การเป็นปกติและทฤษฎีบทสมสัณฐานของกึ่งกรุปการแปลงที่รักษาอันดับบางชนิด

นางสาวเพ็ญนภา รุ่งรัตน์ตระกูล

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REGULARITY AND ISOMORPHISM THEOREMS OF SOME ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

Miss Pennapa Rungrattrakoon

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……………………………………..Dean of the Faculty of Science

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Thesis Committee

....Chairman

(Assistant Professor Patanee Udomkavanich, Ph.D.)

....Thesis Advisor

(Professor Yupaporn Kemprasit, Ph.D.)

....Member

(Assistant Professor Ajchara Harnchoowong, Ph.D.)

เพ็ญนภา รุ่งรัตน์ตระกูล : การเป็นปกติและทฤษฎีบทสมสัณฐานของกึ่งกรุปการแปลงที่รักษาอันดับบางชนิด (REGULARITY AND ISOMORPHISM THEOREMS OF SOME ORDER–PRESERVING TRANSFORMATION SEMIGROUPS)
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สำหรับโพเซต X ให้ $OT(X)$, $OP(X)$ และ $OI(X)$ แทนกึ่งกรุปการแปลงเต็มที่รักษาอันดับบน X กึ่งกรุปการแปลงบางส่วนที่รักษาอันดับบน X และกึ่งกรุปการแปลงบางส่วนหนึ่งต่อหนึ่งที่รักษาอันดับบน X ตาม ลำดับ ความจริงในเรื่องการเป็นปกติเกี่ยวกับกึ่งกรุปการแปลงที่รักษาอันดับต่อไปนี้เป็นที่ทราบกันแล้ว สำหรับเซตย่อยอันดับทุกส่วน X ของ Z ใดๆ $OT(X)$ เป็นกึ่งกรุปปกติ และสำหรับช่วง X ใน R $OT(X)$ เป็นกึ่งกรุปปกติ ก็ต่อเมื่อ X เป็นช่วงปิดและมีขอบเขต กึ่งกรุป $OP(X)$ และ $OI(X)$ เป็นกึ่งกรุปปกติสำหรับเซตอันดับทุกส่วน X ใดๆ ทฤษฎีบทสมสัณฐานที่น่าสนใจบทหนึ่งเกี่ยวกับกึ่งกรุปการแปลงเต็มที่รักษาอันดับคือ สำหรับโพเซต X และ Y ใดๆ $OT(X) \cong OT(Y)$ ก็ต่อเมื่อ X และ Y ไม่สมสัณฐานอันดับกันก็ปฏิสมสัณฐานอันดับกัน

จุดมุ่งหมายของเราคือให้ผลที่มากขึ้นเกี่ยวกับการเป็นปกติและทฤษฎีบทสมสัณฐานของกึ่งกรุปการแปลงที่รักษาอันดับ ขั้นแรกเราแสดงให้เห็นว่า สำหรับช่วง X ในฟิลด์ย่อย F ของ R ซึ่ง $|X| > 1$ $OT(X)$ เป็นกึ่งกรุปปกติ ก็ต่อเมื่อ $F = R$ และ X เป็นช่วงปิดและมีขอบเขต ขั้นต่อไปเราได้พิจารณากึ่งกรุปย่อยของกึ่งกรุป $OT(X)$, $OP(X)$ และ $OI(X)$ ตามลำดับต่อไปนี้ $OT(X, X') = \{\alpha \in OT(X) \mid \operatorname{ran} \alpha \subseteq X'\}$, $OP(X, X') = \{\alpha \in OP(X) \mid \operatorname{ran} \alpha \subseteq X'\}$ และ $OI(X, X') = \{\alpha \in OI(X) \mid \operatorname{ran} \alpha \subseteq X'\}$ โดยที่ X' เป็นเซตย่อยอันดับทุกส่วนของเซตอันดับทุกส่วน X เราจะบอกลักษณะว่าเมื่อใด $OT(X, X')$ เป็นกึ่งกรุปปกติในเทอมของ X, X' และการเป็นปกติของ $OT(X)$ ทั้งยังพิสูจน์ด้วยว่า $X = X'$ เป็นเงื่อนไขที่จำเป็นและเพียงพอสำหรับ $OP(X, X')$ และ $OI(X, X')$ ที่จะเป็นกึ่งกรุปปกติ ทฤษฎีบทสมสัณฐานของกึ่งกรุปการแปลงที่รักษาอันดับที่น่าสนใจที่ได้ในงานวิจัยนี้มีดังต่อไปนี้ ถ้า $OT(X, X') \cong OT(Y, Y')$ แล้ว X' และ Y' ไม่สมสัณฐานอันดับกันก็ปฏิสมสัณฐานอันดับกัน ถ้า $OP(X, X') \cong OP(Y, Y')$ แล้ว $|X'| = |Y'|$ และ X' และ Y' ไม่สมสัณฐานอันดับกันก็ปฏิสมสัณฐานอันดับกัน ยิ่งไปกว่านั้นสำหรับ $|X'| > 1$ และ $|Y'| > 1$ $OI(X, X') \cong OI(Y, Y')$ ก็ต่อเมื่อ มีสมสัณฐานอันดับหรือปฏิสมสัณฐานอันดับ $\theta : X \rightarrow Y$ โดยที่ $X'\theta = Y'$ ทฤษฎีบทสมสัณฐานแรกของเรานี้เป็นการขยายทฤษฎีบทสมสัณฐานที่รู้กันแล้วดังกล่าวข้างต้นในกรณีของเซตอันดับทุกส่วน เราได้แสดงด้วยว่าบทกลับของทฤษฎีบทสมสัณฐานสองทฤษฎีบทแรกไม่จริงโดยทั่วไป อย่างไรก็ตาม ผลตามมาที่น่าสนใจของทฤษฎีบทสมสัณฐานสองทฤษฎีบทนี้คือสำหรับเซตอันดับทุกส่วน X และ Y ใดๆ $OP(X) \cong OP(Y)$ $[OI(X) \cong OI(Y)]$ ก็ต่อเมื่อ X และ Y ไม่สมสัณฐานอันดับกันก็ปฏิสมสัณฐานอันดับกัน

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For a poset X, let $OT(X)$, $OP(X)$ and $OI(X)$ denote respectively the full order-preserving transformation semigroup on X, the order-preserving partial transformation semigroup on X and the order-preserving one-to-one partial transformation semigroup on X. The following facts of regularity of order-preserving transformation semigroups are known. For any subchain X of $\mathbb{Z}$, $OT(X)$ is regular, and for an interval X in $\mathbb{R}$, $OT(X)$ is regular if and only if X is closed and bounded. The semigroups $OP(X)$ and $OI(X)$ are regular for any chain X. An interesting isomorphism theorem of full order-preserving transformation semigroups is that for posets X and Y, $OT(X) \cong OT(Y)$ if and only if X and Y are either order-isomorphic or anti-order-isomorphic.

Our purpose is to give more results of regularity and isomorphism theorems of order-preserving transformation semigroups. First, we show that for a nontrivial interval X in a subfield F of $\mathbb{R}$, $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded. Next, the following respective subsemigroups of $OT(X)$, $OP(X)$ and $OI(X)$ are considered. $OT(X, X') = \{\alpha \in OT(X) \mid \operatorname{ran} \alpha \subseteq X'\}$, $OP(X, X') = \{\alpha \in OP(X) \mid \operatorname{ran} \alpha \subseteq X'\}$ and $OI(X, X') = \{\alpha \in OI(X) \mid \operatorname{ran} \alpha \subseteq X'\}$ where X' is a subchain of a chain X. We characterize when $OT(X, X')$ is regular in terms of X, X' and the regularity of $OT(X)$. It is proved that $X = X'$ is necessary and sufficient for $OP(X, X')$ and $OI(X, X')$ to be regular. The interesting isomorphism theorems of order-preserving transformation semigroups obtained in this research are as follows: If $OT(X, X') \cong OT(Y, Y')$, then X' and Y' are either order-isomorphic or anti-order-isomorphic. If $OP(X, X') \cong OP(Y, Y')$, then $|X'| = |Y'|$ and X' and Y' are either order-isomorphic or anti-order-isomorphic. Moreover, for $|X'| > 1$ and $|Y'| > 1$, $OI(X, X') \cong OI(Y, Y')$ if and only if there is an order-isomorphism or an anti-order-isomorphism $\theta : X \to Y$ such that $X'\theta = Y'$. Our first isomorphism theorem is an extension of the above known isomorphism theorem for the case of chains. We also show that the converses of our first two isomorphism theorems are not generally true. However, interesting consequences of these two isomorphism theorems are as follows: For any chains X and Y, $OP(X) \cong OP(Y)$ [$OI(X) \cong OI(Y)$] if and only if X and Y are either order-isomorphic or anti-order-isomorphic.

Department **Mathematics** Student's signature.......................................

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote respectively the set of all real numbers, the set of all rational numbers, the set of all integers and the set of all natural numbers and the partial order on any nonempty subset of $\mathbb{R}$ means the natural partial order on $\mathbb{R}$.

For a set X, let $|X|$ denote the cardinality of X and $\mathcal{P}(X)$ denote the power set of X. In this reseach, we use the Generalized Continuum Hypothesis. Then for any sets X and Y, if $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$, then $|X| = |Y|$.

An element a of a semigroup is called an *idempotent* if $a^2 = a$. The set of all idempotents of a semigroup S is denoted by $E(S)$, that is,

$$E(S) = \{x \in S \mid x^2 = x\}.$$

An *idempotent semigroup* or a *band* is a semigroup S in which $x^2 = x$ for every $x \in S$, that is, $E(S) = S$. An element a of a semigroup S is said to be *regular* if $a = aba$ for some $b \in S$ and S is called a *regular semigroup* if every element of S is regular. If $a, b \in S$ are such that $a = aba$, then $a = a(bab)a$ and $bab = (bab)a(bab)$. Hence for $a \in S$, a is regular if and only if there is an element $c \in S$ such that $a = aca$ and $c = cac$, and c is called an *inverse* of a in S. Thus S is a regular semigroup if and only if every element of S has an inverse in S. Then every idempotent semigroup is regular.

Let X be a set. We call a map α from a subset of X into X *a partial transformation* of X, and if domain of α is X, then α is a *transformation* of X. We let 0 denote the mapping with empty domain. Then 0 is a partial transformation of X which called the *empty transformation*.

The domain and the range of a partial transformation of X will be denoted respectively by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ and the image of x in the domain of α under α is written by $x\alpha$. The identity mapping on a nonempty set A is denoted by 1_A and for $x \in X$ and $\varnothing \neq A \subseteq X$, let A_x denote the constant map whose domain and range are A and $\{x\}$, respectively.

Let $P(X)$, $T(X)$ and $I(X)$ denote the set of all partial transformations of X, the set of all transformations of X and the set of all 1-1 partial transformations of X, respectively, that is,

$$\begin{aligned} P(X) &= \{\alpha : A \to X \mid A \subseteq X\}, \\ T(X) &= \{\alpha \in P(X) \mid \operatorname{dom}\alpha = X\}, \\ I(X) &= \{\alpha \in P(X) \mid \alpha \text{ is 1-1}\}. \end{aligned}$$

We can see that all of $P(X)$, $T(X)$ and $I(X)$ contain 1_X and 0 is contained in $P(X)$ and $I(X)$ but not in $T(X)$ if $X \neq \varnothing$ and $T(X)$ and $I(X)$ are subsets of $P(X)$.

