CHAPTER II

Γ-SEMINEARRINGS

This chapter is split into three major parts. In the first section, we give a definition of a Γ-seminearring along with examples of Γ-seminearrings. Then some general properties of Γ-seminearrings are investigated. Next, ideals of Γ-seminearrings are introduced in the second section. Also, notion of zero-symmetric nearrings and distributively generated nearrings are adopted to zero-symmetric Γ-seminearrings and distributively generated Γ-seminearrings in order to obtain more results. In the last section, we present Γ-homomorphisms and study related properties.

2.1 Definitions and Examples

The begining of this section is assigned to introduce common concept of Γ-seminearrings.

Definition 2.1.1. Let $(R, +)$ be a semigroup and Γ a nonempty set. Then R is called a *(right)* Γ*-seminearring* if there exists a mapping from $R \times \Gamma \times R$ into R (sending $(a, \alpha, b) \mapsto a\alpha b$) satisfying the following conditions:

(i) the *right ditributivity*: $(a + b)\alpha c = a\alpha c + b\alpha c$ for all $a, b, c \in R$ and $\alpha \in \Gamma$,

(ii) the *associativity*: $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

A *left* Γ*-seminearring* can be defined analogously.

Let R be a Γ-seminearring. If the semigroup R contains an identity, then R is called a Γ*-seminearring with identity.* In this thesis, the identity of the semigroup R is denoted by 0_R or simply 0.

We remark here that a right Γ-seminearring satisfies the right distributive law but not necessarily the left distributive law.

Recall that a nonempty set R with two binary operations $+$ and $\cdot$ is called a seminearring if and only if $(R,+)$ and $(R,\cdot)$ are semigroups and $(a+b)c = ac+bc$ for all $a,b,c \in R$. It is obvious that for a seminearring $(R,+,\cdot)$, the semigroup R under $+$ is an R-seminearring under the mapping from $R \times R \times R$ into R defined by $(a,\alpha,b) \mapsto a\cdot\alpha\cdot b$ for all $a,b,\alpha \in R$. This gives a trivial example of Γ-seminearrings. In addition, any Γ-nearrings and Γ-semirings (see definitions in Chapter I page 4) are Γ-seminearrings from their definitions.

Next example shows that a seminearring can be constructed from a given Γ-seminearring.

Example 2.1.1. Let R be a Γ-seminearring and α be a fixed element in Γ. Define $a \cdot b = a\alpha b$ for all $a,b \in R$. Then $(R,+,\cdot)$ is a seminearring.

To see this fact, first, we show that $(R,\cdot)$ is a semigroup. Let $x,y,z \in R$. Since R is a Γ-seminearring, $x\cdot y = x\alpha y \in R$, and $(x\cdot y)\cdot z = (x\alpha y)\alpha z = x\alpha(y\alpha z) = x\cdot(y\cdot z)$. Thus $(R,\cdot)$ is a semigroup. Next, $(x+y)\cdot z = (x+y)\alpha z = x\alpha z + y\alpha z = x\cdot z + y\cdot z$. Hence $(R,+,\cdot)$ is a seminearring as claimed.

Other examples of Γ-seminearrings are provided in the next.

Example 2.1.2. Let V and W be finite-dimension vecter spaces over a same field F where $\dim(V) = n$ and $\dim(W) = m$. Let

$$R = \mathcal{L}(V,W) := \{f : V \to W \mid f \text{ is a linear transformation}\} \quad \text{and}$$

$$\Gamma = \mathcal{L}(W,V) := \{\alpha : W \to V \mid \alpha \text{ is a linear transformation}\}.$$

Then $(R,+)$ and $(\Gamma,+)$ are abelian groups where $+$ is the usual addition of functions. Define the mapping from $R\times\Gamma\times R$ into R by $(f,\alpha,g)\mapsto f\circ\alpha\circ g$ for all $f,g\in R$ and $\alpha\in\Gamma$ where $\circ$ is the usual composition of functions. Then R is a Γ-seminearring.

First of all, since $(R,+)$ and $(\Gamma,+)$ are abelian groups, $(R,+)$ is a semigroup and $\Gamma\neq\varnothing$. For the convenience of this proof, we write $f\alpha g$ instead of $f\circ\alpha\circ g$ for any $f,g\in R$ and $\alpha\in\Gamma$. Next, let $f,g,h\in R$, $\alpha,\beta\in\Gamma$ and $x\in V$. It is obvious that $f\alpha g\in R$ and $(f\alpha g)\beta h=f\alpha(g\beta h)$. Furthermore,

$$\begin{aligned}\big((f+g)\alpha h\big)(x)&=(f+g)\big(\alpha(h(x))\big)=f\big(\alpha(h(x))\big)+g\big(\alpha(h(x))\big)\\&=(f\alpha h)(x)+(g\alpha h)(x)=\big(f\alpha h+g\alpha h\big)(x),\end{aligned}$$

i.e., $(f+g)\alpha h=f\alpha h+g\alpha h$. Hence R is a Γ-seminearring as desired.

Now consider further in Example 2.1.2. Let R and Γ be defined as in Example 2.1.2. Then it can be shown that $\big(f\alpha(g+h)\big)(x)=(f\alpha)\big(g(x)+h(x)\big)=(f\alpha g)(x)+(f\alpha h)(x)=\big(f\alpha g+f\alpha h\big)(x)$ for all $f,g,h\in R$, $\alpha,\beta\in\Gamma$ and $x\in V$ because $f\alpha$ is a linear transformation. Thus R is also a Γ-nearring, a Γ-semiring and a Γ-ring.

Let V,W,F,m,n be defined as in Example 2.1.2. Moreover, let $M_{k,l}(F)$ be the set of all $k\times l$ matrices over F. It is known that $\mathcal{L}(V,W)$ is isomorphic to $M_{m,n}(F)$ and $\mathcal{L}(W,V)$ is isomorphic to $M_{n,m}(F)$. Consequently, we can conclude that $M_{m,n}(F)$ is a Γ-seminearring, a Γ-nearring, a Γ-semiring and a Γ-ring where $\Gamma=M_{n,m}(F)$.

So far, Γ-seminearrings do exist but they also are other known Γ-structures. It is worth to give an example of Γ-seminearring which is not Γ-nearring, a Γ-semiring or a Γ-ring. As a result, this will ensure us that a Γ-seminearring is actually a generalization of a Γ-nearring, a Γ-semiring and a Γ-ring.

Example 2.1.3. Let A be a nonempty set, $(B, *)$ be a semigroup which is not a group,

$$R = \{f \,|\, f : A \to B\} \quad \text{and} \quad \Gamma = \{\alpha \,|\, \alpha : B \to A\}.$$

Then $\Gamma \neq \varnothing$ and $(R, +)$ is a semigroup where $(f + g)(x) = f(x) * g(x)$ for all $x \in A$. Define a mapping from $R \times \Gamma \times R$ into R by

$$(f, \alpha, g) \mapsto f \circ \alpha \circ g \qquad \text{for all } f, g \in R \text{ and } \alpha \in \Gamma$$

where $\circ$ is the usual composition of functions. Then R is a Γ-seminearring which is not a Γ-nearring, a Γ-semiring or a Γ-ring.

To be certain that this is true, first, the nonemptyness of A and B implies that both R and Γ are not empty. For each $f, g, h \in R$, it can be shown that $f + g \in R$ and $(f + g) + h = f + (g + h)$ because B is a semigroup. Thus $(R, +)$ forms a semigroup. Moreover, the fact that $f\alpha g \in R$, $(f + g)\alpha h = f\alpha h + g\alpha h$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f, g, h \in R$ and $\alpha, \beta \in \Gamma$ can be obtained in the similar way of the proof of Example 2.1.2. Hence R is a Γ-seminearring.

Note that $(R, +)$ is not a group which is a consequence of the fact that $(B, *)$ is not a group. As a result, R is not a Γ-nearring. Furthermore, the property that $f\alpha(g + h) = f\alpha g + f\alpha h$ for any $f, g, h \in R$ and $\alpha \in \Gamma$ may not hold which implies that R is not a Γ-semiring. Accordingly, R is not a Γ-ring.

The last example gives an example of a left Γ-seminearring whose semigroup part is noncommutative. Moreover, knowledge in set theory is required.

