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ALGEBRAIC PROPERTIES OF GYROGROUPS

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TEERAPONG SUKSUMRAN : ALGEBRAIC PROPERTIES OF GYROGROUPS.
ADVISOR : ASST. PROF. KENG WIBOONTON, Ph.D., 58 pp.

In the first part of the dissertation, we give an algebraic proof that the open unit ball $\mathbb{B}$ of Euclidean space $\mathbb{R}^n$, equipped with Einstein addition $\oplus_E$, forms a uniquely 2-divisible gyrocommutative gyrogroup or, equivalently, a B-loop, using the Clifford algebra formalism. As a consequence, we obtain a compact formula for Einstein addition in terms of Möbius addition. We then give a characterization of associativity and commutativity of vectors in $\mathbb{B}$ with respect to Einstein addition. In the second part of the dissertation, we study gyrogroups from an algebraic point of view. We extend some well-known theorems in group theory to gyrogroups, including Cayley's theorem, the isomorphism theorems, and Lagrange's theorem. We also prove that gyrogroups of particular order satisfy the Cauchy property.

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THESIS OVERVIEW

1.1 Background and outline

In 1988, Abraham A. Ungar made the discovery of the first gyrogroup structure, the so-called *Einstein gyrogroup*, by studying the parametrization of Lorentz transformations [32]. Later, he also found another gyrogroup, the so-called *Möbius gyrogroup*, by studying the group of conformal mappings of the complex plane that preserve the complex unit disk [33].

In [38], *Einstein velocity addition* on the set of relativistically admissible velocities,

$$\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 \colon \|\mathbf{v}\| < c\},$$

is given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right\}, \tag{1.1}$$

where c is a positive constant representing the speed of light in vacuum and $\gamma_{\mathbf{u}}$ is the *Lorentz factor* given by $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}$.

The system $(\mathbb{R}^3_c, \oplus_E)$ does not have the group structure since Einstein addition is neither associative nor commutative. Nevertheless, Ungar shows that $(\mathbb{R}^3_c, \oplus_E)$ is rich in structure, namely it forms a *gyrogroup*—a nonassociative group-like structure. He also introduces space rotations, $\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]$, called *gyroautomorphisms*, to repair the breakdown of associativity in $(\mathbb{R}^3_c, \oplus_E)$:

$$\begin{aligned} \mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) &= (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]\mathbf{w} \\ (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w} &= \mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathrm{gyr}\,[\mathbf{v}, \mathbf{u}]\mathbf{w}) \end{aligned}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3_c$. Taking the key features of $(\mathbb{R}^3_c, \oplus_E)$, Ungar formulates an abstract definition of a gyrogroup. He declares that $(\mathbb{R}^3_c, \oplus_E)$ is a *gyrocommutative* gyrogroup, which is an extension of an abelian group, where the gyrogroup axioms can be checked using computer algebra.

The Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_E)$ has strong connections with the Lorentz transformations, as described in Chapters 10 and 11 of [39]. Recall that the *Lorentz boost* parametrized by a relativistically admissible velocity $\mathbf{u}$ in $\mathbb{R}^3_c$, denoted by $L(\mathbf{u})$, is given by

$$L(\mathbf{u})\begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \gamma_{\mathbf{u}}\left(t+\frac{1}{c^2}\langle \mathbf{u},\mathbf{x}\rangle\right) \\ \gamma_{\mathbf{u}} t\mathbf{u}+\mathbf{x}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u},\mathbf{x}\rangle\mathbf{u} \end{bmatrix}, \qquad t>0,\, \mathbf{x}=t\mathbf{v},\, \mathbf{v}\in\mathbb{R}^3_c.$$

The Lorentz boosts are linear transformations of Minkowski space. Further, the composite of two Lorentz boosts is not a pure Lorentz boost, but a Lorentz boost followed by a rotation of Minkowski space:

$$L(\mathbf{u})\circ L(\mathbf{v}) = L(\mathbf{u}\oplus_E \mathbf{v})\circ \mathrm{Gyr}\,[\mathbf{u},\mathbf{v}], \tag{1.2}$$

where $\mathrm{Gyr}\,[\mathbf{u},\mathbf{v}]$ is described by

$$\mathrm{Gyr}\,[\mathbf{u},\mathbf{v}]\begin{bmatrix} t \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} t \\ \mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{w} \end{bmatrix}, \qquad t>0,\, \mathbf{w}\in\mathbb{R}^3_c.$$

Another example of a gyrogroup is the *Möbius gyrogroup*, which consists of the complex unit disk, $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$, with *Möbius addition*, $\oplus_M$, defined by

$$a\oplus_M b = \frac{a+b}{1+\bar{a}b} \tag{1.3}$$

for all $a,b\in\mathbb{D}$. The complex version of Möbius addition is extended to the Euclidean version by Ungar [40]:

$$\mathbf{u}\oplus_M \mathbf{v} = \frac{(1+2\langle \mathbf{u},\mathbf{v}\rangle+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\langle \mathbf{u},\mathbf{v}\rangle+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{1.4}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}$. Here, $\mathbb{B}$ denotes the open unit ball of $\mathbb{R}^n$, $\mathbb{B}=\{\mathbf{v}\in\mathbb{R}^n\colon \|\mathbf{v}\|<1\}$.

The Einstein and Möbius gyrogroups play the central role in gyrogroup theory as they provide concrete models for the abstract theory. Because the formula (1.4) for the Euclidean version of Möbius addition is very complicated, J. Lawson [23] and M. Ferreira and G. Ren [9] use the Clifford algebra formalism to study the Möbius gyrogroup and to simplify Möbius addition:

$$\mathbf{u}\oplus_M \mathbf{v} = (\mathbf{u}+\mathbf{v})(1-\mathbf{u}\mathbf{v})^{-1}, \tag{1.5}$$

where the product and inverse on the right hand side of Equation (1.5) are taken in the Clifford algebra of negative Euclidean space. Using the Clifford algebra formalism together with the compact formula (1.5) for Möbius addition, we provide a solid proof that $(\mathbb{B},\oplus_E)$ forms a gyrocommutative gyrogroup satisfying the uniquely 2-divisible property (Einstein gyrogroup as a B-loop, *Reports on Mathematical Physics*, vol. 76 (2015), pp. 63–74).

As noted above, a gyrogroup is a group-like structure, but not a group since its binary operation is neither associative nor commutative, in general. However, gyrogroups share remarkable analogies with groups. In fact, every group may be viewed as a gyrogroup with *trivial* gyroautomorphisms. From this point of view, we extend some well-known theorems in group theory to gyrogroups, including Cayley's theorem, the isomorphism theorems, and a portion of Lagrange's theorem. We also present an abstract version of the composition law (1.2) of Lorentz boosts for an arbitrary gyrogroup (Isomorphism theorems for gyrogroups and L-subgyrogroups, *Journal of Geometry and Symmetry in Physics*, vol. 37 (2015), pp. 67–83).

Gyrogroup theory is closely related to loop theory in the sense that a gyrogroup and a left Bol loop with the property that left inner mappings are automorphisms are algebraically identical, and that certain loops give rise to gyrogroups. Furthermore, a gyrocommutative gyrogroup and a Bruck loop (also called a K-loop) are algebraically identical. Using Lagrange's theorem for finite Bruck loops [1], we extend our result in [31] by proving the full Lagrange theorem for finite gyrogroups. As a consequence of Lagrange's theorem, we prove that gyrogroups of particular order satisfy the Cauchy property (Lagrange's theorem for gyrogroups and the Cauchy property, *Quasigroups and Related Systems*, vol. 22, no. 2 (2014), pp. 283–294).

The three articles mentioned above were published as partial fulfillments of the requirements for the Degree of Doctor of Philosophy Program in Mathematics, at Chulalongkorn University.

1.2 Significance of the study

The importance of studying of gyrogroups lies in the fact that gyrogroup theory is related to several fields, including mathematical physics, non-Euclidean geometry, group theory, loop theory, and abstract algebra. For instance, the gyrogroup structure appears as an algebraic structure that regulates Einstein velocity addition [38, 41]. It is also an algebraic structure that underlies the qubit density matrices, which play an important role in quantum mechanics [21, 35].

A certain gyrogroup gives rise to a vector space-like structure, called a *gyrovector space*, which forms the algebraic setting for hyperbolic geometry, just as a vector space forms the algebraic setting for Euclidean geometry [39, 42]. In fact, the Möbius gyrovector space is associated with the Poincaré model of conformal geometry on the open unit ball in n-dimensional Euclidean space $\mathbb{R}^n$, and the Einstein gyrovector space is associated with the Beltrami-Klein model of hyperbolic geometry on the open unit ball in $\mathbb{R}^n$ [30].

1.3 Purpose of the study

The main goal of the dissertation is to find out connections between the Möbius and Einstein gyrogroups, to investigate algebraic properties of gyrogroups, and to generalize group-theoretic theorems to the case of gyrogroups.

1.4 Scope of the study

We will examine an algebraic aspect of gyrogroups. Specifically, we will generalize the following well-known theorems in abstract algebra to gyrogroups:

- Cayley's theorem;
- the isomorphism theorems;
- Lagrange's theorem.

1.5 Expected advantage

We will obtain more insight into the gyrogroup structure from the algebraic viewpoint.

PART II

THE MANUSCRIPTS OF DISSERTATION

Article I: Einstein gyrogroup as a B-loop

Published in Reports on Mathematical Physics, vol. 76 (2015), pp. 63–74.

Article II: Isomorphism theorems for gyrogroups and L-subgyrogroups

Published in Journal of Geometry and Symmetry in Physics, vol. 37 (2015), pp. 67–83.

Article III: Lagrange's theorem for gyrogroups and the Cauchy property

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CHAPTER II

EINSTEIN GYROGROUP AS A B-LOOP*

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Abstract. Using the Clifford algebra formalism, we give an algebraic proof that the open unit ball $\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n \colon \|\mathbf{v}\| < 1\}$ of $\mathbb{R}^n$ equipped with Einstein addition $\oplus_E$ forms a B-loop or, equivalently, a uniquely 2-divisible gyrocommutative gyrogroup. We obtain a compact formula for Einstein addition in terms of Möbius addition. We then give a characterization of associativity and commutativity of vectors in $\mathbb{B}$ with respect to Einstein addition.
Keywords: Einstein velocity addition, gyrogroup, B-loop.
2010 MSC: 83A05, 20N05, 15A66.
Journal: Reports on Mathematical Physics, vol. 76 (2015), pp. 63–74.