For $\alpha, \beta \in P(X)$, let $\alpha\beta$ be the composition of α and β, that is, $\alpha\beta = 0$ if $\operatorname{ran}\alpha \cap \operatorname{dom}\beta = \varnothing$, and otherwise, $\alpha\beta = \alpha|_{(\operatorname{ran}\alpha\cap\operatorname{dom}\beta)\alpha^{-1}}\beta|_{(\operatorname{ran}\alpha\cap\operatorname{dom}\beta)}$, the composition of α restricted to $(\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}$ and β restricted to $\operatorname{ran}\alpha \cap \operatorname{dom}\beta$. Then $P(X)$ under the composition defined above is a semigroup having $T(X)$ and $I(X)$ as subsemigroups. Observe that for $\alpha, \beta \in P(X)$,

$$\begin{aligned} \operatorname{dom}(\alpha\beta) &= (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha, \\ \operatorname{ran}(\alpha\beta) &= (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta, \end{aligned}$$

$$\text{for } x \in X,\ x \in \operatorname{dom}(\alpha\beta) \iff x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta.$$

The semigroups $P(X)$, $T(X)$ and $I(X)$ are called the *partial transformation semigroup* on X, the *full transformation semigroup* on X and the *1-1 partial transformation semigroup* or *the symmetric inverse semigroup* on X, respectively. It is well-known that all the semigroups $P(X)$, $T(X)$ and $I(X)$ are regular ([3], page 4). Moreover, for $\alpha \in P(X)$, $\alpha^2 = \alpha$ if and only if $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Hence $X_a \in E(T(X))$ for all $a \in X$ and for a nonempty subset A of X and $x \in X$, $A_x \in E(P(X))$ if and only if $x \in A$. In particular,

$$\begin{aligned} E(T(X)) &= \{\alpha \in T(X) \mid x\alpha = x \text{ for all } x \in \operatorname{ran}\alpha\}, \\ E(I(X)) &= \{1_A \mid \varnothing \neq A \subseteq X\} \cup \{0\}. \end{aligned}$$

For convenience, we may use a bracket notation to define a mapping in $P(X)$. For examples,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stands for $\alpha \in P(X)$ defined by $\operatorname{dom}\alpha = \{a, b\}$ and $a\alpha = c$ and $b\alpha = d$,

$\begin{pmatrix} A & x \\ a & x \end{pmatrix}_{x \in X \smallsetminus A}$ stands for $\beta \in T(X)$ defined by $x\alpha = \begin{cases} a & \text{if } x \in A, \\ x & \text{if } x \in X \smallsetminus A. \end{cases}$

By the above notation representing an element of $P(X)$, we have that for any $\alpha \in P(X) \smallsetminus \{0\}$, $\alpha = \begin{pmatrix} x \\ x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha}$.

The full transformation semigroup $T(X)$ is considered very important. In 1975, J. S. V. Symons [8] considered the semigroup $T(X, X')$, $\varnothing \neq X' \subseteq X$, under composition comprised of all mappings in $T(X)$ whose ranges are contained in X', that is,

$$T(X, X') = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq X'\}.$$

Then $T(X,X')$ is a subsemigroup of $T(X)$ containing X_a for every $a \in X'$. Since $T(X,X) = T(X)$, $T(X,X')$ can be counted as a generalization of $T(X)$. J. S. V. Symons studied in [8] the automorphisms of $T(X,X')$ and being isomorphic of two $T(X,X')$. In fact, in 1966, K. D. Magrill Jr. [7] has studied the semigroup

$$\overline{T}(X,X') \ = \ \{\alpha \in T(X) \mid X'\alpha \subseteq X'\}$$

which clearly contains $T(X,X')$ defined above. Also, if $X' = X$, then $\overline{T}(X,X') = T(X)$, then $\overline{T}(X,X')$ can be also considered as a generalization of $T(X)$.

In this research, the semigroups $P(X,X')$ and $I(X,X')$ are defined similarly, that is,

$$\begin{aligned} P(X,X') \ &= \ \{\alpha \in P(X) \mid \operatorname{ran}\alpha \subseteq X'\}, \\ I(X,X') \ &= \ \{\alpha \in I(X) \mid \operatorname{ran}\alpha \subseteq X'\}. \end{aligned}$$

Then $P(X,X')$ and $I(X,X')$ are respectively subsemigroups of $P(X)$ and $I(X)$ containing 0 and $1_{X'}$. Also, since $P(X,X) = P(X)$ and $I(X,X) = I(X)$, we can also count $P(X,X')$ as a generalization of $P(X)$ and $I(X,X')$ as a generalization of $I(X)$.

By a *subchain* of a poset X we mean a subposet of X which is also a chain.

For posets X and Y, the map $\varphi : X \to Y$ is said to be *order-preserving* if

$$\text{for all } \ a,b \in X,\ a \leq b \text{ in } X \ \Rightarrow \ a\varphi \leq b\varphi \text{ in } Y,$$

and we call φ an *order-isomorphism* of X onto Y if φ is a bijection of X onto Y and both φ and φ^{-1} are order-preserving. Hence a bijection $\varphi : X \to Y$ is an order-isomorphism if and only if

$$\text{for all } \ a,b \in X,\ a \leq b \text{ in } X \ \Leftrightarrow \ a\varphi \leq b\varphi \text{ in } Y.$$

The posets X and Y are said to be *order-isomorphic* if there is an order-isomorphism of X onto Y. It is clear that if X and Y are chains, then φ is an order-isomorphism of X onto Y if and only if φ is an order-preserving bijection of X onto Y. Naturally, a bijection $\varphi : X \to Y$ satisfying the condition

$$\text{for } a, b \in X,\ a \leq b \text{ in } X \ \Leftrightarrow\ b\varphi \leq a\varphi \text{ in } Y$$

is called an *anti-order-isomorphism.* We say that X and Y are *anti-order-isomorphic* if there is an anti-order-isomorphism from X onto Y.

For a poset X, we say that $\alpha \in P(X)$ is *order-preserving* if

$$\text{for } a, b \in \operatorname{dom}\alpha,\ a \leq b \ \Rightarrow\ a\alpha \leq b\alpha$$

and let $OP(X)$ denote the set of all order-preserving transformations in $P(X)$, that is,

$$OP(X) = \{\alpha \in P(X) \mid \alpha \text{ is order-preserving}\}.$$

Then $OP(X)$ is clearly a subsemigroup of $P(X)$ containing 0 and 1_X and $OP(X)$ is called the *order-preserving partial transformation semigroup* on X. Similarly, we define

$$OT(X) = \{\alpha \in T(X) \mid \alpha \text{ is order-preserving}\},$$

$$OI(X) = \{\alpha \in I(X) \mid \alpha \text{ is order-preserving}\}.$$

Also, $OT(X)$ and $OI(X)$ are respectively subsemigroups of $T(X)$ and $I(X)$, $1_X \in OT(X)$ and $0, 1_X \in OI(X)$. The semigroups $OT(X)$ and $OI(X)$ are called the *full order-preserving transformation semigroup* on X and the *order-preserving 1-1 partial transformation semigroup* on X, respectively.

Intervals in a chain are defined naturally as follows : A nonempty subset Y of a chain X is called an *interval* in X if for a, b, $x \in X$, a, $b \in Y$ and $a \leq x \leq b$ imply that $x \in Y$. We say that an interval Y in X is a *nontrivial interval* if Y contains more than one element. Since every subfield F of $\mathbb{R}$ contains $\mathbb{Q}$, it follows that every nontrivial interval X of F is infinite.

It is well-known [3, page 203] that the semigroup $OT(X)$ is regular if X is a finite chain. In 2000, Y. Kemprasit and T. Changphas [5] extended this results by showing that $OT(X)$ is regular for any chain which is order-isomorphic to a subchain of $\mathbb{Z}$. In particular, the following result is obtained.

Theorem 1.1. ([5]). *For any nonempty subset X of $\mathbb{Z}$, $OT(X)$ is a regular semigroup.*

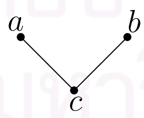
Moreover, they also proved that for an interval X in $\mathbb{R}$, being closed and bounded of X is necessary and sufficient for $OT(X)$ to be regular and for any chain X, $OP(X)$ and $OI(X)$ are always regular.

Theorem 1.2. ([5]). *For an interval X in $\mathbb{R}$, $OT(X)$ is a regular semigroup if and only if X is closed and bounded.*

Theorem 1.3. ([5]). *For any chain X, the semigroups $OP(X)$ and $OI(X)$ are regular.*

The following example shows that Theorem 1.3 need not be true if X is a poset which is not a chain.

Example 1.4. Let X be a poset defined by the Hasse diagram as follows :



Define $\alpha = \begin{pmatrix} a & b \\ c & b \end{pmatrix}$. Then $\alpha \in OI(X)$ and suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in OP(X)$. Then $c = a\alpha = a\alpha\beta\alpha = (c\beta)\alpha$ and $b = b\alpha = b\alpha\beta\alpha = (b\beta)\alpha$, so by the definition of α, $c\beta = a$ and $b\beta = b$. But $c < b$ and $c\beta$ and $b\beta$ are not comparable, so β is not order-preserving. This is a contradiction. This shows that both $OP(X)$ and $OI(X)$ are not regular.

In passsing, we note here that in 1970, J. M. Howie [4] showed that if X

is a finite chain, $OT(X)$ is also idempotent generated or equivalently, for every $\alpha \in OT(X)$, $\alpha = \delta_1\delta_2 \dots \delta_k$ for some $\delta_1, \delta_2, \dots, \delta_k \in E(OT(X))$. In 1981, C. C. Edwards and M. Anderson [1] considered the semigroup $S(X)$ consisting of all order-preserving transformations α whose domains are *final segments* in a chain X, that is, $x \in \operatorname{dom} \alpha$ and $x \leq y \in X$ imply $y \in \operatorname{dom} \alpha$ and they observed that $S(X)$ need not be regular. V. H. Fernandes noted in [2] in 1997 that $OI(X)$ is a regular semigroup if X is a finite chain. This result becomes a special case of Theorem 1.3.

An important isomorphism theorem of full order-preserving transformation semigroups given in the book named " Semigroups " written by E. S. Lyapin [6] is as follows :

Theorem 1.5. ([6, page 222-223]) *For posets X and Y, $OT(X) \cong OT(Y)$ if and only if X and Y are either order-isomorphic or anti-order-isomorphic.*

The converse of Theorem 1.5 is obtained from the following natural fact. It is mentioned that it is easy in [6], page 222 and the isomorphism of $OT(X)$ onto $OT(Y)$ is not provided.

Proposition 1.6. *Let X and Y be posets and $\varphi : X \to Y$. If φ is either an order-isomorphism or an anti-order-isomorphism, then the map $\alpha \mapsto \varphi^{-1}\alpha\varphi$ is an isomorphism of $OT(X)$ onto $OT(Y)$.*

Proof. Let $\alpha \in OT(X)$ and let $a, b \in X$ be such that $a \leq b$. If φ is an order-isomorphism, then φ^{-1}, α and φ are order-preserving, and thus $\varphi^{-1}\alpha\varphi$ is order-preserving. If $\beta \in OT(Y)$, then $\varphi\beta\varphi^{-1} \in OT(X)$ and $\varphi^{-1}(\varphi\beta\varphi^{-1})\varphi = \beta$. Since φ is a bijection, $\alpha \mapsto \varphi^{-1}\alpha\varphi$ is a 1-1 map.

For the case that φ is an anti-order-isomophism, we have that for $\alpha \in OT(X)$,

$$\begin{aligned} \text{for } c, d \in Y, c \le d \ &\Rightarrow\ c\varphi^{-1} \ge d\varphi^{-1} \\ &\Rightarrow\ c\varphi^{-1}\alpha \ge d\varphi^{-1}\alpha \\ &\Rightarrow\ c\varphi^{-1}\alpha\varphi \le d\varphi^{-1}\alpha\varphi. \end{aligned}$$

Hence $\alpha \mapsto \varphi^{-1}\alpha\varphi$ is a map from $OT(X)$ onto $OT(Y)$. We show analogously as above that map is also onto and 1-1 $\square$

It is easily seen that for finite chains X and Y, X and Y are order-isomorphic [anti-order-isomorphic] if and only if $|X| = |Y|$. Hence from Theorem 1.5, we have

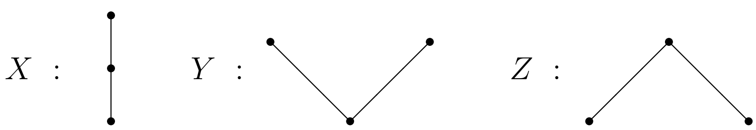
Corollary 1.7. *For finite chains X and Y, $OT(X) \cong OT(Y)$ if and only if $|X| = |Y|$.*

Example 1.8. (1) For $n \in \mathbb{N}$, the map $x \mapsto nx$ [$x \mapsto -nx$] is an order-isomorphism [anti-order-isomorphism] of $\mathbb{Z}$ onto $n\mathbb{Z}$, so by Theorem 1.5, $OT(\mathbb{Z}) \cong OT(n\mathbb{Z})$.

