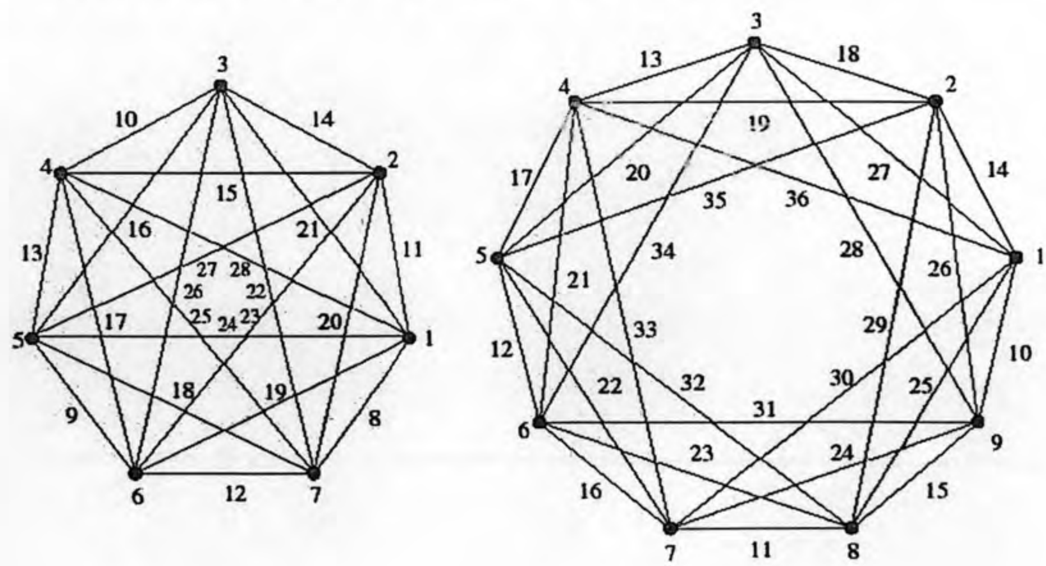

Figure 3.6 : Super vertex-magic graphs $C_7(1,2,3)$ and $C_9(1,2,3)$

Due to Theorem 3.12, the labeling of a circulant graph $C_n(1,2,3)$ where $n \geq 7$ and n is odd, can be generalized to a labeling of a circulant graph $C_n(1,2,m)$ where $n \geq 7$, n is odd, and $m \in \{3, 4, ..., \frac{n-1}{2}\}$ as described in the following theorem.

Theorem 3.14. *If G is a circulant graph* $C_n(1,2,m)$ *where* $n \geq 7$, *n is odd, and* $m \in \{3, 4, ..., \frac{n-1}{2}\}$, *then G is a super vertex-magic graph.*

Proof. Let G be a circulant graph $C_n(1,2,m)$, $n \geq 7$, n is odd, and $m \in \{3, 4, ..., \frac{n-1}{2}\}$.

Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$.

Let $\lambda : V(G) \cup E(G) \to \{1,2, ..., 4n\}$ be a labeling defined by

$$\lambda(v_i) = \begin{cases} m-i & \text{if } i = 0,1,2, ..., m-1, \\ n+m-i & \text{if } i = m, m+1, ..., n-1, \end{cases} \qquad .....(1)$$

$$\left.\begin{array}{ll} \lambda(v_i v_{i+1}) = \begin{cases} 2n & \text{if } i = 0, \\ \dfrac{3n+i}{2} & \text{if } i = 1,3, ..., n-2, \\ \dfrac{2n+i}{2} & \text{if } i = 2,4, ..., n-1, \end{cases} & \\ \lambda(v_i v_{i+2}) = 3n - i & \text{if } i = 0,1,2, ..., n-1, \\ \lambda(v_i v_{i+m}) = 3n + i + 1 & \text{if } i = 0,1,2, ..., n-1. \end{array}\right\} \qquad .....(2)$$

Claim that λ is a super vertex-magic total labeling of the graph G.

Since $v = n$ and $e = 3n$, we will show that

(i) $\lambda(V(G)) = \{1, 2, \ldots, n\}$,

(ii) $\lambda(E(G)) = \{n+1, n+2, \ldots, 4n\}$,

and (iii) $w_\lambda(v_i) = \frac{31n+7}{2}$ for all $i = 0,1,2, ..., n-1$.

From (1), for $i = 0,1,2, ..., n-1$, we have $1 \leq \lambda(v_i) \leq n$.

Therefore $\lambda(V(G)) = \{1, 2, \ldots, n\}$, (i) holds.

From (2), for $i=0,1,2,\dots,n-1$, we have $n+1 \le \lambda(v_iv_{i+1}) \le 2n$,

$2n+1 \le \lambda(v_iv_{i+2}) \le 3n$, and $3n+1 \le \lambda(v_iv_{i+m}) \le 4n$.

Therefore $\lambda(E(G)) = \{n+1, n+2, \dots, 4n\}$, (ii) holds.

Since all number 1 through $4n$ are used exactly once, λ is a bijection.

Note that for each $i \in \{0,1,\dots,n-1\}$, the 6 edges incident with vertex v_i are

$v_iv_{i+1}, v_iv_{i+2}, v_iv_{i+m}, v_{i-1}v_i, v_{i-2}v_i, v_{i-m}v_i$, where all indices are taken modulo n.

Since v_0 is also incident with v_1, so we consider the following 6 cases.

Case 1 : $i=0$.

We have $\lambda(v_0)=m$, $\lambda(v_0v_1)=2n$, $\lambda(v_0v_2)=3n$, $\lambda(v_0v_m)=3n+1$,

$$\lambda(v_{n-1}v_0)=\frac{2n+(n-1)}{2}=\frac{3n-1}{2},\ \lambda(v_{n-2}v_0)=3n-(n-2)=2n+2,$$

$$\lambda(v_{n-m}v_0)=3n+(n-m)+1=4n-m+1.$$

Thus

$$w_\lambda(v_0)=\lambda(v_0)+\lambda(v_0v_1)+\lambda(v_0v_2)+\lambda(v_0v_m)+\lambda(v_{n-1}v_0)+\lambda(v_{n-2}v_0)+\lambda(v_{n-m}v_0)=\frac{31n+7}{2}.$$

Case 2 : $i=1$.

We have $\lambda(v_1)=m-1$, $\lambda(v_1v_2)=\frac{3n+1}{2}$, $\lambda(v_1v_3)=3n-1$, $\lambda(v_1v_{1+m})=3n+2$,

$$\lambda(v_0v_1)=2n,\ \lambda(v_{n-1}v_1)=3n-(n-1)=2n+1,$$

$$\lambda(v_{n-m+1}v_1)=3n+(n-m+1)+1=4n-m+2.$$

Thus

$$w_\lambda(v_1)=\lambda(v_1)+\lambda(v_1v_2)+\lambda(v_1v_3)+\lambda(v_1v_{1+m})+\lambda(v_0v_1)+\lambda(v_{n-1}v_1)+\lambda(v_{n-m+1}v_1)=\frac{31n+7}{2}.$$

Case 3 : $2 \le i \le m-1$ and i is odd.

We have $\lambda(v_i)=m-i$, $\lambda(v_iv_{i+1})=\frac{3n+i}{2}$, $\lambda(v_iv_{i+2})=3n-i$, $\lambda(v_iv_{i+m})=3n+i+1$,

$$\lambda(v_{i-1}v_i)=\frac{2n+(i-1)}{2},\ \lambda(v_{i-2}v_i)=3n-(i-2)=3n-i+2,$$

$$\lambda(v_{n+i-m}v_i)=3n+(n+i-m)+1=4n-m+i+1.$$

Thus

$$w_\lambda(v_i)=\lambda(v_i)+\lambda(v_iv_{i+1})+\lambda(v_iv_{i+2})+\lambda(v_iv_{i+m})+\lambda(v_{i-1}v_i)+\lambda(v_{i-2}v_i)+\lambda(v_{n+i-m}v_i)=\frac{31n+7}{2}.$$

Case 4 : $2 \le i \le m-1$ and i is even.

We have $\lambda(v_i) = m - i$, $\lambda(v_i v_{i+1}) = \dfrac{2n+i}{2}$, $\lambda(v_i v_{i+2}) = 3n - i$, $\lambda(v_i v_{i+m}) = 3n + i + 1$,

$$\lambda(v_{i-1}v_i) = \frac{3n+(i-1)}{2},\ \lambda(v_{i-2}v_i) = 3n-(i-2) = 3n-i+2,$$

$$\lambda(v_{n+i-m}v_i) = 3n+(n+i-m)+1 = 4n-m+i+1.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_i v_{i+1}) + \lambda(v_i v_{i+2}) + \lambda(v_i v_{i+m}) + \lambda(v_{i-1}v_i) + \lambda(v_{i-2}v_i) + \lambda(v_{n+i-m}v_i) = \frac{31n+7}{2}.$$

Case 5 : $m \le i \le n-1$ and i is odd.

We have $\lambda(v_i) = n - m + i$, $\lambda(v_i v_{i+1}) = \dfrac{3n+i}{2}$, $\lambda(v_i v_{i+2}) = 3n - i$, $\lambda(v_i v_{i+m}) = 3n + i + 1$,

$$\lambda(v_{i-1}v_i) = \frac{2n+(i-1)}{2},\ \lambda(v_{i-2}v_i) = 3n-(i-2) = 3n-i+2,$$

$$\lambda(v_{i-m}v_i) = 3n+(i-m)+1 = 3n-m+i+1.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_i v_{i+1}) + \lambda(v_i v_{i+2}) + \lambda(v_i v_{i+m}) + \lambda(v_{i-1}v_i) + \lambda(v_{i-2}v_i) + \lambda(v_{i-m}v_i) = \frac{31n+7}{2}.$$

Case 6 : $m \le i \le n-1$ and i is even.

We have $\lambda(v_i) = n + m - i$, $\lambda(v_i v_{i+1}) = \dfrac{2n+i}{2}$, $\lambda(v_i v_{i+2}) = 3n - i$, $\lambda(v_i v_{i+m}) = 3n + i + 1$,

$$\lambda(v_{i-1}v_i) = \frac{3n+(i-1)}{2},\ \lambda(v_{i-2}v_i) = 3n-(i-2) = 3n-i+2,$$

$$\lambda(v_{i-m}v_i) = 3n+(i-m)+1 = 3n-m+i+1.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_i v_{i+1}) + \lambda(v_i v_{i+2}) + \lambda(v_i v_{i+m}) + \lambda(v_{i-1}v_i) + \lambda(v_{i-2}v_i) + \lambda(v_{i-m}v_i) = \frac{31n+7}{2}.$$

Hence $w_\lambda(v_i) = \dfrac{31n+7}{2}$ for all $i = 0, 1, 2, ..., n-1$.

Therefore λ is a super vertex-magic total labeling of G. □

Example 3.15. Super vertex-magic graphs $C_{13}(1,2,4)$ and $C_{13}(1,2,5)$ with the magic constant 205 is shown in Figure 3.7.