Example 2.1.4. Let $R = \aleph_1$ and $\Gamma = \aleph_0$. Then clearly $\aleph_0 \neq \varnothing$ and $(R, \oplus)$ is a noncommutative semigroup where $\oplus$ is the ordinal sum. Define a mapping from $R \times \Gamma \times R$ into R by

$$(a, \alpha, b) \mapsto a \otimes \alpha \otimes b \quad \text{for all } a, b \in R \text{ and } \alpha \in \Gamma$$

where $\otimes$ is the ordinal product. Then R is a left Γ-seminearring which is not a Γ-semiring, a Γ-nearring and a Γ-ring. Details of the proof can be read in the appendix.

Next, we give the definition of sub Γ-seminearrings.

Definition 2.1.2. Let R be a Γ-seminearring under the mapping from $R\times\Gamma\times R$ into R, say f. A subsemigroup A of R is called a *sub Γ-seminearring* of R if A is a Γ-seminearring under the restriction of f to $A\times\Gamma\times A$.

Note that a nonempty subset A of a Γ-seminearring R is a sub Γ-seminearring of R if and only if $a+b$, $a\alpha b\in A$ for all $a,b\in A$ and $\alpha\in\Gamma$.

Remark 2.1.1. Let R be a Γ-seminearring. Then R is also a semigroup. Thus, when we would like to emphasize that R is considered as a semigroup, it will be stated so. Besides, we say that A is a *subsemigroup of R* whenever A is considered as a subsemigroup of the semigroup R.

We give an example of a sub Γ-seminearring.

Example 2.1.5. As in Example 2.1.3, if $(C,*)$ is a subsemigroup of $(B,*)$, then $\{f\,|\,f:A\to C\}$ is obviously a sub Γ-seminearring of R.

Throughout this thesis, let $\mathbb{N}$ be the set of all positive integers. Let R be a Γ-seminearring. If A and B are nonempty subsets of R, we denote by $A\Gamma B$ the subset of R consisting of all finite sums of the form $\sum_{i=1}^{m} a_i\alpha_i b_i$ where $m\in\mathbb{N}$, $a_i\in A$, $b_i\in B$ and $\alpha_i\in\Gamma$ for all i, i.e.,

$$A\Gamma B=\left\{\sum_{i=1}^{m} a_i\alpha_i b_i \;\middle|\; m\in\mathbb{N}, a_i\in A, b_i\in B, \alpha_i\in\Gamma \text{ for all } i\right\}.$$

Conveniently, we write $\sum a_i\alpha_i b_i$ instead of such the finite sum $\sum_{i=1}^{m} a_i\alpha_i b_i$.

Moreover, $x\Gamma B$ and $A\Gamma x$ are written instead of $\{x\}\Gamma B$ and $A\Gamma\{x\}$, respectively, for all $x \in R$. Similarly, $A\alpha B$ is used for $A\{\alpha\}B$ for all $\alpha \in \Gamma$.

In particular, by the right distributivity, $R\alpha x = \{r\alpha x \mid r \in R\}$ for all $x \in R$ and $\alpha \in \Gamma$. However, it is not true in general that $x\alpha R = \{x\alpha r \mid r \in R\}$ where $x \in R$ and $\alpha \in \Gamma$.

It is interesting to know whether the subset $A\Gamma B$ of R forms any algebraic structures for any nonempty subsets A and B of R.

Proposition 2.1.1. *Let A and B be nonempty subsets of a Γ-seminearring R. Then $A\Gamma B$ is a subsemigroup of R.*

Proof. Since A, Γ and B are nonempty sets, $A\Gamma B \neq \varnothing$. For each $x = \sum r_i\alpha_i s_i$, $y = \sum u_j\beta_j v_j \in A\Gamma B$, we see that $x+y = \sum r_i\alpha_i s_i + \sum u_j\beta_j v_j \in A\Gamma B$. Therefore, $A\Gamma B$ is a subsemigroup of R. □

However, if A and B are nonempty subsets of a Γ-seminearring R, then $A\Gamma B$ may not be a sub Γ-seminearring of R because, for each $\sum a_i\alpha_i b_i, \sum c_j\gamma_j d_j \in A\Gamma B$ and $\beta \in \Gamma$, $\big(\sum a_i\alpha_i b_i\big)\beta\big(\sum c_j\gamma_j d_j\big) = \sum\Big((a_i\alpha_i b_i)\beta\big(\sum c_j\gamma_j d_j\big)\Big)$ where $(a_i\alpha_i b_i)\beta\big(\sum c_j\gamma_j d_j\big)$ may be not contained in $A\Gamma B$. Although, if A and B are subsemigroups of R, it cannot be concluded either that $A\Gamma B$ is a sub Γ-seminearring of R. Besides, we see later in Proposition 2.2.2 when $A\Gamma B$ is a sub Γ-seminearring of R.

Proposition 2.1.2. *Let R be a Γ-seminearring. If A, B and C are nonempty subsets of R, then $(A\Gamma B)\,\Gamma C \subseteq A\Gamma\,(B\Gamma C)$.*

Proof. Let A, B and C be nonempty subsets of R. Let $x \in (A\Gamma B)\,\Gamma C$. Then $x = \sum\left(\sum a_j\beta_j b_j\right)\alpha_i c_i$ where $a_j \in A$, $b_j \in B$, $c_i \in C$ and $\alpha_i, \beta_j \in \Gamma$ for all i, j.

Thus

$$
\begin{aligned}
x &= \sum \left(\sum a_j\beta_j b_j\right) \alpha_i c_i \\
&= \sum\sum \left((a_j\beta_j b_j)\, \alpha_i c_i \right) && \text{because of the right distributivity} \\
&= \sum\sum \left(a_j\beta_j\, (b_j\alpha_i c_i) \right) && \text{because of the associativity} \\
&\in A\Gamma\,(B\Gamma C) && \text{since } a_j\beta_j\,(b_j\alpha_i c_i) \in A\Gamma\,(B\Gamma C) \text{ for all } i, j.
\end{aligned}
$$

□

In general, if A, B and C are nonempty subsets of a Γ-seminearring R, then it is not necessary that $A\Gamma\,(B\Gamma C)$ is contained in $(A\Gamma B)\,\Gamma C$ because it is not reasonable to say that

$$\sum a_i\alpha_i\big(\sum b_j\beta_j c_j\big) = \sum\sum a_i\alpha_i(b_j\beta_j c_j)$$

for any element $\sum a_i\alpha_i\big(\sum b_j\beta_j c_j\big) \in A\Gamma(B\Gamma C)$.

As a result, $(A\Gamma B)\,\Gamma C = A\Gamma\,(B\Gamma C)$ does not necassarily hold.

Definition 2.1.3. Let R be a Γ-seminearring. An element $x \in R$ is called a *left (right) zero* of R if $x\alpha y = x$ $(y\alpha x = x)$ for all $y \in R$ and $\alpha \in \Gamma$. Furthermore, if x is both a left and a right zero of R, then x is called a *zero* of R.

In addition, if all elements of R are entirely left (right) zeros, then R is called a *left (right) zero* Γ-seminearring.

Proposition 2.1.3. *Let R be a Γ-seminearring.*

(i) If R has a left zero and a right zero, then R has a zero.

(ii) If R has a zero, then that zero is unique.

Proof. (i) Let x and y be a left zero and a right zero of R, respectively. Fix an element $\alpha \in \Gamma$. Then $x = x\alpha y = y$. Thus R has a zero.

(ii) This is obvious from the proof of (i) that the zero of R is unique.

□

If R is a Γ-seminearring with identity 0, i.e., 0 is the identity of the semigroup R, then 0 is not necessary a left zero or a right zero of R in general. However, if the semigroup R also satisfies either the left cancellation or the right cancellation, then 0 is a left zero of R since either $0\alpha x + 0 = 0\alpha x = (0+0)\alpha x = 0\alpha x + 0\alpha x$ or $0 + 0\alpha x = 0\alpha x + 0\alpha x$ for all $x \in R$ and $\alpha \in \Gamma$ but 0 need not be a right zero of R since $0 = x\alpha 0$ may not holds for some $x \in R$ and $\alpha \in \Gamma$.

Definition 2.1.4. A Γ-seminearring R with identity 0 is called *zero-symmetric* if $0\alpha x = 0 = x\alpha 0$ for all $x \in R$ and $\alpha \in \Gamma$.

If R is a zero-symmetric Γ-seminearring, then it follows directly from the definition that the identity of the semigroup R and the zero of the Γ-seminearring R are identical.

Definition 2.1.5. Let R be a Γ-seminearring. An element $e \in R$ is called a *left (right)* Γ*-identity* of R if $e\alpha x = x$ ($x\alpha e = x$) for all $x \in R$ and $\alpha \in \Gamma$. Moreover, if e is both a left and a right Γ-identity of R, then e is called a Γ*-identity* of R.