2.1 Introduction

Gyrogroup theory, introduced by Abraham A. Ungar, is related to various fields, including mathematical physics. For instance, the gyrogroup structure appears as an algebraic structure that encodes Einstein's velocity addition law [11, 14]. It is also an algebraic structure that underlies the qubit density matrices, which play an important role in quantum mechanics [6, 10]. For a connection to Thomas precession, see [15]. Of particular importance is the following composition law of Lorentz boosts,

$$L(\mathbf{u}) \circ L(\mathbf{v}) = L(\mathbf{u} \oplus_E \mathbf{v}) \circ \mathrm{Gyr}[\mathbf{u}, \mathbf{v}],$$

where $L(\mathbf{u})$ and $L(\mathbf{v})$ stand for Lorentz boosts parameterized by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3_c$ and $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$ is a rotation of spacetime coordinates induced by the Einstein gyroautomorphism generated by $\mathbf{u}$

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and $\mathbf{v}$ [12, p. 448]. Connections between Einstein addition, Möbius addition, and hyperbolic geometry are described in [7, 9]. For a connection to loops, see [5].

In [11], Einstein velocity addition, $\oplus_E$, on the set of relativistically admissible velocities, $\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 \colon \|\mathbf{v}\| < c\}$, is given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}} \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right\},$$

where c is a positive constant representing the speed of light in vacuum and $\gamma_{\mathbf{u}}$ is the Lorentz factor given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}.$$

The system $(\mathbb{R}^3_c, \oplus_E)$ does not form a group since $\oplus_E$ is neither associative nor commutative. Nevertheless, $(\mathbb{R}^3_c, \oplus_E)$ is rich in structure and encodes a group-like structure, namely the gyrogroup structure. Ungar declared that $(\mathbb{R}^3_c, \oplus_E)$ forms a gyrocommutative gyrogroup, the so-called *Einstein gyrogroup*, where the gyrogroup axioms can be checked using computer algebra. It seems to us that no solid proof of this result is given in the literature. For this reason, we use the Clifford algebra formalism to prove this result. It turns out that *Einstein gyroautomorphisms*, also known as *Thomas gyrations*, can be expressed in a simple form using Clifford algebra operations.

Another example of a gyrogroup is the *Möbius gyrogroup*, which consists of the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ and Möbius addition

$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b}, \qquad a, b \in \mathbb{D}.$$

In [13], the complex Möbius addition is extended to the Euclidean one,

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2 \|\mathbf{v}\|^2}, \qquad \mathbf{u}, \mathbf{v} \in \mathbb{B}.$$

Here, $\mathbb{B}$ denotes the open unit ball of $\mathbb{R}^n$,

$$\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n \colon \|\mathbf{v}\| < 1\}.$$

Because the formula for the Euclidean version of Möbius addition is very complicated, Lawson [8], Ferreira and Ren [2] used the Clifford algebra formalism to study the Möbius gyrogroup and to simplify Möbius addition,

$$\mathbf{u} \oplus_M \mathbf{v} = (\mathbf{u} + \mathbf{v})(1 - \mathbf{u}\mathbf{v})^{-1}, \tag{2.1}$$

where the product and inverse on the right hand side of Equation (2.1) are performed in the Clifford algebra of *negative* Euclidean space.

With the compact formula (2.1) for Möbius addition in hand, we give an algebraic proof that the unit ball of $\mathbb{R}^n$ with Einstein addition does form a B-loop or a gyrocommutative gyrogroup with the uniquely 2-divisible property. As a consequence, we give a characterization of associativity and commutativity of the elements of Einstein gyrogroup $(\mathbb{B}, \oplus_E)$.

2.2 Preliminaries

Let $(G, \oplus)$ be a magma. Denote the group of automorphisms of G with respect to $\oplus$ by $\mathrm{Aut}\,(G, \oplus)$.

Definition 2.2.1 ([12]). A magma $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

(G1) There is an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$. (left identity)

(G2) For each $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$. (left inverse)

(G3) For all $a, b \in G$, there is an automorphism $\mathrm{gyr}\,[a, b] \in \mathrm{Aut}\,(G, \oplus)$ such that for all $c \in G$,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b]c. \qquad \text{(left gyroassociative law)}$$

(G4) For all $a, b \in G$, $\mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[a \oplus b, b]$. (left loop property)

Definition 2.2.2 ([12]). A gyrogroup $(G, \oplus)$ having the additional property that

$$a \oplus b = \mathrm{gyr}\,[a, b](b \oplus a) \qquad \text{(gyrocommutative law)}$$

for all $a, b \in G$ is called a *gyrocommutative gyrogroup*.

We remark that the axioms in Definition 2.2.1 imply the right counterparts. The map $\mathrm{gyr}\,[a, b]$ is called the *gyroautomorphism generated by a and b*. We refer the reader to [12] for a deep discussion of gyrogroups.

Definition 2.2.3. A magma L has the *uniquely 2-divisible property* if the squaring map $x \mapsto x^2$ is a bijection from L to itself.

Definition 2.2.4. A loop L has the A_ℓ*-property* if the left inner mapping

$$\ell(a, b) := L_{ab}^{-1} \circ L_a \circ L_b$$

defines an automorphism of L for all $a, b \in L$. Here, L_a denotes the left multiplication map by a defined by $L_a \colon x \mapsto ax$ for $x \in L$.

A loop L is called a *K-loop* or *Bruck loop* if every element of L has a unique inverse and L satisfies the left Bol identity (I) and the automorphic inverse property (II):

(I) $a(b(ac)) = (a(ba))c$

(II) $(ab)^{-1} = a^{-1}b^{-1}$

for all $a,b,c \in L$. A loop L is called a *B-loop* if it is a uniquely 2-divisible K-loop. In the literature, it is known that gyrogroups and left Bol loops with the A_ℓ-property are equivalent, and that uniquely 2-divisible gyrocommutative gyrogroups and B-loops are equivalent.

In order to prove that the unit ball of $\mathbb{R}^n$ with Einstein addition forms a uniquely 2-divisible gyrocommutative gyrogroup, we make use of the following theorem.

Theorem 2.2.5 (Theorem 1, [1]). *Let $(G,\oplus)$ be a gyrogroup, X an arbitrary space, and $\phi : X \to G$ a bijection between G and X. Then X endowed with the induced operation*

$$a \oplus_X b := \phi^{-1}(\phi(a) \oplus \phi(b))$$

for $a,b \in X$ becomes a gyrogroup.

Proposition 2.2.6. *Let $(G,\oplus)$ be a gyrogroup. If $(G,\oplus)$ is gyrocommutative, then so is the induced gyrogroup $(X,\oplus_X)$. If $(G,\oplus)$ is uniquely 2-divisible, then so is $(X,\oplus_X)$.*

Proof. The proof of the first statement is straightforward. Let D_G and D_X denote the doubling maps of G and X, respectively. Assume that G is uniquely 2-divisible, that is, D_G is bijective. For all $x \in X$,

$$D_X(x) = x \oplus_X x = \phi^{-1}(\phi(x) \oplus \phi(x)) = \phi^{-1}(D_G(\phi(x))) = (\phi^{-1} \circ D_G \circ \phi)(x).$$

It follows that $D_X = \phi^{-1} \circ D_G \circ \phi$ and hence D_X is bijective, which proves that X is uniquely 2-divisible. ■

2.3 Quadratic spaces and Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ of characteristic different from 2. A *quadratic form Q* on V is a map $Q \colon V \to \mathbb{F}$ such that

(1) $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in \mathbb{F}$, $v \in V$ and

(2) the map $B \colon V \times V \to \mathbb{F}$ defined by $B(u,v) = 1/2\,(Q(u+v) - Q(u) - Q(v))$ is a symmetric bilinear form on V.

Note that any symmetric bilinear form B on V gives rise to a quadratic form Q by defining $Q(v) = B(v,v)$ for $v \in V$. A *quadratic space* is a vector space equipped with a quadratic form on which the associated bilinear form is nondegenerate. Let (V,Q) be a quadratic space with the corresponding bilinear form B. A basis $\{e_1, e_2, \ldots, e_n\}$ of V is *orthogonal* if $B(e_i, e_j) = 0$ for all $i \neq j$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal basis for (V,Q). The *Clifford algebra* of (V,Q), written $C\ell_Q$, is a unital associative algebra over $\mathbb{F}$ with a basis

$$\{e_I \colon I = \emptyset \text{ or } I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}\},$$

where $e_\emptyset := 1$ and $e_I := e_{i_1} e_{i_2} \cdots e_{i_k}$ for $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. From this a typical element of $C\ell_Q$ is of the form $\sum_I \lambda_I e_I$, $\lambda_I \in \mathbb{F}$. Vector addition and scalar multiplication of $C\ell_Q$ are defined pointwise, and multiplication is performed by using the distributive law without assuming commutativity subject to the defining relations

$$e_i^2 = Q(e_i)1 \quad \text{and} \quad e_i e_j = -e_j e_i$$

for $i \neq j$.

In $C\ell_Q$, one has relations $v^2 = Q(v)1$ and $uv + vu = 2B(u,v)1$ for all $u, v \in V$. The base field $\mathbb{F}$ is embedded into $C\ell_Q$ by the map $\lambda \mapsto \lambda 1$, and V is naturally embedded into $C\ell_Q$ by inclusion. For a deep discussion of Clifford algebras, we refer the reader to [4].

There are three standard maps of a Clifford algebra. One is an involutive algebra automorphism, and the others are involutive algebra anti-automorphisms. Table 2.1 summarizes basic properties of such maps.

MAP	**TYPE**	**ON V**
reversion $\rho(a) = \tilde{a}$	anti-automorphism	id_V
grade involution $\tau(a) = \hat{a}$	automorphism	$-\mathrm{id}_V$
Clifford conjugation $\kappa(a) = \bar{a}$	anti-automorphism	$-\mathrm{id}_V$

TABLE 2.1: Three standard maps of $C\ell_Q$

The *Clifford* or *Lipschitz group* of $C\ell_Q$, written $\Gamma(Q)$, is defined via the grade involution as

$$\Gamma(Q) = \{g \in C\ell_Q^\times \colon \forall v \in V, \hat{g} v g^{-1} \in V\}.$$

In the finite-dimensional case, $\Gamma(Q)$ does form a subgroup of the group of units of $C\ell_Q$. Further, the grade involution descends to a group automorphism of $\Gamma(Q)$, and the reversion and Clifford conjugation descend to group anti-automorphisms of $\Gamma(Q)$.