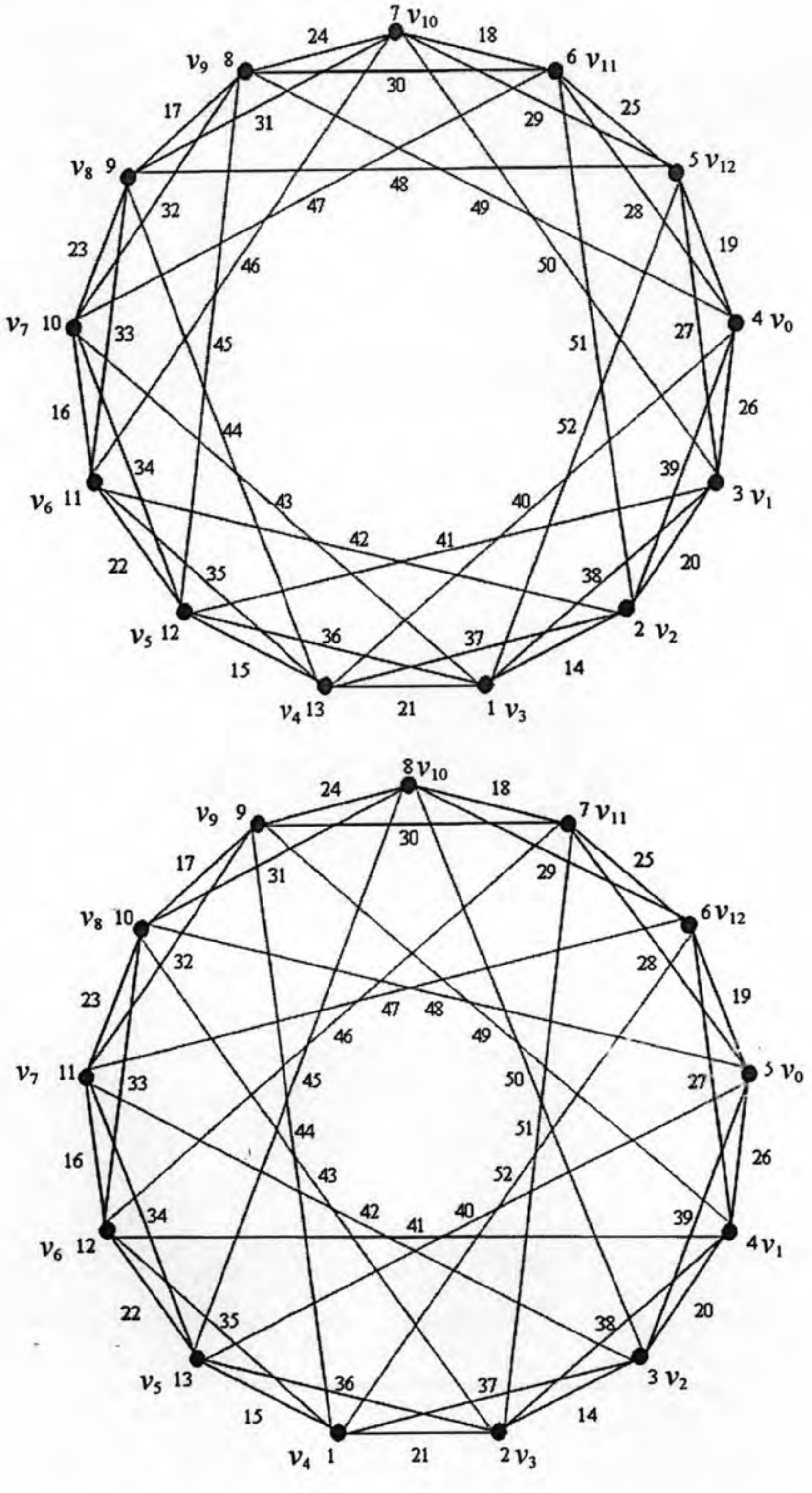

Figure 3.7 : Super vertex-magic graphs $C_{13}(1,2,4)$ and $C_{13}(1,2,5)$

Theorem 3.16. *If G is a circulant graph* $C_n(1,3,m)$ *where* $n \geq 9$, *n is odd, and* $m \in \{4,5,\ldots,\frac{n-1}{2}\}$, *then G is a super vertex-magic graph.*

Proof. Let G be a circulant graph $C_n(1,3,m)$, $n \geq 9$, n is odd, and $m \in \{4,5,\ldots,\frac{n-1}{2}\}$.

Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$.

Let $\lambda : V(G) \cup E(G) \to \{1,2,\ldots,4n\}$ be a labeling defined by

$$\lambda(v_i) = \begin{cases} m-i-1 & \text{if } i = 0,1,2,\ldots,m-2, \\ n+m-i-1 & \text{if } i = m-1,m,\ldots,n-1, \end{cases} \qquad \ldots..(3)$$

$$\left.\begin{aligned} \lambda(v_iv_{i+1}) &= \begin{cases} 2n & \text{if } i = 0, \\ \dfrac{3n+i}{2} & \text{if } i = 1,3,\ldots,n-2, \\ \dfrac{2n+i}{2} & \text{if } i = 2,4,\ldots,n-1, \end{cases} \\ \lambda(v_iv_{i+3}) &= \begin{cases} 3n-i-1 & \text{if } i = 0,1,2,\ldots,n-2, \\ 3n & \text{if } i = n-1, \end{cases} \\ \lambda(v_iv_{i+m}) &= \begin{cases} 3n+i+2 & \text{if } i = 0,1,2,\ldots,n-2, \\ 3n+1 & \text{if } i = n-1, \end{cases} \end{aligned}\right\} \qquad \ldots..(4)$$

Claim that λ is a super vertex-magic total labeling of the graph G.

Since $v = n$ and $e = 3n$, we will show that

(i) $\lambda(V(G)) = \{1, 2, \ldots, n\}$,

(ii) $\lambda(E(G)) = \{n+1, n+2, \ldots, 4n\}$,

and (iii) $w_\lambda(v_i) = \frac{31n+7}{2}$ for all $i = 0,1,2,\ldots,n-1$.

From (3), For $i = 0,1,2,\ldots,n-1$, we have $1 \leq \lambda(v_i) \leq n$.

Therefore $\lambda(V(G)) = \{1, 2, \ldots, n\}$, (i) holds.

From (4), For $i = 0,1,2,\ldots,n-1$, we have $n+1 \leq \lambda(v_iv_{i+1}) \leq 2n$,

$2n+1 \leq \lambda(v_iv_{i+3}) \leq 3n$, and $3n+1 \leq \lambda(v_iv_{i+m}) \leq 4n$.

Therefore $\lambda(E(G)) = \{n+1, n+2, \ldots, 4n\}$, (ii) holds.

Since all number 1 through $4n$ are used exactly once, λ is a bijection.

Note that for each $i \in \{0,1,...,n-1\}$, the 6 edges incident with vertex v_i are $v_iv_{i+1}, v_iv_{i+3}, v_iv_{i+m}, v_{i-1}v_i, v_{i-3}v_i, v_{i-m}v_i$, where all indicies are taken modulo n. Since v_0 is also incident with v_1, v_{n-1} is incident with v_2, and v_{n-1} is incident with v_{m-1}, so we consider the following 10 cases.

Case 1 : $i = 0$.

We have $\lambda(v_0) = m-1$, $\lambda(v_0v_1) = 2n$, $\lambda(v_0v_3) = 3n-1$, $\lambda(v_0v_m) = 3n+2$,

$$\lambda(v_{n-1}v_0) = \frac{2n+(n-1)}{2} = \frac{3n-1}{2},\ \lambda(v_{n-3}v_0) = 3n-(n-3)-1 = 2n+2,$$

$$\lambda(v_{n-m}v_0) = 3n+(n-m)+2 = 4n-m+2.$$

Thus

$$w_\lambda(v_0) = \lambda(v_0)+\lambda(v_0v_1)+\lambda(v_0v_3)+\lambda(v_0v_m)+\lambda(v_{n-1}v_0)+\lambda(v_{n-3}v_0)+\lambda(v_{n-m}v_0) = \frac{31n+7}{2}.$$

Case 2 : $i = 1$.

We have $\lambda(v_1) = m-2$, $\lambda(v_1v_2) = \frac{3n+1}{2}$, $\lambda(v_1v_4) = 3n-2$, $\lambda(v_1v_{1+m}) = 3n+3$,

$$\lambda(v_0v_1) = 2n,\ \lambda(v_{n-2}v_1) = 3n-(n-2)-1 = 2n+1,$$

$$\lambda(v_{n-m+1}v_1) = 3n+(n-m+1)+2 = 4n-m+3.$$

Thus

$$w_\lambda(v_1) = \lambda(v_1)+\lambda(v_1v_2)+\lambda(v_1v_4)+\lambda(v_1v_{1+m})+\lambda(v_0v_1)+\lambda(v_{n-2}v_1)+\lambda(v_{n-m+1}v_1) = \frac{31n+7}{2}.$$

Case 3 : $i = 2$.

We have $\lambda(v_2) = m-3$, $\lambda(v_2v_3) = n+1$, $\lambda(v_2v_5) = 3n-3$, $\lambda(v_2v_{2+m}) = 3n+4$,

$$\lambda(v_1v_2) = \frac{3n+1}{2},\ \lambda(v_{n-1}v_2) = 3n,$$

$$\lambda(v_{n-m+2}v_2) = 3n+(n-m+2)+2 = 4n-m+4.$$

Thus

$$w_\lambda(v_2) = \lambda(v_2)+\lambda(v_2v_3)+\lambda(v_2v_5)+\lambda(v_2v_{2+m})+\lambda(v_1v_2)+\lambda(v_{n-1}v_2)+\lambda(v_{n-m+2}v_2) = \frac{31n+7}{2}.$$

Case 4 : $3 \le i \le m-2$ and i is odd.

We have $\lambda(v_i) = m-i-1$, $\lambda(v_iv_{i+1}) = \frac{3n+i}{2}$, $\lambda(v_iv_{i+3}) = 3n-i-1$,

$$\lambda(v_iv_{i+m}) = 3n+i+2,\ \lambda(v_{i-1}v_i) = \frac{2n+(i-1)}{2},$$

$$\lambda(v_{i-3}v_i) = 3n-(i-3)-1 = 3n-i+2,$$

$$\lambda(v_{n+i-m}v_i) = 3n+(n+i-m)+2 = 4n-m+i+2.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i)+\lambda(v_iv_{i+1})+\lambda(v_iv_{i+3})+\lambda(v_iv_{i+m})+\lambda(v_{i-1}v_i)+\lambda(v_{i-3}v_i)+\lambda(v_{n+i-m}v_i) = \frac{31n+7}{2}.$$

Case 5 : $3 \le i \le m-2$ and i is even.

