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SURJECTIONS AND WEAK FORMS OF THE AXIOM OF CHOICE

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If there is an injection from a nonempty set A into a set B, then there is always a surjection from B onto A. Without the Axiom of Choice (AC), the converse of this statement is not necessarily true. There are many other consequences of AC concerning surjections. We study these consequences and introduce some new weak forms as well as equivalent forms of AC and show some relationships among them.

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CONTENTS

CHAPTER I
INTRODUCTION

The Zermelo-Fraenkel set theory (ZF) is the most widely accepted axiomatic system of mathematics. It is based on a collection of axioms introduced by *Ernst Zermelo* (1871 - 1953) and *Abraham Fraenkel* (1891 - 1965). For more details on ZF see [2] and [7] for example.

The Axiom of Choice (AC), initiated by *Zermelo*, is an important axiom in mathematics and is independent from ZF. Many important theorems in many areas of mathematics are obtained by AC. AC has many equivalent forms, for example:

- *Well-Ordering Theorem*: every set can be well-ordered.

- *Existence of Bases*: every vector space has a basis.

- *Zorn's Lemma*: every nonempty partially ordered set in which every chain has an upper bound contains a maximal element.

- *Tychonoff's Theorem*: the product of a family of compact spaces is compact in the product topology.

Many consequences of AC are very useful in mathematics, for example: every infinite set has a denumerable subset, a countable union of countable sets is countable, and an arbitrary product of nonempty sets is nonempty. However, AC is a controversial axiom because of its nonconstructive method. AC may lead to some paradoxes, for instance: the existence of non-measurable subsets of $\mathbb{R}$ and *Banach-Tarski Paradox*. So we sometimes use weaker forms of AC in order to avoid such paradoxes. For more details on AC see [8].

An important issue about AC concerns surjections. If there is an injection from a nonempty set X into a set Y, then there is always a surjection from Y onto

X. Without AC, the converse of this statement is not necessarily true. This is a consequence of AC. There are also other consequences of AC concerning surjections.

In this work, we introduce some new equivalent forms as well as some new weak forms of AC concerning surjections and investigate some relationships among them.

The thesis is organized as follows. Chapter II gives preliminaries which provide all basic concepts needed for this work. Chapter III discusses consequences of AC concerning surjections. Our new equivalent forms and consequences of AC are introduced. We also give some relationships among these new forms and other relevant consequences of AC.

CHAPTER II
PRELIMINARIES

In this chapter, we give background on some concepts in set theory which are needed for later work. All basic notions and definitions in set theory are defined in the usual way. All theorems in this chapter are basic theorems which can be proved in Zermelo-Fraenkel set theory (ZF). So their proofs will be omitted. The details of Section 2.1-2.8 can be found in any axiomatic set theory textbooks, see [2] for example. For Section 2.9 see [5] and [10].

Throughout this work, we use $a, b, c, \ldots, A, B, C, \ldots$ and these letters with subscripts for sets. We write $\mathcal{P}(X)$ for the power set of X, $R[X]$ for the image of X under a relation R, R^{-1} for the inverse relation, $dom(R)$ for the domain and $ran(R)$ for the range of a relation R, $\bigcup X$ for the union of X, $X \times Y$ for the Cartesian product and $X \dot{\cup} Y$ for the disjoint union of X and Y.

2.1 Cardinal Numbers

Intuitively, the **cardinality** of a set is the number of all elements of a set. It is defined so that any two sets have the same cardinality if there is a bijection between them. We denote the cardinality of X by $|X|$ and call it a **cardinal (number)**. We say X is **equinumerous** to Y, denoted by $X \approx Y$, if there is a bijection from X onto Y. Therefore for any sets X and Y,

$$|X| = |Y| \text{ if and only if } X \approx Y.$$

2.2 Finite Sets and Infinite Sets

Each **natural number** is constructed so that it is the set of all smaller natural numbers, namely, $0 = \varnothing, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}, \ldots$ and so on.

Let ω denote the set of all natural numbers. The basic properties of natural numbers will be omitted and will be used in the ordinary way. For full details on natural numbers see [2].

Definition 2.2.1. A set is **finite** if it is equinumerous to some natural number. A set is **infinite** if it is not finite.

Theorem 2.2.2. *Every finite set is equinumerous to a unique natural number.*

So for any finite set X, we define the cardinal number of X to be the unique natural number which is equinumerous to X. A cardinal is finite if it is the cardinality of a finite set. Otherwise, it is an infinite cardinal.

2.3 Cardinal Arithmetic

Definition 2.3.1. Let κ and λ be cardinal numbers, say $\kappa = |X|$ and $\lambda = |Y|$. We define

1. $\kappa + \lambda = |X \cup Y|$ where $X \cap Y = \varnothing$,
2. $\kappa \cdot \lambda = |X \times Y|$, and
3. $\kappa^\lambda = |{}^Y X|$, where ${}^Y X$ is the set of all functions from Y into X.

Remark. We may write $\kappa\lambda$ for $\kappa \cdot \lambda$.

Note. For any cardinal κ, $\kappa + 0 = \kappa$, $\kappa \cdot 0 = 0$, $\kappa \cdot 1 = \kappa$, $\kappa^0 = 1$, $\kappa^1 = \kappa$ and $0^\kappa = 0$ if $\kappa \neq 0$.

Theorem 2.3.2. *For any set X, $\mathcal{P}(X) \approx {}^X 2$.*

Remark. For any sets X and Y, there is always a set X' such that $|X'| = |X|$ and $X' \cap Y = \varnothing$, for example: $X' = X \times \{z\}$ for some $z \notin \bigcup\bigcup Y$.

Theorem 2.3.3. *Let κ, λ and μ be cardinal numbers.*

1. $\kappa + \lambda = \lambda + \kappa$ *and* $\kappa \cdot \lambda = \lambda \cdot \kappa$.
2. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ *and* $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$.
3. $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \mu)$.
4. $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$.
5. $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$.
6. $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$.

2.4 Ordering Cardinal Numbers

We say X is **dominated** by Y, written $X \preceq Y$, if there is an injection from X into Y, and we write $X \preceq^* Y$ if $X = \varnothing$ or there is a surjection from Y onto X. We write $X \prec Y$ if $X \preceq Y$ but $X \not\approx Y$, and $X \prec^* Y$ if $X \preceq^* Y$ but $X \not\approx Y$.