Let $\eta : C\ell_Q \to C\ell_Q$ be the map defined by

$$\eta(a) = a\bar{a}$$

for $a \in C\ell_Q$. It can be proved that $\eta(g)$ belongs to $\mathbb{F}^\times 1 := \{\lambda 1 \colon \lambda \in \mathbb{F}^\times\}$ for all $g \in \Gamma(Q)$ and hence the restriction of η to $\Gamma(Q)$ is a group homomorphism.

Proposition 2.3.1. *The restriction of η to $\Gamma(Q)$ is a group homomorphism from $\Gamma(Q)$ to $\mathbb{F}^\times 1$. Furthermore, η is multiplicative over the set of products of vectors in V in the sense that*

$$\eta(v_1 v_2 \cdots v_k) = \eta(v_1)\eta(v_2)\cdots\eta(v_k)$$

for all $v_1, v_2, \ldots, v_k \in V$.

Proof. This is because $\eta(g)$ and $\eta(v)$ are scalar multiples of unity for all $g \in \Gamma(Q)$ and $v \in V$. ■

Invertibility of elements of the form $1 + uv$

In this subsection, we provide a necessary and sufficient condition for invertibility of elements of the form $1 + uv$, where u and v are vectors in a quadratic space. Let (V, Q) be a quadratic space with the corresponding bilinear form B. From now on, the term *vector* is reserved for the elements of V.

Lemma 2.3.2. *If u, v and w are vectors, then so are uvu and $uvw + wvu$.*

Proof. This follows from the fact that $uv + vu = 2B(u, v)1$ for all $u, v \in V$. ■

Proposition 2.3.3. *If u and v are vectors, then either*

(1) *$1 + uv$ is a product of vectors or*

(2) *$1 + uv$ belongs to $\Gamma(Q)$ and $\eta(1 + uv) = 1$.*

Proof. Recall that if w is a nonisotropic vector, then w is invertible and $w^{-1} = w/Q(w)$ is again a vector. If u or v is nonisotropic, then $1 + uv$ is a product of vectors. We may therefore assume that u and v are isotropic. If $B(u, v) \neq 0$, then $u + v$ is invertible and

$$1 + uv = (u + v + 2B(u, v)u)(u + v)^{-1}$$

is a product of vectors. If $B(u,v)=0$, then $\eta(1+uv)=1+2B(u,v)1+Q(u)Q(v)1=1$. By the lemma, $\tau(1+uv)w(1+uv)^{-1}=w+wvu+uvw+uvwvu$ belongs to V for all $w\in V$. Hence, $1+uv\in\Gamma(Q)$. ■

Proposition 2.3.4. *For all $u,v\in V$, $1+uv\in\Gamma(Q)$ if and only if $\eta(1+uv)\neq 0$.*

Proof. ($\Rightarrow$) If $1+uv\in\Gamma(Q)$, then $\eta(1+uv)\in\mathbb{F}^{\times}1$. Hence, $\eta(1+uv)\neq 0$.

($\Leftarrow$) Suppose that $\eta(1+uv)\neq 0$. By Proposition 2.3.3, either $1+uv$ already belongs to $\Gamma(Q)$ or $1+uv$ is a product of vectors. In the latter case, $1+uv=w_1w_2\cdots w_k$ for some $w_1,w_2,\ldots,w_k$ in V. Because $0\neq\eta(1+uv)=\eta(w_1w_2\cdots w_k)=\eta(w_1)\eta(w_2)\cdots\eta(w_k)$, none of $\eta(w_i)$ are zeros. Thus, $w_1,w_2,\ldots$, and w_k are all nonisotropic vectors and hence $1+uv$ belongs to $\Gamma(Q)$. ■

2.4 Negative Euclidean space

The *negative Euclidean space* consists of the underlying vector space $\mathbb{R}^n$ with a nondegenerate symmetric bilinear form

$$B(\mathbf{u},\mathbf{v})=-\langle\mathbf{u},\mathbf{v}\rangle,\qquad \mathbf{u},\mathbf{v}\in\mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean inner product of $\mathbb{R}^n$. Its associated quadratic form is given by $Q(\mathbf{v})=-\|\mathbf{v}\|^2$ for $\mathbf{v}\in\mathbb{R}^n$.

For convenience, let $C\ell_n$ denote the Clifford algebra of negative Euclidean space, let Γ_n denote the Clifford group of $C\ell_n$, and let $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. From now on, we identify elements of $\mathbb{R}1$ with real numbers, that is, $r1\leftrightarrow r$ for $r\in\mathbb{R}$.

Proposition 2.4.1. *In the Clifford algebra $C\ell_n$, the following properties hold.*

(1) $\mathbf{uv}+\mathbf{vu}=-2\langle\mathbf{u},\mathbf{v}\rangle$ *for all* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$.

(2) $\mathbf{v}^2=-\|\mathbf{v}\|^2$ *for all* $\mathbf{v}\in\mathbb{R}^n$.

(3) $\mathbf{e}_i^2=-1$, $\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_j\mathbf{e}_i$ *for* $1\leq i,j\leq n$ *and* $i\neq j$.

(4) $1-\mathbf{uv}\in\Gamma_n$ *and* $(1-\mathbf{uv})^{-1}=\dfrac{1-\mathbf{vu}}{\eta(1-\mathbf{uv})}$ *for all* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ *with* $\|\mathbf{u}\|\|\mathbf{v}\|\neq 1$.

(5) $\eta(\mathbf{w}(1-\mathbf{uv})^{-1})=\dfrac{\eta(\mathbf{w})}{\eta(1-\mathbf{uv})}$ *for all* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$ *with* $\|\mathbf{u}\|\|\mathbf{v}\|\neq 1$.

Proof. Items (1)–(3) follow from the defining relations in $C\ell_n$.

(4) The Cauchy-Schwarz inequality gives

$$\begin{aligned}\eta(1-\mathbf{uv}) &= 1+2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2\\ &\geq 1-2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{u}\|^2\|\mathbf{v}\|^2\\ &= (1-\|\mathbf{u}\|\|\mathbf{v}\|)^2.\end{aligned}$$

It follows that if $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ with $\|\mathbf{u}\|\|\mathbf{v}\|\neq 1$, then $\eta(1-\mathbf{uv})>0$ and hence $1-\mathbf{uv}\in\Gamma_n$ by Proposition 2.3.4. Since $1-\mathbf{uv}\in\Gamma_n$, we have

$$(1-\mathbf{uv})^{-1} = \frac{\overline{1-\mathbf{uv}}}{\eta(1-\mathbf{uv})} = \frac{1-\mathbf{vu}}{\eta(1-\mathbf{uv})}.$$

(5) Let $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$ with $\|\mathbf{u}\|\|\mathbf{v}\|\neq 1$. If $\mathbf{w}=\mathbf{0}$, equality holds trivially. We may therefore assume that $\mathbf{w}\neq\mathbf{0}$. Hence, $\mathbf{w}\in\Gamma_n$. By Item (4), $1-\mathbf{uv}\in\Gamma_n$ and so

$$\eta(\mathbf{w}(1-\mathbf{uv})^{-1}) = \eta(\mathbf{w})\eta((1-\mathbf{uv})^{-1}) = \frac{\eta(\mathbf{w})}{\eta(1-\mathbf{uv})}$$

since η is a group homomorphism of Γ_n. ■

2.5 Möbius and Einstein gyrogroups on $\mathbb{R}^n$

Using relations $\mathbf{v}^2=-\|\mathbf{v}\|^2$ and $\mathbf{uv}+\mathbf{vu}=-2\langle\mathbf{u},\mathbf{v}\rangle$ in the Clifford algebra of negative Euclidean space, Lawson [8] verified that

$$(\mathbf{u}+\mathbf{v})(1-\mathbf{uv})^{-1} = \frac{(1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\langle\mathbf{u},\mathbf{v}\rangle+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} = \mathbf{u}\oplus_M\mathbf{v}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}$. Hence, the Euclidean version of Möbius addition has a compact formula analogous to the complex case. He also dealt with the group of Möbius transformations of $\mathbb{R}^n$ that preserve the open unit ball to prove that $(\mathbb{B},\oplus_M)$ is indeed a B-loop.

In light of the proof of Proposition 2.4.1, $\eta(1-\mathbf{uv})\geq 0$ for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$. Hence, the notation

$$|1-\mathbf{uv}| := \sqrt{\eta(1-\mathbf{uv})}$$

is meaningful whenever $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$. Further, $|\mathbf{v}|=\|\mathbf{v}\|$ for all $\mathbf{v}\in\mathbb{R}^n$.

Theorem 2.5.1 ([8]). *The Möbius loop on the open unit ball in $\mathbb{R}^n$ forms a B-loop whose operation is given in terms of the Clifford algebra $\mathrm{C}\ell_n$ by*

$$\mathbf{u}\oplus_M\mathbf{v} = (\mathbf{u}+\mathbf{v})(1-\mathbf{uv})^{-1}. \tag{2.2}$$

The left inner mappings are given by $\ell(\mathbf{u},\mathbf{v})\mathbf{w}=q\mathbf{w}q^{-1}$, where $q=\dfrac{1-\mathbf{uv}}{|1-\mathbf{uv}|}$.

Combining Equation (2.2) with Proposition 2.4.1 (5) gives

$$\eta(\mathbf{u} \oplus_M \mathbf{v}) = \frac{\eta(\mathbf{u}+\mathbf{v})}{\eta(1-\mathbf{u}\mathbf{v})} \tag{2.3}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$.

From now on, we work in the Clifford algebra of negative Euclidean space, $C\ell_n$. In light of Theorem 2.2.5 and Proposition 2.2.6, we express Einstein addition via Möbius addition to deduce that $(\mathbb{B}, \oplus_E)$ forms a uniquely 2-divisible gyrocommutative gyrogroup.