We have $\lambda(v_i) = m-i-1$, $\lambda(v_iv_{i+1}) = \dfrac{2n+i}{2}$, $\lambda(v_iv_{i+3}) = 3n-i-1$,

$$\lambda(v_iv_{i+m}) = 3n+i+2,\ \lambda(v_{i-1}v_i) = \frac{3n+(i-1)}{2},$$

$$\lambda(v_{i-3}v_i) = 3n-(i-3)-1 = 3n-i+2,$$

$$\lambda(v_{n+i-m}v_i) = 3n+(n+i-m)+2 = 4n-m+i+2.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i)+\lambda(v_iv_{i+1})+\lambda(v_iv_{i+3})+\lambda(v_iv_{i+m})+\lambda(v_{i-1}v_i)+\lambda(v_{i-3}v_i)+\lambda(v_{n+i-m}v_i) = \frac{31n+7}{2}.$$

Case 6 : $i = m-1$ and i is odd.

We have $\lambda(v_{m-1}) = n+m-(m-1)-1 = n$, $\lambda(v_{m-1}v_m) = \dfrac{3n+(m-1)}{2}$,

$$\lambda(v_{m-1}v_{m+2}) = 3n-(m-1)-1 = 3n-m,$$

$$\lambda(v_{m-1}v_{2m-1}) = 3n+(m-1)+2 = 3n+m+1,\ \lambda(v_{m-2}v_{m-1}) = \frac{2n+(m-2)}{2},$$

$$\lambda(v_{m-4}v_{m-1}) = 3n-(m-4)-1 = 3n-m+3,\ \lambda(v_{n-1}v_{m-1}) = 3n+1.$$

Thus

$$\begin{aligned} w_\lambda(v_{m-1}) &= \lambda(v_{m-1})+\lambda(v_{m-1}v_m)+\lambda(v_{m-1}v_{m+2})+\lambda(v_{m-1}v_{2m-1})+\lambda(v_{m-2}v_{m-1}) \\ &\quad +\lambda(v_{m-4}v_{m-1})+\lambda(v_{n-1}v_{m-1}) \\ &= \frac{31n+7}{2}. \end{aligned}$$

Case 7 : $i = m-1$ and i is even.

We have $\lambda(v_{m-1}) = n+m-(m-1)-1 = n$, $\lambda(v_{m-1}v_m) = \dfrac{2n+(m-1)}{2}$,

$$\lambda(v_{m-1}v_{m+2}) = 3n-(m-1)-1 = 3n-m,$$

$$\lambda(v_{m-1}v_{2m-1}) = 3n+(m-1)+2 = 3n+m+1,\ \lambda(v_{m-2}v_{m-1}) = \frac{3n+(m-2)}{2},$$

$$\lambda(v_{m-4}v_{m-1}) = 3n-(m-4)-1 = 3n-m+3,\ \lambda(v_{n-1}v_{m-1}) = 3n+1.$$

Thus

$$w_\lambda(v_{m-1}) = \lambda(v_{m-1}) + \lambda(v_{m-1}v_m) + \lambda(v_{m-1}v_{m+2}) + \lambda(v_{m-1}v_{2m-1}) + \lambda(v_{m-2}v_{m-1})$$
$$+\lambda(v_{m-4}v_{m-1}) + \lambda(v_{n-1}v_{m-1})$$
$$= \frac{31n+7}{2}.$$

Case 8 : $m \le i < n-1$ and i is odd.

We have $\lambda(v_i) = n+m-i-1$, $\lambda(v_iv_{i+1}) = \frac{3n+i}{2}$, $\lambda(v_iv_{i+3}) = 3n-i-1$,

$$\lambda(v_iv_{i+m}) = 3n+i+2,\ \lambda(v_{i-1}v_i) = \frac{2n+(i-1)}{2},$$
$$\lambda(v_{i-3}v_i) = 3n-(i-3)-1 = 3n-i+2,$$
$$\lambda(v_{i-m}v_i) = 3n+(i-m)+2 = 3n-m+i+2.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_iv_{i+1}) + \lambda(v_iv_{i+3}) + \lambda(v_iv_{i+m}) + \lambda(v_{i-1}v_i) + \lambda(v_{i-3}v_i) + \lambda(v_{i-m}v_i) = \frac{31n+7}{2}.$$

Case 9 : $m \le i < n-1$ and i is even.

We have $\lambda(v_i) = n+m-i-1$, $\lambda(v_iv_{i+1}) = \frac{2n+i}{2}$, $\lambda(v_iv_{i+3}) = 3n-i-1$,

$$\lambda(v_iv_{i+m}) = 3n+i+2,\ \lambda(v_{i-1}v_i) = \frac{3n+(i-1)}{2},$$
$$\lambda(v_{i-3}v_i) = 3n-(i-3)-1 = 3n-i+2,$$
$$\lambda(v_{i-m}v_i) = 3n+(i-m)+2 = 3n-m+i+2.$$

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_iv_{i+1}) + \lambda(v_iv_{i+3}) + \lambda(v_iv_{i+m}) + \lambda(v_{i-1}v_i) + \lambda(v_{i-3}v_i) + \lambda(v_{i-m}v_i) = \frac{31n+7}{2}.$$

Case 10 : $i = n-1$.

We have $\lambda(v_{n-1}) = n+m-(n-1)-1 = m$, $\lambda(v_{n-1}v_0) = \frac{2n+(n-1)}{2} = \frac{3n-1}{2}$,

$$\lambda(v_{n-1}v_2) = 3n,\ \lambda(v_{n-1}v_{n+m-1}) = 3n+1,\ \lambda(v_{n-2}v_{n-1}) = \frac{3n+(n-2)}{2} = 2n-1,$$
$$\lambda(v_{n-4}v_{n-1}) = 3n-(n-4)-1 = 2n+3,$$
$$\lambda(v_{n-m-1}v_{n-1}) = 3n+(n-m-1)+2 = 4n-m+1.$$

Thus

$$w_\lambda(v_{n-1}) = \lambda(v_{n-1}) + \lambda(v_{n-1}v_0) + \lambda(v_{n-1}v_2) + \lambda(v_{n-1}v_{n+m-1}) + \lambda(v_{n-2}v_{n-1})$$
$$+\lambda(v_{n-4}v_{n-1}) + \lambda(v_{n-m-1}v_{n-1})$$
$$= \frac{31n+7}{2}.$$

Hence $w_\lambda(v_i) = \dfrac{31n+7}{2}$ for all $i = 0,1,2,...,n-1$.

Therefore λ is a super vertex-magic total labeling of G. □

Example 3.17. Super vertex-magic graphs $C_{13}(1,3,4)$ and $C_{13}(1,3,5)$ with the magic constant 205 is shown in Figure 3.8.

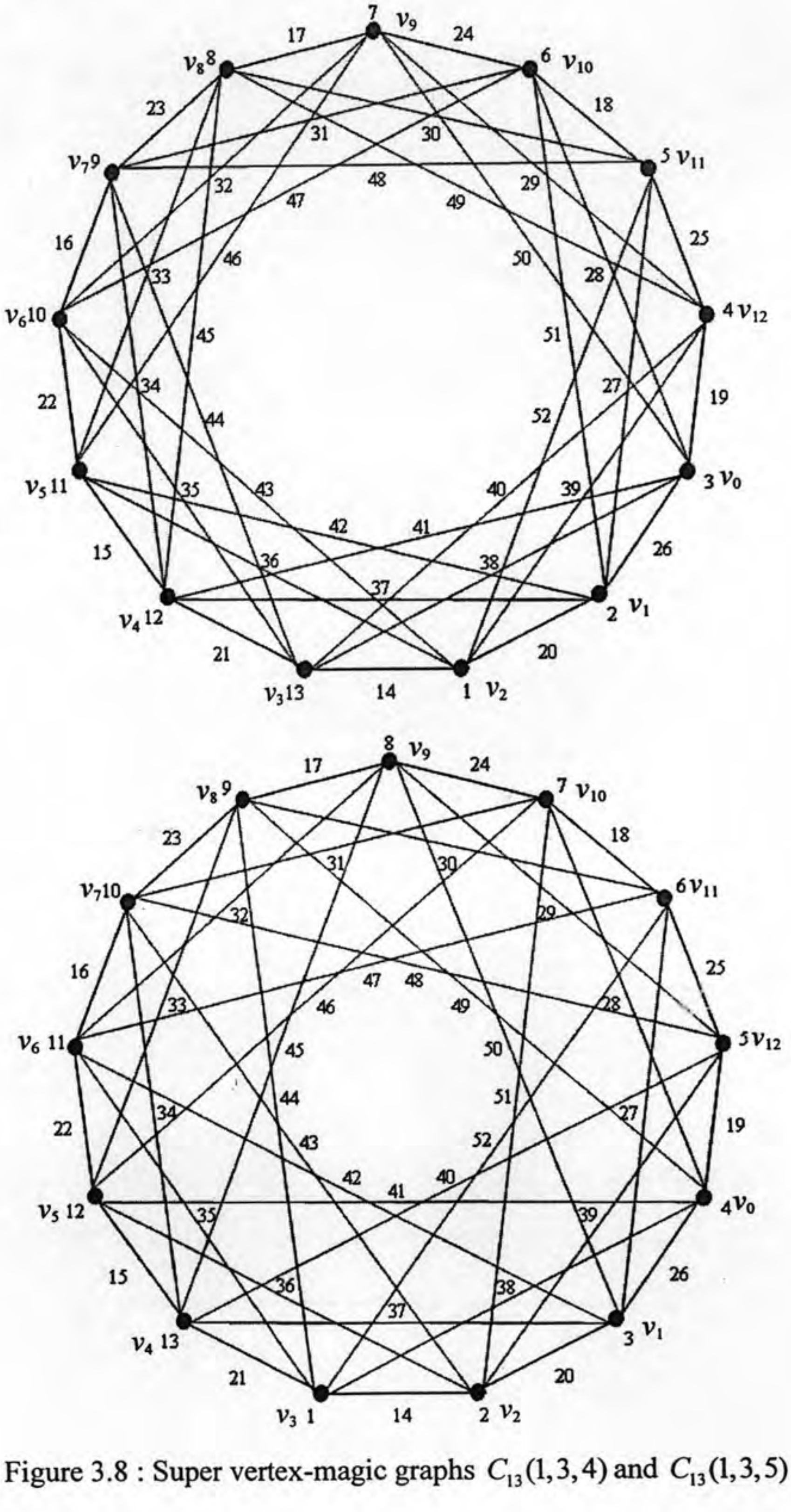

Figure 3.8 : Super vertex-magic graphs $C_{13}(1,3,4)$ and $C_{13}(1,3,5)$

Theorem 3.18.

(i) *Let* G_1 *be a circulant graph* $kC_n(1,2,m)$ *where* $n \geq 7$, *n is odd, and* $m \in \{3, 4, ..., \frac{n-1}{2}\}$. G_1 *is a super vertex-magic graph iff k is odd.*

(ii) *Let* G_2 *be a circulant graph* $kC_n(1,3,m)$ *where* $n \geq 9$, *n is odd, and* $m \in \{4,5,...,\frac{n-1}{2}\}$. G_2 *is a super vertex-magic graph iff k is odd.*

Proof.