Theorem 2.4.1. *For any sets X and Y, if $X \preceq Y$, then $X \preceq^* Y$.*

Remark. We will discuss later that the converse of the above theorem is not necessarily true without the Axiom of Choice.

Definition 2.4.2. A set X is said to be **countable** if $X \preceq \omega$ and X is **denumerable** if $X \approx \omega$.

Remark. Every subset of a countable set is countable and every infinite subset of a denumerable set is denumerable.

Theorem 2.4.3. ***(Cantor-Bernstein Theorem)***

For any sets X and Y, if $X \preceq Y$ and $Y \preceq X$, then $X \approx Y$.

Theorem 2.4.4. ***(Cantor's Theorem)***

For any set X, $X \not\approx \mathcal{P}(X)$.

Note. For any set X, since $X \preceq \mathcal{P}(X)$, $X \prec \mathcal{P}(X)$.

Theorem 2.4.5. *For any sets X and Y, if $X \preceq^* Y$, then $\mathcal{P}(X) \preceq \mathcal{P}(Y)$.*

Definition 2.4.6. For any sets X and Y, we define

$|X|$ to be **less than or equal** to $|Y|$, written $|X| \leq |Y|$, if $X \preceq Y$,

$|X|$ to be **less than** $|Y|$, written $|X| < |Y|$, if $|X| \leq |Y|$ but $|X| \neq |Y|$,

$|X| \leq^* |Y|$ if $X \preceq^* Y$, and

$|X| <^* |Y|$ if $|X| \leq^* |Y|$ but $|X| \neq |Y|$.

Remark. For any cardinal numbers κ and λ, $\kappa \leq \lambda$ implies $\kappa \leq^* \lambda$.

Theorem 2.4.7. *If X is finite and Y is infinite, then $|X| < |Y|$. As a result, an infinite set has finite subsets of all cardinalities.*

Theorem 2.4.8. *Let κ, λ and μ be cardinal numbers.*

1. *(Reflexivity) $\kappa \leq \kappa$.*

2. *(Transitivity) If $\kappa \leq \lambda$ and $\lambda \leq \mu$, then $\kappa \leq \mu$.*

3. *(Antisymmetry) If $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$.*

Corollary 2.4.9. *Let κ, λ and μ be cardinal numbers.*

1. *If $\kappa < \lambda \leq \mu$, then $\kappa < \mu$.*

2. *If $\kappa \leq \lambda < \mu$, then $\kappa < \mu$.*

Theorem 2.4.10. *For any cardinals κ, λ and μ, if $\kappa \leq^* \lambda$ and $\lambda \leq^* \mu$, then $\kappa \leq^* \mu$.*

Corollary 2.4.11. *Let κ, λ and μ be cardinal numbers.*

1. *If $\kappa <^* \lambda \leq^* \mu$, then $\kappa <^* \mu$.*

2. *If $\kappa \leq^* \lambda <^* \mu$, then $\kappa <^* \mu$.*

Theorem 2.4.12. *Let κ, λ, μ and ν be cardinal numbers.*

1. *If $\kappa \leq \lambda$ and $\mu \leq \nu$, then $\kappa + \mu \leq \lambda + \nu$.*

2. *If* $\kappa \leq \lambda$ *and* $\mu \leq \nu$*, then* $\kappa \cdot \mu \leq \lambda \cdot \nu$.

3. *If* $\kappa \leq^* \lambda$ *and* $\mu \leq^* \nu$*, then* $\kappa + \mu \leq^* \lambda + \nu$.

4. *If* $\kappa \leq^* \lambda$ *and* $\mu \leq^* \nu$*, then* $\kappa \cdot \mu \leq^* \lambda \cdot \nu$.

Theorem 2.4.13. *Let* κ, λ *and* μ *be cardinal numbers. If* $\kappa \leq \lambda$*, then*

1. $\kappa^\mu \leq \lambda^\mu$,

2. $\mu^\kappa \leq \mu^\lambda$ *whenever* $\mu \neq 0$ *or* $\kappa \neq 0$.

Theorem 2.4.14. *For any cardinals* κ *and* λ*, if both* κ *and* λ *are greater than* 1*, then* $\kappa + \lambda \leq \kappa \cdot \lambda$.

2.5 Well-Ordered Sets

A **(strict) partial ordering** on a set X is a binary relation on X which is irreflexive and transitive. A **linear ordering** on X is a partial ordering on X whose every two members are comparable. A **well-ordering** R on X is a linear ordering on X such that every nonempty subset of X has an R-least element. We say a set X is **well-ordered** if there is a well-ordering on X.

In this section, we give some important theorems concerning well-ordered sets used in this work.

Theorem 2.5.1. *If* X *is well-ordered and* $Y \preceq^* X$*, then* Y *can be well-ordered and* $Y \preceq X$.

Theorem 2.5.2. *If* X *and* Y *are well-ordered, then so is* $X \cup Y$.

Theorem 2.5.3. *Let* X *and* Y *be well-ordered sets such that* X *or* Y *is infinite. Then*

1. $|X| + |Y| = \max\{|X|, |Y|\}$.

2. $|X| \cdot |Y| = \max\{|X|, |Y|\}$ *if* X *and* Y *are nonempty.*

Corollary 2.5.4. *For any infinite well-ordered set* X*,* $|X| = 2|X|$.

2.6 Ordinals

Definition 2.6.1. A set X is **transitive** if $\forall x \in X, x \subseteq X$.

Definition 2.6.2. For any set X, let $\in_X = \{(x_1, x_2) \in X \times X \mid x_1 \in x_2\}$.

Definition 2.6.3. A set α is an **ordinal (number)** if α is transitive and $\in_\alpha$ is a well-ordering on α.

Throughout this work, we use $\alpha, \beta, \gamma, \ldots$ for ordinals.

Example. Every natural number and ω are ordinals.

Note that every member of an ordinal is also an ordinal and the relation $\in$ on the class of ordinals is a well ordering. Thus we order ordinals as follows.