(2) Let $\mathbb{Z}^+$ and $\mathbb{Z}^-$ be the set of positive integers and the set of negative integers, respectively (that is, $\mathbb{Z}^+ = \mathbb{N}$). Since the map $x \mapsto -x$ is an anti-order-isomorphism of $\mathbb{Z}^+$ onto $\mathbb{Z}^-$, from Theorem 1.5, $OT(\mathbb{Z}^+) \cong OT(\mathbb{Z}^-)$.

(3) Since $\mathbb{Z}$ has neither a maximum nor a minimum while $\mathbb{Z}^+$ has a minimum, we deduce that $\mathbb{Z}$ and $\mathbb{Z}^+$ are neither order-isomorphic nor anti-order-isomorphic. Hence $OT(\mathbb{Z})$ and $OT(\mathbb{Z}^+)$ are not isomorphic.

(4) Let X, Y and Z be posets as shown by the following Hasse diagrams.



Then X is neither order-isomorphic nor anti-order-isomorphic to Y and Z but Y and Z are anti-order-isomorphic. We therefore have from Theorem 1.5 that

$OT(X) \ncong OT(Y) \cong OT(Z)$. Observe that $|X| = |Y| = |Z| = 3$. This example also shows that Corollary 1.7 is not generally true for finite posets.

Based on the semigroup $T(X, X')$ introduced by J. S. V. Synmons [8] and those $P(X, X')$ and $I(X, X')$ mentioned previously for a set X and $\varnothing \neq X' \subseteq X$, the following semigroups $OT(X, X')$, $OP(X, X')$ and $OI(X, X')$ are defined similarly to generalize $OT(X)$, $OP(X)$ and $OI(X)$, respectively where X' is a subposet of a poset X. That is,

$$\begin{aligned} OT(X, X') &= \{\alpha \in OT(X) \mid \operatorname{ran}\alpha \subseteq X'\}, \\ OP(X, X') &= \{\alpha \in OP(X) \mid \operatorname{ran}\alpha \subseteq X'\} \text{ and} \\ OI(X, X') &= \{\alpha \in OI(X) \mid \operatorname{ran}\alpha \subseteq X'\} \end{aligned}$$

which are respectively subsemigroups of $OT(X)$, $OP(X)$ and $OI(X)$. Also, $OT(X, X) = OT(X)$, $OP(X, X) = OP(X)$ and $OI(X, X) = OI(X)$. Notice that $X_a \in OT(X, X')$ for every $a \in X'$, $0 \in OP(X, X')$, $0 \in OI(X, X')$, $A_x \in OP(X, X')$ for every nonempty subset A of X and every $x \in X'$, and $A_x \in OI(X, X')$ if and only if $|A| = 1$.

Due to Theorem 1.1 and Theorem 1.2, it is natural to ask when $OT(X)$ is regular if X is an interval in $\mathbb{Q}$. To answer this question, a more extensive result is obtained in our study. We extend Theorem 1.2 by showing that for a nontrivial interval X in a subfield F of $\mathbb{R}$, $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded. An interesting consequence is that $OT(X)$ is not regular for any nontrivial interval X in $\mathbb{Q}$. This is our first purpose of Chapter II. A characterization of when $OT(X, X')$ is regular is given in terms of X and X' and the regularity of $OT(X)$ is our second purpose of Chapter II where X is a chain and X' is a subchain of X. From Theorem 1.3, one might expect that for any chain X and any subchain X' of X, $OP(X, X')$ and $OI(X, X')$ are regular. We show in the last part of this chapter that this is not true except $X = X'$. It is shown that $X = X'$ is a necessary and sufficient for $OP(X, X')$ and $OI(X, X')$ to be regular. Note that the sufficiency part is Theorem 1.3.

In Chapter III, many isomorphism theorems of $OT(X, X')$, $OP(X, X')$ and $OI(X, X')$ are provided where X' is a subchain of a chain X. The main isomorphism theorems obtained in this chapter are as follows : If $OT(X, X') \cong OT(Y, Y')$, then X' and Y' are either order-isomorphic or anti-order-isomorphic. This result generalizes Theorem 1.5 for chains. If $OP(X, X') \cong OP(Y, Y')$, then $|X| = |Y|$ and X' and Y' are either order-isomorphic or anti-order-isomorphic. Also, $OI(X, X') \cong OI(Y, Y')$ if and only if either $|X| = |Y|$ and $|X'| = |Y'| = 1$ or there is an order-isomorphism or an anti-order-isomorphism $\theta : X \to Y$ such that $X'\theta = Y'$. The converse of the first two isomorphism theorems are also shown to be not generally true. Some interesting consequences of our second and third isomorphism theorems are as follows : For chains X and Y, $OP(X) \cong OP(Y)[OI(X) \cong OI(Y)]$ if and only if X and Y are either order-isomorphic or anti-order-isomorphic.

CHAPTER II

REGULAR ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

This chapter deals with the regularity of our target order-preserving transformation semigroups. We characterize when they are regular.

2.1 Regularity of $OT(X)$ with X an Interval in a Subfield of $\mathbb{R}$

The purpose of this section is to extend Theorem 1.2 by showing that for a nontrivial interval X in a subfield F of $\mathbb{R}$ under usual addition and multiplication, $OT(X)$ is regular if and only if $F = \mathbb{R}$ and X is closed and bounded. Notice that if $|X| = 1$, then $|OT(X)| = 1$, so $OT(X)$ is trivially regular. First, we note that every subfield of $\mathbb{R}$ with usual addition and multiplication contains $\mathbb{Q}$ and there are infinitely many subfields of $\mathbb{R}$, namely, $\mathbb{Q}(\sqrt{p}) = \{ \, x + y\sqrt{p} \mid x, y \in \mathbb{Q} \, \}$ where $p \in P$ and P is the set of all positive prime numbers. In particular, the set of all algebraic numbers in $\mathbb{R}$ is a well-known proper subfield of $\mathbb{R}$.

To obtain the main result mentioned above, Theorem 1.2 and the following lemma are our main tools.

Lemma 2.1.1. *If F is a proper subfield of $\mathbb{R}$ and X is a nontrivial interval in F, then the semigroup $OT(X)$ is not regular.*

Proof. Let X be an interval in a proper subfield F of $\mathbb{R}$ such that $|X| > 1$.

Let $a, b \in X$ such that $a < b$. Since $\mathbb{Q} \subseteq F \subsetneq \mathbb{R}$, there exists an irrational number $c \in \mathbb{R} \setminus F$. Then $a - c < b - c$, and so $a - c < d < b - c$ for some $d \in \mathbb{Q}$. Consequently, $a < c + d < b$. But $d \in \mathbb{Q} \subseteq F$ and $c \notin F$, so $c + d \notin F$. Put $e = c + d$. Then

$$X = ((-\infty, a) \cap X) \cup ([a, e) \cap X) \cup ((e, \infty) \cap X) \tag{1}$$

and $\dfrac{a+e}{2} < e$. Define $\alpha : \mathbb{R} \to \mathbb{R}$ by

$$x\alpha = \begin{cases} x & \textit{if } x < a, \\ \dfrac{a+x}{2} & \textit{if } a \leq x \leq e, \\ x & \textit{if } x > e. \end{cases} \tag{2}$$

Then α is a 1-1 order-preserving map and $\operatorname{ran} \alpha = (-\infty, \dfrac{a+e}{2}] \cup (e, \infty)$. Let $\beta = \alpha_{|X}$, the restriction of α to X. Then β is 1-1 and order-preserving. Also, from (1) and (2), we have

$$\begin{aligned} \operatorname{ran} \beta &= ((-\infty, a) \cap X)\alpha \cup ([a, e) \cap X)\alpha \cup ((e, \infty) \cap X)\alpha \\ &= ((-\infty, a) \cap X) \cup ([a, e) \cap X)\alpha \cup ((e, \infty) \cap X). \end{aligned} \tag{3}$$

Since F is a field, $\mathbb{Q} \subseteq F$ and $a \in F$, it follows that $\dfrac{a+x}{2}$, $2x - a \in F$ for all $x \in F$. We claim that $([a, e) \cap X)\alpha = [a, \dfrac{a+e}{2}) \cap X$. Let $x \in [a, e) \cap X$. Then $a \leq x < e < b$ and $x \in X \subseteq F$, so

$$a \leq \frac{a+x}{2} = x\alpha < \frac{a+e}{2} < \frac{a+b}{2} < b$$

which implies that $x\alpha \in [a, \dfrac{a+e}{2}) \cap X$ since X is an interval in F. Thus $([a, e) \cap X)\alpha \subseteq [a, \dfrac{a+e}{2}) \cap X$. For the reverse inclusion, let $y \in [a, \dfrac{a+e}{2}) \cap X$. Then $a \leq 2y - a < e < b$, and hence $2y - a \in [a, e) \cap X$ and $(2y - a)\alpha = y$ by (2). Therefore we have the claim. It then follows from (3) that

$$\operatorname{ran} \beta = ((-\infty, a) \cap X) \cup ([a, \frac{a+e}{2}) \cap X) \cup ((e, \infty) \cap X). \tag{4}$$

Hence we have that $\beta \in OT(X)$. Suppose that $\beta = \beta\gamma\beta$ for some $\gamma \in OT(X)$. Since β is 1-1, $\beta\gamma = 1_X$. Consequently,

$$(\operatorname{ran}\beta)\gamma = X, \tag{5}$$

$$\gamma_{|\operatorname{ran}\beta} = \beta^{-1} \text{ which is a 1-1 map from } \operatorname{ran}\beta \text{ onto } X. \tag{6}$$

Let $f \in (\frac{a+e}{2}, e) \cap \mathbb{Q}$. Then $a < \frac{a+e}{2} < f < e < b$ and $f \in F$, so $f \in X$. We have from (5) that

$$g\gamma = f\gamma \quad \text{for some } g \in \operatorname{ran}\beta. \tag{7}$$

From (4), $g < \frac{a+e}{2}$ or $g > e$.

Case 1 : $g < \frac{a+e}{2}$. Then $g < \frac{a+e}{2} < f$. Let $p \in \mathbb{Q}$ be such that $g < p < \frac{a+e}{2}$. Thus $p \in X$ since $f, g \in X$. By (4), $p \in \operatorname{ran}\beta$. Since γ is order-preserving, $g\gamma \leq p\gamma \leq f\gamma$. We have from (7) that $g\gamma = p\gamma$ which is contrary to (6) because of $g, p \in \operatorname{ran}\beta$ with $g < p$.

Case 2 : $g > e$. Then $f < e < g$. Let $q \in \mathbb{Q}$ be such that $e < q < g$. Therefore $q \in X$ since $f < q < g$ and $f, g \in X$, and so $q \in \operatorname{ran}\beta$ from (4). Hence $f\gamma \leq q\gamma \leq g\gamma$ since $\gamma \in OT(X)$ and hence $q\gamma = g\gamma$ by (7). This contradicts (6).