Let G_1 be a circulant graph $kC_n(1,2,m)$, $n \geq 7$, n is odd, and $m \in \{3, 4, ..., \frac{n-1}{2}\}$.

Let G_2 be a circulant graph $kC_n(1,3,m)$, $n \geq 9$, n is odd, and $m \in \{4,5,...,\frac{n-1}{2}\}$.

We have $|V(G_1)|=|V(G_2)|=nk$, $|E(G_1)|=|E(G_2)|=3nk$, and G_1, G_2 are 6-regular.

It suffices to show only G_1 is a super vertex-magic graph, because G_2 possesses similar properties compared with G_1.

(⇒) Assume that G_1 is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(4nk)(4nk+1)}{nk} - \frac{nk+1}{2}$$
$$= 16nk + 4 - \frac{nk+1}{2} = \frac{31nk+7}{2}.$$

Since $\frac{31nk+7}{2}$ is an integer and n is odd, k is odd.

(⇐) Assume that k is odd.

Since k is odd and the circulant graph $C_n(1,2,m)$ is 6-regular, $\frac{(k-1)(r+1)}{2} = \frac{7(k-1)}{2}$ is an integer.

By Theorem 2.12, G_1 and G_2 are super vertex-magic graphs. □

Theorem 3.19. *If G is a circulant graph* $C_n(1,2,3,4)$ *where* $n \geq 9$ *and n is odd, then G is a super vertex-magic graph.*

Proof. Let G be a circulant graph $C_n(1,2,3,4)$, $n \geq 9$, and n is odd.

Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$.

Let $\lambda : V(G) \cup E(G) \to \{1,2, ...,5n\}$ be a labeling defined by

$$\lambda(v_i) = \begin{cases} 3-i & \text{if } i = 0,1,2, \\ n+3-i & \text{if } i = 3,4, \ldots, n-1, \end{cases} \qquad \ldots..(5)$$

$$\left.\begin{aligned} \lambda(v_i v_{i+1}) &= \begin{cases} \dfrac{4n-i}{2} & \text{if } i = 0,2, \ldots, n-1, \\ \dfrac{3n-i}{2} & \text{if } i = 1,3, \ldots, n-2, \end{cases} \\ \lambda(v_i v_{i+2}) &= 2n+i+1 \qquad \text{if } i = 0,1, \ldots, n-1, \\ \lambda(v_i v_{i+3}) &= \begin{cases} 4n-i-1 & \text{if } i = 0,1,2, \ldots, n-2, \\ 4n & \text{if } i = n-1, \end{cases} \\ \lambda(v_i v_{i+4}) &= \begin{cases} 4n+i+2 & \text{if } i = 0,1,2, \ldots, n-2, \\ 4n+1 & \text{if } i = n-1, \end{cases} \end{aligned}\right\} \qquad \ldots..(6)$$

Claim that λ is a super vertex-magic total labeling of the graph G.

Since $v = n$ and $e = 4n$, we will show that

(i) $\lambda(V(G)) = \{1, 2, \ldots, n\}$,

(ii) $\lambda(E(G)) = \{n+1, n+2, \ldots, 5n\}$,

and (iii) $w_\lambda(v_i) = \dfrac{49n+9}{2}$ for all $i = 0,1, \ldots, n-1$.

From (5), For $i = 0,1,\ldots,n-1$, we have $1 \le \lambda(v_i) \le n$.

Therefore $\lambda(V(G)) = \{1, 2, \ldots, n\}$, (i) holds.

From (6), For $i = 0,1,2,\ldots,n-1$, we have $n+1 \le \lambda(v_i v_{i+1}) \le 2n$

$2n+1 \le \lambda(v_i v_{i+2}) \le 3n$, $3n+1 \le \lambda(v_i v_{i+3}) \le 4n$, and $4n+1 \le \lambda(v_i v_{i+4}) \le 5n$.

Therefore $\lambda(E(G)) = \{n+1, n+2, \ldots, 5n\}$, (ii) holds.

Since all number 1 through $5n$ are used exactly once, λ is a bijection.

Note that for each $i \in \{0,1, \ldots, n-1\}$, the 8 edges incident with vertex v_i are

$v_i v_{i+1}, v_i v_{i+2}, v_i v_{i+3}, v_i v_{i+4}, v_{i-1} v_i, v_{i-2} v_i, v_{i-3} v_i, v_{i-4} v_i$, where all indicies are taken modulo n.

Since v_{n-1} is also incident with v_3, so we consider the following 7 cases.

Case 1 : $i = 0$.

We have $\lambda(v_0) = 3$, $\lambda(v_0v_1) = 2n$, $\lambda(v_0v_2) = 2n+1$, $\lambda(v_0v_3) = 4n-1$,

$$\lambda(v_0v_4) = 4n+2,\ \lambda(v_{n-1}v_0) = \frac{4n-(n-1)}{2} = \frac{3n+1}{2},$$

$$\lambda(v_{n-2}v_0) = 2n+(n-2)+1 = 3n-1,\ \lambda(v_{n-3}v_0) = 4n-(n-3)-1 = 3n+2,$$

$$\lambda(v_{n-4}v_0) = 4n+(n-4)+2 = 5n-2.$$

Thus

$$\begin{aligned} w_\lambda(v_0) &= \lambda(v_0) + \lambda(v_0v_1) + \lambda(v_0v_2) + \lambda(v_0v_3) + \lambda(v_0v_4) + \lambda(v_{n-1}v_0) + \lambda(v_{n-2}v_0) \\ &\quad + \lambda(v_{n-3}v_0) + \lambda(v_{n-4}v_0) \\ &= \frac{49n+9}{2}. \end{aligned}$$

Case 2 : $i = 1$.

We have $\lambda(v_1) = 2$, $\lambda(v_1v_2) = \frac{3n-1}{2}$, $\lambda(v_1v_3) = 2n+2$, $\lambda(v_1v_4) = 4n-2$,

$$\lambda(v_1v_5) = 4n+3,\ \lambda(v_0v_1) = 2n,\ \lambda(v_{n-1}v_1) = 2n+(n-1)+1 = 3n,$$

$$\lambda(v_{n-2}v_1) = 4n-(n-2)-1 = 3n+1,\ \lambda(v_{n-3}v_1) = 4n+(n-3)+2 = 5n-1.$$

Thus

$$\begin{aligned} w_\lambda(v_1) &= \lambda(v_1) + \lambda(v_1v_2) + \lambda(v_1v_3) + \lambda(v_1v_4) + \lambda(v_1v_5) + \lambda(v_0v_1) + \lambda(v_{n-1}v_1) \\ &\quad + \lambda(v_{n-2}v_1) + \lambda(v_{n-3}v_1) \\ &= \frac{49n+9}{2}. \end{aligned}$$

Case 3 : $i = 2$.

We have $\lambda(v_2) = 1$, $\lambda(v_2v_3) = 2n-1$, $\lambda(v_2v_4) = 2n+3$, $\lambda(v_2v_5) = 4n-3$,

$$\lambda(v_2v_6) = 4n+4,\ \lambda(v_1v_2) = \frac{3n-1}{2},\ \lambda(v_0v_2) = 2n+1,\ \lambda(v_{n-1}v_2) = 4n,$$

$$\lambda(v_{n-2}v_2) = 4n+(n-2)+2 = 5n.$$

Thus

$$\begin{aligned} w_\lambda(v_2) &= \lambda(v_2) + \lambda(v_2v_3) + \lambda(v_2v_4) + \lambda(v_2v_5) + \lambda(v_2v_6) + \lambda(v_1v_2) + \lambda(v_0v_2) \\ &\quad + \lambda(v_{n-1}v_2) + \lambda(v_{n-2}v_2) \\ &= \frac{49n+9}{2}. \end{aligned}$$

Case 4 : $i=3$.

We have $\lambda(v_3)=n$, $\lambda(v_3v_4)=\frac{3n-3}{2}$, $\lambda(v_3v_5)=2n+4$, $\lambda(v_3v_6)=4n-4$, $\lambda(v_3v_7)=4n+5$, $\lambda(v_2v_3)=2n-1$, $\lambda(v_1v_3)=2n+2$, $\lambda(v_0v_3)=4n-1$, $\lambda(v_{n-1}v_3)=4n+1$.

Thus

$$w_\lambda(v_3)=\lambda(v_3)+\lambda(v_3v_4)+\lambda(v_3v_5)+\lambda(v_3v_6)+\lambda(v_3v_7)+\lambda(v_2v_3)+\lambda(v_1v_3)+\lambda(v_0v_3)+\lambda(v_{n-1}v_3)$$
$$=\frac{49n+9}{2}.$$

Case 5 : $4\le i<n-1$ and i is odd.

We have $\lambda(v_i)=n+3-i$, $\lambda(v_iv_{i+1})=\frac{3n-i}{2}$, $\lambda(v_iv_{i+2})=2n+i+1$, $\lambda(v_iv_{i+3})=4n-i-1$, $\lambda(v_iv_{i+4})=4n+i+2$, $\lambda(v_{i-1}v_i)=\frac{4n-(i-1)}{2}=\frac{4n-i+1}{2}$, $\lambda(v_{i-2}v_i)=2n+(i-2)+1=2n+i-1$, $\lambda(v_{i-3}v_i)=4n-(i-3)-1=4n-i+2$, $\lambda(v_{i-4}v_i)=4n+(i-4)+2=4n+i-2$.

Thus

$$w_\lambda(v_i)=\lambda(v_i)+\lambda(v_iv_{i+1})+\lambda(v_iv_{i+2})+\lambda(v_iv_{i+3})+\lambda(v_iv_{i+4})+\lambda(v_{i-1}v_i)+\lambda(v_{i-2}v_i)$$
$$+\lambda(v_{i-3}v_i)+\lambda(v_{i-4}v_i)$$
$$=\frac{49n+9}{2}.$$

Case 6 : $4\le i<n-1$ and i is even.

We have $\lambda(v_i)=n+3-i$, $\lambda(v_iv_{i+1})=\frac{4n-i}{2}$, $\lambda(v_iv_{i+2})=2n+i+1$, $\lambda(v_iv_{i+3})=4n-i-1$, $\lambda(v_iv_{i+4})=4n+i+2$, $\lambda(v_{i-1}v_i)=\frac{3n-(i-1)}{2}=\frac{3n-i+1}{2}$, $\lambda(v_{i-2}v_i)=2n+(i-2)+1=2n+i-1$, $\lambda(v_{i-3}v_i)=4n-(i-3)-1=4n-i+2$, $\lambda(v_{i-4}v_i)=4n+(i-4)+2=4n+i-2$.

Thus

$$w_\lambda(v_i)=\lambda(v_i)+\lambda(v_iv_{i+1})+\lambda(v_iv_{i+2})+\lambda(v_iv_{i+3})+\lambda(v_iv_{i+4})+\lambda(v_{i-1}v_i)+\lambda(v_{i-2}v_i)$$
$$+\lambda(v_{i-3}v_i)+\lambda(v_{i-4}v_i)$$
$$=\frac{49n+9}{2}.$$

Case 7 : $i = n-1$.