However, we often simply call such an element e a *left (right) identity* or an *identity* of R if it is clear from the context that the Γ-identity is mentioned not the identity of the semigroup R.

Proposition 2.1.4. *Let R be a Γ-seminearring.*

(i) If R has a left identity and a right identity, then R has an identity.

(ii) If R has an identity, then that identity is unique.

Proof. (i) Let e and f be a left identity and a right identity of R, respectively. Fix an element $\alpha \in \Gamma$. Then $f = e\alpha f = e$. Thus R has an identity.

(ii) This is obvious from the proof of (i) that the identity of R is unique.

□

For a Γ-seminearring having an identity, we denote that unique identity by 1.

This section is ended with, providing that R is a Γ-seminearring, the relationship between the identity of the semigroup R and the Γ-identity of R.

Proposition 2.1.5. *Let R be a Γ-seminearring with Γ-identity 1 and have more than one element. Moreover, let 0 be the identity of the semigroup R. If the semigroup R satisfies either the left or the right cancellation, then $0 \neq 1$.*

Proof. Without loss of generality, assume that the semigroup R satisfies the left cancellation. Suppose that $0 = 1$. Since R has more than one element, $R \neq \{0\}$ so that there exists an element $a \in R$ such that $a \neq 0$. Then $a = 1\alpha a = 0\alpha a = 0$ which is a contradiction. As a result, $0 \neq 1$. □

2.2 Ideals of Γ-seminearrings

The concept of ideals of algebraic structures plays quite important roles so that it makes sense to define and to study ideals of Γ-seminearrings.

Definition 2.2.1. A subset I of a Γ-seminearring R is called a *left (right) ideal* of R if I is a subsemigroup of R and $r\alpha x \in I$ ($x\alpha r \in I$) for all $r \in R$, $x \in I$ and $\alpha \in \Gamma$. If I is both a left and a right ideal of R, then I is called an *ideal* of R.

Example 2.2.1. Let A be a nonempty set, $(B, *)$ a semigroup which is not a group, C a subsemigroup of B, $R = \{f \mid f : A \to B\}$ and $\Gamma = \{\alpha \mid \alpha : B \to A\}$. Then R is a Γ-seminearring from Example 2.1.3. Moreover, $\{f \mid f : A \to C\}$ is a right ideal of R.

Let $I = \{f \mid f : A \to C\}$. Then I is a sub Γ-seminearring of R by Example 2.1.5. To show that I is a right ideal of R, let $r \in R$, $f \in I$, $\alpha \in \Gamma$ and $x \in A$. Then $(f\alpha r)(x) = f(\alpha(r(x))) \in f(\alpha(r(A))) \subseteq f(\alpha(B)) \subseteq f(A) \subseteq C$ so that $f\alpha r \in I$.

In Example 2.2.1, note that I may not be an ideal of R since $r\alpha f$ need not be an element of I where $r \in R$, $f \in I$ and $\alpha \in \Gamma$.

Example 2.2.2. Let $(A, *)$ be a semigroup, B be a subsemigroup of A which is not a group, C be a subsemigroup of B,

$$R = \{f : A \to B \mid f|_C : C \to C\} \quad \text{and} \quad \Gamma = \{\alpha : B \to A \mid \alpha|_C : C \to C\}.$$

Then $\Gamma \neq \varnothing$ and $(R, +)$ is a semigroup where $(f + g)(x) = f(x) * g(x)$ for all $x \in A$. Furthermore, it can be shown that R is also a Γ-seminearring under the mapping from $R \times \Gamma \times R$ into R by $(f, \alpha, g) \mapsto f \circ \alpha \circ g$ for all $f, g \in R$ and $\alpha \in \Gamma$ where $\circ$ is the usual composition of functions.

Moreover, $I = \{f \mid f : A \to C\}$ is an ideal of R. It can see that I is a right ideal of R from the above example. To show that I is an ideal of R, let $r \in R$, $f \in I$, $\alpha \in \Gamma$ and $x \in A$. Then $(r\alpha f)(x) = r(\alpha(f(x))) \in r(\alpha(f(A))) \subseteq r(\alpha(C)) \subseteq r(C) \subseteq C$ so that $r\alpha f \in I$.

Proposition 2.2.1. *Let R be a Γ-seminearring. If I is a left ideal, a right ideal or an ideal of R, then I is a sub Γ-seminearring of R.*

Proof. This is obvious. □

Proposition 2.2.2. *Let R be a Γ-seminearring. If A and B are a right ideal and a left ideal of R, respectively, then $A\Gamma B$ is a sub Γ-seminearring of R.*

Proof. Assume that A and B are a right ideal and a left ideal of R, respectively. By Theorem 2.1.1, $A\Gamma B$ is a subsemigroup of R. Moreover, for each $\sum a_i\alpha_i b_i, \sum c_j\gamma_j d_j \in A\Gamma B$ and $\beta \in \Gamma$, $\big(\sum a_i\alpha_i b_i\big)\beta\big(\sum c_j\gamma_j d_j\big) \in A\Gamma B$ since A and B are a right ideal and a left ideal of R, respectively. Therefore, $A\Gamma B$ is a sub Γ-seminearring of R. □

Proposition 2.2.3. *Let I be a subsemigroup of a Γ-seminearring R. Then I is a left (right) ideal of R if and only if $R\Gamma I \subseteq I$ $(I\Gamma R \subseteq I)$.*

Proof. If I is a left (right) ideal of R, then $R\Gamma I \subseteq I$ $(I\Gamma R \subseteq I)$ since each $r_i\alpha_i x \in I$ so that $\sum r_i\alpha_i x \in I$ $\big(\sum x\alpha_i r_i \in I\big)$ where $r_i \in R$, $x \in I$ and $\alpha_i \in \Gamma$.

Next, it is enough to assume that $R\Gamma I \subseteq I$. For each $r \in R$, $x \in I$ and $\alpha \in \Gamma$, we see that $r\alpha x \in R\Gamma I \subseteq I$. Thus I is a left ideal of R. □

Let R be a Γ-seminearring. Then R must be a semigroup. If the semigroup R also contains the identity 0, then it is interesting to know whether $\{0\}$ is an ideal of R.

Proposition 2.2.4. *Let R be a Γ-seminearring with identity 0. Then R is zero-symmetric if and only if $\{0\}$ is an ideal of R.*

The ideal $\{0\}$ is called the *zero ideal* of R.

Proof. It is clear that if R is zero-symmetric, then $\{0\}$ is an ideal of R.

Conversely, assume that $\{0\}$ is an ideal of R. For each $x \in R$ and $\alpha \in \Gamma$, we see that $0\alpha x,\ x\alpha 0 \in \{0\}$ so that $0\alpha x = 0 = x\alpha 0$. Therefore R is zero-symmetric. □

We investigate variety of left ideals, right ideals and ideals of Γ-seminearrings.

Theorem 2.2.5. *Let R be a Γ-seminearring.*

(i) For each $a \in R$ and $\alpha \in \Gamma$, $R\alpha a$ $(a\alpha R)$ is a left (right) ideal of R.

(ii) If A is a nonempty subset of R and B is a right ideal of R, then $A\Gamma B$ is a right ideal of R.

(iii) If A and B are left (right) ideals of R such that $A \cap B \neq \varnothing$, then $A \cap B$ is a left (right) ideal of R.

Proof. (i) Let $a \in R$ and $\alpha \in \Gamma$. Then $R\alpha a$ and $a\alpha R$ are subsemigroups of R. Obviously, if $x, r \in R$ and $\beta \in \Gamma$, then $r\beta(x\alpha a) = (r\beta x)\alpha a \in R\alpha a$. Therefore $R\alpha a$ is a left ideal of R.

To show that $a\alpha R$ is a right ideal of R, let $r \in R$, $y \in a\alpha R$ and $\beta \in \Gamma$. Then $y = \sum a\alpha r_i$ where $r_i \in R$ for all i. Thus $y\beta r = (\sum a\alpha r_i)\beta r = \sum(a\alpha r_i)\beta r = \sum a\alpha(r_i\beta r) \in a\alpha R$. Therefore $a\alpha R$ is a right ideal of R.

(ii) Let A be a nonempty subset of R and B a right ideal of R. Then $A\Gamma B$ is a subsemigroup of R from Proposition 2.1.1. Next, let $r \in R$, $x \in A\Gamma B$ and $\beta \in \Gamma$. Thus $x = \sum a_i\alpha_i b_i$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$ for all i. Moreover, $x\beta r = (\sum a_i\alpha_i b_i)\beta r = \sum(a_i\alpha_i b_i)\beta r = \sum a_i\alpha_i(b_i\beta r) \in A\Gamma B$. Thus $A\Gamma B$ is a right ideal of R.