This shows that β is not a regular element of $OT(X)$, and hence $OT(X)$ is not a regular semigroup. $\square$

Theorem 2.1.2. *For a nontrivial interval X in a subfield F of $\mathbb{R}$, $OT(X)$ is a regular semigroup if and only if $F = \mathbb{R}$ and X is closed and bounded.*

Proof. Let F be a subfield of $\mathbb{R}$ and X a nontrivial interval in F. Assume that the semigroup $OT(X)$ is regular. By Lemma 2.1.1, $F = \mathbb{R}$, and hence X is closed and bounded by Theorem 1.2.

The converse follows directly from Theorem 1.2. $\square$

The following corollary is a direct consequence of Theorem 2.1.2.

Corollary 2.1.3. *The semigroup $OT(X)$ is not regular for any nontrivial interval X in $\mathbb{Q}$.*

2.2 Regularity of $OT(X, X')$

For a poset X, we let $\min X$ and $\max X$ denote respectively the minimum and the maximum of X if they exist.

We give necessary and sufficient conditions for $OT(X, X')$ to be regular where X is a chain and X' is a subchain of X. For our required result, the following lemmas are needed.

Lemma 2.2.1. *If X is a poset whose minimum and maximum exist, then $OT(X, \{\min X, \max X\})$ is an idempotent semigroup (band).*

Proof. Let $\alpha \in OT(X, \{\min X, \max X\})$. Then $\operatorname{ran}\alpha = \{\min X\}$, $\operatorname{ran}\alpha = \{\max X\}$ or $\operatorname{ran}\alpha = \{\min X, \max X\}$. If $\operatorname{ran}\alpha = \{\min X\}$, then $\alpha = X_{\min X}$. Also, $\alpha = X_{\max X}$ if $\operatorname{ran}\alpha = \{\max X\}$. If $\operatorname{ran}\alpha = \{\min X, \max X\}$, then $(\min X)\alpha = \min X$ and $(\max X)\alpha = \max X$ since α is order-preserving. These imply that $x\alpha = x$ for all $x \in \operatorname{ran}\alpha$, and hence $\alpha^2 = \alpha$. $\square$

Lemma 2.2.2. *Let X be a chain. If $X' \subsetneq X$ and $|X'| \geq 3$, then the semigroup $OT(X, X')$ is not regular.*

Proof. Let $a, b, c \in X'$ be such that $a < b < c$ and let $d \in X \setminus X'$. Define $\alpha : X \to X'$ by

$$x\alpha = \begin{cases} a & \text{if } x < d, \\ b & \text{if } x = d \\ c & \text{if } x > d. \end{cases}$$

Then $\alpha \in OT(X, X')$. Let $\beta \in T(X)$ be such that $\alpha = \alpha\beta\alpha$. Thus

$$b \;=\; d\alpha \;=\; d\alpha\beta\alpha \;=\; (b\beta)\alpha$$

which implies by the definition of α that $b\beta = d$. But $d \in X \smallsetminus X'$, so $\beta \notin OT(X, X')$. Hence α is not a regular element of $OT(X, X')$. $\square$

Lemma 2.2.3. *Let X be a chain and assume that X has no minimum or maximum. If $X' \subseteq X$ and $|X'| = 2$, then the semigroup $OT(X, X')$ is not regular.*

Proof. Let $X' = \{a, b\}$ be such that $a < b$.

Case 1 : X has no minimum. Then there is an element $c \in X$ such that $c < a$. Let $\alpha : X \to X'$ be defined by

$$x\alpha = \begin{cases} a & \text{if } x < a, \\ b & \text{if } x \geq a. \end{cases}$$

Then $\alpha \in OT(X, X')$. If $\beta \in T(X)$ is such that $\alpha = \alpha\beta\alpha$, then

$$a \;=\; c\alpha \;=\; c\alpha\beta\alpha \;=\; (a\beta)\alpha,$$

so $a\beta < a$ from the definition of α and hence $\operatorname{ran}\beta \nsubseteq X'$. This shows that α is not a regular element of $OT(X, X')$.

Case 2 : X has no maximum. Then $b < d$ for some $d \in X$. Let $\lambda : X \to X'$ be defined by

$$x\lambda = \begin{cases} a & \text{if } x \leq b, \\ b & \text{if } x > b. \end{cases}$$

Then $\lambda \in OT(X, X')$. If $\mu \in T(X)$ is such that $\lambda = \lambda\mu\lambda$, then we have

$$b \;=\; d\lambda \;=\; d\lambda\mu\lambda \;=\; (b\mu)\lambda,$$

which implies that $b\mu > b$ and thus $\operatorname{ran}\mu \nsubseteq X'$. We thus deduce that λ is not a regular element of $OT(X, X')$. $\square$

Lemma 2.2.4. *Let X be a chain whose minimum and maximum exist, $X' \subseteq X$ and $|X'| \geq 2$. If the semigroup $OT(X, X')$ is regular, then* $\min X, \max X \in X'$.

Proof. Let $a, b \in X'$ be such that $a < b$. Define $\alpha, \beta : X \to X'$ by

$$x\alpha = \begin{cases} a & \textit{if } x = \min X, \\ b & \textit{if } x > \min X, \end{cases} \quad , \quad x\beta = \begin{cases} a & \textit{if } x < \max X, \\ b & \textit{if } x = \max X. \end{cases}$$

Then $\alpha, \beta \in OT(X, X')$. By the regularity of $OT(X, X')$, $\alpha = \alpha\lambda\alpha$ and $\beta = \beta\mu\beta$ for some $\lambda, \mu \in OT(X, X')$. Consequently,

$$a = (\min X)\alpha = (\min X)\alpha\lambda\alpha = (a\lambda)\alpha,$$

$$b = (\max X)\beta = (\max X)\beta\mu\beta = (b\mu)\beta.$$

We therefore deduce from the definitions of α and β that $a\lambda = \min X$ and $b\mu = \max X$. But since $\operatorname{ran}\lambda \subseteq X'$ and $\operatorname{ran}\mu \subseteq X'$, it follows that $\min X, \max X \in X'$, as required. $\square$

Theorem 2.2.5. *Let X be a chain and X' a subchain of X. Then the semigroup $OT(X, X')$ is regular if and only if one of the following statements holds.*

(i) $|X'| = 1$.

(ii) $X' = X$ *and* $OT(X)$ *is regular.*

(iii) The minimum and the maximum of X exist and $X' = \{\min X, \max X\}$.

Proof. If (i) holds, then $|OT(X, X')| = 1$, so $OT(X, X')$ is regular. If (ii) holds, then $OT(X, X') = OT(X)$ which is regular. It follows from Lemma 2.2.1 that $OT(X, X')$ is regular if (iii) is true. Therefore the sufficiency part is proved.

To prove neccessity, assume that the semigroup $OT(X, X')$ is regular and (i) and (ii) are false. Then $|X'| \geq 2$ and either $X' \subsetneq X$ or $OT(X)$ is not regular.

Case 1 : $|X'| \geq 2$ and $X' \subsetneq X$. Since $OT(X, X')$ is regular, it follows from Lemma 2.2.2 that $|X'| < 3$ and thus $|X'| = 2$. We therefore deduce from Lemma 2.2.3, the minimum and the maximum of X must exist. Also, by Lemma 2.2.4, $\min X, \max X \in X'$. Since $|X'| = 2$, $X' = \{\min X, \max X\}$. Hence (iii) holds.

Case 2 : $|X'| \geq 2$ and $OT(X)$ is not regular. Since $OT(X, X')$ is regular and $OT(X)$ is not regular, it follows that $X' \subsetneq X$. Thus $|X'| < 3$ because of Lemma 2.2.2. Hence $|X'| = 2$. Since $OT(X, X')$ is regular, we conclude from Lemma 2.2.3 that both the minimum and the maximum of X must exist. Then by Lemma 2.2.4, $\min X, \max X \in X'$. But $|X'| = 2$, thus $X' = \{\min X, \max X\}$ and hence (iii) holds. $\square$

The following corollary is a direct consequence of Theorem 1.1 and Theorem 2.2.5

Corollary 2.2.6. *Let X and X' be nonempty subsets of $\mathbb{Z}$ such that $X' \subseteq X$. Then the semigroup $OT(X, X')$ is regular if and only if one of the following statements holds.*

(i) $|X'| = 1$.

(ii) $X' = X$.

(iii) X is finite and $X' = \{\min X, \max X\}$.

Also, Theorem 2.1.2 and Theorem 2.2.5 yield the following result.

Corollary 2.2.7. *Let X be a nontrivial interval of a subfield F of $\mathbb{R}$ and X' a nonempty subset of X. Then $OT(X, X')$ is a regular semigroup if and only if one of the following statements holds.*

(i) $|X'| = 1$.

(ii) $X' = X$, $F = \mathbb{R}$ *and* X *is closed and bounded.*

(iii) The minimum and the maximum of X *exist and* $X' = \{\min X, \max X\}$.

Example 2.2.8. By Corollary 2.2.7, we have that $OT(\,[0,1] \cap \mathbb{Q}, \{\frac{1}{2}\}\,)$ and $OT(\,[0,1] \cap \mathbb{Q}, \{0,1\}\,)$ are regular while $OT(\,[0,1] \cap \mathbb{Q}, \{0, \frac{1}{2}\}\,)$ is not regular.

2.3 Regularity of $OP(X, X')$ and $OI(X, X')$

Recall that for any chain X, $OP(X)$ and $OI(X)$ are always regular (Theorem 1.3). We shall show that for any proper subchain X' of X, both $OP(X, X')$ and $OI(X, X')$ are not regular semigroups.

Theorem 2.3.1. . *Let* X *be a chain and* X' *a nonempty subchain of* X *and let* $S(X, X')$ *be* $OP(X, X')$ *or* $OI(X, X')$. *Then the semigroup* $S(X, X')$ *is regular if and only if* $X' = X$.

Proof. Assume that $S(X, X')$ is a regular semigroup. To prove that $X' = X$, suppose on the contrary that $X' \subsetneq X$. Let $a \in X \smallsetminus X'$ and $b \in X'$. Then $\begin{pmatrix} a \\ b \end{pmatrix} \in S(X, X')$, so

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \alpha \begin{pmatrix} a \\ b \end{pmatrix}$$

for some $\alpha \in S(X, X')$. Thus $\begin{pmatrix} a \\ b \end{pmatrix} \alpha \begin{pmatrix} a \\ b \end{pmatrix} \neq 0$ which implies that $b \in \operatorname{dom} \alpha$ and $b\alpha = a$. But $\alpha \in S(X, X')$, so $a \in \operatorname{ran} \alpha \subseteq X'$. This is a contrary to the choice of a.

The converse follows directly from Theorem 1.3. $\square$

Remark 2.3.2. We can see from the proof of Theorem 2.3.1 that the following result is true. For any posets X and any proper subposet X' of X, the semigroups $OP(X, X')$ and $OI(X, X')$ are not regular.

The next theorem yields the result that the domain of every regular element of $OI(X,X')$ does not contain any element of $X \smallsetminus X'$. Moreover, the set of all regular elements of $OI(X,X')$ and the set of all regular elements of $OI(X')$ are identical.

Theorem 2.3.3. *Let X be a poset and X' a subposet of X.*
(i) For $\alpha \in OI(X,X')$, if α is a regular element of $OI(X,X')$, then $\operatorname{dom}\alpha \subseteq X'$.
(ii) $\{\alpha \in OI(X,X') \mid \alpha \text{ is regular in } OI(X,X')\}$

$$= \{\alpha \in OI(X') \mid \alpha \text{ is regular in } OI(X')\}.$$

Proof. (i) Let $\alpha \in OI(X,X')$. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in OI(X,X')$. Then $\operatorname{ran}\alpha\beta \subseteq X'$ and

$$1_{\operatorname{dom}\alpha} = \alpha\alpha^{-1} = \alpha\beta\alpha\alpha^{-1} = \alpha\beta 1_{\operatorname{dom}\alpha}.$$

Consequently,

$$\begin{aligned}
\operatorname{dom}\alpha &= \operatorname{ran}(1_{\operatorname{dom}\alpha}) \\
&= \operatorname{ran}(\alpha\beta 1_{\operatorname{dom}\alpha}) \\
&= ((\operatorname{ran}\alpha\beta) \cap \operatorname{dom}(1_{\operatorname{dom}\alpha}))1_{\operatorname{dom}\alpha} \\
&= ((\operatorname{ran}\alpha\beta) \cap \operatorname{dom}\alpha)1_{\operatorname{dom}\alpha} \\
&= \operatorname{ran}\alpha\beta \cap \operatorname{dom}\alpha \\
&\subseteq \operatorname{ran}\alpha\beta \subseteq X'.
\end{aligned}$$

Hence (i) is proved.