For each $\mathbf{v} \in \mathbb{B}$, set

$$r_{\mathbf{v}} = \frac{1}{1+\sqrt{1-\|\mathbf{v}\|^2}}. \tag{2.4}$$

Then

$$r_{\mathbf{v}} = \frac{1-\sqrt{1-\|\mathbf{v}\|^2}}{\|\mathbf{v}\|^2}$$

and

$$r_{\mathbf{v}} = \frac{1}{1+\sqrt{1+\mathbf{v}^2}}$$

in $C\ell_n$. According to the Lorentz factor normalized to $c = 1$, we have

$$r_{\mathbf{v}} = \frac{1}{1+\gamma_{\mathbf{v}}^{-1}} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}.$$

It follows that $0 < r_{\mathbf{v}} < 1$. In fact, $r_{\mathbf{v}}$ is a solution to the quadratic equation

$$\|\mathbf{v}\|^2 x^2 - 2x + 1 = 0$$

in the variable x. Hence,

$$\frac{2r_{\mathbf{v}}}{1-r_{\mathbf{v}}^2\mathbf{v}^2} = 1. \tag{2.5}$$

Let Ψ be the map defined on $\mathbb{B}$ by

$$\Psi(\mathbf{v}) = r_{\mathbf{v}}\mathbf{v}, \qquad \mathbf{v} \in \mathbb{B}. \tag{2.6}$$

Since $0 < r_{\mathbf{v}} < 1$, we have $\|\Psi(\mathbf{v})\| = \|r_{\mathbf{v}}\mathbf{v}\| = r_{\mathbf{v}}\|\mathbf{v}\| < 1$. Hence, $\Psi(\mathbb{B}) \subseteq \mathbb{B}$.

Let Φ be the map defined on $\mathbb{B}$ by

$$\Phi(\mathbf{v}) = \mathbf{v} \oplus_M \mathbf{v}. \tag{2.7}$$

From Equation (2.2), we have

$$\Phi(\mathbf{v}) = \frac{2\mathbf{v}}{1-\mathbf{v}^2} = \frac{2\mathbf{v}}{1+\|\mathbf{v}\|^2}.$$

The map Φ is called the *doubling map* and is of importance for the study of Möbius and Einstein gyrogroups, see for instance [7].

In the case $\|\mathbf{v}\| = 0$, $\mathbf{v} = \mathbf{0}$ and hence $\|\Phi(\mathbf{v})\| = \|\Phi(\mathbf{0})\| = \|\mathbf{0}\| = 0$. In the case $0 < \|\mathbf{v}\| < 1$,

$$\|\Phi(\mathbf{v})\| = \frac{2}{\frac{1}{\|\mathbf{v}\|} + \|\mathbf{v}\|} < 1$$

since $\dfrac{1}{\|\mathbf{v}\|} + \|\mathbf{v}\| > 2$. It follows that $\Phi(\mathbb{B}) \subseteq \mathbb{B}$.

Proposition 2.5.2. *The maps Ψ and Φ are bijections from $\mathbb{B}$ to itself and are inverses of each other.*

Proof. Let $\mathbf{v} \in \mathbb{B}$. Since $1 - \|\Phi(\mathbf{v})\|^2 = 1 - \dfrac{4\|\mathbf{v}\|^2}{(1+\|\mathbf{v}\|^2)^2} = \left(\dfrac{1-\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2}\right)^2$, we have

$$1 + \sqrt{1 - \|\Phi(\mathbf{v})\|^2} = 1 + \frac{1-\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2} = \frac{2}{1+\|\mathbf{v}\|^2} = \frac{2}{1-\mathbf{v}^2}.$$

It follows that $(\Psi \circ \Phi)(\mathbf{v}) = \Psi(\Phi(\mathbf{v})) = r_{\Phi(\mathbf{v})}\Phi(\mathbf{v}) = \dfrac{1}{1+\sqrt{1+\Phi(\mathbf{v})^2}}\dfrac{2\mathbf{v}}{1-\mathbf{v}^2} = \mathbf{v}$.

From Equation (2.5), we have

$$(\Phi \circ \Psi)(\mathbf{v}) = \Phi(\Psi(\mathbf{v})) = \frac{2\Psi(\mathbf{v})}{1-\Psi(\mathbf{v})^2} = \frac{2r_{\mathbf{v}}}{1-r_{\mathbf{v}}^2\mathbf{v}^2}\mathbf{v} = \mathbf{v}.$$

This proves $\Psi \circ \Phi = \mathrm{id}_{\mathbb{B}}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathbb{B}}$. Hence, Φ and Ψ are bijections, $\Phi^{-1} = \Psi$, and $\Psi^{-1} = \Phi$. ∎

Proposition 2.5.3. *The unit ball $\mathbb{B}$ with the induced operation*

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \Psi^{-1}(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})), \qquad \mathbf{u}, \mathbf{v} \in \mathbb{B},$$

forms a uniquely 2-divisible gyrocommutative gyrogroup.

Proof. The proposition follows from Theorem 2.2.5 and Proposition 2.2.6 applied to $(\mathbb{B}, \oplus_M)$ and Ψ. ∎

In fact, the induced addition $\oplus_{\mathbb{B}}$ is nothing but Einstein addition, as shown in the following theorem.

Theorem 2.5.4. *For all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$,*

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \mathbf{u} \oplus_E \mathbf{v}.$$

In particular, $(\mathbb{B}, \oplus_E)$ forms a uniquely 2-divisible gyrocommutative gyrogroup. In terms of the Clifford algebra $C\ell_n$, Einstein addition can be rewritten as

$$\mathbf{u} \oplus_E \mathbf{v} = 2(r_{\mathbf{u}}\mathbf{u} \oplus_M r_{\mathbf{v}}\mathbf{v})\left(1 - (r_{\mathbf{u}}\mathbf{u} \oplus_M r_{\mathbf{v}}\mathbf{v})^2\right)^{-1} \tag{2.8}$$

and the Einstein gyroautomorphisms are given by

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = q\mathbf{w}q^{-1}, \qquad q = \frac{1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}}{|1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}|},$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{B}$.

Proof. Since $\Psi^{-1} = \Phi$, we have

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \Phi(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) = \frac{2}{1 - [\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})]^2}[\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})].$$

Note that $\eta(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) = [\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})]\overline{\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})} = -[\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})]^2$ since $\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}) \in \mathbb{R}^n$. Equations (2.2) and (2.3) and Proposition 2.4.1 together imply

$$\begin{aligned}\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} &= \frac{2}{1 + \eta(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}))}[\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})] \\ &= \frac{2}{1 + \dfrac{\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v}))}{\eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))}}[\Psi(\mathbf{u}) + \Psi(\mathbf{v})][1 - \Psi(\mathbf{u})\Psi(\mathbf{v})]^{-1} \\ &= \frac{2[\Psi(\mathbf{u}) + \Psi(\mathbf{v})][1 - \Psi(\mathbf{v})\Psi(\mathbf{u})]}{\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v})) + \eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))}.\end{aligned} \tag{2.9}$$

Since $1 - r_{\mathbf{u}}^2\mathbf{u}^2 - r_{\mathbf{v}}^2\mathbf{v}^2 + r_{\mathbf{u}}^2\mathbf{u}^2 r_{\mathbf{v}}^2\mathbf{v}^2 = (1 - r_{\mathbf{u}}^2\mathbf{u}^2)(1 - r_{\mathbf{v}}^2\mathbf{v}^2) = (2r_{\mathbf{u}})(2r_{\mathbf{v}}) = 4r_{\mathbf{u}}r_{\mathbf{v}}$, we have

$$\begin{aligned}\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v})) + \eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v})) &= 1 - r_{\mathbf{u}}^2\mathbf{u}^2 - r_{\mathbf{v}}^2\mathbf{v}^2 + r_{\mathbf{u}}^2\mathbf{u}^2 r_{\mathbf{v}}^2\mathbf{v}^2 + 4r_{\mathbf{u}}r_{\mathbf{v}}\langle\mathbf{u},\mathbf{v}\rangle \\ &= 4r_{\mathbf{u}}r_{\mathbf{v}}(1 + \langle\mathbf{u},\mathbf{v}\rangle).\end{aligned} \tag{2.10}$$

We also have

$$\begin{aligned}&\frac{1}{2r_{\mathbf{u}}r_{\mathbf{v}}}[\Psi(\mathbf{u}) + \Psi(\mathbf{v})][1 - \Psi(\mathbf{v})\Psi(\mathbf{u})] \\ &= \frac{\mathbf{u}}{2r_{\mathbf{v}}} - \frac{r_{\mathbf{u}}}{2}\mathbf{u}\mathbf{v}\mathbf{u} + \frac{\mathbf{v}}{2r_{\mathbf{u}}} - \frac{r_{\mathbf{v}}}{2}\mathbf{v}^2\mathbf{u} \\ &= \frac{1}{2}\left(\frac{1}{r_{\mathbf{v}}} - r_{\mathbf{v}}\mathbf{v}^2\right)\mathbf{u} - \frac{r_{\mathbf{u}}}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})\mathbf{u} + \frac{1}{2}\left(r_{\mathbf{u}}\mathbf{u}^2 + \frac{1}{r_{\mathbf{u}}}\right)\mathbf{v} \\ &= \mathbf{u} + \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle\mathbf{u},\mathbf{v}\rangle\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}.\end{aligned} \tag{2.11}$$

We obtain the third equation of (2.11) because $\dfrac{1}{r_{\mathbf{w}}} = 1 + \sqrt{1 - \|\mathbf{w}\|^2} = 1 + \dfrac{1}{\gamma_{\mathbf{w}}}$ and

$$r_{\mathbf{w}}\mathbf{w}^2 = \frac{1 - \sqrt{1 - \|\mathbf{w}\|^2}}{\|\mathbf{w}\|^2}(-\|\mathbf{w}\|^2) = \sqrt{1 - \|\mathbf{w}\|^2} - 1 = \frac{1}{\gamma_{\mathbf{w}}} - 1$$

for all $\mathbf{w} \in \mathbb{B}$. Combining Equations (2.9) – (2.11) gives

$$\begin{aligned} \mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} &= \frac{4r_{\mathbf{u}}r_{\mathbf{v}}\left\{\mathbf{u} + \dfrac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u} + \dfrac{1}{\gamma_{\mathbf{u}}}\mathbf{v}\right\}}{4r_{\mathbf{u}}r_{\mathbf{v}}(1+\langle \mathbf{u}, \mathbf{v}\rangle)} \\ &= \frac{1}{1+\langle \mathbf{u}, \mathbf{v}\rangle}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}\right\} \\ &= \mathbf{u} \oplus_E \mathbf{v}. \end{aligned}$$

The second part of the theorem follows from the result that

$$\operatorname{gyr}_E[\mathbf{u}, \mathbf{v}] = \ell(\Phi(\mathbf{u}), \Phi(\mathbf{v}))$$

and Theorem 2.5.1. ■

Equation (2.8) shows a close relationship between elements of Einstein and Möbius gyrogroups. See also [12, Equation (6.297)] and [1, Proposition 6]. In terms of *Einstein scalar multiplication* [12, p. 218], given by

$$r \otimes_E \mathbf{v} = \frac{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad r \in \mathbb{R}, \mathbf{0} \neq \mathbf{v} \in \mathbb{B}, \tag{2.12}$$

Equations (2.6) and (2.7) can be rewritten as

$$\Psi(\mathbf{v}) = \frac{1}{2} \otimes_E \mathbf{v} \quad \text{and} \quad \Phi(\mathbf{v}) = 2 \otimes_E \mathbf{v},$$

which reflects the fact that Ψ and Φ are inverses of each other.