Definition 2.6.4. For any ordinals α and β, we say

α is **less than** β, written $\alpha < \beta$, if $\alpha \in \beta$, and

α is **less than or equal** to β, written $\alpha \leq \beta$, if $\alpha < \beta$ or $\alpha = \beta$.

Notation. Let $\alpha + 1 = \alpha \cup \{\alpha\}$.

Note that $\alpha + 1$ is also an ordinal.

Definition 2.6.5. Let α be an ordinal.

α is a **successor ordinal** if $\alpha = \beta + 1$ for some ordinal β.

α is a **limit ordinal** if $\alpha \neq 0$ and α is not a successor ordinal.

Theorem 2.6.6. *Every well-ordered set is isomorphic to a unique ordinal (up to a well-ordering of the set).*

It follows from the above theorem that every well-ordered set is equinumerous to some ordinals. We define the cardinality of a well-ordered set as the least ordinal equinumerous to it.

Example. $|\omega + 1| = |\omega| = \omega$. Thus ω is a cardinal but $\omega + 1$ is not.

2.7 Alephs

Definition 2.7.1. The cardinality of an infinite well-ordered set is called an **aleph**.

Note that every aleph is an ordinal.

Theorem 2.7.2. *Any two alephs, or two well-ordered sets, are always comparable, i.e. for any well-ordered sets X and Y, $|X| \leq |Y|$ or $|Y| \leq |X|$.*

Theorem 2.7.3. ***(Hartogs' Theorem)***

For any set X, there is the least aleph $\aleph$ such that $\aleph \nleq |X|$.

Definition 2.7.4. For any set X, the **Hartogs number** of X, denoted by $\aleph(X)$, is a least aleph $\aleph$ such that $\aleph \nleq |X|$.

Notation. For any ordinal α, let $\alpha^+ = \aleph(\alpha)$.

In an analogous way, for any set X, there exists a least aleph $\aleph^*(X)$ such that $\aleph^*(X) \nleq^* |X|$. Such an aleph is called the **Lindenbaum number** of X.

Remark. For any set X, $\aleph(X) \leq \aleph^*(X)$.

Theorem 2.7.5. *If A is a set of ordinals, then $\bigcup A$ is an ordinal which is the least upper bound of A. Moreover, if A is a set of alephs, then $\bigcup A$ is an aleph.*

Definition 2.7.6. Define an aleph $\aleph_\alpha$ recursively as follows:

$$\begin{aligned} \aleph_0 &= \omega, \\ \aleph_{\alpha+1} &= \aleph_\alpha^+, \text{ and} \\ \aleph_\lambda &= \bigcup_{\beta<\lambda} \aleph_\beta \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Theorem 2.7.7. *If X is an infinite well-ordered set, then $|X| = \aleph_\alpha$ for some ordinal α.*

2.8 The Axiom of Choice

The Axiom of Choice (AC) is an important axiom in mathematics and is independent from ZF. That is AC and its negation cannot be proved from ZF. There are many equivalent forms of AC. We list some of them which are used in this work below.

Well-Ordering Theorem: every set can be well-ordered.

Trichotomy Principle: for any sets A and B, $A \preceq B$ or $B \preceq A$.

Dual Trichotomy Principle: for any sets A and B, $A \preceq^* B$ or $B \preceq^* A$.

Existence of Choice Functions: if A is a pairwise disjoint collection of nonempty sets, then A has a choice function, i.e. a function $f : A \to \bigcup A$ such that $f(B) \in B$ for any $B \in A$.

Existence of Maximal Functions: for any relation R, there is a function $F \subseteq R$ such that $dom(F) = dom(R)$.

2.9 Dedekind Infinite Sets and Weakly Dedekind Infinite Sets

Without AC, it is no longer true that any two cardinals are comparable. Moreover, for an infinite set X, we cannot guarantee whether $\omega \preceq X$ or not. Therefore, in the absence of AC, the following definitions are needed.

Definition 2.9.1. A set X is said to be **Dedekind infinite** if $\omega \preceq X$. Otherwise, X is **Dedekind finite**.

Definition 2.9.2. A set X is said to be **weakly Dedekind infinite** if $\omega \preceq^* X$. Otherwise, X is **weakly Dedekind finite**.

Since $X \preceq Y$ implies $X \preceq^* Y$ for any sets X and Y, every Dedekind infinite set is always weakly Dedekind infinite. Equivalently, every weakly Dedekind finite set is always Dedekind finite. Without AC, a Dedekind finite set and a weakly Dedekind finite set need not be finite. If AC holds, all these concepts of infinity and finiteness are the same.

A cardinal number is (weakly) Dedekind infinite if it is the cardinality of a (weakly) Dedekind infinite set, and it is (weakly) Dedekind finite if it is not (weakly) Dedekind infinite.

Theorem 2.9.3. *The following statements are equivalent.*

1. *X is Dedekind infinite.*

2. *$X \approx X \cup A$ for any finite set A.*

3. *There is a proper subset Y of X such that $Y \approx X$.*

Theorem 2.9.4. *The following statements are equivalent.*

1. *X is weakly Dedekind infinite.*

2. *$\mathcal{P}(X)$ is Dedekind infinite.*

Theorem 2.9.5. *Let X be an infinite set. If there is a proper subset Y of X such that $X \preceq^* Y$, then X is weakly Dedekind infinite.*

CHAPTER III
THE AXIOM OF CHOICE AND SOME WEAK FORMS CONCERNING SURJECTIONS

We are interested in the Axiom of Choice and some weak forms related to the *Partition Principle*, the *Cantor-Bernstein Theorem*, the *Trichotomy Principle*, and weakly Dedekind finite sets. These are divided into three sections as follows.

3.1 On the Partition Principle and the Cantor-Bernstein Theorem

The *Dual Cantor-Bernstein Theorem* and the *Partition Principle* are both famous consequences of AC concerning surjections. They state that

Dual Cantor-Bernstein Theorem (CB*): for any sets X and Y, if $X \preceq^* Y$ and $Y \preceq^* X$, then $X \approx Y$.

Partition Principle (PP): for any sets X and Y, if $X \preceq^* Y$, then $X \preceq Y$.