(ii) Let $\alpha \in OI(X,X')$ be a regular element. Then α has an inverse in $OI(X,X')$, say β. Thus $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. It then follows from (i) that $\operatorname{dom}\alpha \subseteq X'$ and $\operatorname{dom}\beta \subseteq X'$. Hence $\beta \in OI(X')$, so α is a regular in $OI(X')$. This shows that

$$\{\alpha \in OI(X, X') \mid \alpha \text{ is regular in } OI(X, X')\}$$
$$\subseteq \{\alpha \in OI(X') \mid \alpha \text{ is regular in } OI(X')\}.$$

The reverse inclusion is obvious since $OI(X') \subseteq OI(X, X')$, so (ii) is obtained. $\square$

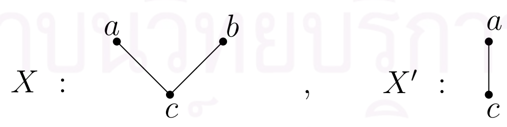
If X' is a chain, then by Theorem 1.3, $OI(X')$ is a regular semigroup. Also, if X' is *isolated*, that is, any two distinct elements of X' are not comparable, then $OI(X') = I(X')$ which is a regular semigroup. Due to these facts and Theorem 2.3.3(ii), the following consequence is obtained.

Corollary 2.3.4. *Let X be a poset and X' a subposet of X. Assume that X' is a chain or X' is isolated. Then*

$$\{\alpha \in OI(X, X') \mid \alpha \text{ is regular in } OI(X, X')\} = OI(X').$$

We note that Theorem 2.3.3 is not true if we replace $OI(X, X')$ by $OP(X, X')$ as shown by the following example.

Example 2.3.5. Let X be a poset and X' a subposet of X as shown by the following Hasse diagrams :



Define $\alpha \in OP(X, X')$ by $\alpha = \begin{pmatrix} b & c \\ c & c \end{pmatrix}$. Then $\operatorname{dom} \alpha = \{b, c\} \nsubseteq X'$ but $\begin{pmatrix} c \\ c \end{pmatrix} \in OP(X, X')$ and

$$\begin{pmatrix} b & c \\ c & c \end{pmatrix}\begin{pmatrix} c \\ c \end{pmatrix}\begin{pmatrix} b & c \\ c & c \end{pmatrix} = \begin{pmatrix} b & c \\ c & c \end{pmatrix}.$$

Example 2.3.6. Let X and X' be defined as in Example 2.3.5. Since X' is a subchain of X, by Corollary 2.3.4, $\{\alpha \in OI(X, X') \mid \alpha \text{ is regular in } OI(X, X')\} = OI(X')$. It is clear that

$$OI(X') = \left\{ 0, 1_{X'}, \begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} c \\ a \end{pmatrix} \right\},$$

so the number of all regular elements of $OI(X, X')$ is 6.

Remark 2.3.7. The assumption that X' is a chain or X' is isolated in Corollary 2.3.4 cannot be omitted. This clearly follows from the fact if X is a poset which is neither a chain nor an isolated poset, then $OI(X)$ need not be regular. Example 1.4 is an example for this case.

CHAPTER III

ISOMORPHISM THEOREMS OF ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

The purpose is to provide isomorphism theorems of any two of $OT(X, X')$, of $OP(X, X')$ and of $OI(X, X')$ for chains. In particular, Theorem 1.5 for chains is extended.

3.1 Some Elementary Results

In this section, some elementary results are provided and they will be referred later.

Proposition 3.1.1. *Let X be a chain and X' a subchain of X. Then $OT(X, X')$ has an identity if and only if $|X'| = 1$ or $X' = X$.*

Proof. Assume that $OT(X, X')$ has an identity, say η. Then $\alpha\eta = \eta\alpha = \alpha$ for all $\alpha \in OT(X, X')$. Suppose that $|X'| > 1$ and $X' \subsetneq X$. Let $a \in X \smallsetminus X'$. Then $a\eta \in X'$ and either $a\eta < a$ or $a < a\eta$. Since $|X'| > 1$, there is some $b \in X'$ such that $b \neq a\eta$. Then either $b < a\eta$ or $a\eta < b$.

Case 1 : $b < a\eta < a$. Let $\alpha \in OT(X, X')$ be defined by

$$x\alpha = \begin{cases} a\eta & \text{if } x \geq a, \\ b & \text{if } x < a. \end{cases}$$

Then $\eta\alpha = \alpha$, so $b = (a\eta)\alpha = a\alpha = a\eta$ which is a contradiction.

Case 2 : $a\eta < a$ and $a\eta < b$. Let $\beta \in OT(X, X^{'})$ be defined by

$$x\beta = \begin{cases} b & \textit{if } x \geq a, \\ a\eta & \textit{if } x < a. \end{cases}$$

Thus $\eta\beta = \beta$, and hence $a\eta = (a\eta)\beta = a\beta = b$, a contradiction.

Case 3 : $a < a\eta$ and $b < a\eta$. Let $\gamma \in OT(X, X^{'})$ be defined by

$$x\gamma = \begin{cases} a\eta & \textit{if } x > a, \\ b & \textit{if } x \leq a. \end{cases}$$

Then $\eta\gamma = \gamma$. This is contrary to that $a\eta = (a\eta)\gamma = a\gamma = b$.

Case 4 : $a < a\eta < b$. Let $\lambda \in OT(X, X^{'})$ be defined by

$$x\lambda = \begin{cases} b & \textit{if } x > a, \\ a\eta & \textit{if } x \leq a. \end{cases}$$

Therefore $\eta\lambda = \lambda$. This is a contradiction because $b = (a\eta)\lambda = a\lambda = a\eta$.

The converse is trivial. □

Proposition 3.1.2. *Let X be a poset and $X^{'}$ a subposet of X.*

(i) $OP(X, X^{'})$ has an identity if and only if $X^{'} = X$.

(ii) $OI(X, X^{'})$ has an identity if and only if $X^{'} = X$.

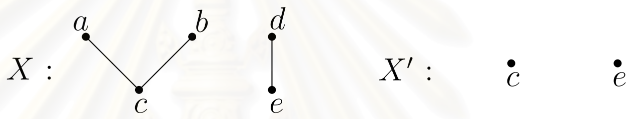
Proof. Let $S(X, X^{'})$ be $OP(X, X^{'})$ or $OI(X, X^{'})$ and let η be the identity of $S(X, X^{'})$. Then $\alpha\eta = \eta\alpha = \alpha$ for all $\alpha \in S(X, X^{'})$. Let $a \in X^{'}$ be fixed. Then $\begin{pmatrix} x \\ a \end{pmatrix} \in S(X, X^{'})$ for all $x \in X$, and hence $\eta\begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} x \\ a \end{pmatrix}$ for every $x \in X$.

This implies that $x \in \operatorname{dom} \eta\begin{pmatrix} x \\ a \end{pmatrix} \subseteq \operatorname{dom} \eta$ and $x\eta = x$ for all $x \in X$, thus $\operatorname{ran} \eta = X \subseteq X'$, that is, $X = X'$.

The converses of (i) and (ii) are trivial. $\square$

Due to Proposition 3.1.2, it is natural to ask whether Proposition 3.1.1 is still true if X is any poset. The following example gives a negative answer.

Example 3.1.3. Let X be a poset and X' a subposet of X defined by the Hasse diagram as follows :



Define $\eta,\ \alpha \in OT(X, X')$ as follows :

$$\eta = \begin{pmatrix} \{a, b, c\} & \{d, e\} \\ c & e \end{pmatrix}, \quad \alpha = \begin{pmatrix} \{a, b, c\} & \{d, e\} \\ e & c \end{pmatrix}.$$

Clearly, $OT(X, X') = \{X_c, X_e, \eta, \alpha\}$. Also, the multiplication on $OT(X, X')$ is as follows :

$\cdot$	X_c	X_e	η	α
X_c	X_c	X_e	X_c	X_e
X_e	X_c	X_e	X_e	X_c
η	X_c	X_e	η	α
α	X_c	X_e	α	η

This table shows that η is the identity of $OT(X, X')$.

From the proof of Proposition 1.6, the following result is obtained similarly.

Proposition 3.1.4. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. If $\varphi : X \to Y$ is an order-isomorphism or an anti-order-isomorphism such that $X'\varphi = Y'$, then the map $\overline{\varphi} : OP(X, X') \to OP(Y, Y')$ defined by $\alpha\overline{\varphi} = \varphi^{-1}\alpha\varphi$ for all $\alpha \in OP(X, X')$ is an isomorphism such that $(OT(X, X'))\overline{\varphi} = OT(Y, Y')$ and $(OI(X, X'))\overline{\varphi} = OI(Y, Y')$.*

3.2 Isomorphism Theorems of $OT(X, X')$

The purpose of this section is to generalize Theorem 1.5. To obtain the required theorem, the following lemma is required.

Lemma 3.2.1. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. If φ is an isomorphism of $OT(X, X')$ onto $OT(Y, Y')$, then the following statements hold.*

(i) For every $a \in X'$, there is an element $\overline{a} \in Y'$ such that $X_a\varphi = Y_{\overline{a}}$.

(ii) The map $a \mapsto \overline{a}$ is a bijection of X' onto Y'.

Proof. (i) Let $a \in X'$. Then $X_a \in OT(X, X')$ and $X_a\varphi \in OT(Y, Y')$. Let $\overline{a} \in \operatorname{ran}(X_a\varphi)$. Therefore $\overline{a} \in Y'$ and $Y_{\overline{a}} \in OT(Y, Y')$, so $\alpha\varphi = Y_{\overline{a}}$ for some $\alpha \in OT(X, X')$. Hence $\alpha X_a = X_a$. Since $X_a \in E(OT(X, X'))$, $X_a\varphi \in E(OT(Y, Y'))$. But $\overline{a} \in \operatorname{ran}(X_a\varphi)$, so $\overline{a}(X_a\varphi) = \overline{a}$. Consequently, $Y_{\overline{a}}(X_a\varphi) = Y_{\overline{a}}$ and thus

$$X_a\varphi \;=\; (\alpha X_a)\varphi \;=\; (\alpha\varphi)(X_a\varphi) \;=\; Y_{\overline{a}}(X_a\varphi) \;=\; Y_{\overline{a}}.$$

(ii) Since φ is one-to-one, the map $a \mapsto \overline{a}$ is a one-to-one map of X' into Y'. Because $\varphi^{-1} : OT(Y, Y') \to OT(X, X')$ is an isomorphism, from (i), we have that for any $b \in Y'$, $Y_b\varphi^{-1} = X_a$ for some $a \in X'$, so $Y_b = X_a\varphi = Y_{\overline{a}}$ which implies that $\bar{a} = b$. Hence (ii) holds. □

Theorem 3.2.2. *Let X and Y be chains, X' a subchain of X and Y' a subchain of Y. If $OT(X, X') \cong OT(Y, Y')$, then X' and Y' are either order-isomorphic or anti-order-isomorphic.*

Proof. Let $\varphi : OT(X, X') \to OT(Y, Y')$ be an isomorphism. By Lemma 3.2.1, for each $a \in X'$, there is an element $\overline{a} \in Y'$ be such that $X_a\varphi = Y_{\overline{a}}$. Define $\theta : X' \to Y'$ by $a\theta = \overline{a}$ for all $a \in X'$. Then by Lemma 3.2.1(ii), θ is a bijection from X' onto Y'. To show that θ is either order-isomorphism or anti-order-isomorphism, let $a, b, c, d \in X'$ such that $a < b$ and $c < d$. Since X' and Y' are chains and θ is one-to-one, it follows that $\overline{a} < \overline{b}$ or $\overline{a} > \overline{b}$ and $\overline{c} < \overline{d}$ or $\overline{c} > \overline{d}$. Define $\alpha : X \to X'$ by

$$x\alpha = \begin{cases} c & \textit{if } x < b, \\ d & \textit{if } x \geq b. \end{cases}$$

Then $\alpha \in OT(X, X')$, $X_a\alpha = X_c$ and $X_b\alpha = X_d$. Consequently,

$$\begin{aligned} Y_{\overline{a}}(\alpha\varphi) &= (X_a\varphi)(\alpha\varphi) = (X_a\alpha)\varphi = X_c\varphi = Y_{\overline{c}}, \\ Y_{\overline{b}}(\alpha\varphi) &= (X_b\varphi)(\alpha\varphi) = (X_b\alpha)\varphi = X_d\varphi = Y_{\overline{d}}, \end{aligned}$$

which imply that $\overline{a}(\alpha\varphi) = \overline{c}$ and $\overline{b}(\alpha\varphi) = \overline{d}$. Since $\alpha\varphi$ is order-preserving, we deduce that $\overline{a} < \overline{b}$ implies $\overline{c} < \overline{d}$ and $\overline{a} > \overline{b}$ implies $\overline{c} > \overline{d}$.