Although the result that Einstein addition can be expressed via Möbius addition is known, see Friedman and Scarr [3, Equation (2.13)], we obtain these results using a *different* technique. In fact, Friedman and Scarr obtained the result using the *principle of special relativity*, whereas we use an algebraic approach.

We end this section with the following characterization of associativity and commutativity of the elements of Einstein gyrogroup $(\mathbb{B}, \oplus_E)$.

Theorem 2.5.5. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{B}$*,*

$$\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}$$

if and only if $\langle \mathbf{u}, \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle = 0$ *or* $\mathbf{u}\|\mathbf{v}$.

Proof. $(\Rightarrow)$ Suppose that $\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}$. Since $\Psi\colon (\mathbb{B}, \oplus_E) \to (\mathbb{B}, \oplus_M)$ is a gyrogroup isomorphism, we have

$$\Psi(\mathbf{u}) \oplus_M (\Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{w})) = (\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) \oplus_M \Psi(\mathbf{w}).$$

By Lemma 10 of [2], $\langle \Psi(\mathbf{u}), \Psi(\mathbf{w}) \rangle = \langle \Psi(\mathbf{v}), \Psi(\mathbf{w}) \rangle = 0$ or $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$. By Equation (2.6), $\langle r_{\mathbf{u}}\mathbf{u}, r_{\mathbf{w}}\mathbf{w} \rangle = 0 = \langle r_{\mathbf{v}}\mathbf{v}, r_{\mathbf{w}}\mathbf{w} \rangle$ or $r_{\mathbf{u}}\mathbf{u} \| r_{\mathbf{v}}\mathbf{v}$, which implies the desired statement.

($\Leftarrow$) If $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = 0$, then $\langle \Psi(\mathbf{u}), \Psi(\mathbf{w}) \rangle = \langle r_{\mathbf{u}}\mathbf{u}, r_{\mathbf{w}}\mathbf{w} \rangle = 0$. Similarly, $\langle \Psi(\mathbf{v}), \Psi(\mathbf{w}) \rangle = 0$. Hence, $\Psi(\mathbf{u}) \oplus_M (\Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{w})) = (\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) \oplus_M \Psi(\mathbf{w})$. Applying Φ to both sides of the equation gives $\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w}$ since $\Phi = \Psi^{-1}$ and Φ preserves the operations. If $\mathbf{u} \| \mathbf{v}$, then $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$ and so the same reasoning applies. ■

Theorem 2.5.6. *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{B}$,

$$\mathbf{u} \oplus_E \mathbf{v} = \mathbf{v} \oplus_E \mathbf{u}$$

if and only if $\mathbf{u} \| \mathbf{v}$.

Proof. If $\mathbf{u} \oplus_E \mathbf{v} = \mathbf{v} \oplus_E \mathbf{u}$, then $\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}) = \Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{u})$. By Lemma 11 of [2], $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$. Hence, $\mathbf{u} \| \mathbf{v}$. Conversely, if $\mathbf{u} \| \mathbf{v}$, then $\Psi(\mathbf{u}) \| \Psi(\mathbf{v})$, which implies

$$\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}) = \Psi(\mathbf{v}) \oplus_M \Psi(\mathbf{u}).$$

Applying Φ to both sides of the equation gives $\mathbf{u} \oplus_E \mathbf{v} = \mathbf{v} \oplus_E \mathbf{u}$. ■

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References

[1] Ferreira, M.: *Hypercomplex Analysis and Applications*, I. Sabadini and F. Sommen (eds.), chap. Gyrogroups in projective hyperbolic Clifford analysis, pp. 61–80. Trends in Mathematics. Springer, Basel (2011)

[2] Ferreira, M., Ren, G.: Möbius gyrogroups: A Clifford algebra approach. *J. Algebra* **328**, 230–253 (2011)

[3] Friedman, Y., Scarr, T.: *Physical Applications of Homogeneous Balls*, *Progress in Mathematical Physics*, vol. 40. Birkhäuser, Boston (2005)

[4] Grove, L.C.: *Classical Groups and Geometric Algebra*, *Graduate Studies in Mathematics*, vol. 39. AMS, Providence, RI (2001)

[5] Issa, A.N.: Gyrogroups and homogeneous loops. *Rep. Math. Phys.* **44**(3), 345–358 (1999)

[6] Kim, S.: Distances of qubit density matrices on Bloch sphere. *J. Math. Phys.* **52**(10), 1–8 (2011)

[7] Kim, S., Lawson, J.: Unit balls, Lorentz boosts, and hyperbolic geometry. *Results. Math.* **63** (2013)

[8] Lawson, J.: Clifford algebras, Möbius transformations, Vahlen matrices, and B-loops. *Comment. Math. Univ. Carolin.* **51**(2), 319–331 (2010)

[9] Sönmez, N., Ungar, A.A.: The Einstein relativistic velocity model of hyperbolic geometry and its plane separation axiom. *Adv. Appl. Clifford Algebras* **23**, 209–236 (2013)

[10] Ungar, A.A.: The hyperbolic geometric structure of the density matrix for mixed state qubits. *Found. Phys.* **32**(11), 1671–1699 (2002)

[11] Ungar, A.A.: Einstein's velocity addition law and its hyperbolic geometry. *Comput. Math. Appl.* **53**, 1228–1250 (2007)

[12] Ungar, A.A.: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*. World Scientific, Hackensack, NJ (2008)

[13] Ungar, A.A.: From Möbius to gyrogroups. *Amer. Math. Monthly* **115**(2), 138–144 (2008)

[14] Ungar, A.A.: Einstein's Special Relativity: The hyperbolic geometric viewpoint. In: PIRT Conference Proceedings, 1–35 (2009).

[15] Ungar, A.A.: *Essays in Mathematics and its Applications*, P. M. Pardalos and T. M. Rassias (eds.), chap. Gyrations: The missing link between classical mechanics with its underlying Euclidean geometry and relativistic mechanics with its underlying hyperbolic geometry, pp. 463–504. Springer, Berlin Heidelberg (2012)

CHAPTER III

ISOMORPHISM THEOREMS FOR GYROGROUPS AND L-SUBGYROGROUPS*

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Abstract. We extend well-known results in group theory to gyrogroups, especially the isomorphism theorems. We prove that an arbitrary gyrogroup G induces the gyrogroup structure on the symmetric group of G so that Cayley's Theorem is obtained. Introducing the notion of L-subgyrogroups, we show that an L-subgyrogroup partitions G into left cosets. Consequently, if H is an L-subgyrogroup of a finite gyrogroup G, then the order of H divides the order of G.
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3.1 Introduction

Let c be a positive constant representing the speed of light in vacuum and let $\mathbb{R}^3_c$ denote the c-ball of relativistically admissible velocities, $\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 \colon \|\mathbf{v}\| < c\}$. In [13], Einstein velocity addition $\oplus_E$ in the c-ball is given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} \right\}$$

where $\gamma_{\mathbf{u}}$ is the Lorentz factor given by $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}$.

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The system $(\mathbb{R}^3_c, \oplus_E)$ does not form a group since $\oplus_E$ is neither associative nor commutative. Nevertheless, Ungar showed that $(\mathbb{R}^3_c, \oplus_E)$ is rich in structure and encodes a group-like structure, namely the gyrogroup structure. He introduced space rotations $\operatorname{gyr}[\mathbf{u},\mathbf{v}]$, called *gyroautomorphisms*, to repair the breakdown of associativity in $(\mathbb{R}^3_c, \oplus_E)$

$$\begin{aligned} \mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) &= (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \operatorname{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} \\ (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w} &= \mathbf{u} \oplus_E (\mathbf{v} \oplus_E \operatorname{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) \end{aligned}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^3_c$. The resulting system forms a gyrocommutative gyrogroup, called the *Einstein gyrogroup*, which has been intensively studied in [3, 5, 9, 11, 13, 14, 16, 17].

There are close connections between the Einstein gyrogroup and the Lorentz transformations, as described in [14, Chapter 11] and [12]. A Lorentz transformation without rotation is called a *Lorentz boost*. Let $L(\mathbf{u})$ and $L(\mathbf{v})$ denote Lorentz boosts parametrized by $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3_c$. The composite of two Lorentz boosts is not a pure Lorentz boost, but a Lorentz boost followed by a space rotation

$$L(\mathbf{u}) \circ L(\mathbf{v}) = L(\mathbf{u} \oplus_E \mathbf{v}) \circ \operatorname{Gyr}[\mathbf{u},\mathbf{v}] \tag{3.1}$$

where $\operatorname{Gyr}[\mathbf{u},\mathbf{v}]$ is a rotation of spacetime coordinates induced by the Einstein gyroautomorphism $\operatorname{gyr}[\mathbf{u},\mathbf{v}]$. In this paper, we present an abstract version of the composition law (3.1) of Lorentz boosts.