By the *Cantor-Bernstein Theorem*, we can see that PP implies CB*.

Nowadays, it is still an open problem whether CB* is equivalent to AC or not (see [9]). In [1], *Bernhard Banaschewski* and *Gregory H. Moore* introduced a natural refinement of CB*.

Refined CB*: for any sets X and Y, if $f : X \to Y$ and $g : Y \to X$ are surjections, then there is a bijection $h : X \to Y$ such that $h \subseteq f \cup g^{-1}$.

It has been shown in [1] that the *Refined* CB* is an equivalent form of AC.

It is still an open problem whether PP is equivalent to AC or not (see [9]). We now introduce a refinement of PP and show that it is equivalent to AC.

Refined PP: for any sets X and Y, if $f : Y \to X$ is a surjection, then there is an injection $g : X \to Y$ such that $g \subseteq f^{-1}$.

Theorem 3.1.1. *The Refined PP $\Leftrightarrow$ AC.*

Proof. ($\Leftarrow$) Assume AC. Then every set can be well-ordered. Let $f : Y \to X$ be a surjection. Fix a well-ordering on Y. Since f is surjective, for any $x \in X$, $f^{-1}[\{x\}]$ is nonempty and so $f^{-1}[\{x\}]$ has a least element.

Define a function $g : X \to Y$ by $g(x) =$ *the least element of* $f^{-1}[\{x\}]$. Since f is a function, g is an injection. It is easy to see that $g \subseteq f^{-1}$.

($\Rightarrow$) Assume the *Refined PP*. We will show that for any relation R, there is a function $F \subseteq R$ such that $dom(F) = dom(R)$. This is an equivalent form of AC.

Let R be a relation. Define a function $f : R \to dom(R)$ by $f(a, b) = a$. Clearly, f is surjective. By the assumption, there is a function $g : dom(R) \to R$ which is injective and $g \subseteq f^{-1}$. Let $F = ran(g)$. Then $F \subseteq R$ and so $dom(F) \subseteq dom(R)$. To show that they are equal, let $x \in dom(R)$. Then $g(x) \in R$, say $g(x) = (a, b)$. Since $(x, g(x)) \in g \subseteq f^{-1}$, $(g(x), x) \in f$. Hence $x = f(g(x)) = f(a, b) = a \in dom(F)$ since $(a, b) \in ran(g) = F$. Thus $dom(F) = dom(R)$.

To show that F is a function, let $(a, b_1), (a, b_2) \in F$. Then there are $x_1, x_2 \in dom(R)$ such that $g(x_1) = (a, b_1)$ and $g(x_2) = (a, b_2)$. Since $g \subseteq f^{-1}$, we obtain that $x_1 = f(a, b_1) = a = f(a, b_2) = x_2$ and then $b_1 = b_2$. Hence F is a function. $\square$

Next, we will concentrate on some statements concerning $\aleph_\alpha$.

$\aleph_\alpha$-CB*: for any set X, if $\aleph_\alpha \leq^* |X|$ and $|X| \leq^* \aleph_\alpha$, then $|X| = \aleph_\alpha$.

$\aleph_\alpha$-PP: for any set X, if $\aleph_\alpha \leq^* |X|$, then $\aleph_\alpha \leq |X|$.

$\aleph_\alpha$-AC: if X is a pairwise disjoint collection of nonempty sets such that $|X| = \aleph_\alpha$, then X has a choice function.

Since $|X| \leq^* \aleph_\alpha$ implies that X can be well-ordered, it is easy to see that $\aleph_\alpha$-CB* is just a special case of the *Cantor-Bernstein Theorem* which is provable without AC. On the other hand, $\aleph_\alpha$-PP and $\aleph_\alpha$-AC follow from AC and it has been shown that "$\aleph_\alpha$-*PP*" follows from "$\aleph_\alpha$-*AC*", while "$\forall\alpha, \aleph_\alpha$-*AC*" is a consequence of PP that is weaker than PP (see [6]).

We now strengthen $\aleph_\alpha$-PP to the *Refined* $\aleph_\alpha$-PP.

Refined $\aleph_\alpha$-PP: for any set X, if $f : X \to \aleph_\alpha$ is a surjection, then there is an injection $g : \aleph_\alpha \to X$ such that $g \subseteq f^{-1}$.

Theorem 3.1.1 shows that the *Refined* PP is equivalent to AC. We now show that "the *Refined* $\aleph_\alpha$-*PP*" is equivalent to "$\aleph_\alpha$-*AC*".

Theorem 3.1.2. *The Refined $\aleph_\alpha$-PP $\Leftrightarrow$ $\aleph_\alpha$-AC.*

Proof. ($\Rightarrow$) Assume the *Refined* $\aleph_\alpha$-*PP*. Let X be a pairwise disjoint collection of nonempty sets such that $|X| = \aleph_\alpha$. Say $X = \{X_\beta\}_{\beta\in\aleph_\alpha}$.