(iii) Assume that A and B are left ideals of R with $A \cap B \neq \varnothing$. Clearly, $A \cap B$ is a subsemigroup of R. Let $r \in R$, $x \in A \cap B$ and $\alpha \in \Gamma$. Then $r\alpha x \in A$

and $r\alpha x \in B$ since A and B are left ideals of R so that $r\alpha x \in A \cap B$. Hence $A \cap B$ is a left ideal of R.

The proof for the case of right ideals is obtained similarly.

□

We can see from Theorem 2.2.5(iii) that $A \cap B \neq \varnothing$ may not hold if A and B are both left ideals or both right ideals of R. However, if A and B is a left ideal and a right ideal of R, respectively (or A and B is a right ideal and a left ideal of R, respectively), then $A \cap B \neq \varnothing$ since $b\alpha a \in A \cap B$ for all $a \in A$, $b \in B$ and $\alpha \in \Gamma$ ($a\alpha b \in A \cap B$ for all $a \in A$, $b \in B$ and $\alpha \in \Gamma$).

Corollary 2.2.6. *Let R be a Γ-seminearring. Then $a\Gamma R$, $(a\Gamma R)\Gamma R$ and $(R\Gamma a)\Gamma R$ are right ideals of R for any $a \in R$.*

Proof. This follows directly from Theorem 2.2.5 (ii) since R is a right ideal of R.

□

Let R be a Γ-seminearring, $a \in R$ and $\alpha \in \Gamma$. We see that $a\alpha R$ and $a\Gamma R$ are right ideals of R. However, $R\alpha a$ is a left ideal of R but $R\Gamma a$ need not be a left ideal of R since it is not necessary that $r\beta\left(\sum r_i\alpha_i a\right) = \sum r\beta(r_i\alpha_i a)$ for all $r, r_i \in R$ and $\beta, \alpha_i \in \Gamma$. Nevertheless, if R satisfies the left distributivity, then $R\Gamma a$ is; definitely, a left ideal of R because $r\beta\left(\sum r_i\alpha_i a\right) = \sum r\beta(r_i\alpha_i a) = \sum(r\beta r_i)\alpha_i a \in R\Gamma a$ for all $r, r_i \in R$ and $\beta, \alpha_i \in \Gamma$. However, we can weaken the condition that R satisfies the left distributivity and still obtain the same result. The idea of distributively generated Γ-seminearrings is needed.

Conveniently, we write na instead of $\overbrace{a + \cdots + a}^{n}$ for each $n \in \mathbb{N}$ and for each element a of a Γ-seminearring.

Definition 2.2.2. Let R be a Γ-seminearring under the mapping from $R \times \Gamma \times R$ into R, say f, and D be the set of all *distributive elements* of R, i.e., $D = \{d \in R \mid d\alpha(a+b) = d\alpha a + d\alpha b$ for all $a, b \in R$ and $\alpha \in \Gamma\}$. Then R is called *distributively generated* (or *d.g.* for short) if D is a nonempty subset of R which $f|_{D\times\Gamma\times D} : D \times \Gamma \times D \to D$ and $(\langle D\rangle, +) = (R, +)$ where

$$\langle D\rangle = \left\{ \sum n_i d_i \,|\, n \in \mathbb{N} \text{ and } d_i \in D \text{ for all } i \right\}$$

where $\sum n_i d_i \in \langle D\rangle$ is a finite sum.

In fact, $\langle D\rangle = \left\{ \sum d_i \mid d_i \in D \right\}$ where all d_i's in $\sum d_i$ may not be distinct. In addition, $\left(\langle D\rangle, + \right) = (R, +)$ means that every element in R can be written as a finite sum of distributive elements.

The following proposition shows the importance of the distributively generated property on the associative property of a Γ-seminearring.

Proposition 2.2.7. *Let R be a distributively generated Γ-seminearring and B and C be both nonempty subsets of R. Then $R\Gamma(B\Gamma C) = (R\Gamma B)\Gamma C$.*

Proof. It suffices to show only that $R\Gamma\,(B\Gamma C) \subseteq (R\Gamma B)\,\Gamma C$ as a result of Proposition 2.1.2. Let $x \in R\Gamma\,(B\Gamma C)$. Then $x = \sum r_i \alpha_i \left(\sum b_j \beta_j c_j\right)$ for some $r_i \in R$, $b_j \in B$, $c_j \in C$ and $\alpha_i, \beta_j \in \Gamma$ for all i, j. Since R is d.g., each r_i can be written as $\sum d_{ki}$ where all d_{ki}'s are distributive elements of R. Then

$$\begin{aligned}
x &= \sum \left(\sum d_{ki}\right) \alpha_i \left(\sum b_j \beta_j c_j\right) \\
&= \sum \sum \left(d_{ki}\alpha_i \left(\sum b_j \beta_j c_j\right)\right) \\
&= \sum \sum \sum \left(d_{ki}\alpha_i \left(b_j \beta_j c_j\right)\right) && \text{since each } d_{ki} \text{ is a distributive element of } R \\
&= \sum \sum \sum \left(\left(d_{ki}\alpha_i b_j\right) \beta_j c_j\right) \\
&\in (R\Gamma B)\,\Gamma C && \text{since } \left(d_{ki}\alpha_i b_j\right)\beta_j c_j \in (R\Gamma B)\,\Gamma C \text{ for all } i, j, k.
\end{aligned}$$

Therefore the claim is proved so $R\Gamma(B\Gamma C) = (R\Gamma B)\Gamma C$. □

As in Proposition 2.2.7, in fact, $R\Gamma(b\Gamma C) = (R\Gamma b)\Gamma C$, $R\Gamma(B\Gamma c) = (R\Gamma B)\Gamma c$ and $R\Gamma(b\Gamma c) = (R\Gamma b)\Gamma c$ where b and c are any elements of R.

Furthermore, for each element a of R, we have $(a\Gamma B)\Gamma C \subseteq a\Gamma(B\Gamma C)$ by Proposition 2.1.2; however, $a\Gamma(B\Gamma C) \subseteq (a\Gamma B)\Gamma C$ holds when a is a distributive element of R which can be shown in the same manner as the proof of Proposition 2.2.7.

Now, we are ready to provide when $R\Gamma a$ is a left ideal of a Γ-seminearring R containing a.

Theorem 2.2.8. *Let R be a distibutively generated Γ-seminearring.*

(i) If A is a left ideal of R and B is a nonempty subset of R, then $A\Gamma B$ is a left ideal of R.

(ii) If A is a left ideal and B is a right ideal of R, then $A\Gamma B$ is an ideal of R.

Proof. (i) Let A be a left ideal of R and B be a nonempty subset of R. Then $A\Gamma B$ is a subsemigroup of R. Moreover, let $r \in R$, $x \in A\Gamma B$ and $\beta \in \Gamma$. Then $x = \sum a_i\alpha_i b_i$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$ for all i. Since R is d.g., we have $r = \sum d_k$ where d_k is a distributive element for all k. Thus $r\beta x = (\sum d_k)\beta(\sum a_i\alpha_i b_i) = \sum \big(d_k\beta(\sum a_i\alpha_i b_i)\big)$ $= \sum\sum \big(d_k\beta(a_i\alpha_i b_i)\big)$ because each d_k is a distributive element of R. Then $r\beta x = \sum\sum \big(d_k\beta(a_i\alpha_i b_i)\big) = \sum\sum \big((d_k\beta a_i)\alpha_i b_i\big) \in A\Gamma B$ since A is a left ideal of R. Thus $A\Gamma B$ is a left ideal of R.

(ii) This is a result of (i) and Theorem 2.2.5 (ii).

□

Corollary 2.2.9. *Let R be a distibutively generated Γ-seminearring and $a \in R$. Then $R\Gamma a$ and $R\Gamma(R\Gamma a)$ are left ideals of R. Furthermore, $R\Gamma R$, $(R\Gamma a)\Gamma R$ and $R\Gamma(a\Gamma R)$ are ideals of R.*

Proof. This follows directly from Theorem 2.2.8 and Corollary 2.2.6. □

Besides, in Corollary 2.2.9, the ideals $(R\Gamma a)\Gamma R$ and $R\Gamma(a\Gamma R)$ are actually the same by Proposition 2.2.7.

For a given Γ-seminearring, the principal left ideals, principal right ideals and principal ideals are able to be studied.