Another example of a gyrogroup is the *Möbius gyrogroup*, which consists of the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ with Möbius addition

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b} \tag{3.2}$$

for $a, b \in \mathbb{D}$. The Möbius gyroautomorphisms are given by

$$\operatorname{gyr}[a,b]z = \frac{1+a\bar{b}}{1+\bar{a}b}z, \qquad z \in \mathbb{D}. \tag{3.3}$$

Let $\mathbb{B}$ denote the open unit ball of n-dimensional Euclidean space $\mathbb{R}^n$ (or more generally of a real inner product space). In [15], Ungar extended Möbius addition from the complex unit disk to the unit ball

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1+2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{3.4}$$

for $\mathbf{u},\mathbf{v} \in \mathbb{B}$. The unit ball together with Möbius addition forms a gyrocommutative gyrogroup, which has been intensively studied in [1, 4, 6, 7, 14–16].

The factorization of Möbius gyrogroups was comprehensively studied by Ferreira and Ren in [1, 4], in which they showed that any Möbius subgyrogroup partitions the Möbius gyrogroup into left cosets. The fact that any subgyrogroup of an arbitrary gyrogroup partitions the gyrogroup is not stated in the literature, and this is indeed the case, as shown in Theorem 3.4.2. This result leads to the introduction of *L-subgyrogroups*. We prove that an L-subgyrogroup partitions the gyrogroup into left cosets and consequently obtain a portion of *Lagrange's Theorem*: if H is an L-subgyrogroup of a finite gyrogroup G, then the order of H divides the order of G. We also prove the isomorphism theorems for gyrogroups, in full analogy with their group counterparts.

3.2 Basic properties of gyrogroups

A pair $(G,\oplus)$ consisting of a nonempty set G and a binary operation $\oplus$ on G is called a *magma*. Let $(G,\oplus)$ be a magma. A bijection from G to itself is called an *automorphism* of G if $\varphi(a\oplus b)=\varphi(a)\oplus\varphi(b)$ for all $a,b\in G$. The set of all automorphisms of G is denoted by $\mathrm{Aut}\,(G,\oplus)$. Ungar formulated the formal definition of a gyrogroup as follows.

Definition 3.2.1 ([14]). A magma $(G,\oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms

(G1) $\exists 0\in G\forall a\in G,\, 0\oplus a=a$

(G2) $\forall a\in G\exists b\in G,\, b\oplus a=0$

(G3) $\forall a,b\in G\exists \mathrm{gyr}\,[a,b]\in \mathrm{Aut}\,(G,\oplus)\forall c\in G,\, a\oplus(b\oplus c)=(a\oplus b)\oplus \mathrm{gyr}\,[a,b]c$

(G4) $\forall a,b\in G,\, \mathrm{gyr}\,[a,b]=\mathrm{gyr}\,[a\oplus b,b]$.

The axioms in Definition 3.2.1 imply the right counterparts.

Theorem 3.2.2 ([14]). *A magma $(G,\oplus)$ forms a gyrogroup if and only if it satisfies the following properties:*

(g1) $\exists 0\in G\forall a\in G, 0\oplus a=a$ *and* $a\oplus 0=a$ (two-sided identity)

(g2) $\forall a\in G\exists b\in G, b\oplus a=0$ *and* $a\oplus b=0.$ (two-sided inverse)

For $a,b,c\in G$, define

$$\mathrm{gyr}\,[a,b]c=\ominus(a\oplus b)\oplus(a\oplus(b\oplus c)), \qquad \text{(gyrator identity)}$$

then

(g3) $\mathrm{gyr}\,[a,b]\in \mathrm{Aut}\,(G,\oplus)$ (gyroautomorphism) (g3a) $a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b]c$ (left gyroassociative law)

(g3b) $(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b,a]c)$ (right gyroassociative law)

(g4a) $\text{gyr}[a,b] = \text{gyr}[a \oplus b, b]$ (left loop property)

(g4b) $\text{gyr}[a,b] = \text{gyr}[a, b \oplus a]$. (right loop property)

The map $\text{gyr}[a,b]$ is called the *gyroautomorphism generated by elements a and b*. By Theorem 3.2.2, any gyroautomorphism is completely determined by its generators via the *gyrator identity*. A gyrogroup G having the additional property that

$$a \oplus b = \text{gyr}[a,b](b \oplus a) \qquad \text{(gyrocommutative law)}$$

for all $a, b \in G$ is called a *gyrocommutative gyrogroup*.

Many of group theoretic theorems are generalized to the gyrogroup case with the aid of gyroautomorphisms, see [11, 14] for more details. Some theorems are listed here for easy reference. To shorten notation, we write $a \ominus b$ instead of $a \oplus (\ominus b)$.

Theorem 3.2.3 (Theorem 2.11, [11]). *Let G be a gyrogroup. Then*

$$(\ominus a \oplus b) \oplus \text{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c \tag{3.5}$$

for all $a, b, c \in G$.

Theorem 3.2.4 (Theorem 2.25, [11]). *For any two elements a and b of a gyrogroup,*

$$\ominus (a \oplus b) = \text{gyr}[a,b](\ominus b \ominus a). \tag{3.6}$$

Theorem 3.2.5 (Theorem 2.27, [11]). *The gyroautomorphisms of any gyrogroup G are even*

$$\text{gyr}[\ominus a, \ominus b] = \text{gyr}[a,b] \tag{3.7}$$

and inversive symmetric

$$\text{gyr}^{-1}[a,b] = \text{gyr}[b,a] \tag{3.8}$$

for all $a, b \in G$.

Using Theorem 3.2.5, one can prove the following proposition.

Proposition 3.2.6. *Let G be a gyrogroup and let* $X \subseteq G$. *Then the following are equivalent:*

(1) $\text{gyr}[a,b](X) \subseteq X$ *for all* $a,b \in G$

(2) $\text{gyr}[a,b](X) = X$ *for all* $a,b \in G$.

The *gyrogroup cooperation* $\boxplus$ is defined by the equation

$$a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b, \qquad a,b \in G. \tag{3.9}$$

Like groups, every linear equation in a gyrogroup G has a unique solution in G.

Theorem 3.2.7 (Theorem 2.15, [11]). *Let G be a gyrogroup and let $a,b \in G$. The unique solution of the equation $a \oplus x = b$ in G for the unknown x is $x = \ominus a \oplus b$, and the unique solution of the equation $x \oplus a = b$ in G for the unknown x is $x = b \boxplus (\ominus a)$.*

The following cancellation laws in gyrogroups are derived as a consequence of Theorem 3.2.7.

Theorem 3.2.8 ([11]). *Let G be a gyrogroup. For all $a,b,c \in G$,*

(1) $a \oplus b = a \oplus c$ *implies* $b = c$ (general left cancellation law)

(2) $\ominus a \oplus (a \oplus b) = b$ (left cancellation law)

(3) $(b \ominus a) \boxplus a = b$ (right cancellation law I)

(4) $(b \boxplus (\ominus a)) \oplus a = b$. (right cancellation law II)

It is known in the literature that every gyrogroup forms a left Bol loop with the A_ℓ-property, where the gyroautomorphisms correspond to *left inner mappings* or *precession maps*. In fact, gyrogroups and left Bol loops with the A_ℓ-property are equivalent, see for instance [8].

To prove an analog of Cayley's theorem for gyrogroups, we will make use of the following theorem

Theorem 3.2.9 (Theorem 1, [2]). *Let G be a gyrogroup, let X be an arbitrary set, and let $\phi: X \to G$ be a bijection. Then X endowed with the induced operation*

$$a \oplus_X b := \phi^{-1}(\phi(a) \oplus \phi(b))$$

for $a,b \in X$ becomes a gyrogroup.

3.3 Cayley's theorem

Recall that for $a \in \mathbb{D}$, the map τ_a that sends a complex number z to $a \oplus_M z$ defines a Möbius transformation or conformal mapping on $\mathbb{D}$, known as a *Möbius translation*. In the literature, the following composition law of Möbius translations is known

$$\tau_a \circ \tau_b = \tau_{a \oplus_M b} \circ \operatorname{gyr}[a,b] \tag{3.10}$$

for all $a, b \in \mathbb{D}$. In this section, we extend the composition law (3.10) to an arbitrary gyrogroup G. We also show that the symmetric group of G admits the gyrogroup structure induced by G, thus obtaining an analog of Cayley's theorem for gyrogroups.

Throughout this section, G and H are arbitrary gyrogroups.

For each $a \in G$, the *left gyrotranslation by a* and the *right gyrotranslation by a* are defined on G by

$$L_a\colon x \mapsto a \oplus x \quad \text{and} \quad R_a\colon x \mapsto x \oplus a. \tag{3.11}$$

Theorem 3.3.1. *Let G be a gyrogroup.*

(1) *The left gyrotranslations are permutations of G.*

(2) *Denote the set of all left gyrotranslations of G by $\overline{G}$. The map $\psi\colon G \to \overline{G}$ defined by $\psi(a) = L_a$ is bijective. The inverse map $\phi := \psi^{-1}$ fulfills the condition in Theorem* 3.2.9. *In this case, the induced operation $\oplus_{\overline{G}}$ is given by*

$$L_a \oplus_{\overline{G}} L_b = L_{a \oplus b}$$

for all $a, b \in G$.

(3) *For all $a, b, c \in G$,*

$$L_a \circ L_b = L_{a \oplus b} \circ \operatorname{gyr}[a,b] \tag{3.12}$$

and

$$\operatorname{gyr}_{\overline{G}}[L_a, L_b]L_c = L_{\operatorname{gyr}[a,b]c}. \tag{3.13}$$

Proof. Let $a, b \in G$.

(1) That L_a is injective follows from the general left cancellation law. That L_a is surjective follows from Theorem 3.2.7.

(2) That ψ is bijective is clear. By Theorem 3.2.9, the induced operation is given by

$$L_a \oplus_{\overline{G}} L_b = \psi(\psi^{-1}(L_a) \oplus \psi^{-1}(L_b)) = \psi(a \oplus b) = L_{a \oplus b}.$$

(3) By the left cancellation law, $L_a^{-1} = L_{\ominus a}$. By the gyrator identity,

$$\operatorname{gyr}[a,b] = L_{\ominus(a\oplus b)} \circ L_a \circ L_b$$

and hence $\operatorname{gyr}[a,b] = L_{a\oplus b}^{-1} \circ L_a \circ L_b$. It follows that $L_a \circ L_b = L_{a\oplus b} \circ \operatorname{gyr}[a,b]$. Equation (3.13) follows from the gyrator identity. ■

Let $\operatorname{Stab}(0)$ denote the set of permutations of G leaving the gyrogroup identity fixed

$$\operatorname{Stab}(0) = \{\rho \in \operatorname{Sym}(G) \colon \rho(0) = 0\}.$$

It is clear that $\operatorname{Stab}(0)$ is a subgroup of the symmetric group, $\operatorname{Sym}(G)$, and we have the following inclusions

$$\{\operatorname{gyr}[a,b] \colon a,b \in G\} \subseteq \operatorname{Aut}(G) \leqslant \operatorname{Stab}(0) \leqslant \operatorname{Sym}(G).$$

The next theorem enables us to introduce a binary operation $\oplus$ on the symmetric group of G so that $\operatorname{Sym}(G)$ equipped with $\oplus$ becomes a gyrogroup containing an isomorphic copy of G.