We have $\lambda(v_{n-1}) = 4$, $\lambda(v_{n-1}v_0) = \dfrac{4n-(n-1)}{2} = \dfrac{3n+1}{2}$, $\lambda(v_{n-1}v_1) = 2n+(n-1)+1 = 3n$,

$\lambda(v_{n-1}v_2) = 4n$, $\lambda(v_{n-1}v_3) = 4n+1$, $\lambda(v_{n-2}v_{n-1}) = \dfrac{3n-(n-2)}{2} = n+1$,

$\lambda(v_{n-3}v_{n-1}) = 2n+(n-3)+1 = 3n-2$, $\lambda(v_{n-4}v_{n-1}) = 4n-(n-4)-1 = 3n+3$,

$\lambda(v_{n-5}v_{n-1}) = 4n+(n-5)+2 = 5n-3$.

Thus

$$w_\lambda(v_{n-1}) = \lambda(v_{n-1}) + \lambda(v_{n-1}v_0) + \lambda(v_{n-1}v_1) + \lambda(v_{n-1}v_2) + \lambda(v_{n-1}v_3) + \lambda(v_{n-2}v_{n-1}) + \lambda(v_{n-3}v_{n-1})$$
$$+\lambda(v_{n-4}v_{n-1}) + \lambda(v_{n-5}v_{n-1})$$
$$= \frac{49n+9}{2}.$$

Hence $w_\lambda(v_i) = \dfrac{49n+9}{2}$ for all $i = 0, 1, \ldots, n-1$.

Therefore λ is a super vertex-magic total labeling of G. □

Example 3.20. The super vertex-magic graph $C_{13}(1,2,3,4)$ with the magic constant 323 is shown in Figure 3.9.

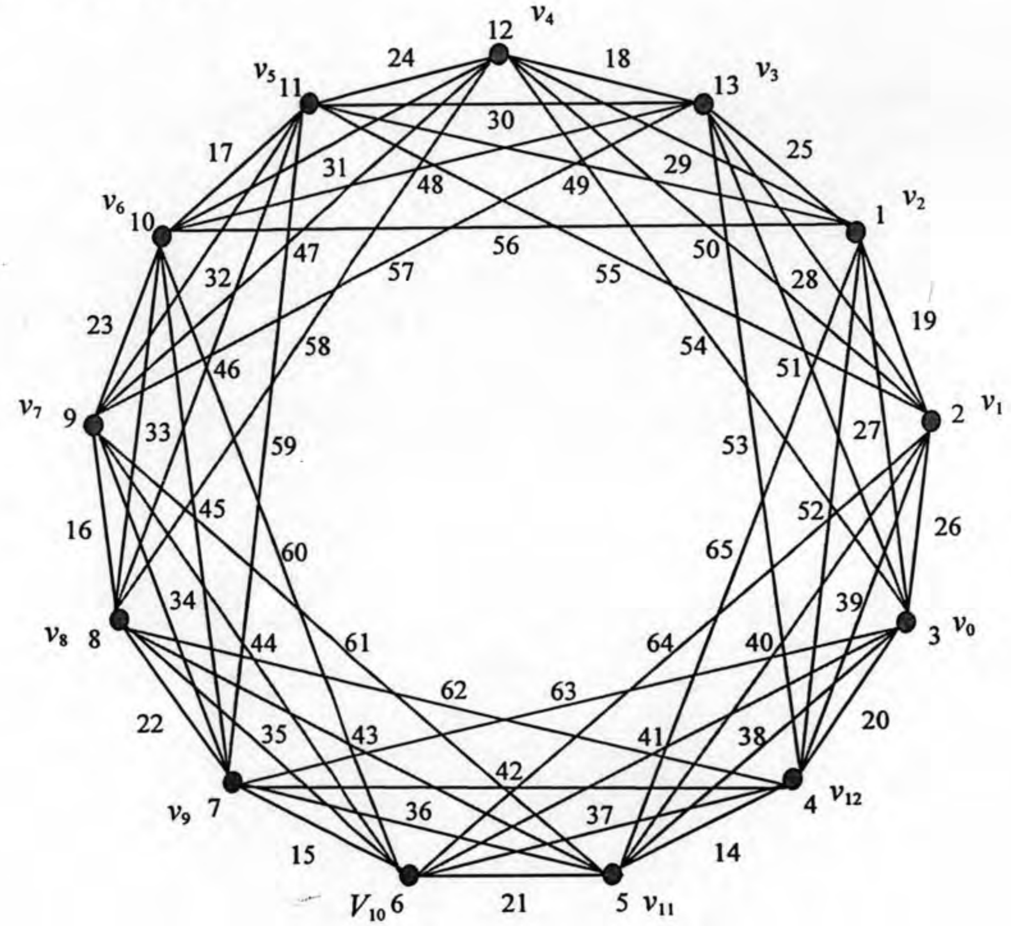

Figure 3.9 : Super vertex-magic graph $C_{13}(1,2,3,4)$

Theorem 3.21. *If G is a circulant graph* $kC_n(1,2,3,4)$ *where* $n \geq 9$ *and n is odd, then G is a super vertex-magic graph iff k is odd.*

Proof. Let G be a circulant graph $kC_n(1,2,3,4)$, $n \geq 9$, and n is odd.

We have $v = nk$ and $e = 4nk$.

(⇒) Assume that G is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(5nk)(5nk+1)}{nk} - \frac{nk+1}{2}$$
$$= 25nk + 5 - \frac{nk+1}{2} = \frac{49nk+9}{2}.$$

Since $\frac{49nk+9}{2}$ is an integer and n is odd, k is odd.

($\Leftarrow$) Assume that k is odd.

Since k is odd and the circulant graph $C_n(1,2,3,4)$ is 8-regular, $\frac{(k-1)(r+1)}{2} = \frac{9(k-1)}{2}$ is an integer.

By Theorem 2.12, G is a super vertex-magic graph. □

Theorem 3.22. *If G is a circulant graph* $C_n(1,2,3,4,5)$ *where* $n \geq 11$ *and n is odd. then G is a super vertex-magic graph.*

Proof. Let G be a circulant graph $C_n(1,2,3,4,5)$, $n \geq 11$, and n be odd.

Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$.

Let $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., 6n\}$ be a labeling defined by

$$\lambda(v_i) = \begin{cases} 4-i & \text{if } i = 0,1,2,3 \\ n+4-i & \text{if } i = 4,5, ..., n-1, \end{cases} \qquad .....(7)$$

$$\left.\begin{aligned} \lambda(v_i v_{i+1}) &= \begin{cases} 2n & \text{if } i = 0, \\ \frac{3n+i}{2} & \text{if } i = 1,3, ..., n-2, \\ \frac{2n+i}{2} & \text{if } i = 2,4, ..., n-1, \end{cases} \\ \lambda(v_i v_{i+2}) &= 3n - i \qquad \text{if } i = 0,1,2, ..., n-1, \\ \lambda(v_i v_{i+3}) &= 3n + i + 1 \qquad \text{if } i = 0,1,2, ..., n-1. \\ \lambda(v_i v_{i+4}) &= \begin{cases} 5n - i - 1 & \text{if } i = 0,1,2, ..., n-2, \\ 5n & \text{if } i = n-1, \end{cases} \\ \lambda(v_i v_{i+5}) &= \begin{cases} 5n + i + 2 & \text{if } i = 0,1,2, ..., n-2, \\ 5n+1 & \text{if } i = n-1, \end{cases} \end{aligned}\right\} \qquad .....(8)$$

Claim that λ is a super vertex-magic total labeling of the graph G.

Since $v = n$ and $e = 5n$, we will show that

(i) $\lambda(V(G)) = \{1, 2, \ldots, n\}$,

(ii) $\lambda(E(G)) = \{n+1, n+2, ..., 6n\}$,

and (iii) $w_\lambda(v_i) = \frac{71n+11}{2}$ for all $i = 0,1, ..., n-1$.

From (7), For $i=0,1,2,...,n-1$, we have $1 \le \lambda(v_i) \le n$.

Therefore $\lambda(V(G)) = \{1, 2, \ldots, n\}$, (i) holds.

From (8), For $i=0,1,2,...,n-1$, we have $n+1 \le \lambda(v_i v_{i+1}) \le 2n$

$2n+1 \le \lambda(v_i v_{i+2}) \le 3n$, $3n+1 \le \lambda(v_i v_{i+3}) \le 4n$, $4n+1 \le \lambda(v_i v_{i+4}) \le 5n$,

and $5n+1 \le \lambda(v_i v_{i+5}) \le 6n$.

Therefore $\lambda(E(G)) = \{n+1, n+2, ..., 6n\}$, (ii) holds.

Since all number 1 through $6n$ are used exactly once, λ is a bijection.

Note that for each $i \in \{0,1,...,n-1\}$, the 10 edges incident with vertex v_i are $v_i v_{i+1}, v_i v_{i+2}, v_i v_{i+3}, v_i v_{i+4}, v_i v_{i+5}, v_{i-1}v_i, v_{i-2}v_i, v_{i-3}v_i, v_{i-4}v_i, v_{i-5}v_i$, where all indicies are taken modulo n.

Since v_{n-1} is also incident with v_4, so we consider the following 8 cases.

Case 1 : $i=0$.

We have $\lambda(v_0)=4$, $\lambda(v_0v_1)=2n$, $\lambda(v_0v_2)=3n$, $\lambda(v_0v_3)=3n+1$,

$\lambda(v_0v_4)=5n-1$, $\lambda(v_0v_5)=5n+2$, $\lambda(v_{n-1}v_0)=\dfrac{2n+(n-1)}{2}=\dfrac{3n-1}{2}$,

$\lambda(v_{n-2}v_0)=3n-(n-2)=2n+2$, $\lambda(v_{n-3}v_0)=3n+(n-3)+1=4n-2$,

$\lambda(v_{n-4}v_0)=5n-(n-4)-1=4n+3$, $\lambda(v_{n-5}v_0)=5n+(n-5)+2=6n-3$.

Thus

$$\begin{aligned} w_\lambda(v_0) &= \lambda(v_0)+\lambda(v_0v_1)+\lambda(v_0v_2)+\lambda(v_0v_3)+\lambda(v_0v_4)+\lambda(v_0v_5)+\lambda(v_{n-1}v_0)+\lambda(v_{n-2}v_0) \\ &\quad +\lambda(v_{n-3}v_0)+\lambda(v_{n-4}v_0)+\lambda(v_{n-5}v_0) \\ &= \frac{71n+11}{2}. \end{aligned}$$

Case 2 : $i=1$.

We have $\lambda(v_1)=3$, $\lambda(v_1v_2)=\dfrac{3n+1}{2}$, $\lambda(v_1v_3)=3n-1$, $\lambda(v_1v_4)=3n+2$,

$\lambda(v_1v_5)=5n-2$, $\lambda(v_1v_6)=5n+3$, $\lambda(v_0v_1)=2n$,

$\lambda(v_{n-1}v_1)=3n-(n-1)=2n+1$, $\lambda(v_{n-2}v_1)=3n+(n-2)+1=4n-1$,

$\lambda(v_{n-3}v_1)=5n-(n-3)-1=4n+2$, $\lambda(v_{n-4}v_1)=5n+(n-4)+2=6n-2$.