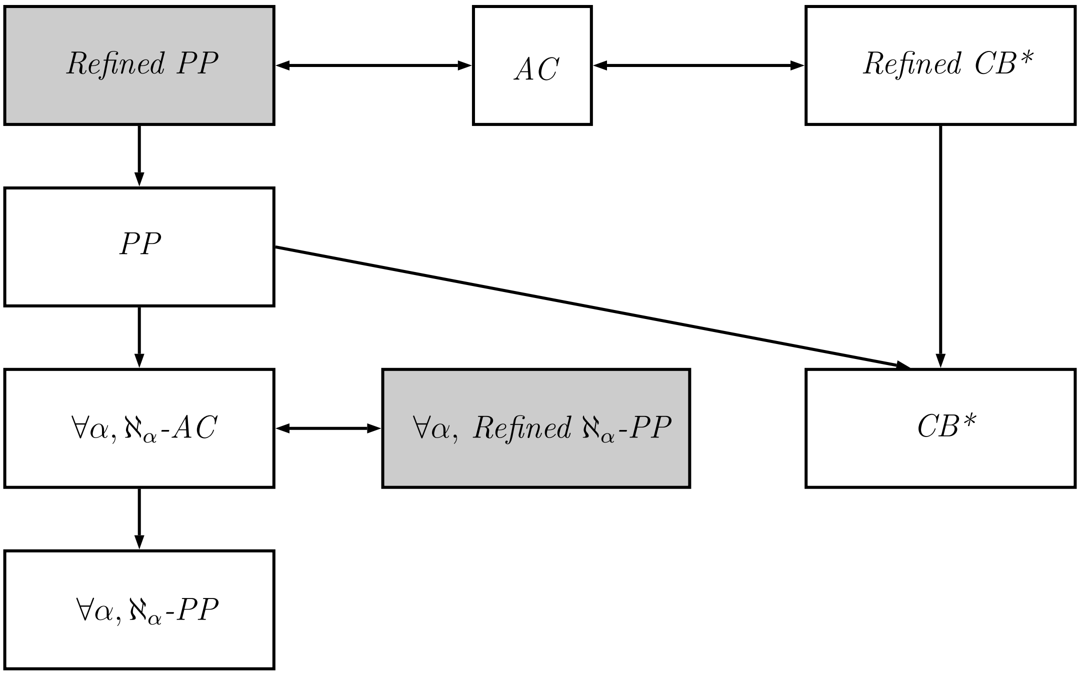
Define a function $f : \bigcup X \to \aleph_\alpha$ by $f(a) = \beta$ if $a \in X_\beta$. Since any two members of X are disjoint, f is well-defined. Since each member of X is nonempty, f is surjective. By the assumption, there is an injection $g : \aleph_\alpha \to \bigcup X$ such that $g \subseteq f^{-1}$.

Define a function $h : X \to \bigcup X$ by $h(X_\beta) = g(\beta)$. We will show that h is a choice function for X. Let $\beta \in \aleph_\alpha$. Then $(\beta, h(X_\beta)) = (\beta, g(\beta)) \in g$. Since $g \subseteq f^{-1}$, $(h(X_\beta), \beta) \in f$. So $f(h(X_\beta)) = \beta$. It follows from the definition of f that $h(X_\beta) \in X_\beta$.

(⇐) Assume $\aleph_\alpha$-AC. Let X be a set and $f : X \to \aleph_\alpha$ be a surjection. Set $M = \{f^{-1}[\{\beta\}] \mid \beta \in \aleph_\alpha\}$. Since f is a surjection, M is a pairwise disjoint collection of nonempty sets where $|M| = \aleph_\alpha$. By the assumption, there is a choice function F for M.

Define a function $g : \aleph_\alpha \to X$ by $g(\beta) = F(f^{-1}[\{\beta\}])$. Since any two members of M are disjoint and $F(f^{-1}[\{\beta\}]) \in f^{-1}[\{\beta\}]$ for all $\beta \in \aleph_\alpha$, F is injective and so is g. To show that $g \subseteq f^{-1}$, let $\beta \in \aleph_\alpha$. So $g(\beta) = F(f^{-1}[\{\beta\}]) \in f^{-1}[\{\beta\}]$ and then $f(g(\beta)) = \beta$. Hence $(\beta, g(\beta)) \in f^{-1}$ and thus $g \subseteq f^{-1}$. □

We now have some deductive relations of the above statements as shown in the diagram below.



3.2 On the Trichotomy Principle

The *Trichotomy Principle* is an equivalent form of AC. The main idea of the *Trichotomy Principle* lies in the existence of an injection connecting arbitrary two sets. It states that

Trichotomy Principle: for any sets X and Y, $X \preceq Y$ or $Y \preceq X$.

In 2008, *David Feldman* and *Mehmet Orhon* extended the idea of the *Trichotomy Principle* to the *k-Trichotomy Principle* (see [3]).

k-Trichotomy Principle: every family which is of cardinality k contains two distinct sets X and Y such that $X \preceq Y$.

Notice that the *2-Trichotomy Principle* and the *Trichotomy Principle* are the same statement and the principle for $k > 2$ seems to be weaker. Surprisingly, it has been shown in [3] that for any natural number $k \geq 2$, the *k-Trichotomy Principle* is also equivalent to AC.

The following is another form of the *Trichotomy Principle* which concerns surjections.

Dual Trichotomy Principle: for any sets X and Y, $X \preceq^* Y$ or $Y \preceq^* X$.

Since $X \preceq Y$ implies $X \preceq^* Y$ for any sets X and Y, it is obvious that the *Dual Trichotomy Principle* follows from the *Trichotomy Principle.* Since, without AC, we cannot guarantee that $X \preceq^* Y$ implies $X \preceq Y$ for arbitrary sets X and Y, the *Dual Trichotomy Principle* seems to be weaker than AC. It is a surprise that the *Dual Trichotomy Principle* is also equivalent to AC (see [11]).

We now extend the idea of the *k-Trichotomy Principle* to the *k-Dual Trichotomy Principle* or *k*-Trichotomy Principle.*

k*-Trichotomy Principle: every family which is of cardinality k contains two distinct sets X and Y such that $X \preceq^* Y$.

In this section, k is a natural number greater than or equal to 2. We will show that the *k*-Trichotomy Principle* is equivalent to AC.

Throughout this work, for a set X and $l \in \omega$, we write lX for the Cartesian product $l \times X$. First, we will show that the *k*-Trichotomy Principle* implies that every infinite set is weakly Dedekind infinite.

Lemma 3.2.1. *Assume k*-Trichotomy Principle.*

Let $A_1, A_2, \ldots, A_k$ be sets such that $A_1 \preceq A_2 \preceq \cdots \preceq A_k$. Then there exist $m, n \leq k$ where $n < m$ and a well-ordered set R such that $R \preceq A_m \preceq^* A_n \dot{\cup} R$.

Proof. Define $\mu_k = \aleph^*(A_k)$ and $\mu_j = \aleph^*(\mu_{j+1})$ for all $1 \leq j < k$. Then $\mu_1 > \mu_2 > \cdots > \mu_k$. Consider a family $\{A_i \dot{\cup} \mu_i\}_{i=1}^k$ and apply the *k*-Trichotomy Principle*, we obtain $m \neq n$ and $A_m \dot{\cup} \mu_m \preceq^* A_n \dot{\cup} \mu_n$, i.e. there exists a surjective map $f : A_n \dot{\cup} \mu_n \to A_m \dot{\cup} \mu_m$.