Theorem 3.3.2. *For each $\sigma \in \operatorname{Sym}(G)$, σ can be written uniquely as $\sigma = L_a \circ \rho$, where $a \in G$ and $\rho \in \operatorname{Stab}(0)$.*

Proof. Suppose that $L_a \circ \rho = L_b \circ \eta$, where $a,b \in G$ and $\rho, \eta \in \operatorname{Stab}(0)$. Then $a = (L_a \circ \rho)(0) = (L_b \circ \eta)(0) = b$, which implies $L_a = L_b$ and so $\rho = \eta$. This proves the uniqueness of factorization. Let σ be an arbitrary permutation of G. Choose $a = \sigma(0)$ and set $\rho = L_{\ominus a} \circ \sigma$. Note that $\rho(0) = L_{\ominus a}(a) = \ominus a \oplus a = 0$. Hence, $\rho \in \operatorname{Stab}(0)$. Since $L_{\ominus a} = L_a^{-1}$, $\sigma = L_a \circ \rho$. This proves the existence of factorization. ■

The following *commutation relation* determines how to commute a left gyrotranslation and an automorphism of G

$$\rho \circ L_a = L_{\rho(a)} \circ \rho \tag{3.14}$$

whenever ρ is an automorphism of G.

Let σ and τ be permutations of G. By Theorem 3.3.2, σ and τ have factorizations $\sigma = L_a \circ \gamma$ and $\tau = L_b \circ \delta$, where $a,b \in G$ and $\gamma, \delta \in \operatorname{Stab}(0)$. Define an operation $\oplus$ on $\operatorname{Sym}(G)$ by

$$\sigma \oplus \tau = L_{a\oplus b} \circ (\gamma \circ \delta). \tag{3.15}$$

Because of the uniqueness of factorization, $\oplus$ is a binary operation on $\operatorname{Sym}(G)$. In fact, $(\operatorname{Sym}(G), \oplus)$ forms a gyrogroup.

Theorem 3.3.3. $\mathrm{Sym}(G)$ *is a gyrogroup under the operation defined by* (3.15)*, and*

$$L_a \oplus L_b = L_a \oplus_{\overline{G}} L_b = L_{a\oplus b}$$

for all $a, b \in G$*. In particular, the map* $a \mapsto L_a$ *defines an injective gyrogroup homomorphism from* G *into* $\mathrm{Sym}(G)$.

Proof. Suppose that $\sigma = L_a \circ \gamma$, $\tau = L_b \circ \delta$ and $\rho = L_c \circ \lambda$, where $a, b, c \in G$ and $\gamma, \delta, \lambda \in \mathrm{Stab}(0)$. The identity map id_G acts as a left identity of $\mathrm{Sym}(G)$ and $L_{\ominus a} \circ \gamma^{-1}$ is a left inverse of σ with respect to $\oplus$. The gyroautomorphisms of $\mathrm{Sym}(G)$ are given by

$$\mathrm{gyr}[\sigma, \tau]\rho = (\mathrm{gyr}[L_a, L_b]L_c) \circ \lambda = L_{\mathrm{gyr}[a,b]c} \circ \lambda.$$

Since G satisfies the left gyroassociative law and the left loop property, so does $\mathrm{Sym}(G)$. ■

By Theorem 3.3.3, the following version of Cayley's theorem for gyrogroups is immediate.

Corollary 3.3.4 (Cayley's theorem). *Every gyrogroup is isomorphic to a subgyrogroup of the gyrogroup of permutations.*

Proof. The map $a \mapsto L_a$ defines a gyrogroup isomorphism from G onto $\overline{G}$ and $\overline{G}$ is a subgyrogroup of $\mathrm{Sym}(G)$. ■

3.4 L-subgyrogroups

Throughout this section, G is an arbitrary gyrogroup.

A nonempty subset H of G is a *subgyrogroup* if H forms a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H for all $a, b \in H$. If H is a subgyrogroup of G, then we write $H \leqslant G$ as in the group case.

Proposition 3.4.1 (The subgyrogroup criterion). *A nonempty subset* H *of* G *is a subgyrogroup if and only if* $\ominus a \in H$ *and* $a \oplus b \in H$ *for all* $a, b \in H$.

Proof. Axioms (G1), (G2), (G4) hold trivially. Let $a, b \in H$. By the gyrator identity,

$$\mathrm{gyr}[a, b](H) \subseteq H.$$

Since the gyroautomorphisms are inversive symmetric (Theorem 3.2.5), we also have the reverse inclusion. Thus, the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H and so axiom (G3) holds. ■

Let H be a subgyrogroup of G. In contrast to groups, the relation

$$a \sim b \quad \text{if and only if} \quad \ominus a \oplus b \in H \tag{3.16}$$

does not, in general, define an equivalence relation on G. Nevertheless, we can modify (3.16) to obtain an equivalence relation on G. From this point of view, any subgyrogroup of G partitions G. This leads to the introduction of L-subgyrogroups.

Let H be a subgyrogroup of G. Define a relation $\sim_H$ on G by letting

$$a \sim_H b \quad \text{if and only if} \quad \ominus a \oplus b \in H \text{ and } \mathrm{gyr}\,[\ominus a, b](H) = H. \tag{3.17}$$

Theorem 3.4.2. *The relation* $\sim_H$ *defined by* (3.17) *is an equivalence relation on* G.

Proof. Let $a,b,c \in G$. Since $\ominus a \oplus a = 0 \in H$ and $\mathrm{gyr}\,[\ominus a, a] = \mathrm{id}_G$, $a \sim_H a$. Hence, $\sim_H$ is reflexive. Suppose that $a \sim_H b$. By Theorem 3.2.4, $\mathrm{gyr}\,[\ominus a, b](\ominus b \oplus a) = \ominus(\ominus a \oplus b)$. Hence, $\ominus b \oplus a = \mathrm{gyr}^{-1}[\ominus a, b](\ominus(\ominus a \oplus b))$, which implies $\ominus b \oplus a \in H$ since $\mathrm{gyr}^{-1}[\ominus a, b](H) = H$. By Theorem 3.2.5,

$$\mathrm{gyr}\,[\ominus a, b] = \mathrm{gyr}\,[\ominus a, \ominus(\ominus b)] = \mathrm{gyr}\,[a, \ominus b] = \mathrm{gyr}^{-1}[\ominus b, a].$$

Hence, $\mathrm{gyr}\,[\ominus b, a] = \mathrm{gyr}^{-1}[\ominus a, b]$. Since $\mathrm{gyr}\,[\ominus a, b](H) = H$, $\mathrm{gyr}\,[\ominus b, a](H) = H$ as well. This proves $b \sim_H a$ and so $\sim_H$ is symmetric. Suppose that $a \sim_H b$ and $b \sim_H c$. By Theorem 3.2.3, $\ominus a \oplus c = (\ominus a \oplus b) \oplus \mathrm{gyr}\,[\ominus a, b](\ominus b \oplus c)$ and so $\ominus a \oplus c \in H$. Using the composition law (3.12) and the commutation relation (3.14), we have

$$\mathrm{gyr}\,[\ominus a, c] = \mathrm{gyr}\,[\ominus a \oplus b, \mathrm{gyr}\,[\ominus a, b](\ominus b \oplus c)] \circ \mathrm{gyr}\,[\ominus a, b] \circ \mathrm{gyr}\,[\ominus b, c].$$

This implies $\mathrm{gyr}\,[\ominus a, c](H) = H$ and so $a \sim_H c$. This proves $\sim_H$ is transitive. ■

Let $a \in G$. Let $[a]$ denote the equivalence class of a determined by $\sim_H$. Theorem 3.4.2 says that $\{[a]\colon a \in G\}$ is a partition of G. Set $a \oplus H := \{a \oplus h\colon h \in H\}$, called the *left coset of* H *induced by* a.

Proposition 3.4.3. *For each* $a \in G$, $[a] \subseteq a \oplus H$.

Proof. If $x \in [a]$, by (3.17), $\ominus a \oplus x \in H$. Hence, $x = a \oplus (\ominus a \oplus x) \in a \oplus H$. ■

Proposition 3.4.3 leads to the notion of L-subgyrogroups.

Definition 3.4.4. A subgyrogroup H of G is said to be an L-subgyrogroup, denoted by $H \leqslant_L G$, if $\mathrm{gyr}\,[a, h](H) = H$ for all $a \in G$ and $h \in H$.

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	1	0	6	7	5	4	11	10	8	9	15	14	12	13
3	3	2	0	1	7	6	4	5	10	11	9	8	14	15	13	12
4	4	5	6	7	3	2	0	1	15	14	12	13	9	8	11	10
5	5	4	7	6	2	3	1	0	14	15	13	12	8	9	10	11
6	6	7	5	4	0	1	2	3	13	12	15	14	10	11	9	8
7	7	6	4	5	1	0	3	2	12	13	14	15	11	10	8	9
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	9	8	14	15	13	12	3	2	0	1	7	6	4	5
11	11	10	8	9	15	14	12	13	2	3	1	0	6	7	5	4
12	12	13	14	15	11	10	8	9	6	7	5	4	0	1	2	3
13	13	12	15	14	10	11	9	8	7	6	4	5	1	0	3	2
14	14	15	13	12	8	9	10	11	4	5	6	7	3	2	0	1
15	15	14	12	13	9	8	11	10	5	4	7	6	2	3	1	0

TABLE 3.1: Addition table for the gyrogroup K_{16}, (cf [10]).