Therefore the theorem is proved. $\square$

Theorem 1.5 for chains follows directly from Theorem 3.2.2 and Proposition 3.1.4.

Corollary 3.2.3. For chains X and Y, $OT(X) \cong OT(Y)$ if and only if X and Y are either order-isomorphic or anti-order-isomorphic.

The converse of Theorem 3.2.2 is not generally true as shown by the following example.

Example 3.2.4. Let X be any chain and $X^{'}$ a proper subchain of X containing more than one element. By Proposition 3.1.1, $OT(X, X^{'})$ has no identity. But $OT(X^{'}, X^{'}) = OT(X^{'})$ has an identity, thus $OT(X, X^{'}) \ncong OT(X^{'}) = OT(X', X^{'})$.

From this example, it is natural to ask whether it is true that for a chain X and subchains X_1, X_2 of X, if X_1 and X_2 are either order-isomorphic or anti-order-isomorphic, then $OT(X, X_1) \cong OT(X, X_2)$. The following example gives a negative answer. The map $x \mapsto 2x$ is an order-isomorphism from $\mathbb{Z}$ onto $2\mathbb{Z}$. Also, $x \mapsto -2x$ is an anti-order-isomorphism from $\mathbb{Z}$ onto $2\mathbb{Z}$. Since $OT(\mathbb{Z}, 2\mathbb{Z})$ has no identity by Proposition 3.1.1, it follows that $OT(\mathbb{Z}, \mathbb{Z}) = OT(\mathbb{Z}) \ncong OT(\mathbb{Z}, 2\mathbb{Z})$.

In fact, Example 3.2.4 follows from the following general fact.

Corollary 3.2.5. *Let* X *and* Y *be chains and* $X^{'}$ *a subchain of* X*. Then* $OT(X, X^{'}) \cong OT(Y)$ *if and only if*
(i) $|X^{'}| = |Y| = 1$ *or*
(ii) $X^{'} = X$ *and* X *and* Y *are either order-isomorphic or anti-order-isomorphic.*

Proof. Suppose that $OT(X, X^{'}) \cong OT(Y)$. We then have from Theorem 3.2.2 that $X^{'}$ and Y are either order-isomorphic or anti-order-isomorphic. Then $|X^{'}| = |Y|$. Since $OT(X, X^{'})$ must have an identity, by Proposition 3.1.1, $|X^{'}| = 1$ or $X^{'} = X$. Hence $|X^{'}| = |Y| = 1$ or $X^{'} = X$ and X and Y are either order-isomorphic or anti-order-isomorphic.

If (i) holds, then $|OT(X, X^{'})| = |OT(Y)| = 1$, and thus $OT(X, X^{'}) \cong OT(Y)$. From Corollary 3.2.3, (ii) implies that $OT(X, X^{'}) \cong OT(Y)$. $\square$

3.3 Isomorphism Theorems of $OP(X, X')$

The aim of this section is to show that for chains X and Y, a subchain X' of X and a subchain Y' of Y, if $OP(X, X') \cong OP(Y, Y')$, then $|X| = |Y|$ and X' and Y' are either order-isomorphic or anti-order-isomorphic.

The following two lemmas are required.

Lemma 3.3.1. Let X and Y be posets, $a \in X$ and $b \in Y$. Then $OP(X, \{a\}) \cong OP(Y, \{b\})$ if and only if $|X| = |Y|$.

Proof. Assume that $|X| = |Y|$. Then $|X \smallsetminus \{a\}| = |Y \smallsetminus \{b\}|$. Let $\varphi : X \to Y$ be a bijection such that $a\varphi = b$. Then

$$\begin{aligned} \mathcal{P}(Y) \;&=\; \{A\varphi \mid A \in \mathcal{P}(X)\} \text{ where } A\varphi \;=\; \{x\varphi \mid x \in A\}, \\ \text{and for } A \in \mathcal{P}(X),\; a \in A \;&\Leftrightarrow\; b \in A\varphi. \end{aligned} \tag{1}$$

It is clearly seen that

$$\begin{aligned} OP(X, \{a\}) &= \{A_a \mid A \in \mathcal{P}(X) \smallsetminus \{\phi\}\} \cup \{0\}, \qquad (2) \\ OP(Y, \{b\}) &= \{(A\varphi)_b \mid A \in \mathcal{P}(X) \smallsetminus \{\phi\}\} \cup \{0\}. \end{aligned}$$

Define $\overline{\varphi} : OP(X, \{a\}) \to OP(Y, \{b\})$ by

$$0\overline{\varphi} \;=\; 0 \text{ and } A_a\overline{\varphi} = (A\varphi)_b \quad \text{for all } A \in \mathcal{P}(X) \smallsetminus \{\phi\}.$$

Then $\overline{\varphi}$ is a bijection by (2) and we have from (1) that for $A, B \in \mathcal{P}(X) \smallsetminus \{\phi\}$,

$$\begin{aligned} a \in B &\Longrightarrow A_aB_a = A_a \text{ and } (A\varphi)_b(B\varphi)_b = (A\varphi)_b, \\ a \notin B &\Longrightarrow A_aB_a = 0 \text{ and } (A\varphi)_b(B\varphi)_b = 0. \end{aligned}$$

Hence $\overline{\varphi}$ is an isomorphism.

For the converse, assume that $OP(X, \{a\}) \cong OP(Y, \{b\})$. Then $|OP(X, \{a\})| =$

$|OP(Y,\{b\})|$. We therefore deduce from (1) and (2) that $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$. This implies that $|X| = |Y|$. $\square$

Lemma 3.3.2. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. If $\varphi : OP(X,X') \to OP(Y,Y')$ is an isomorphism, then the following statements hold.*

(i) For each $a \in X'$, there is an element $\overline{a} \in Y'$ such that $OP(X,\{a\})\varphi = OP(Y,\{\overline{a}\})$.

(ii) The map $\theta : X' \to Y'$ defined by $a\theta = \overline{a}$ for all $a \in X'$ is a bijection.

(iii) For each nonempty subset A of X, there is a unique nonempty subset $\overline{A}$ of Y such that $A_a\varphi = \overline{A}_{\overline{a}}$ for every $a \in X'$.

Proof. (i) Let $a \in X'$. Then $X_a\varphi \in E(OP(Y,Y')) \smallsetminus \{0\}$. Let $\overline{a} \in \operatorname{ran}(X_a\varphi)$. Then $\overline{a}(X_a\varphi) = \overline{a}$ and

$$((Y_{\overline{a}}\varphi^{-1})X_a)\varphi \ = \ Y_{\overline{a}}(X_a\varphi) \ = \ Y_{\overline{a}}.$$

Hence $(Y_{\overline{a}}\varphi^{-1})X_a = Y_{\overline{a}}\varphi^{-1}$ which implies that $\operatorname{ran}(Y_{\overline{a}}\varphi^{-1}) = \{a\}$. Thus $Y_{\overline{a}}\varphi^{-1} = Z_a$ for some $\phi \neq Z \subseteq X$ with $a \in Z$, and so $Z_a\varphi = Y_{\overline{a}}$. It then follows that

$$(X_a\varphi)Y_{\overline{a}} \ = \ (X_a\varphi)(Z_a\varphi) \ = \ (X_aZ_a)\varphi \ = \ X_a\varphi.$$

This implies that $\operatorname{ran}(X_a\varphi) = \{\overline{a}\}$. Next, to show that $OP(X,\{a\})\varphi = OP(Y,\{\overline{a}\})$, let $\phi \neq A \subseteq X$. Since $A_aX_a = A_a$, $(A_a\varphi)(X_a\varphi) = A_a\varphi$. But $\operatorname{ran}(X_a\varphi) = \{\overline{a}\}$, so $\operatorname{ran}(A_a\varphi) = \{\overline{a}\}$. We therefore have that $A_a\varphi = \overline{A}_{\overline{a}}$ for some $\phi \neq \overline{A} \subseteq Y$. This proves that

$$OP(X,\{a\})\varphi \subseteq OP(Y,\{\overline{a}\}). \tag{1}$$

Since $\varphi^{-1} : OP(Y,Y') \to OP(X,X')$ is an isomorphism, from (1), we can deduce that there is an element $b \in X'$ such that

$$OP(Y,\{\overline{a}\})\varphi^{-1} \subseteq OP(X,\{b\}). \tag{2}$$

It then follows from (1) and (2) that

$$OP(X,\{a\})\varphi \subseteq OP(Y,\{\overline{a}\}) \subseteq OP(X,\{b\})\varphi. \tag{3}$$

But φ is a one-to-one map, so $OP(X,\{a\}) \subseteq OP(X,\{b\})$. Consequently, $a = b$, and hence (3) yields

$$OP(X,\{a\})\varphi = OP(Y,\{\overline{a}\}). \tag{4}$$

(ii) If $a, b \in X$ are such that $\overline{a} = \overline{b}$, from (i), $OP(X,\{a\}) = OP(X,\{b\})$ since φ is one-to-one. Thus $a = b$. This shows that θ is a one-to-one map from X' into Y'. Since $\varphi^{-1} : OP(Y,Y') \to OP(X,X')$ is an isomorphism, from (i), we have similarly that

$$\text{for every } c \in Y', \text{ there is an element } c' \in X' \text{such that}$$
$$OP(Y,\{c\})\varphi^{-1} = OP(X,\{c'\}). \tag{5}$$

If $d \in Y'$, then from (5), we have $OP(Y,\{d\})\varphi^{-1} = OP(X,\{d'\})$, so

$$OP(X,\{d'\})\varphi = OP(Y,\{d\}). \tag{6}$$

Since $d' \in X'$, we have from (i) that

$$OP(X,\{d'\})\varphi = OP(Y,\{\overline{d'}\}). \tag{7}$$

Hence (6) and (7) yield $OP(Y,\{d\}) = OP(Y,\{\overline{d'}\})$, and thus $d = \overline{d'} = d'\theta$. This proves that $\theta : X' \to Y'$ is a bijection, as required.