Thus

$$w_\lambda(v_1) = \lambda(v_1) + \lambda(v_1v_2) + \lambda(v_1v_3) + \lambda(v_1v_4) + \lambda(v_1v_5) + \lambda(v_1v_6) + \lambda(v_0v_1) + \lambda(v_{n-1}v_1) + \lambda(v_{n-2}v_1) + \lambda(v_{n-3}v_1) + \lambda(v_{n-4}v_1) = \frac{71n+11}{2}.$$

Case 3 : $i = 2$.

We have $\lambda(v_2) = 2$, $\lambda(v_2v_3) = n+1$, $\lambda(v_2v_4) = 3n-2$, $\lambda(v_2v_5) = 3n+3$, $\lambda(v_2v_6) = 5n-3$, $\lambda(v_2v_7) = 5n+4$, $\lambda(v_1v_2) = \frac{3n+1}{2}$, $\lambda(v_0v_2) = 3n$, $\lambda(v_{n-1}v_2) = 3n+(n-1)+1 = 4n$, $\lambda(v_{n-2}v_2) = 5n-(n-2)-1 = 4n+1$, $\lambda(v_{n-3}v_2) = 5n+(n-3)+2 = 6n-1$.

Thus

$$w_\lambda(v_2) = \lambda(v_2) + \lambda(v_2v_3) + \lambda(v_2v_4) + \lambda(v_2v_5) + \lambda(v_2v_6) + \lambda(v_2v_7) + \lambda(v_1v_2) + \lambda(v_0v_2) + \lambda(v_{n-1}v_2) + \lambda(v_{n-2}v_2) + \lambda(v_{n-3}v_2) = \frac{71n+11}{2}.$$

Case 4 : $i = 3$.

We have $\lambda(v_3) = 1$, $\lambda(v_3v_4) = \frac{3n+3}{2}$, $\lambda(v_3v_5) = 3n-3$, $\lambda(v_3v_6) = 3n+4$, $\lambda(v_3v_7) = 5n-4$, $\lambda(v_3v_8) = 5n+5$, $\lambda(v_2v_3) = n+1$, $\lambda(v_1v_3) = 3n-1$, $\lambda(v_0v_3) = 3n+1$, $\lambda(v_{n-1}v_3) = 5n$, $\lambda(v_{n-2}v_3) = 5n+(n-2)+2 = 6n$.

Thus

$$w_\lambda(v_3) = \lambda(v_3) + \lambda(v_3v_4) + \lambda(v_3v_5) + \lambda(v_3v_6) + \lambda(v_3v_7) + \lambda(v_3v_8) + \lambda(v_2v_3) + \lambda(v_1v_3) + \lambda(v_0v_3) + \lambda(v_{n-1}v_3) + \lambda(v_{n-2}v_3) = \frac{71n+11}{2}.$$

Case 5 : $i = 4$.

We have $\lambda(v_4) = n$, $\lambda(v_4v_5) = n+2$, $\lambda(v_4v_6) = 3n-4$, $\lambda(v_4v_7) = 3n+5$,

$\lambda(v_4v_8) = 5n-5$, $\lambda(v_4v_9) = 5n+6$, $\lambda(v_3v_4) = \dfrac{3n+3}{2}$, $\lambda(v_2v_4) = 3n-2$,

$\lambda(v_1v_4) = 3n+2$, $\lambda(v_0v_4) = 5n-1$, $\lambda(v_{n-1}v_4) = 5n+1$.

Thus

$$w_\lambda(v_4) = \lambda(v_4) + \lambda(v_4v_5) + \lambda(v_4v_6) + \lambda(v_4v_7) + \lambda(v_4v_8) + \lambda(v_4v_9) + \lambda(v_3v_4) + \lambda(v_2v_4)$$
$$+\lambda(v_1v_4) + \lambda(v_0v_4) + \lambda(v_{n-1}v_4)$$
$$= \frac{71n+11}{2}.$$

Case 6 : $5 \le i < n-2$ and i is odd.

We have $\lambda(v_i) = n+4-i$, $\lambda(v_iv_{i+1}) = \dfrac{3n+i}{2}$, $\lambda(v_iv_{i+2}) = 3n-i$, $\lambda(v_iv_{i+3}) = 3n+i+1$,

$\lambda(v_iv_{i+4}) = 5n-i-1$, $\lambda(v_iv_{i+5}) = 5n+i+2$, $\lambda(v_{i-1}v_i) = \dfrac{2n+(i-1)}{2}$,

$\lambda(v_{i-2}v_i) = 3n-(i-2) = 3n-i+2$, $\lambda(v_{i-3}v_i) = 3n+(i-3)+1 = 3n+i-2$,

$\lambda(v_{i-4}v_i) = 5n-(i-4)-1 = 5n-i+3$, $\lambda(v_{i-5}v_i) = 5n+(i-5)+2 = 5n+i-3$.

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_iv_{i+1}) + \lambda(v_iv_{i+2}) + \lambda(v_iv_{i+3}) + \lambda(v_iv_{i+4}) + \lambda(v_iv_{i+5}) + \lambda(v_{i-1}v_i)$$
$$+\lambda(v_{i-2}v_i) + \lambda(v_{i-3}v_i) + \lambda(v_{i-4}v_i) + \lambda(v_{i-5}v_i)$$
$$= \frac{71n+11}{2}.$$

Case 7 : $5 \le i < n-2$ and i is even.

We have $\lambda(v_i) = n+4-i$, $\lambda(v_iv_{i+1}) = \dfrac{2n+i}{2}$, $\lambda(v_iv_{i+2}) = 3n-i$, $\lambda(v_iv_{i+3}) = 3n+i+1$,

$\lambda(v_iv_{i+4}) = 5n-i-1$, $\lambda(v_iv_{i+5}) = 5n+i+2$, $\lambda(v_{i-1}v_i) = \dfrac{3n+(i-1)}{2}$,

$\lambda(v_{i-2}v_i) = 3n-(i-2) = 3n-i+2$, $\lambda(v_{i-3}v_i) = 3n+(i-3)+1 = 3n+i-2$,

$\lambda(v_{i-4}v_i) = 5n-(i-4)-1 = 5n-i+3$, $\lambda(v_{i-5}v_i) = 5n+(i-5)+2 = 5n+i-3$.

Thus

$$w_\lambda(v_i) = \lambda(v_i) + \lambda(v_iv_{i+1}) + \lambda(v_iv_{i+2}) + \lambda(v_iv_{i+3}) + \lambda(v_iv_{i+4}) + \lambda(v_iv_{i+5}) + \lambda(v_{i-1}v_i)$$
$$+\lambda(v_{i-2}v_i) + \lambda(v_{i-3}v_i) + \lambda(v_{i-4}v_i) + \lambda(v_{i-5}v_i)$$
$$= \frac{71n+11}{2}.$$

Case 8 : $i = n$-1.

We have $\lambda(v_{n-1}) = n+4-(n-1) = 5$, $\lambda(v_{n-1}v_0) = \frac{2n+(n-1)}{2} = \frac{3n-1}{2}$,

$\lambda(v_{n-1}v_1) = 3n-(n-1) = 2n+1$, $\lambda(v_{n-1}v_2) = 3n+(n-1)+1 = 4n$,

$\lambda(v_{n-1}v_3) = 5n$, $\lambda(v_{n-1}v_4) = 5n+1$, $\lambda(v_{n-2}v_{n-1}) = \frac{3n+(n-2)}{2} = 2n-1$,

$\lambda(v_{n-3}v_{n-1}) = 3n-(n-3) = 2n+3$, $\lambda(v_{n-4}v_{n-1}) = 3n+(n-4)+1 = 4n-3$,

$\lambda(v_{n-5}v_{n-1}) = 5n-(n-5)-1 = 4n+4$, $\lambda(v_{n-6}v_{n-1}) = 5n+(n-6)+2 = 6n-4$.

Thus

$$w_\lambda(v_{n-1}) = \lambda(v_{n-1}) + \lambda(v_{n-1}v_0) + \lambda(v_{n-1}v_1) + \lambda(v_{n-1}v_2) + \lambda(v_{n-1}v_3) + \lambda(v_{n-1}v_4) + \lambda(v_{n-2}v_{n-1})$$
$$+\lambda(v_{n-3}v_{n-1}) + \lambda(v_{n-4}v_{n-1}) + \lambda(v_{n-5}v_{n-1}) + \lambda(v_{n-6}v_{n-1})$$
$$= \frac{71n+11}{2}.$$

Hence $w_\lambda(v_i) = \frac{71n+11}{2}$ for all $i = 0, 1, \ldots, n-1$.

Therefore λ is a super vertex-magic total labeling of G. □

Example 3.23. The super vertex-magic graph $C_{13}(1,2,3,4,5)$ with the magic constant 467 is shown in Figure 3.10.

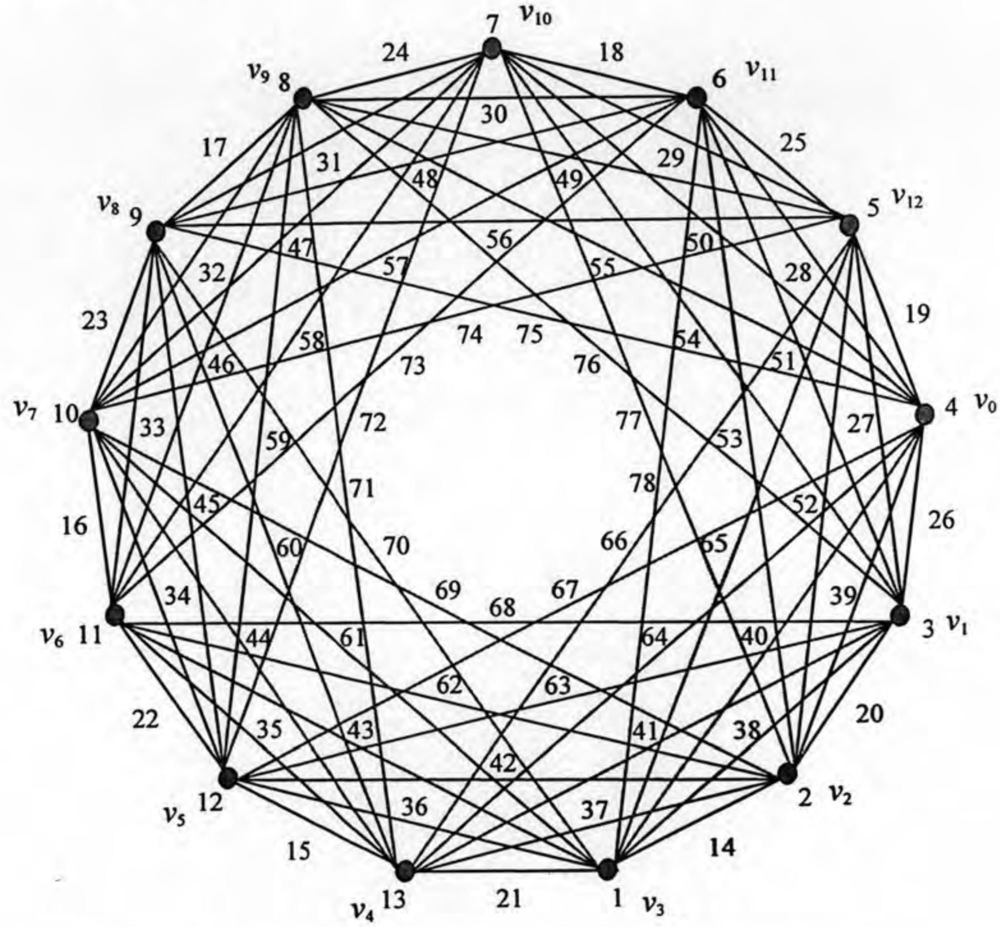

Figure 3.10 : Super vertex-magic graph $C_{13}(1,2,3,4,5)$

Theorem 3.24. *If G is a circulant graph $kC_n(1,2,3,4,5)$ where $n \geq 11$ and n is odd, then G is a super vertex-magic graph iff k is odd.*

Proof. Let G be a circulant graph $kC_n(1,2,3,4,5)$, $n \geq 11$, and n is odd.