Definition 2.2.3. Let R be a Γ-seminearring and $a \in R$. Then the smallest left (right) ideal containing a is called *the principal left (right) ideal generated by* a and is denoted by $\langle a|$ ($|a\rangle$). Similarly, the smallest ideal of R containing a is called *the principal ideal generated by* a and is denoted by $\langle a \rangle$.

Theorem 2.2.10. *Let R be a Γ-seminearring. Then for each element a of R,*

$$|a\rangle = \left\{pa \;\middle|\; p \in \mathbb{N}\right\} \cup \left\{ \sum_{j=1}^{l} a\beta_j s_j \;\middle|\; l \in \mathbb{N}, s_j \in R \text{ and } \beta_j \in \Gamma \text{ for all } j \right\}$$
$$\cup \left\{ na + \sum_{i=1}^{m} a\alpha_i r_i \;\middle|\; n, m \in \mathbb{N},\, r_i \in R \text{ and } \alpha_i \in \Gamma \text{ for all } i \right\}.$$

Proof. Let $L = \{pa \mid p \in \mathbb{N}\} \cup \{\sum_{j=1}^{l} a\beta_j s_j \mid l \in \mathbb{N},\ s_j \in R \text{ and } \beta_j \in \Gamma \text{ for all } j\} \cup \{na + \sum_{i=1}^{m} a\alpha_i r_i \mid n, m \in \mathbb{N},\ r_i \in R \text{ and } \alpha_i \in \Gamma \text{ for all } i\}$. First, we show that L is a right ideal of R containing a. Notice that $L \neq \varnothing$ since $a = 1a \in L$. Moreover, it is clear that $x + y \in L$ for any $x, y \in L$. Next, let $r \in R$, $x \in L$ and $\alpha \in \Gamma$. Then $x = pa$ or $x = \sum a\beta_j s_j$ or $x = na + \sum a\alpha_i r_i$ where $p, n \in \mathbb{N}$, $s_j, r_i \in R$ and $\beta_j, \alpha_i \in \Gamma$ for all i, j. Without loss of generality, only the case when $x = na + \sum a\alpha_i r_i$ is considered here since the proofs of other cases are obtained similarly. We see that

$$\begin{aligned} x\alpha r &= \Big(na + \sum a\alpha_i r_i\Big)\alpha r \\ &= (na)\alpha r + \sum \big((a\alpha_i r_i)\alpha r\big) \\ &= (\overbrace{a + \cdots + a}^{n})\alpha r + \sum \big(a\alpha_i(r_i\alpha r)\big) \\ &= (\overbrace{a\alpha r + \cdots + a\alpha r}^{n}) + \sum \big(a\alpha_i(r_i\alpha r)\big) \\ &= n(a\alpha r) + \sum \big(a\alpha_i(r_i\alpha r)\big) \in L \end{aligned}$$

since both of $n(a\alpha r)$ and $\sum \big(a\alpha_i(r_i\alpha r)\big)$ are of the form $\sum a\beta_j s_j$ where $s_j \in R$ and $\beta_j \in \Gamma$ for all j. This shows that L is a right ideal of R containing a as desired. Hence $|a\rangle \subseteq L$.

Next, to show that $L \subseteq |a\rangle$, let $x \in L$. Then $x = pa$ or $x = \sum a\beta_j s_j$ or $x = na + \sum a\alpha_i r_i$ for some $p, n \in \mathbb{N}$, $s_j, r_i \in R$ and $\beta_j, \alpha_i \in \Gamma$ for all i, j. We consider when $x = na + \sum a\alpha_i r_i$ only. Since $|a\rangle$ is a right ideal of R containing a, this implies that $na, a\alpha_i r_i \in |a\rangle$ for all i so that $x = na + \sum a\alpha_i r_i \in |a\rangle$. Thus $L \subseteq |a\rangle$.

Consequently, $L = |a\rangle$. □

Theorem 2.2.11. *Let R be a distributively generated Γ-seminearring. For each element a of R,*

$$\begin{aligned} \langle a| = \Big\{pa \;\Big|\; p \in \mathbb{N}\Big\} &\cup \left\{ \sum_{j=1}^{l} s_j\beta_j a \;\middle|\; l \in \mathbb{N}, s_j \in R \text{ and } \beta_j \in \Gamma \text{ for all } j \right\} \\ &\cup \left\{ na + \sum_{i=1}^{m} r_i\alpha_i a \;\middle|\; n, m \in \mathbb{N}, r_i \in R \text{ and } \alpha_i \in \Gamma \text{ for all } i \right\} \quad \text{and} \end{aligned}$$

$$\langle a\rangle = A_1 \cup A_2 \cup A_3 \quad \text{where} \quad A_1 = \{pa \,|\, p \in \mathbb{N}\},$$

$$A_2 = \Big\{ \sum a\alpha_i r_i + \sum s_j\beta_j a + \sum u_k\gamma_k a\lambda_k v_k \ \Big|\ r_i, s_j, u_k, v_k \in R, \\ \alpha_i, \beta_j, \gamma_k, \lambda_k \in \Gamma \textit{ for all } i, j, k \Big\}$$

$$A_3 = \Big\{ na + \sum a\alpha_i r_i + \sum s_j\beta_j a + \sum u_k\gamma_k a\lambda_k v_k \ \Big|\ n \in \mathbb{N},\ r_i, s_j, u_k, v_k \in R, \\ \alpha_i, \beta_j, \gamma_k, \lambda_k \in \Gamma \textit{ for all } i, j, k \Big\}.$$

Here, those $\sum a\alpha_i r_i$, $\sum s_j\beta_j a$ and $\sum u_k\gamma_k a\lambda_k v_k$ are finite sums whose lengths are not necessary the same.

Proof. We prove that $\langle a\rangle = A_1 \cup A_2 \cup A_3$ only since the first result can be shown similarly. Let $I = A_1 \cup A_2 \cup A_3$. We show that I is an ideal of R containing a so that $\langle a\rangle \subseteq I$, then we show that $I \subseteq \langle a\rangle$.

Clearly, $a \in A_1$ and $x + y \in I$ for all $x, y \in I$. Let $r \in R$, $x \in I$ and $\alpha \in \Gamma$. This implies that $x \in A_1$ or $x \in A_2$ or $x \in A_3$. Without loss of generality, we consider only the case that $x \in A_3$. Then $x = na + \sum a\alpha_i r_i + \sum s_j\beta_j a + \sum u_k\gamma_k a\lambda_k v_k$ for some $n \in \mathbb{N}$, $r_i, s_j, u_k, v_k \in R$ and $\alpha_i, \beta_j, \gamma_k, \lambda_k \in \Gamma$ for all i, j, k. The fact that $x\alpha r \in A_2 \subseteq I$ is obtained in the same manner of the proof showing that "$x\alpha r \in L$" in Theorem 2.2.10. It remains to explain that $r\alpha x \in I$. Since R is d.g., we can write r as $\sum d_l$ where each d_l is a distributive element of R. Then

$$\begin{aligned}
r\alpha x &= \Big(\sum d_l\Big)\alpha\Big(na + \sum a\alpha_i r_i + \sum s_j\beta_j a + \sum u_k\gamma_k a\lambda_k v_k\Big) \\
&= \sum \Big(d_l\alpha\Big(na + \sum a\alpha_i r_i + \sum s_j\beta_j a + \sum u_k\gamma_k a\lambda_k v_k\Big)\Big) \\
&= \sum \Big(d_l\alpha(na) + d_l\alpha\Big(\sum a\alpha_i r_i\Big) + d_l\alpha\Big(\sum s_j\beta_j a\Big) + d_l\alpha\Big(\sum u_k\gamma_k a\lambda_k v_k\Big)\Big) \\
&= \sum \Big(d_l\alpha(\overbrace{a + \cdots + a}^{n}) + \sum d_l\alpha(a\alpha_i r_i) + \sum d_l\alpha(s_j\beta_j a) + \sum d_l\alpha(u_k\gamma_k a\lambda_k v_k)\Big) \\
&= \sum \Big((\overbrace{d_l\alpha a + \cdots + d_l\alpha a}^{n}) + \sum d_l\alpha a\alpha_i r_i + \sum (d_l\alpha s_j)\beta_j a + \sum (d_l\alpha u_k)\gamma_k a\lambda_k v_k\Big) \\
&= \sum \Big(n(d_l\alpha a) + \sum d_l\alpha a\alpha_i r_i + \sum (d_l\alpha s_j)\beta_j a + \sum (d_l\alpha u_k)\gamma_k a\lambda_k v_k\Big) \\
&\in I
\end{aligned}$$

since each of $n(d_l\alpha a)$, $\sum d_l\alpha a\alpha_i r_i$, $\sum(d_l\alpha s_j)\beta_j a$ and $\sum(d_l\alpha u_k)\gamma_k a\lambda_k v_k$ is an element of A_2. We conclude that I is an ideal of R containing a. Thus $\langle a\rangle \subseteq I$.