(iii) Let A be a nonempty subset of X and $a \in X'$. Since $OP(X,\{a\})\varphi = OP(Y,\{\overline{a}\})$ by (i) and $A_a \in OP(X,\{a\})$, there is a nonempty subset $\overline{A}$ of Y such that $A_a\varphi = \overline{A}_{\overline{a}}$. Let $b \in X'$. We then have similarly that $A_b\varphi = B_{\overline{b}}$ for some $\phi \neq B \subseteq Y$. We shall show that $B = \overline{A}$. Since $A_aX_b = A_b$, we have $(A_a\varphi)(X_b\varphi) = A_b\varphi$. Thus $\overline{A}_{\overline{a}}(X_b\varphi) = B_{\overline{b}}$ which implies that $\overline{a} \in \operatorname{dom}(X_b\varphi)$ and $\overline{a}(X_b\varphi) = \overline{b}$. Hence $\overline{A}_{\overline{b}} = B_{\overline{b}}$, so $B = \overline{A}$.

Therefore the proof is complete. □

Theorem 3.3.3. Let X and Y be chains, $X^{'}$ a subchain of X and $Y^{'}$ a subchain of Y. If $OP(X,X^{'}) \cong OP(Y,Y^{'})$, then $|X| = |Y|$ and $X^{'}$ and $Y^{'}$ are either order-isomorphic or anti-order-isomorphic.

Proof. Let $\varphi : OP(X,X^{'}) \to OP(Y,Y^{'})$ be an isomorphism. From Lemma 3.3.2(i), for each $a \in X'$, there is an element $\overline{a} \in Y'$ such that $OP(X,\{a\})\varphi = OP(Y,\{\overline{a}\})$ and by Lemma 3.3.2(ii), $\theta : X' \to Y'$ defined by $a\theta = \overline{a}$ for all $a \in X'$ is a bijection. It then follows that for $a \in X'$, $OP(X,\{a\}) \cong OP(Y,\{\overline{a}\})$. By Lemma 3.3.1, $|X| = |Y|$.

Next, we shall show that θ is an order-isomorphism or an anti-order-isomorphism. Let $a,b,c,d \in X'$ be such that $a < b$ and $c < d$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in OP(X,X')$. We have from Lemma 3.3.2(iii) that there are nonempty subsets A and B of Y such that

$$\begin{pmatrix} a \\ a \end{pmatrix}\varphi = A_{\overline{a}}\ ,\ \begin{pmatrix} a \\ c \end{pmatrix}\varphi = A_{\overline{c}}\ ,\ \begin{pmatrix} b \\ b \end{pmatrix}\varphi = B_{\overline{b}}\ ,\ \begin{pmatrix} b \\ d \end{pmatrix}\varphi = B_{\overline{d}}.$$

But

$$\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \begin{pmatrix} b \\ b \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix},$$

so we have

$$A_{\overline{a}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi\right) = A_{\overline{c}} \text{ and } B_{\overline{b}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi\right) = B_{\overline{d}}\ .$$

Consequently,

$$\bar{a}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi\right) = \bar{c} \text{ and } \bar{b}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi\right) = \bar{d}.$$

Since X' and Y' are chains, θ is one-to-one and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi \in OP(Y,Y')$, it follows that $\bar{a} < \bar{b}$ implies $\bar{c} < \bar{d}$ and $\bar{a} > \bar{b}$ implies $\bar{c} > \bar{d}$. This shows that θ is either an order-isomorphism or an anti-order-isomorphism of X' onto Y', as required. $\square$

The following interesting isomorphism theorem is a direct consequence of Proposition 3.1.4 and Theorem 3.3.3.

Corollary 3.3.4. *For chains X and Y, $OP(X) \cong OP(Y)$ if and only if X and Y are either order-isomorphic or anti-order-isomorphic.*

The next example shows that the converse of Theorem 3.3.3 need not be true.

Example 3.3.5. The maps $x \mapsto 2x$ and $x \mapsto -2x$ are respectively an order-isomorphism and an anti-order-isomorphism from $\mathbb{Z}$ onto $2\mathbb{Z}$. Since $OP(\mathbb{Z})$ and $OP(2\mathbb{Z})$ have an identity and by Proposition 3.1.2, $OP(\mathbb{Z}, 2\mathbb{Z})$ has no identity, we deducd that $OP(\mathbb{Z}) \ncong OP(\mathbb{Z}, 2\mathbb{Z}) \ncong OP(2\mathbb{Z})$. In fact, $OP(\mathbb{Z}) \cong OP(2\mathbb{Z})$ by Corollary 3.3.4.

The following corollary gives a general fact of Example 3.3.5 . It is obtained directly from Proposition 3.1.2(i) and Corollary 3.3.4.

Corollary 3.3.6. *Let X and Y be chains and X' a subchain of X . Then $OP(X, X') \cong OP(Y)$ if and only if $X' = X$ and X and Y are either order-isomorphic or and anti-order-isomorphic.*

Remark 3.3.7. From Theorem 3.3.3 and Proposition 3.1.4 one might expect that the if part of Proposition 3.1.4 may be neccessary and sufficient conditions for $OP(X, X')$ and $OP(Y, Y')$ to be isomorphic for chains X and Y, $\varnothing \neq X' \subseteq X$ and $\varnothing \neq Y' \subseteq Y$. Lemma 3.3.1 shows that this is not true . For example, $OP([0,2], \{1\}) \cong OP((0,2), \{1\})$ by Lemma 3.3.1 since $|[0,2]| = |(0,2)| = \aleph_1$. Since $[0,2]$ has a minimum and a maximum while $(0,2)$ has neither a minimum and a maximum, we have that $[0, 2]$ and $(0, 2)$ are neither order-isomorphic nor anti-order-isomorphic.

3.4 Isomorphism Theorems of $OI(X, X')$

Our purpose of this section is to give neccessary and sufficient conditions for $OI(X, X')$ and $OI(Y, Y')$ being isomorphic where X and Y are any chains, X' is a subchain of X and Y' is a subchain of Y.

The following lemma is a main tool to obtain our required result. We first note that for a subposet X' of a poset X and for $\alpha \in OI(X, X')$, $\alpha \in E(OI(X, X'))$ if and only if $\alpha = 0$ or $\alpha = 1_A$ for some nonempty subset $A \subseteq X'$.

Lemma 3.4.1. *Let X and Y be posets, X' a subposet of X and Y' a subposet of Y. If $\varphi : OI(X, X') \to OI(Y, Y')$ is an isomorphism, then the following statements hold.*

(i) For every $x \in X$, there is defined an element $\overline{x} \in Y$ subject to :

$$\begin{pmatrix} x \\ a \end{pmatrix}\varphi = \begin{pmatrix} \overline{x} \\ \overline{a} \end{pmatrix} \quad \text{for all } x \in X \text{ and } a \in X'.$$

(ii) The map $\theta : X \to Y$ defined by $x\theta = \overline{x}$ for all $x \in X$ is a bijection such that $X'\theta = Y'$.

(iii) For every $\alpha \in OI(X, X')$, $\quad \alpha\varphi = \begin{pmatrix} \overline{x} \\ \overline{x\alpha} \end{pmatrix}_{x \in \mathrm{dom}\,\alpha}$.

Proof. (i) Let $a_0 \in X'$ be fixed. Since $0 \neq \begin{pmatrix} a_0 \\ a_0 \end{pmatrix} \in E(OI(X, X'))$, $0 \neq \begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\varphi \in E(OI(Y, Y'))$. Then $\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\varphi = 1_B$ for some nonempty subset B of Y'. Let $b_0 \in B$. Then $0 \neq \begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1} \in E(OI(X, X'))$. But

$$\left(\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\left(\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1}\right)\right)\varphi = 1_B\begin{pmatrix} b_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_0 \end{pmatrix},$$

$$\left(\left(\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1}\right)\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\right)\varphi \;=\; \begin{pmatrix} b_0 \\ b_0 \end{pmatrix}1_B \;=\; \begin{pmatrix} b_0 \\ b_0 \end{pmatrix},$$

so

$$\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\left(\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1}\right) = \begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1} = \left(\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1}\right)\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}.$$

Consequently, dom $\left(\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1}\right) = \{a_0\} =$ ran $\left(\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1}\right)$. Hence $\begin{pmatrix} b_0 \\ b_0 \end{pmatrix}\varphi^{-1} = \begin{pmatrix} a_0 \\ a_0 \end{pmatrix}$, and so $\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\varphi = \begin{pmatrix} b_0 \\ b_0 \end{pmatrix}$. This also proves the following fact.

$$\text{For every } a \in X',\ \begin{pmatrix} a \\ a \end{pmatrix}\varphi \;=\; \begin{pmatrix} b \\ b \end{pmatrix} \quad \text{for some } \ b \in Y'. \tag{1}$$

Next, let $x \in X$. Then

$$0 \neq \begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi \;=\; \left(\begin{pmatrix} x \\ a_0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_0 \end{pmatrix}\right)\varphi \;=\; \left(\begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi\right)\begin{pmatrix} b_0 \\ b_0 \end{pmatrix},$$

so ran $\left(\begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi\right) = \{b_0\}$. Since $\begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi$ is one-to-one, there exists an element $\overline{x} \in Y$ such that $\begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi = \begin{pmatrix} \overline{x} \\ b_0 \end{pmatrix}$. Now, we have that for every $x \in X$, there exists an element $\bar{x} \in Y$ subject to :

$$\begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi = \begin{pmatrix} \bar{x} \\ b_0 \end{pmatrix} \text{ for all } x \in X. \tag{2}$$

To prove that $\begin{pmatrix} x \\ a \end{pmatrix}\varphi = \begin{pmatrix} \overline{x} \\ \overline{a} \end{pmatrix}$ for all $x \in X$ and $a \in X'$, let $x \in X$ and $a \in X'$ be arbitrary fixed. Then $\begin{pmatrix} a \\ a_0 \end{pmatrix}\varphi = \begin{pmatrix} \bar{a} \\ b_0 \end{pmatrix}$ by (2) and hence

$$\begin{pmatrix} \overline{a} \\ b_0 \end{pmatrix} \;=\; \begin{pmatrix} a \\ a_0 \end{pmatrix}\varphi \;=\; \left(\begin{pmatrix} a \\ a \end{pmatrix}\begin{pmatrix} a \\ a_0 \end{pmatrix}\right)\varphi \;=\; \left(\begin{pmatrix} a \\ a \end{pmatrix}\varphi\right)\begin{pmatrix} \overline{a} \\ b_0 \end{pmatrix}.$$

This implies that $\overline{a} \in$ dom $\left(\begin{pmatrix} a \\ a \end{pmatrix}\varphi\right)$ and $\overline{a} \in$ ran $\left(\begin{pmatrix} a \\ a \end{pmatrix}\varphi\right) \subseteq Y'$. It then follows from (1) that

$$\begin{pmatrix} a \\ a \end{pmatrix}\varphi = \begin{pmatrix} \overline{a} \\ \overline{a} \end{pmatrix}. \tag{3}$$

Since

$$\begin{pmatrix} x \\ a \end{pmatrix}\varphi = \left(\begin{pmatrix} x \\ a \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix}\right)\varphi = \left(\begin{pmatrix} x \\ a \end{pmatrix}\varphi\right)\begin{pmatrix} \overline{a} \\ \overline{a} \end{pmatrix}$$

by (3), we have that ran $\left(\begin{pmatrix} x \\ a \end{pmatrix}\varphi\right) = \{\overline{a}\}$. Also, since

$$\begin{pmatrix} \overline{x} \\ b_0 \end{pmatrix} = \begin{pmatrix} x \\ a_0 \end{pmatrix}\varphi = \left(\begin{pmatrix} x \\ a \end{pmatrix}\begin{pmatrix} a \\ a_0 \end{pmatrix}\right)\varphi = \left(\begin{pmatrix} x \\ a \end{pmatrix}\varphi\right)\begin{pmatrix} \bar{a} \\ b_0 \end{pmatrix}$$

from (2), it follows that $\overline{x} \in \operatorname{dom}\left(\begin{pmatrix} x \\ a \end{pmatrix}\varphi\right)$. Consequently, $\begin{pmatrix} x \\ a \end{pmatrix}\varphi = \begin{pmatrix} \overline{x} \\ \overline{a} \end{pmatrix}$ because $\begin{pmatrix} x \\ a \end{pmatrix}\varphi$ is a one-to-one map.