We have $v = nk$ and $e = 5nk$.

(⇒) Assume that G is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v} - \frac{v+1}{2} = \frac{(6nk)(6nk+1)}{nk} - \frac{nk+1}{2}$$

$$= 36nk + 6 - \frac{nk+1}{2} = \frac{71nk+11}{2}.$$

Since $\frac{71nk+11}{2}$ is an integer and n is odd, k is odd.

($\Leftarrow$) Assume that k is odd.

Since k is odd and $C_n(1,2,3,4,5)$ is 10-regular, $\frac{(k-1)(r+1)}{2}=\frac{11(k-1)}{2}$ is an integer.

By Theorem 2.12, G is a super vertex-magic graph. □

Theorem 3.25. *Let G is a graph $k(C_3+C_6)$. G is a super vertex-magic graph iff k is odd.*

Proof. Let G be a graph $k(C_3+C_6)$. We have $v = 9k$ and $e = 9k$.

($\Rightarrow$) Assume that G is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{(18k)(18k+1)}{9k}-\frac{9k+1}{2}$$
$$=36k+2-\frac{9k+1}{2}=\frac{63k+3}{2}.$$

Then k is odd.

($\Leftarrow$) Assume that k is odd.

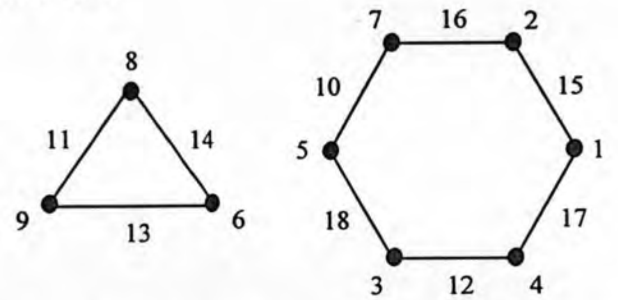

Figure 3.11 : Super vertex-magic graph C_3+C_6 with the magic constant 33

By the total labeling shown in Figure 3.12, C_3+C_6 is a super vertex-magic graph.

Since k is odd and C_3+C_6 is 2-regular, $\frac{(k-1)(r+1)}{2}=\frac{3(k-1)}{2}$ is an integer.

By Theorem 2.12, G is a super vertex-magic graph. □

Theorem 3.26.

(i) *Let G_1 be a graph $k(C_3+C_8)$. G_1 is a super vertex-magic graph iff k is odd.*

(ii) *Let G_2 be a graph $k(C_5+C_6)$. G_2 is a super vertex-magic graph iff k is odd.*

Proof. Let G_1 be a graph $k(C_3+C_8)$ and G_2 be a graph $k(C_5+C_6)$.

We have $|V(G_1)|=|V(G_2)|=11k$, $|E(G_1)|=|E(G_2)|=11k$, and G_1, G_2 are 2-regular.

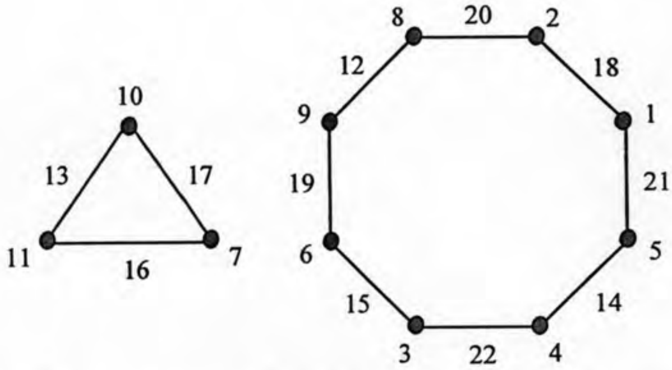

Figure 3.12 : Super vertex-magic graph C_3+C_8 with the magic constant 40

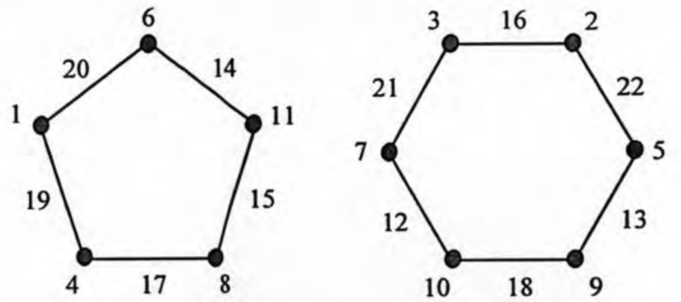

Figure 3.13 : Super vertex-magic graph C_5+C_6 with the magic constant 40

It suffices to show only G_1 is a super vertex-magic graph, because G_2 possesses similar properties compared with G_1.

($\Rightarrow$) Assume that G_1 is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{(22k)(22k+1)}{11k}-\frac{11k+1}{2}$$
$$=44k+2-\frac{11k+1}{2}=\frac{77k+3}{2}.$$

Then k is odd.

($\Leftarrow$) Assume that k is odd.

By the total labeling shown in Figure 3.12, C_3+C_8 is a super vertex-magic graph.

Since k is odd and C_3+C_8 is 2-regular, $\frac{(k-1)(r+1)}{2}=\frac{3(k-1)}{2}$ is an integer.

By Theorem 2.12, G_1 and G_2 are super vertex-magic graphs. □

Theorem 3.27.

(i) *Let* G_1 *be a graph* $k(C_3+C_{10})$. G_1 *is a super vertex-magic graph iff* k *is odd.*

(ii) *Let* G_2 *be a graph* $k(C_3+C_3+C_7)$. G_2 *is a super vertex-magic graph iff* k *is odd.*

Proof. Let G_1 be a graph $k(C_3+C_{10})$ and G_2 be a graph $k(C_3+C_3+C_7)$.

We have $|V(G_1)|=|V(G_2)|=13k$, $|E(G_1)|=|E(G_2)|=13k$, and G_1, G_2 are 2-regular.

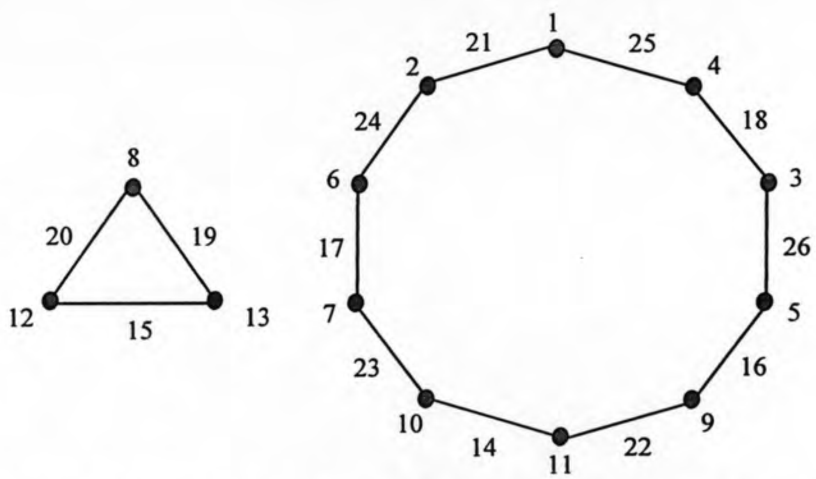

Figure 3.14 : Super vertex-magic graph C_3+C_{10} with the magic constant 47

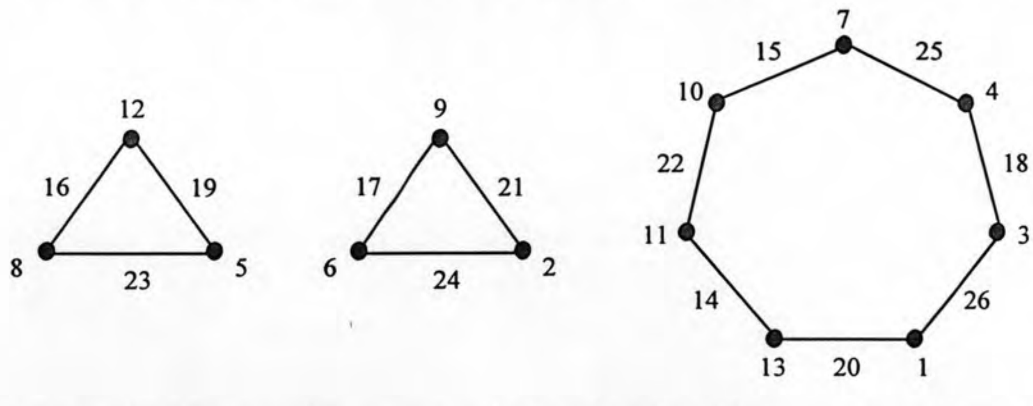

Figure 3.15 : Super vertex-magic graph $C_3+C_3+C_7$ with the magic constant 47

It suffices to show only G_1, a super vertex-magic graph, because G_2 possesses similar properties compared with G_1.

(⇒) Assume that G_1 is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{(26k)(26k+1)}{13k}-\frac{13k+1}{2}$$
$$=52k+2-\frac{13k+1}{2}=\frac{91k+3}{2}.$$

Then k is odd.

(⇐) Assume that k is odd.

By the total labeling shown in Figure 3.14, C_3+C_{10} is a super vertex-magic graph.

Since k is odd and C_3+C_{10} is 2-regular, $\frac{(k-1)(r+1)}{2}=\frac{3(k-1)}{2}$ is an integer.

By Theorem 2.12, G_1 and G_2 are super vertex-magic graphs. □

Theorem 3.28. *Let G be a graph* $k(C_4+C_4+C_7)$. *G is a super vertex-magic graph iff k is odd.*

Proof. Let G be a graph $k(C_4+C_4+C_7)$. We have $v = 15k$ and $e = 15k$.