Finally, in order to prove that $I \subseteq \langle a\rangle$, let $x \in I$. Without loss of generality, $x = na + \sum a\alpha_i r_i + \sum s_j\beta_j a + \sum u_k\gamma_k a\lambda_k v_k$ for some $n \in \mathbb{N}$, $r_i, s_j, u_k, v_k \in R$ and $\alpha_i, \beta_j, \gamma_k, \lambda_k \in \Gamma$ for all i, j, k. Since $\langle a\rangle$ is an ideal of R containing a, this implies that na, $a\alpha_i r_i$, $s_j\beta_j a$ and $u_k\gamma_k a\lambda_k v_k$ are elements of $\langle a\rangle$ for all i, j, k. Then $x \in \langle a\rangle$ so that $I \subseteq \langle a\rangle$.

As a result, $I = \langle a\rangle$. □

The notion of prime ideals of Γ-seminearrings and prime Γ-seminearrings is also reasonable to be investigated.

Definition 2.2.4. Let R be a Γ-seminearring. An ideal P of R is called a *prime ideal* if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of R.

Furthermore, if R is zero-symmetric, then R is called a *prime* Γ*-seminearring* if the zero ideal is prime.

Theorem 2.2.12. *Let R be a Γ-seminearring and P an ideal of R. Then P is prime if and only if $\langle a\rangle \Gamma \langle b\rangle \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$.*

Proof. Let P be prime and $a, b \in R$. Assume that $\langle a\rangle \Gamma \langle b\rangle \subseteq P$. Since P is prime, this implies that $a \in P$ or $b \in P$.

On the other hand, assume that $\langle a\rangle \Gamma \langle b\rangle \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$. To show that P is prime, let A and B be ideals of R such that $A\Gamma B \subseteq P$. Suppose that $A \nsubseteq P$. Then there exists $a \in A$ but $a \notin P$. Let $b \in B$. Thus $\langle a\rangle \Gamma \langle b\rangle \subseteq A\Gamma B \subseteq P$. From the assumption, $a \in P$ or $b \in P$. Since $a \notin P$, this implies that $b \in P$. Thus $B \subseteq P$.

Consequently, P is prime. □

Let P be an ideal of a Γ-seminearring R. The necessary condition for P to be prime can be regarded only from principal left ideals or principal right ideals of R rather than from principal ideals of R.

Theorem 2.2.13. *Let R be a Γ-seminearring and P an ideal of R. Then P is prime if $|a\rangle\Gamma|b\rangle \subseteq P$ implies $a \in P$ or $b \in P$ for any $a, b \in R$.*

Proof. Assume that if $|a\rangle\Gamma|b\rangle \subseteq P$, then $a \in P$ or $b \in P$ for any $a, b \in R$. To show that P is prime, let A and B be ideals of R such that $A\Gamma B \subseteq P$ but $A \nsubseteq P$. Then there exists $a \in A$ but $a \notin P$. Let $b \in B$. Thus $|a\rangle\Gamma|b\rangle \subseteq A\Gamma B \subseteq P$. This implies that $b \in P$ so that $B \subseteq P$. Therefore P is prime. □

Theorem 2.2.14. *Let R be a Γ-seminearring and P an ideal of R. Then P is prime if $\langle a|\Gamma\langle b| \subseteq P$ implies $a \in P$ or $b \in P$ for all $a, b \in R$.*

Proof. The proof is analogous to the one of Theorem 2.2.13. □

Finally, in this section, we show that the nonempty intersection of ideals of a Γ-seminearring with at least one prime ideal is still a prime ideal.

Theorem 2.2.15. *Let R be a Γ-seminearring, B an ideal of R and P a prime ideal of R. Then $B \cap P$ is a prime ideal of B.*

Proof. We can see that $B \cap P$ is a nonempty subset of R. It follows from Theorem 2.2.5(iii) that $B \cap P$ is an ideal of R and then it is an ideal of B. To show that $B \cap P$ is prime, let A and C be ideals of B such that $A\Gamma C \subseteq B \cap P$. This implies that $A\Gamma C \subseteq B$ and $A\Gamma C \subseteq P$. Since P is prime, $A \subseteq P$ or $C \subseteq P$. Thus $A \subseteq B \cap P$ or $C \subseteq B \cap P$. Therefore $B \cap P$ is a prime ideal of B. □

Proposition 2.2.16. *Let R be a Γ-seminearring, B and P prime ideals of R. Then $B \cap P$ is a prime ideal of R.*

Proof. Clearly, $B \cap P$ is a nonempty subset of R. By Theorem 2.2.5(iii), $B \cap P$ is an ideal of R. To show that $B \cap P$ is prime, let A and C be ideals of R such that $A\Gamma C \subseteq B \cap P$. This implies that $A\Gamma C \subseteq B$ and $A\Gamma C \subseteq P$. Since B and P are prime, this implies that $A \subseteq B \cap P$ or $C \subseteq B \cap P$. Therefore $B \cap P$ is a prime ideal of R. □

2.3 Γ-Homomorphisms

The final section is assigned to explore Γ-homomorphisms of Γ-seminearrings. The quotient Γ-seminearrings are defined and the first isomorphism theorem for this new Γ-structure is obtained.

Definition 2.3.1. Let R and S be Γ-seminearrings and θ a map from R into S. Then θ is called a *Γ-homomorphism* if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Besides, the *image* of the Γ-homomorphism θ, denoted by $\operatorname{im}\theta$, is defined in the usual way as follows: $\operatorname{im}\theta = \{\theta(x) \mid x \in R\}$.

Furthermore, the Γ-homomorphism θ is called a *Γ-isomorphism* if θ is also a bijection and in this case we say that R and S are *Γ-isomorphic.*

Throughout this thesis, "θ is a Γ-homomorphism from R into S" means that θ is a Γ-homomorphism from a Γ-seminearring R into a Γ-seminearring S.

Definition 2.3.2. Let R be a Γ-seminearring. Then R is said to be *commutative* if $x\alpha y = y\alpha x$ for all $x, y \in R$ and $\alpha \in \Gamma$.

We give elementary properties of Γ-homomorphisms.

Proposition 2.3.1. *Let θ be a Γ-homomorphism from R into S.*

(i) If A is a sub Γ-seminearring of R, then $\theta(A)$ is a sub Γ-seminearring of S.

(ii) If B is a sub Γ-seminearring of S and $\theta^{-1}(B) \neq \varnothing$, then $\theta^{-1}(B)$ is a sub Γ-seminearring of R.

(iii) $\operatorname{im}\theta$ *is a sub Γ-seminearring of S.*

(iv) If R is commutative, then so is $\operatorname{im}\theta$.

Proof. (i) Assume that A is a sub Γ-seminearring of R. Obviously, $\theta(A)$ is a nonempty subset of S. Let $a, b \in A$ and $\alpha \in \Gamma$. Then $\theta(a) + \theta(b) = \theta(a+b) \in \theta(A)$ and $\theta(a)\alpha\theta(b) = \theta(a\alpha b) \in \theta(A)$ since A is a Γ-seminearring. Thus $\theta(A)$ is a sub Γ-seminearring of S.

(ii) Assume that B is a sub Γ-seminearring of S and $\theta^{-1}(B) \neq \varnothing$. Let $x, y \in \theta^{-1}(B)$ and $\alpha \in \Gamma$. Then $\theta(x), \theta(y) \in B$. Since $\theta(x+y) = \theta(x) + \theta(y) \in B$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y) \in B$ because B is a Γ-seminearring. This implies that $x + y, x\alpha y \in \theta^{-1}(B)$. Thus $\theta^{-1}(B)$ is a sub Γ-seminearring of R.

(iii) This follows from (i) and the fact that $\operatorname{im}\theta = \theta(R)$.

(iv) If R is commutative, then $\theta(a)\alpha\theta(b) = \theta(a\alpha b) = \theta(b\alpha a) = \theta(b)\alpha\theta(a)$ for any $a, b \in R$ and $\alpha \in \Gamma$. Thus $\operatorname{im}\theta$ is commutative.