Let $M := f^{-1}[\mu_m] \cap \mu_n$ and $A' := f^{-1}[\mu_m \setminus f[M]]$. Clearly, $f[A']$ and $f[M]$ are subsets of μ_m such that $f[A'] \dot{\cup} f[M] = \mu_m$. If $|f[A']| < \mu_m$ and $|f[M]| < \mu_m$, then $\mu_m = |f[A']| + |f[M]| = \max\{|f[A']|, |f[M]|\} < \mu_m$ which is not possible. Then $|f[A']| = \mu_m$ or $|f[M]| = \mu_m$. Since $A_n \preceq A_k$, $\aleph^*(A_n) \leq \aleph^*(A_k) = \mu_k \leq \mu_m$, so $\mu_m \not\preceq^* A_n$. Since $A' \subseteq A_n$ and $\mu_m \not\preceq^* A_n$, $\mu_m \not\preceq^* A'$, so $|f[A']| \neq \mu_m$. Hence $|f[M]| = \mu_m$ and then $\mu_m \leq^* |M| \leq \mu_n$. This implies that $n < m$.

Let $P := f^{-1}[A_m] \cap \mu_n$. Since $P \subseteq \mu_n$ is well-ordered, there is $R \subseteq P$ such that $R \approx f[P] \subseteq A_m$. So $R \preceq A_m \preceq^* A_n \dot{\cup} R$. □

Lemma 3.2.2. *Let A be an infinite set, R an infinite well-ordered set and $n > 0$.*

If $R \preceq nA$, then $R \preceq A$.

Proof. It is trivial if $n = 1$. Assume the statement holds for m and $R \preceq (m+1)A$. Then $R \preceq mA \dot{\cup} A$, say by an injection f. Let $X = f[R] \cap mA$ and $Y = f[R] \cap A$.

Since R is an infinite well-ordered set and f is injective,

$$|R| = |f[R]| = |X\dot{\cup}Y| = |X| + |Y| = \max\{|X|, |Y|\}.$$

Hence $R \approx X \subseteq mA$ or $R \approx Y \subseteq A$. Thus $R \preceq mA$ or $R \preceq A$. By the induction hypothesis, we have $R \preceq A$. □

Lemma 3.2.3. *If X is a weakly Dedekind finite set, then so is lX for all $l \in \omega$.*

Proof. The proof proceeds by induction. Let $l \in \omega$ and X be a weakly Dedekind finite set. Assume that lX is weakly Dedekind finite but $(l+1)X$ is weakly Dedekind infinite. Then $\omega \preceq^* (l+1)X \approx lX\dot{\cup}X$, so there is a surjection $f : lX\dot{\cup}X \to \omega$. Since $\omega \not\preceq^* lX$ and every infinite subset of ω is equinumerous to ω, $f[lX]$ is finite. Since f is a surjection, $f[X]$ must be infinite and then $\omega \approx f[X] \preceq^* X$ but X is weakly Dedekind finite, a contradiction. □

Theorem 3.2.4. *The k*-Trichotomy Principle implies that every infinite set is weakly Dedekind infinite.*

Proof. Assume the *k*-Trichotomy Principle*. Suppose there is an infinite weakly Dedekind finite set A. Then $\aleph^*(A) = \aleph_0$. Since A is weakly Dedekind finite, by Lemma 3.2.3, so is lA for all $l \in \omega$. Since $A \preceq 2A \preceq \cdots \preceq kA$, by applying Lemma 3.2.1, $R \preceq mA \preceq^* nA\dot{\cup}R$ for some $m, n \leq k$ such that $n < m$ and some well-ordered set R. Since $n < m$, $nA \preceq mA$. Since mA is weakly Dedekind finite and $nA \subsetneq mA$, $mA \not\preceq^* nA$. So $nA \not\approx mA$ and it follows that $nA \prec mA$. Since $R \preceq mA$, $|R| < \aleph^*(mA) = \aleph_0$, i.e. R is finite. Since A is infinite, $R \prec A$ and so $nA\dot{\cup}R \approx X$ for some $X \subsetneq nA\dot{\cup}A \approx (n+1)A$. Since $(n+1)A$ is weakly Dedekind finite, $nA\dot{\cup}R \not\approx (n+1)A$. It follows that $nA\dot{\cup}R \prec (n+1)A \preceq mA$. Hence we can view $nA\dot{\cup}R$ as a proper subset of mA but $mA \preceq^* nA\dot{\cup}R$ where mA is weakly Dedekind finite, a contradiction. □

We now obtain that if we assume the *k*-Trichotomy Principle*, then every infinite set is weakly Dedekind infinite.

Next, we will show that the *k*-Trichotomy Principle* implies that every Dedekind infinite set can be well-ordered and so can every infinite set.

Theorem 3.2.5. *Assume k^*-Trichotomy Principle.*

If A is a Dedekind infinite set, then $\mathcal{P}(nA) \approx 2\mathcal{P}(nA) \approx \mathcal{P}(2nA)$ for some $n \in \omega$.

Proof. Let A be a Dedekind infinite set. Since $A \preceq 2A \preceq \cdots \preceq kA$, by Lemma 3.2.1, there exist $m, n \in \omega$ and a well-ordered set R such that $R \preceq mA \preceq^* nA\dot{\cup}R$ where $n < m$. Since A is Dedekind infinite, if R is finite, then $nA\dot{\cup}R \approx nA$. Suppose R is infinite. Then $R \approx 2R$. Since $R \preceq mA$, by Lemma 3.2.2, $R \preceq A \preceq nA$. Hence there is a set X such that

$$nA \approx X\dot{\cup}R \approx X\dot{\cup}2R \approx (X\dot{\cup}R)\dot{\cup}R \approx nA\dot{\cup}R.$$