(ii) Since $\begin{pmatrix} x \\ a \end{pmatrix}\varphi = \begin{pmatrix} \overline{x} \\ \overline{a} \end{pmatrix}$ for all $x \in X$ and $a \in X'$ from (i), we deduce that $\theta : X \to Y$ is a map with $X'\theta \subseteq Y'$. If $x_1, x_2 \in X$ are such that $\overline{x}_1 = \overline{x}_2$, then

$$\begin{pmatrix} x_1 \\ a_0 \end{pmatrix}\varphi = \begin{pmatrix} \overline{x}_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} \overline{x}_2 \\ b_0 \end{pmatrix} = \begin{pmatrix} x_2 \\ a_0 \end{pmatrix}\varphi,$$

so $x_1 = x_2$ since φ is one-to-one. Finally, let $y \in Y$ and $b \in Y'$. Then $\begin{pmatrix} y \\ b \end{pmatrix} \in OI(Y, Y')$. Since $\varphi^{-1} : OI(Y, Y') \to OI(X, X')$ is an isomorphism, from (i) by considering φ^{-1} instead of φ, $\begin{pmatrix} y \\ b \end{pmatrix}\varphi^{-1} = \begin{pmatrix} x \\ a \end{pmatrix}$ for some $x \in X$ and $a \in A$. Thus $\begin{pmatrix} \overline{x} \\ \overline{a} \end{pmatrix} = \begin{pmatrix} x \\ a \end{pmatrix}\varphi = \begin{pmatrix} y \\ b \end{pmatrix}$, and hence $\overline{x} = y$ and $\overline{a} = b$. This proves that $\theta : X \to Y$ is a bijection such that $X'\theta = Y'$, as required.

(iii) Let $\alpha \in OI(X, X')$ and $x \in \operatorname{dom}\alpha$. Then

$$\begin{aligned}
(\alpha\varphi)\begin{pmatrix} \overline{x\alpha} \\ \overline{x\alpha} \end{pmatrix} &= \left(\alpha\begin{pmatrix} x\alpha \\ x\alpha \end{pmatrix}\right)\varphi && \text{from (i)} \\
&= \begin{pmatrix} x \\ x\alpha \end{pmatrix}\varphi && \text{since } \alpha \text{ is one-to-one} \\
&= \begin{pmatrix} \overline{x} \\ \overline{x\alpha} \end{pmatrix} && \text{from (i)}
\end{aligned}$$

which implies that $\overline{x} \in \text{dom}\,(\alpha\varphi)$ and $\overline{x}\,(\alpha\varphi) = \overline{x\alpha}$.

Next, let $y \in \text{dom}\,(\alpha\varphi)$. By (ii), $y = \overline{x}$ and $y\,(\alpha\varphi) = \overline{a}$ for some $x \in X$ and $a \in X'$ and so $\begin{pmatrix} \bar{x} \\ \bar{a} \end{pmatrix} = \begin{pmatrix} y \\ y(\alpha\varphi) \end{pmatrix}$ and $\begin{pmatrix} \bar{a} \\ \bar{a} \end{pmatrix} = \begin{pmatrix} y(\alpha\varphi) \\ y(\alpha\varphi) \end{pmatrix}$. Hence

$$\begin{aligned} \begin{pmatrix} x \\ a \end{pmatrix}\varphi &= \begin{pmatrix} \bar{x} \\ \bar{a} \end{pmatrix} && \text{from(i)} \\ &= \begin{pmatrix} y \\ y(\alpha\varphi) \end{pmatrix} \\ &= (\alpha\varphi)\begin{pmatrix} y(\alpha\varphi) \\ y(\alpha\varphi) \end{pmatrix} && \text{since } y \in \text{dom}\,(\alpha\varphi) \text{ and } \alpha\varphi \text{ is one-to-one} \\ &= (\alpha\varphi)\begin{pmatrix} \overline{a} \\ \overline{a} \end{pmatrix} \\ &= \left(\alpha\begin{pmatrix} a \\ a \end{pmatrix}\right)\varphi && \text{from(i).} \end{aligned}$$

Since φ is one-to-one, $\begin{pmatrix} x \\ a \end{pmatrix} = \alpha\begin{pmatrix} a \\ a \end{pmatrix}$, so $x \in \text{dom}\,\alpha$ and $\overline{x} = y \in \text{dom}\,(\alpha\varphi)$. This proves that

$$\alpha\varphi = \begin{pmatrix} \overline{x} \\ \overline{x\alpha} \end{pmatrix}_{x \in \text{dom}\ \alpha}.$$

Therefore the lemma is completely proved. $\square$

Lemma 3.4.2. *Let X and Y be posets, $a \in X$ and $b \in Y$. Then $OI(X, \{a\}) \cong OI(Y, \{b\})$ if and only if $|X| = |Y|$.*

Proof. Assume that $|X| = |Y|$. Then $|X \smallsetminus \{a\}| = |Y \smallsetminus \{b\}|$. Let $\varphi : X \to Y$ be a bijection such that $a\varphi = b$. Observe that

$$OI(X, \{a\}) \;=\; \left\{\begin{pmatrix} x \\ a \end{pmatrix} \mid x \in X\right\} \cup \{0\},$$

$$OI(Y,\{b\}) \;=\; \left\{\binom{y}{b} \mid y\in Y\right\}\cup\{0\} \;=\; \left\{\binom{x\varphi}{b} \mid x\in X\}\cup\{0\}\right\}.$$

Define $\overline{\varphi} : OI(X,\{a\}) \to OI(Y,\{b\})$ by

$$0\overline{\varphi} \;=\; 0 \;\text{ and }\; \binom{x}{a}\overline{\varphi} \;=\; \binom{x\varphi}{b} \quad \text{for all } x\in X.$$

Then $\overline{\varphi}$ is a bijection and for all $x_1, x_2 \in X$,

$$\begin{aligned} x_2 = a \Longrightarrow & \binom{x_1}{a}\binom{x_2}{a} = \binom{x_1}{a} \;\text{ and } \\ & \binom{x_1\varphi}{b}\binom{x_2\varphi}{b} = \binom{x_1\varphi}{b}\binom{b}{a} = \binom{x_1\varphi}{b}, \\ x_2 \neq a \Longrightarrow & \binom{x_1}{a}\binom{x_2}{a} = 0 \quad\text{and}\quad \binom{x_1\varphi}{b}\binom{x_2\varphi}{b} = 0 \quad\text{since}\quad x_2\varphi\neq b. \end{aligned}$$

Hence $\overline{\varphi}$ is a homomorphism.

Conversely, assume that $OI(X,\{a\}) \cong OI(Y,\{b\})$. Then $|OI(X,\{a\})| = |OI(Y,\{b\})|$. But $x \mapsto \binom{x}{a}$ is clearly a bijection from X onto $OI(X,\{a\}) \smallsetminus \{0\}$, so $|X|+1 = |OI(X,\{a\})| = |OI(Y,\{b\})| = |Y|+1$. Hence $|X| = |Y|$. □

Theorem 3.4.3. *Let X and Y be chains, X' a subchain of X and Y' a subchain of Y. Then $OI(X,X') \cong OI(Y,Y')$ if and only if one of the following statements holds.*

(i) $|X| = |Y|$ *and* $|X'| = |Y'| = 1$.

(ii) There exists an order-isomorphism or an anti-order-isomorphism $\theta : X \to Y$ such that $X'\theta = Y'$.

Proof. Let $\varphi : OI(X,X') \to OI(Y,Y')$ be an isomorphism. By Lemma 3.4.1(i), for each $x \in X$, there is an element $\overline{x} \in Y$ satisfying the following property.

$$\binom{x}{a}\varphi = \binom{\overline{x}}{\overline{a}} \text{ for all } x\in X \text{ and } a\in X'.$$

By Lemma 3.4.1(ii), the map $\theta : X \to Y$ defined by $x\theta = \overline{x}$ for all $x \in X$ is a bijection such that $X^{'}\theta = Y^{'}$. Then $|X| = |Y|$ and $|X^{'}| = |Y^{'}|$.

First, we claim that $\theta_{|_{X'}} : X^{'} \to Y^{'}$ is either an order-isomorphism or an anti-order-isomorphism. Let $a, b, c, d \in X^{'}$ be such that $a < b$ and $c < d$. Since θ is a one-to-one map and Y' is a chain, $\overline{a} < \overline{b}$ or $\overline{a} > \overline{b}$ and $\overline{c} < \overline{d}$ or $\overline{c} > \overline{d}$. Define $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\alpha \in OI(X, X^{'})$, so by Lemma 3.4.1(iii), $\alpha\varphi = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \in OI(Y, Y')$. Consequently, $\overline{a} < \overline{b}$ implies $\overline{c} < \overline{d}$ or $\overline{a} > \overline{b}$ implies $\overline{c} > \overline{d}$. Hence we have the claim. Suppose that (i) is fault. Since $|X| = |Y|$ and $|X'| = |Y'|$, we have $|X^{'}| = |Y^{'}| > 1$. Let $a, b \in X'$ be such that $a < b$.

Case 1 : $\theta_{|_{X'}} : X^{'} \to Y^{'}$ is an order-isomorphism. Since $a < b$, we have $\overline{a} < \overline{b}$. If $x_1, x_2 \in X$ are such that $x_1 < x_2$, then $\begin{pmatrix} x_1 & x_2 \\ a & b \end{pmatrix} \in OI(X, X^{'})$, then by Lemma 3.4.1(iii), $\begin{pmatrix} \overline{x_1} & \overline{x_2} \\ \overline{a} & \overline{b} \end{pmatrix} \in OI(Y, Y^{'})$ which implies that $\overline{x_1} < \overline{x_2}$ since $\overline{a} < \overline{b}$. We deduce that θ is an order-isomorphism from X onto Y.

Case 2 : $\theta_{|_{X'}} : X^{'} \to Y^{'}$ is an anti-order-isomorphism. Then $\overline{a} > \overline{b}$ since $a < b$. If $x_1, x_2 \in X$ are such that $x_1 < x_2$, then $\begin{pmatrix} x_1 & x_2 \\ a & b \end{pmatrix} \in OI(X, X^{'})$, then by Lemma 3.4.1(iii), $\begin{pmatrix} \overline{x_1} & \overline{x_2} \\ \overline{a} & \overline{b} \end{pmatrix} \in OI(Y, Y^{'})$, so $\overline{x_2} < \overline{x_1}$ since $\overline{b} < \overline{a}$. Consequently, θ is an anti-order-isomorphism from X onto Y.

The converse follows directly from Lemma 3.4.2 and Proposition 3.1.4.

Therefore the theorem is proved, as desired. $\square$

A direct interesting consequence of Theorem 3.4.3 is the following.

Corollary 3.4.4. For chains X and Y, $OI(X) \cong OI(Y)$ if and only if X and Y are either order-isomorphic or an anti-order-isomorphic.

Also, we have

Corollary 3.4.5. Let X and Y be chains and $X^{'}$ a subchain of X. Then $OI(X, X^{'}) \cong OI(Y)$ if and only if $X^{'} = X$ and X and Y are either isomorphic or anti-order-isomorphic.

Proof. Assume that $OI(X, X^{'}) \cong OI(Y)$. We have by Proposition 3.1.2(ii) that $X^{'} = X$, and hence from Corollary 3.4.4, X and Y are either order-isomorphic or anti-order-isomorphic.

The converse follows from Corollary 3.4.4. $\square$

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VITA

Miss Pennapa Rungrattrakoon was born on October 31, 1982 in Chaiyaphum, Thailand. She got a Bachelor of Science in Mathematics with the first class honour from Srinakarinwirot University in 2002. Then she received a scholarship from the Ministry Staff Development Project in 2002 to further her study for master program in mathematics under the requirement of Srinakarinwirot University. After this graduation, she will prepare herself ready to further her study for a Ph. D. program in mathematics.