□

For Γ-seminearrings R and S having the identities 0_R and 0_S, respectively, and for a Γ-homomorphism θ from R into S, it can be proved that $\theta(0_R)$ and 0_S are identical providing a particular condition.

Proposition 2.3.2. *Let R and S be Γ-seminearrings having the identities 0_R and 0_S, respectively, θ a Γ-homomorphism from R into S. If the semigroup S has the left cancellation or the right cancellation, then $\theta(0_R) = 0_S$.*

Proof. Without loss of generality, assume that the semigroup S has the left cancellation. Then $\theta(0_R) + \theta(0_R) = \theta(0_R + 0_R) = \theta(0_R) = \theta(0_R) + 0_S$. Therefore $\theta(0_R) = 0_S$. □

Unlike the previous fact, a Γ-homomorphism sends a left (right) zero and a left (right) identity to a left (right) zero and a left (right) identity, respectively, without additional condition.

Proposition 2.3.3. *Let θ be a Γ-homomorphism from R into S.*

(i) If $x \in R$ is a left (right) zero of R, then $\theta(x)$ is a left (right) zero of $\theta(R)$.

(ii) If $e \in R$ is a left (right) identity of R, then $\theta(e)$ is a left (right) identity of $\theta(R)$.

Proof. (i) Let $x \in R$ be a left zero, $y \in R$ and $\alpha \in \Gamma$. Then $\theta(x)\alpha\theta(y) = \theta(x\alpha y) = \theta(x)$, i.e., $\theta(x)$ is a left zero of $\theta(R)$.

(ii) Let $e \in R$ be a left identity, $x \in R$ and $\alpha \in \Gamma$. Then $\theta(e)\alpha\theta(x) = \theta(e\alpha x) = \theta(x)$ so that $\theta(e)$ is a left identity of $\theta(R)$.

The proofs for the cases of right zeros and right identities are obtained analogously. □

Corollary 2.3.4. *Let θ be a Γ-homomorphism from R into S.*

(i) If R has the zero, then $\theta(R)$ has the zero.

(ii) If R has the identity, then $\theta(R)$ has the identity.

Proof. This follows directly from Proposition 2.3.3, Proposition 2.1.3 and Proposition 2.1.4. □

More properties of Γ-homomorphisms are provided.

Proposition 2.3.5. *Let θ be a Γ-homomorphism from R into S. If A and B are nonempty subsets of R, then $\theta(A)\Gamma\theta(B) = \theta(A\Gamma B)$.*

Proof. Let A and B be nonempty subsets of R. To show that $\theta(A)\Gamma\theta(B) \subseteq \theta(A\Gamma B)$, let $x \in \theta(A)\Gamma\theta(B)$. Then $x = \sum\big(\theta(a_i)\alpha_i\theta(b_i)\big)$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$ for all i. Since θ is a Γ-homomorphism, $x = \sum\big(\theta(a_i)\alpha_i\theta(b_i)\big) = \sum\theta(a_i\alpha_i b_i) = \theta\big(\sum(a_i\alpha_i b_i)\big) \in \theta(A\Gamma B)$. Thus $\theta(A)\Gamma\theta(B) \subseteq \theta(A\Gamma B)$.

On the other hand, let $x \in \theta(A\Gamma B)$. Then $x = \theta\big(\sum a_i\alpha_i b_i\big)$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$ for all i. Moreover, $x = \theta\big(\sum a_i\alpha_i b_i\big) = \sum\theta(a_i\alpha_i b_i) = \sum\big(\theta(a_i)\alpha_i\theta(b_i)\big) \in \theta(A)\Gamma\theta(B)$ This shows that $\theta(A\Gamma B) \subseteq \theta(A)\Gamma\theta(B)$.

As a result, $\theta(A)\Gamma\theta(B) = \theta(A\Gamma B)$. □

Proposition 2.3.6. *Let θ and ϕ be Γ-homomorphisms from R into S and from S into T, respectively. Then $\phi\circ\theta$ is a Γ-homomorphism from R into T.*

Proof. Let $x, y \in R$ and $\alpha \in \Gamma$. Then

$$(\phi\circ\theta)(x+y) = \phi(\theta(x)+\theta(y)) = \phi(\theta(x)) + \phi(\theta(y)) = (\phi\circ\theta)(x) + (\phi\circ\theta)(y) \text{ and}$$

$$(\phi\circ\theta)(x\alpha y) = \phi\big(\theta(x)\alpha\theta(y)\big) = \phi\big(\theta(x)\big)\alpha\phi\big(\theta(y)\big) = \Big((\phi\circ\theta)(x)\Big)\alpha\Big((\phi\circ\theta)(y)\Big).$$

As a result, $\phi\circ\theta$ is a Γ-homomorphism from R into T. □

Theorem 2.3.7. *Let θ be a Γ-homomorphism from R into S.*

(i) If A is a left (right) ideal of R, then $\theta(A)$ is a left (right) ideal of $\theta(R)$.

(ii) If B is a left (right) ideal of S and $\theta^{-1}(B) \neq \varnothing$, then $\theta^{-1}(B)$ is a left (right) ideal of R.

(iii) If A is an ideal of R, then $\theta(A)$ is an ideal of $\theta(R)$.

(iv) If B is an ideal of S and $\theta^{-1}(B) \neq \varnothing$, then $\theta^{-1}(B)$ is an ideal of R.

Proof. (i) Assume that A is a left ideal of R. Then A is a sub Γ-seminearring of R. By Proposition 2.3.1, $\theta(A)$ is a sub Γ-seminearring of S. Let $r \in R$, $a \in A$ and $\alpha \in \Gamma$. Then $\theta(r)\alpha\theta(a) = \theta(r\alpha a) \in \theta(A)$ since θ is a Γ-homomorphism and A is left ideal of R. Therefore $\theta(A)$ is a left ideal of $\theta(R)$.

The proof for the case right ideals is obtained similarly.

(ii) Assume that B is a left ideal of S and $\theta^{-1}(B) \neq \varnothing$. Then B is a sub Γ-seminearring of S. Proposition 2.3.1 gives that $\theta^{-1}(B)$ is a sub Γ-seminearring of R. Let $r \in R$, $x \in \theta^{-1}(B)$ and $\alpha \in \Gamma$. Then $\theta(r\alpha x) = \theta(r)\alpha\theta(x) \in B$ so that $r\alpha x \in \theta^{-1}(B)$. Therefore $\theta^{-1}(B)$ is a left ideal of R.

The proof for the case right ideals follows analogously from the above proof.

(iii) and (iv) are immediate results of (i) and (ii), respectively. □

It is appropriate place to introduce quotient Γ-seminearrings and explore related propreties.

Theorem 2.3.8. *Let R be a Γ-seminearring and I an ideal of the semigroup R. Then the Rees quotient semigroup R/I is a Γ-seminearring.*

Proof. Recall that R is also a semigroup so that R/I is a semigroup by Theorem 1.2.2. Next, define a mapping $R/I \times \Gamma \times R/I \to R/I$ by $(x+I, \alpha, y+I) \mapsto (x\alpha y) + I$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Let $x_1, x_2, y_1, y_2 \in R$ and $\alpha, \beta \in \Gamma$. Assume that $x_1 + I = x_2 + I$ and $y_1 + I = y_2 + I$. Then $x_1 = x_2$ or $x_1, x_2 \in I$ and $y_1 = y_2$ or $y_1, y_2 \in I$, respectively. If $x_1, x_2 \in I$ and $y_1, y_2 \in I$, then $x_1\alpha y_1$, $x_2\alpha y_2 \in I$, and then $x_1\alpha y_1 + I = x_2\alpha y_2 + I$. For the other cases, they are clear that $x_1\alpha y_1 + I = x_2\alpha y_2 + I$. Moreover, $(x_1 + I)\alpha(y_1 + I) = (x_1\alpha y_1) + I \in R/I$. Thus this mapping is well-defined. Next, the associativity and the right distributivity are concerned. We see that

$$\begin{aligned}
\big((x_1 + I)\alpha(x_2 + I)\big)\beta(y_1 + I) &= \big((x_1\alpha x_2)\beta y_1\big) + I = \big(x_1\alpha(x_2\beta y_1)\big) + I \\
&= (x_1 + I)\alpha\big((x_2 + I)\beta(y_1 + I)\big) \quad \text{and} \\
\big((x_1 + I) + (x_2 + I)\big)\alpha(y_1 + I) &= \big((x_1 + x_2)\alpha y_1\big) + I \\
&= (x_1\alpha y_1 + x_2\alpha y_1) + I \\
&= \big((x_1\alpha y_1) + I\big) + \big((x_2\alpha y_1) + I\big) \\
&= \big((x_1 + I)\alpha(y_1 + I)\big) + \big((x_2 + I)\alpha(y_1) + I)\big).
\end{aligned}$$

Hence R/I is a Γ-seminearring. □

Definition 2.3.3. Let R be a Γ-seminearring and I an ideal of the semigroup R. Then R/I in Theorem 2.3.8 is called the *quotient* Γ*-seminearring* of R by I.