So $mA \preceq^* nA$ and then $\mathcal{P}(mA) \preceq \mathcal{P}(nA)$. Since $nA \preceq mA$, $\mathcal{P}(nA) \preceq \mathcal{P}(mA)$. By the *Cantor-Bernstein Theorem*, $\mathcal{P}(nA) \approx \mathcal{P}(mA)$. Since

$$\begin{aligned}
\mathcal{P}(nA) &\approx \mathcal{P}(mA) \\
&\approx \mathcal{P}(nA\dot{\cup}(m-n)A) \\
&\approx \mathcal{P}(nA) \times \mathcal{P}((m-n)A) \\
&\approx \mathcal{P}(mA) \times \mathcal{P}((m-n)A),
\end{aligned}$$

$\mathcal{P}(mA) \approx \mathcal{P}(nA) \approx \mathcal{P}((m+(m-n))A)$ and

$$\begin{aligned}
\mathcal{P}(nA) &\approx \mathcal{P}(mA) \times \mathcal{P}((m-n)A) \\
&\approx [\mathcal{P}(mA) \times \mathcal{P}((m-n)A)] \times \mathcal{P}((m-n)A) \\
&\approx \mathcal{P}(mA) \times [\mathcal{P}((m-n)A) \times \mathcal{P}((m-n)A)] \\
&\approx \mathcal{P}(mA) \times \mathcal{P}(2(m-n)A) \\
&\approx \mathcal{P}((mA)\dot{\cup}2(m-n)A) \\
&\approx \mathcal{P}((m+2(m-n))A).
\end{aligned}$$

It follows, by induction, that $\mathcal{P}(nA) \approx \mathcal{P}((m+q(m-n))A)$ for all $q \in \omega$. We may choose $q = n$. Since $m > n$ and $n(m-n) \geq n$, $m + n(m-n) > 2n$. Set $l = m + n(m-n)$. Then $\mathcal{P}(nA) \approx \mathcal{P}(lA)$ where $l > 2n$.

Set $r = l - 2n$. Then $n + r = l - n$ and $l + r = 2(l - n)$. So

$$\begin{aligned}\mathcal{P}((l-n)A) &= \mathcal{P}((n+r)A)\\ &\approx \mathcal{P}((nA)\dot{\cup}(rA))\\ &\approx \mathcal{P}(nA) \times \mathcal{P}(rA)\\ &\approx \mathcal{P}(lA) \times \mathcal{P}(rA)\\ &\approx \mathcal{P}((lA)\dot{\cup}(rA))\\ &\approx \mathcal{P}((l+r)A)\\ &= \mathcal{P}(2(l-n)A).\end{aligned}$$

Hence there exists $n' := l - n$ such that $\mathcal{P}(n'A) \approx \mathcal{P}(2n'A)$.

Notice that $\mathcal{P}(n'A) \preceq 2\mathcal{P}(n'A) \preceq \mathcal{P}(n'A) \times \mathcal{P}(n'A) \approx \mathcal{P}(2n'A) \approx \mathcal{P}(n'A)$. Hence we obtain a natural number n' such that $\mathcal{P}(n'A) \approx 2\mathcal{P}(n'A) \approx \mathcal{P}(2n'A)$. □

Definition 3.2.6. For any set A, define $Q(A) = Q^1(A) = A \times \mathcal{P}(A)$ and $Q^{i+1}(A) = Q(Q^i(A))$ for all $i \geq 1$.

Remark. For any nonempty set A, $A \prec Q(A)$.

The following lemma is Lemma 7 in [3].

Lemma 3.2.7. *For a set A and a well-ordered set B, an injection $f : Q(A) \to A\dot{\cup}B$ induces a canonical well-ordering of A.*

Theorem 3.2.8. *The k*-Trichotomy Principle implies that every Dedekind infinite set can be well-ordered.*

Proof. Fix a Dedekind infinite set S. By Theorem 3.2.5, there exists an n such that $\mathcal{P}(nS) \approx 2\mathcal{P}(nS) \approx \mathcal{P}(2nS)$. Set $A = nS$. Then $\mathcal{P}(A) \approx 2\mathcal{P}(A) \approx \mathcal{P}(2A)$.

Claim. For any $p \geq 1$, $\mathcal{P}(2Q^p(A)) \approx \mathcal{P}(Q^p(A))$.

For $p = 1$,

$$\mathcal{P}(2Q(A)) = \mathcal{P}(2(A \times \mathcal{P}(A))) \approx \mathcal{P}(A \times 2\mathcal{P}(A)) \approx \mathcal{P}(A \times \mathcal{P}(A)) = \mathcal{P}(Q(A)).$$

Assume $\mathcal{P}(2Q^p(A)) \approx \mathcal{P}(Q^p(A))$. Since

$\mathcal{P}(Q^p(A)) \preceq 2\mathcal{P}(Q^p(A)) \preceq \mathcal{P}(Q^p(A)) \times \mathcal{P}(Q^p(A)) \approx \mathcal{P}(2Q^p(A)) \approx \mathcal{P}(Q^p(A))$, $\mathcal{P}(Q^p(A)) \approx 2\mathcal{P}(Q^p(A)) \approx \mathcal{P}(2Q^p(A))$. Then

$$\begin{aligned}\mathcal{P}(2Q^{p+1}(A)) &= \mathcal{P}(2(Q(Q^p(A))))\\ &= \mathcal{P}(2(Q^p(A) \times \mathcal{P}(Q^p(A))))\\ &\approx \mathcal{P}(Q^p(A) \times 2\mathcal{P}(Q^p(A)))\\ &\approx \mathcal{P}(Q^p(A) \times \mathcal{P}(Q^p(A)))\\ &= \mathcal{P}(Q^{p+1}(A)).\end{aligned}$$

Since $Q(A) \prec Q^2(A) \prec \cdots \prec Q^k(A)$, by Lemma 3.2.1, there exist $m, n \in \omega$ where $n < m$ and a well-ordered set R such that $Q^m(A) \preceq^* Q^n(A) \dot\cup R$. Let $f : Q^n(A)\dot\cup R \to Q^m(A)$ be a surjection, $X = f[Q^n(A)]$ and $Y = f[R]$. Then $Q^m(A) = X \dot\cup Y$ where $X \preceq^* Q^{m-1}(A)$ and Y is a well-ordered set.