(⇒) Assume that G is a super vertex-magic graph.

By Theorem 1.2.1, the magic constant is

$$\frac{(v+e)(v+e+1)}{v}-\frac{v+1}{2}=\frac{(30k)(30k+1)}{15k}-\frac{15k+1}{2}$$
$$=60k+2-\frac{15k+1}{2}=\frac{105k+3}{2}.$$

Then k is odd.

(⇐) Assume that k is odd.

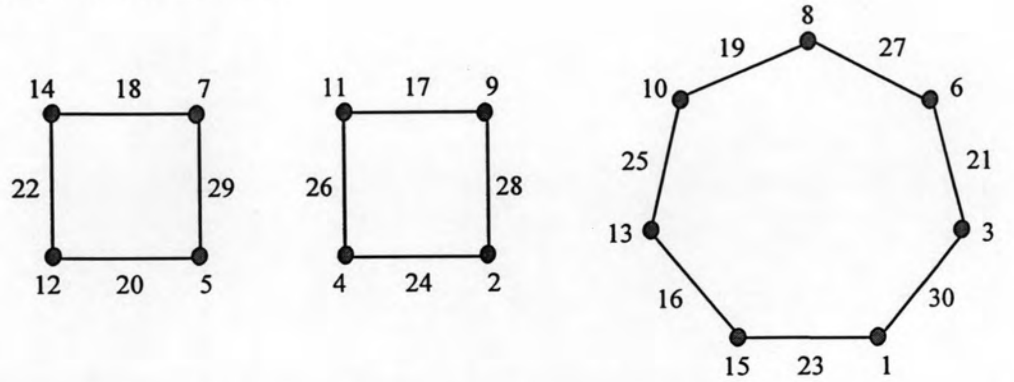

Figure 3.16 : Super vertex-magic graph $C_4 + C_4 + C_7$ with the magic constant 54

By the total labeling shown in Figure 3.16, $C_4 + C_4 + C_7$ is a super vertex-magic graph. Since k is odd and $C_4 + C_4 + C_7$ is 2-regular, $\frac{(k-1)(r+1)}{2} = \frac{3(k-1)}{2}$ is an integer. By Theorem 2.12, G is a super vertex-magic graph. □

Example 3.29. ([6]) Super vertex-magic regular graphs with the magic constant 48 are shown in Figure 3.17.

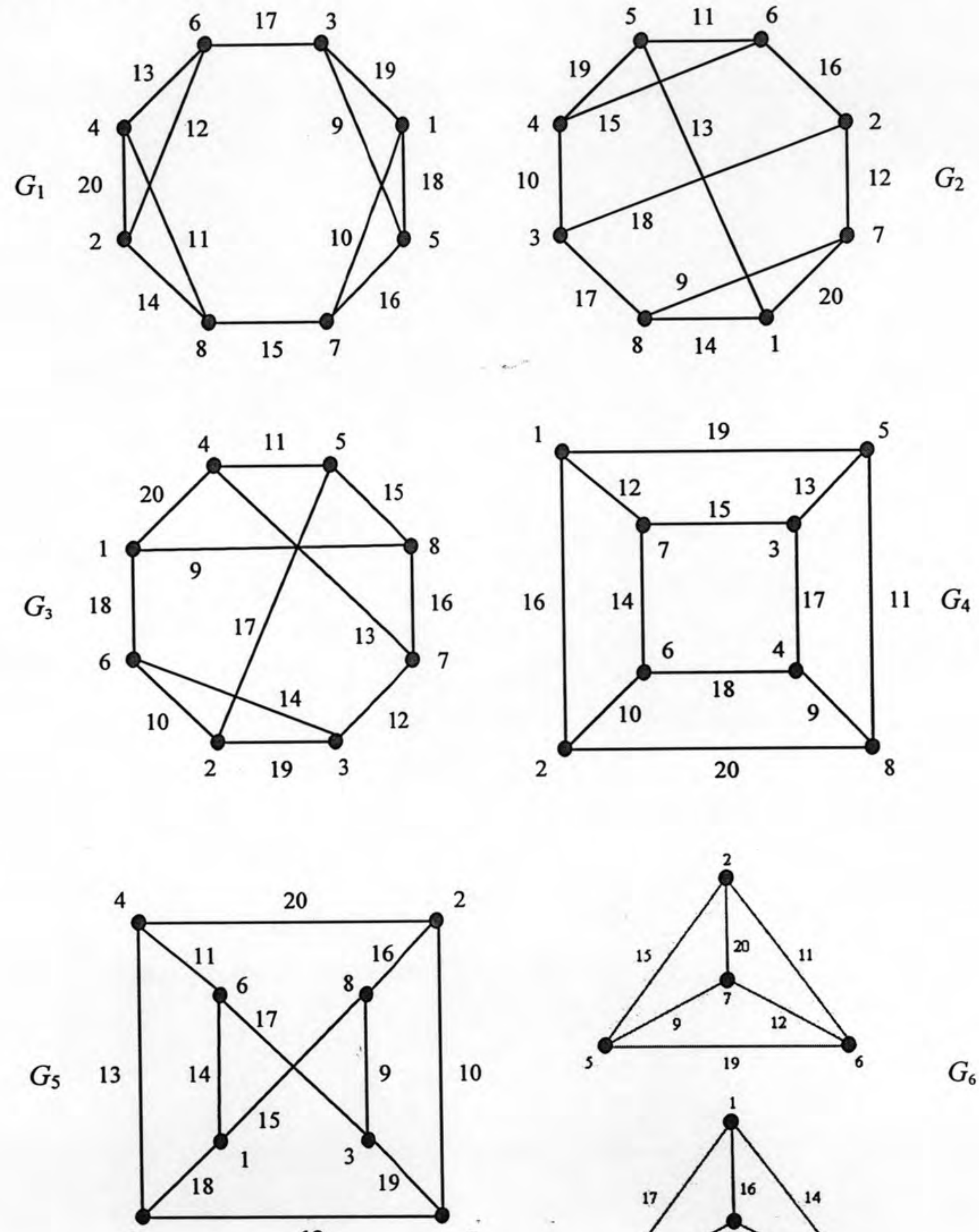

Figure 3.17 : Super vertex-magic regular graphs with the magic constant 48

Theorem 3.30. *If k is a positive integer and* G_1 *or* G_2 *or* G_3 *or* G_4 *or* G_5 *is a graph in Figure* 3.17, *then* kG_1 *or* kG_2 *or* kG_3 *or* kG_4 *or* kG_5 *is a super vertex-magic graph.*

Proof. Let k be a positive integer.

Since each graph is 3-regular, $\frac{(k-1)(r+1)}{2} = \frac{4(k-1)}{2} = 2(k-1)$ is an integer.

By Theorem 2.12, kG_1 or kG_2 or kG_3 or kG_4 or kG_5 or kG_6 are super vertex-magic graphs. □

Example 3.31. ([1]) Super vertex-magic non-regular graphs with the magic constant 66 and 112 are shown in Figure 3.18.

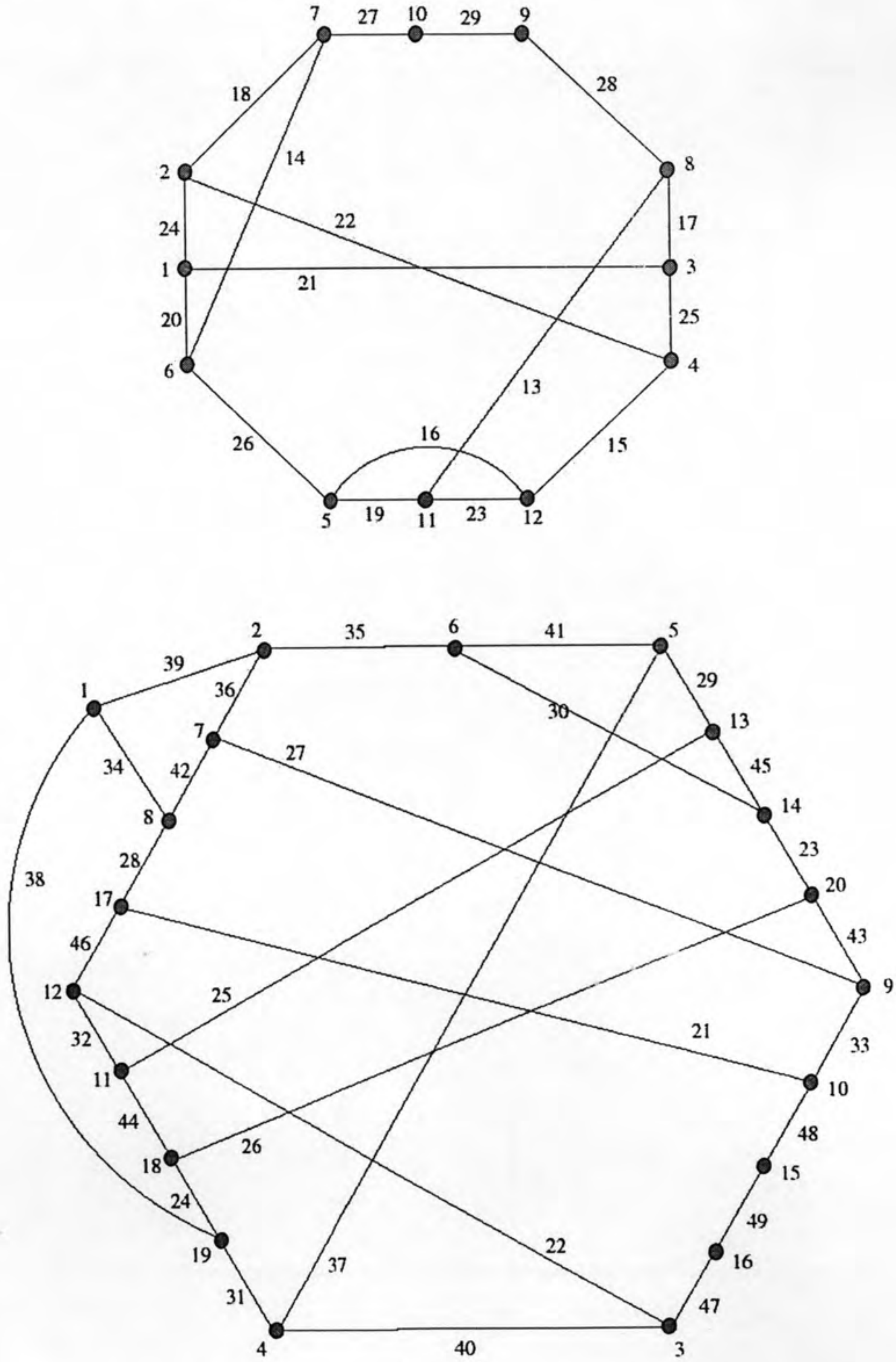

Figure 3.18 : Super vertex-magic non-regular graphs with the magic constant 66 and 112