Theorem 2.3.9. *Let θ be a Γ-homomorphism from R into S. Then the relation $\kappa = \{(x, y) \in R \times R \,|\, \theta(x) = \theta(y)\}$ is a congruence on R.*

Proof. It is obvious that κ is an equivalence relation because of the property of the equal relation.

Next, let $x, y, z \in R$ be such that $(x, y) \in \kappa$. Then $\theta(x) = \theta(y)$. Since θ is a Γ-homomorphism, $\theta(x + z) = \theta(x) + \theta(z) = \theta(y) + \theta(z) = \theta(y + z)$. This shows that $(x + z, y + z) \in \kappa$. Similarly, $(z + x, z + y) \in \kappa$. Therefore, κ is a congruence on R. □

Definition 2.3.4. Let θ be a Γ-homomorphism from R into S. The *kernel* of θ, denoted by $\ker\theta$, is defined to be the congruence

$$\ker\theta = \{(x,y) \in R \times R \,|\, \theta(x) = \theta(y)\}.$$

Theorem 2.3.10. *Let θ be a Γ-homomorphism from R into S. Then $R/\ker\theta$ is a Γ-seminearring.*

Proof. Since $\ker\theta$ is a congruence on R, Theorem 1.2.1 gives that $(R/\ker\theta, +)$ is a semigroup where $(x+\ker\theta)+(y+\ker\theta) = (x+y)+\ker\theta$ for all $x, y \in R$. Define a mapping $R/\ker\theta \times \Gamma \times R/\ker\theta \to R/\ker\theta$ by $(x+\ker\theta, \alpha, y+\ker\theta) \mapsto (x\alpha y)+\ker\theta$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Let $x_1, x_2, y_1, y_2 \in R$ and $\alpha, \beta \in \Gamma$. Assume that $x_1+\ker\theta = x_2+\ker\theta$ and $y_1+\ker\theta = y_2+\ker\theta$. Then $(x_1,x_2), (y_1,y_2) \in \ker\theta$ so that $\theta(x_1) = \theta(x_2)$ and $\theta(y_1) = \theta(y_2)$. Thus $\theta(x_1\alpha y_1) = \theta(x_1)\alpha\theta(y_1) = \theta(x_2)\alpha\theta(y_2) = \theta(x_2\alpha y_2)$. Hence $(x_1\alpha y_1, x_2\alpha y_2) \in \ker\theta$, so $(x_1\alpha y_1)+\ker\theta = (x_2\alpha y_2)+\ker\theta$. We also obtain that $(x_1+\ker\theta)\alpha(y_1+\ker\theta) = (x_1\alpha y_1)+\ker\theta \in R/\ker\theta$. This shows that the mapping is well-defined.

Let $x_1, x_2, y_1 \in R$ and $\alpha, \beta \in \Gamma$. The associativity holds as

$$\begin{aligned}
&\big((x_1+\ker\theta)\alpha(x_2+\ker\theta)\big)\beta(y_1+\ker\theta) \\
&= \big((x_1\alpha x_2)\beta y_1\big)+\ker\theta \\
&= \big(x_1\alpha(x_2\beta y_1)\big)+\ker\theta \\
&= (x_1+\ker\theta)\alpha\big((x_2+\ker\theta)\beta(y_1+\ker\theta)\big)
\end{aligned}$$

so does the right distributivity because

$$\begin{aligned}
&\big((x_1 + \ker\theta) + (x_2 + \ker\theta)\big)\alpha(y_1 + \ker\theta) \\
&= \big((x_1 + x_2)\alpha y_1\big) + \ker\theta \\
&= (x_1\alpha y_1 + x_2\alpha y_1) + \ker\theta \\
&= \big((x_1\alpha y_1) + \ker\theta\big) + \big((x_2\alpha y_1) + \ker\theta\big) \\
&= \big((x_1 + \ker\theta)\alpha(y_1 + \ker\theta)\big) + \big((x_2 + \ker\theta)\alpha(y_1 + \ker\theta)\big).
\end{aligned}$$

Hence $R/\ker\theta$ is a Γ-seminearring. □

Theorem 2.3.11. *Let R be a Γ-seminearring and I an ideal of the semigroup R. Then the map $\varphi : R \to R/I$ defined by $\varphi(r) = r + I$ for all $r \in R$ is a surjective Γ-homomorphism whose kernel is ρ_I.*

This Γ-homomorphism is called the *natural (or canonical) Γ-homomorphism from R onto R/I.*

Proof. We see that R/I forms a Γ-seminearring by Theorem 2.3.8 and the mapping φ is obviously surjective. For all $x, y \in R$ and $\alpha \in \Gamma$,

$$\varphi(x + y) = (x + y) + I = (x + I) + (y + I) = \varphi(x) + \varphi(y) \quad \text{and}$$

$$\varphi(x\alpha y) = (x\alpha y) + I = (x + I)\alpha(y + I) = \varphi(x)\alpha\varphi(y).$$

Hence φ is a Γ-homomorphism.

Next, we show that $\ker\varphi$ and ρ_I are exactly the same as follows:

$$\begin{aligned}
\ker\varphi &= \big\{(x, y) \in R \times R \,|\, \varphi(x) = \varphi(y)\big\} \\
&= \big\{(x, y) \in R \times R \,|\, x + I = y + I\big\} \\
&= \big\{(x, y) \in R \times R \,|\, [x]_{\rho_I} = [y]_{\rho_I}\big\} \\
&= \big\{(x, y) \in R \times R \,|\, x = y \text{ or } x, y \in I\big\} = \rho_I.
\end{aligned}$$

□

Proposition 2.3.12. *Let θ be a Γ-homomorphism from R into S and I an ideal of the semigroup R.*

(i) If R is commutative, then R/I is commutative.

(ii) If R has the identity, then R/I has the identity.

(iii) If R has the zero, then R/I has the zero.

Proof. (i) This follows from Theorem 2.3.11 that $R/I = \varphi(R)$ where φ is the natural Γ-homomorphism and from Propsition 2.3.1(iv).

(ii) and (iii) are benefits of Theorem 2.3.11 and Corollary 2.3.4. □

Finally, the first isomorphism theorem for Γ-seminearrings is given.

Theorem 2.3.13. (***The First Isomorphism Theorem***)

Let θ be a Γ-homomorphism from R into S. Then $R/\ker\theta$ is Γ-isomorphic to $\operatorname{im}\theta$.

Proof. We obtain from Theorem 2.3.10 that $R/\ker\theta$ is a Γ-seminearring. Define $\phi : R/\ker\theta \to \operatorname{im}\theta$ by $\phi(r+\ker\theta) = \theta(r)$ for all $r \in R$. Then ϕ is well-defined since $\theta(r_1) = \phi(r_1 + \ker\theta) = \phi(r_2 + \ker\theta) = \theta(r_2)$ for any elements r_1 and r_2 of R such that $r_1 + \ker\theta = r_2 + \ker\theta$. Let $r, s \in R$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\phi\big((r+\ker\theta)+(s+\ker\theta)\big) &= \phi\big((r+s)+\ker\theta\big) = \theta(r+s)\\ &= \theta(r)+\theta(s) = \phi(r+\ker\theta)+\phi(s+\ker\theta) \quad \text{and}\\ \phi\big((r+\ker\theta)\alpha(s+\ker\theta)\big) &= \phi\big((r\alpha s)+\ker\theta\big) = \theta(r\alpha s)\\ &= \theta(r)\alpha\theta(s) = \phi(r+\ker\theta)\alpha\phi(s+\ker\theta).\end{aligned}$$

Thus ϕ is a Γ-homomorphism.

Moreover, if $\theta(r) = \theta(s)$, then $(r, s) \in \ker\theta$ so that $r+\ker\theta = s+\ker\theta$. Thus ϕ is one-to-one.

In addition, if $\theta(r) \in \operatorname{im}\theta$, then $r + \ker\theta \in R/\ker\theta$ and $\phi(r + \ker\theta) = \theta(r)$. Thus ϕ is onto. Consequently, $R/\ker\theta$ is Γ-isomorphic to $\operatorname{im}\theta$. $\square$