Since $X \preceq^* Q^{m-1}(A)$, $X \preceq \mathcal{P}(X) \preceq \mathcal{P}(Q^{m-1}(A))$. So

$$\begin{aligned}Q(X) &= X \times \mathcal{P}(X)\\ &\preceq \mathcal{P}(Q^{m-1}(A)) \times \mathcal{P}(Q^{m-1}(A))\\ &\approx \mathcal{P}(2Q^{m-1}(A))\\ &\approx \mathcal{P}(Q^{m-1}(A)) && \text{(by the claim)}\\ &\preceq Q^{m-1}(A) \times \mathcal{P}(Q^{m-1}(A))\\ &= Q^m(A)\\ &= X \dot\cup Y,\end{aligned}$$

where Y is a well-ordered set. By Lemma 3.2.7, X can be well-ordered and so $Q^m(A)$ is well-ordered. From $S \preceq nS = A \prec Q(A) \preceq Q^m(A)$, we obtain that S can be well-ordered. □

Theorem 3.2.9. *The k^*-Trichotomy Principle implies AC.*

Proof. Assume the *k^*-Trichotomy Principle*. Let A be an infinite set. By Theorem 3.2.4, A is weakly Dedekind infinite and hence $\mathcal{P}(A)$ is Dedekind infinite. By Theorem 3.2.8, $\mathcal{P}(A)$ can be well-ordered. Since $A \prec \mathcal{P}(A)$, A can also be well-ordered. □

Therefore, we conclude that the following statements are all equivalent.

1. Every set can be well-ordered.
2. *Trichotomy Principle*
3. *Dual Trichotomy Principle*
4. *k-Trichotomy Principle*
5. *k^*-Trichotomy Principle*

For the equivalence of 1 and 2, 1 and 3, and 1 and 4 see [4], [11] and [3], respectively.

3.3 On Weakly Dedekind Finite Sets

It is clear that every finite set is (weakly) Dedekind finite. However, without AC, we cannot guarantee that the converse of this statement is true.

The statement "*every Dedekind finite set is finite*" is a consequence of AC which is weaker than AC (see [10]). In [10], Gregory H. Moore introduced many statements which are equivalent to it. Some of them are listed below.

1. Every Dedekind finite set is finite.

2. Every infinite set has a denumerable subset.

3. If X is Dedekind finite and Y is Dedekind infinite, then $|X| < |Y|$.

4. For any set X, $X \preceq \omega$ or $\omega \preceq X$.

5. If X is Dedekind finite, then so is $\mathcal{P}(X)$.

Since $X \preceq Y$ implies $X \preceq^* Y$ for any sets X and Y, we obtain that every Dedekind infinite set is always weakly Dedekind infinite; equivalently, every weakly Dedekind finite set is always Dedekind finite. It follows that the statement "*every Dedekind finite set is finite*" implies that "*every weakly Dedekind finite set is finite*". In this section, we introduce some equivalent statements of the statement "*every weakly Dedekind finite set is finite*" which correspond to those of the statement "*every Dedekind finite set is finite*".

Theorem 3.3.1. *The following statements are equivalent.*

1. *Every weakly Dedekind finite set is finite.*

2. *Every infinite set has a denumerable partition.*

3. *If X is weakly Dedekind finite and Y is weakly Dedekind infinite, then $|X| < |Y|$.*

4. *For any set X, $X \preceq^* \omega$ or $\omega \preceq^* X$.*

5. *If X is weakly Dedekind finite, then so is $\mathcal{P}(X)$.*

6. *If X is weakly Dedekind finite, then $\mathcal{P}(X)$ cannot be decomposed into two infinite sets.*

Proof. (*1* ⇒ *2*) Let X be an infinite set. By *1.*, X is a weakly Dedekind infinite. Then there is a surjective map $f : X \to \omega$. We have $\{f^{-1}[\{n\}] \mid n \in \omega)\}$ is a countable partition of X.

(*2* ⇒ *1*) Let X be an infinite set with a countable partition $\{A_n \mid n \in \omega\}$. Define a function $f : X \to \omega$ by $f(x) =$ the unique natural number n such that $x \in A_n$. Clearly, f is surjective and hence X is weakly Dedekind infinite.

(*1* ⇒ *3*) Let X be weakly Dedekind finite and Y be weakly Dedekind infinite. By *1.*, X is finite. Since Y is infinite, by Lemma 2.4.7, $|X| < |Y|$.

(*3* ⇒ *1*) Let X be weakly Dedekind finite. Since ω is weakly Dedekind infinite, by *3.*, $|X| < |\omega|$. Hence X is finite.

(*1* ⇒ *4*) Let X be a set. Assume $\omega \not\preceq^* X$. By *1.*, X is finite and then $X \preceq^* \omega$.

(*4* ⇒ *1*) Let X be weakly Dedekind finite. Then $\omega \not\preceq^* X$. By *4.*, $X \prec^* \omega$. Since ω is well-ordered, $X \prec \omega$. Hence X is finite.

(*1* ⇒ *5*) Let X be weakly Dedekind finite. By *1.*, X is finite. So $\mathcal{P}(X)$ is finite and hence $\mathcal{P}(X)$ is weakly Dedekind finite.

(*5* ⇒ *1*) Let X be an infinite set. Define a function $f : \mathcal{P}(X) \to \omega$ by

$$f(A) = \begin{cases} |A| & ; \textit{ A is finite,} \\ 0 & ; \textit{ otherwise.} \end{cases}$$

Since X is infinite, by Lemma 2.4.7, f is surjective. Then $\mathcal{P}(X)$ is weakly Dedekind infinite. By *5.*, X is weakly Dedekind infinite.

(*1* ⇒ *6*) Let X be weakly Dedekind finite. By *1.*, X is finite. So $\mathcal{P}(X)$ is finite and hence it cannot be decomposed into two infinite sets.

(*6* ⇒ *5*) Let X be a set such that $\mathcal{P}(X)$ is weakly Dedekind infinite. Let $f : \mathcal{P}(X) \to \omega$ be a surjection. It is easy to see that $\mathcal{P}(X)$ can be decomposed into two infinite sets $f^{-1}[\mathbb{E}]$ and $f^{-1}[\mathbb{O}]$, where where $\mathbb{E}$ and $\mathbb{O}$ are the sets of even numbers and odd numbers, respectively. By *6.*, X is weakly Dedekind infinite.

□

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