Example 3.4.5. In [10, p. 41], Ungar exhibited the gyrogroup K_{16} whose addition table is presented in Table 3.1. In K_{16}, there is only one nonidentity gyroautomorphism, denoted by A, whose transformation is given in cyclic notation by

$$A = (8\,9)(10\,11)(12\,13)(14\,15). \tag{3.18}$$

The gyration table for K_{16} is presented in Table 3.2. According to (3.18), $H_1 = \{0,1\}$, $H_2 = \{0,1,2,3\}$, and $H_3 = \{0,1,\dots,7\}$ are easily seen to be L-subgyrogroups of K_{16}. In contrast, $H_4 = \{0,8\}$ forms a non-L-subgyrogroup of K_{16} since $\mathrm{gyr}\,[4,8](H_4) \neq H_4$.

The importance of L-subgyrogroups lies in the following results.

Proposition 3.4.6. *If $H \leqslant_L G$, then $[a] = a \oplus H$ for all $a \in G$.*

Proof. Assume that $H \leqslant_L G$. By Proposition 3.4.3, $[a] \subseteq a \oplus H$. If $x = a \oplus h$ for some $h \in H$, then $\ominus a \oplus x = h$ is in H. The left and right loop properties together imply $\mathrm{gyr}\,[\ominus a, x] = \mathrm{gyr}\,[h, a] =$

gyr	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
1	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
2	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
3	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
4	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
5	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
6	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
7	I	I	I	I	I	I	I	I	A	A	A	A	A	A	A	A
8	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
9	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
10	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
11	I	I	I	I	A	A	A	A	I	I	I	I	A	A	A	A
12	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
13	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
14	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I
15	I	I	I	I	A	A	A	A	A	A	A	A	I	I	I	I

TABLE 3.2: Gyration table for K_{16}. Here, A is given by (3.18) and I stands for the identity transformation, (cf [10]).

$\mathrm{gyr}^{-1}[a,h]$. By assumption, $\mathrm{gyr}[a,h](H) = H$, which implies $\mathrm{gyr}[\ominus a, x](H) = \mathrm{gyr}^{-1}[a,h](H) = H$. Hence, $a \sim_H x$ and so $x \in [a]$. This establishes the reverse inclusion. ■

Theorem 3.4.7. *If H is an L-subgyrogroup of a gyrogroup G, then the set*

$$\{a \oplus H : a \in G\}$$

forms a disjoint partition of G.

Proof. This follows directly from Theorem 3.4.2 and Proposition 3.4.6. ■

In light of Theorem 3.4.7, we derive the following version of Lagrange's theorem for L-subgyrogroups.

Theorem 3.4.8 (Lagrange's theorem for L-subgyrogroups). *In a finite gyrogroup G, if $H \leqslant_L G$, then $|H|$ divides $|G|$.*

Proof. Being a finite gyrogroup, G has a finite number of left cosets, namely $a_1 \oplus H$, $a_2 \oplus H$, $\dots$, $a_n \oplus H$. Since $|a_i \oplus H| = |H|$ for $i = 1, 2, \dots, n$, it follows that

$$|G| = \Big|\bigcup_{i=1}^{n} a_i \oplus H\Big| = \sum_{i=1}^{n} |a_i \oplus H| = n|H|,$$

which completes the proof. ■

Let us denote by $[G\colon H]$ the number of left cosets of H in G.

Corollary 3.4.9. *In a finite gyrogroup G, if $H \leqslant_L G$, then $|G| = [G\colon H]|H|$.*

For a *non*-L-subgyrogroup K of G, it is no longer true that the left cosets of K partition G. Moreover, the formula $|G| = [G\colon K]|K|$ is not true, in general.

3.5 Isomorphism theorems

A map $\varphi\colon G \to H$ between gyrogroups is called a *gyrogroup homomorphism* if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. A bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. We say that G and H are *isomorphic gyrogroups*, written $G \cong H$, if there exists a gyrogroup isomorphism from G to H. The next proposition lists basic properties of gyrogroup homomorphisms.

Proposition 3.5.1. *Let $\varphi\colon G \to H$ be a homomorphism of gyrogroups.*

(1) $\varphi(0) = 0$.

(2) $\varphi(\ominus a) = \ominus\varphi(a)$ *for all $a \in G$.*

(3) $\varphi(\mathrm{gyr}[a, b]c) = \mathrm{gyr}[\varphi(a), \varphi(b)]\varphi(c)$ *for all $a, b, c \in G$.*

(4) $\varphi(a \boxplus b) = \varphi(a) \boxplus \varphi(b)$ *for all $a, b \in G$.*

The proof of the following two propositions is routine, using the subgyrogroup criterion and the definition of an L-subgyrogroup.

Proposition 3.5.2. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. If $K \leqslant G$, then $\varphi(K) \leqslant H$. If $K \leqslant_L G$ and if φ is surjective, then $\varphi(K) \leqslant_L H$.*

Proposition 3.5.3. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. If $K \leqslant H$, then $\varphi^{-1}(K) \leqslant G$. If $K \leqslant_L H$, then $\varphi^{-1}(K) \leqslant_L G$.*

Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. The kernel of φ is defined to be the inverse image of the trivial subgyrogroup $\{0\}$ under φ, hence is a subgyrogroup. The kernel of φ is invariant under the gyroautomorphisms of G, that is,

$$\mathrm{gyr}\,[a,b](\ker\varphi) \subseteq \ker\varphi$$

for all $a,b \in G$. By Proposition 3.2.6, $\mathrm{gyr}\,[a,b](\ker\varphi) = \ker\varphi$ for all $a,b \in G$ and so $\ker\varphi$ is an L-subgyrogroup of G. From this the relation (3.17) becomes

$$a \sim_{\ker\varphi} b \quad \text{if and only if} \quad \ominus a \oplus b \in \ker\varphi \tag{3.19}$$

for all $a,b \in G$. More precisely, we have the following proposition.

Proposition 3.5.4. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. For all $a,b \in G$, the following are equivalent*

(1) $a \sim_{\ker\varphi} b$

(2) $\ominus a \oplus b \in \ker\varphi$

(3) $\varphi(a) = \varphi(b)$

(4) $a \oplus \ker\varphi = b \oplus \ker\varphi$.

In view of Proposition 3.5.4, we define a binary operation on the set $G/\ker\varphi$ of left cosets of $\ker\varphi$ in the following natural way

$$(a \oplus \ker\varphi) \oplus (b \oplus \ker\varphi) = (a \oplus b) \oplus \ker\varphi, \qquad a,b \in G. \tag{3.20}$$

The resulting system forms a gyrogroup, called a *quotient gyrogroup*.

Theorem 3.5.5. *If $\varphi\colon G \to H$ is a gyrogroup homomorphism, then $G/\ker\varphi$ with operation defined by* (3.20) *is a gyrogroup.*

Proof. Set $K = \ker\varphi$. The coset $0 \oplus K$ is a left identity of G/K. The coset $(\ominus a) \oplus K$ is a left inverse of $a \oplus K$. For $X = a \oplus K, Y = b \oplus K \in G/K$, the gyroautomorphism generated by X and Y is given by

$$\mathrm{gyr}\,[X,Y](c \oplus K) = (\mathrm{gyr}\,[a,b]c) \oplus K$$

for $c \oplus K \in G/K$. ■

The map $\Pi\colon G \to G/\ker\varphi$ given by $\Pi(a) = a \oplus \ker\varphi$ defines a surjective gyrogroup homomorphism, which will be referred to as the *canonical projection*. In light of Theorem 3.5.5, the first isomorphism theorem for gyrogroups follows.

Theorem 3.5.6 (The first isomorphism theorem). *If $\varphi\colon G\to H$ is a gyrogroup homomorphism, then $G/\ker\varphi\cong\varphi(G)$ as gyrogroups.*

Proof. Set $K=\ker\varphi$. Define $\phi\colon G/K\to\varphi(G)$ by $\phi(a\oplus K)=\varphi(a)$. By Proposition 3.5.4, ϕ is well defined and injective. A direct computation shows that ϕ is a gyrogroup isomorphism from G/K onto $\varphi(G)$. ■

It is known that a subgroup of a group is normal if and only if it is the kernel of some group homomorphism. This characterization of a normal subgroup allows us to define a normal subgyrogroup in a similar fashion, as follows. A subgyrogroup N of a gyrogroup G is *normal in* G, denoted by $N\trianglelefteq G$, if it is the kernel of a gyrogroup homomorphism of G.

Lemma 3.5.7. *Let G be a gyrogroup. If $A\leqslant G$ and $B\trianglelefteq G$, then*

$$A\oplus B:=\{a\oplus b\colon a\in A, b\in B\}$$

forms a subgyrogroup of G.

Proof. By assumption, $B=\ker\phi$, where ϕ is a gyrogroup homomorphism of G. Using Theorem 3.2.7, one can prove that $B\oplus a=a\oplus B$ for all $a\in G$.

Let $x=a\oplus b$, with $a\in A$, $b\in B$. Since $\phi(\mathrm{gyr}[a,b]\ominus a)=\mathrm{gyr}[\phi(a),0]\phi(\ominus a)=\phi(\ominus a)$, we have $\mathrm{gyr}[a,b]\ominus a=\ominus a\oplus b_1$ for some $b_1\in B$. Set $b_2=\mathrm{gyr}[a,b]\ominus b$. Since $b_2\in B$ and $B\oplus(\ominus a)=(\ominus a)\oplus B$, there is an element $b_3\in B$ for which $b_2\ominus a=\ominus a\oplus b_3$. The left and right loop properties together imply

$$\ominus x=\ominus a\oplus(b_3\oplus\mathrm{gyr}[b_3,\ominus a](\mathrm{gyr}[b_2,\ominus a]b_1)),$$

whence $\ominus x$ belongs to $A\oplus B$.

For $x,y\in A\oplus B$, we have $x=a\oplus b$ and $y=c\oplus d$ for some $a,c\in A$, $b,d\in B$. Since

$$\phi(b\oplus\mathrm{gyr}[b,a](c\oplus d))=\phi(b)\oplus\mathrm{gyr}[\phi(b),\phi(a)](\phi(c)\oplus\phi(d))=\phi(c),$$

we have $b\oplus\mathrm{gyr}[b,a](c\oplus d)=c\oplus b_1$ for some $b_1\in B$. The left and right loop properties together imply $x\oplus y=(a\oplus c)\oplus\mathrm{gyr}[a,c]b_1$, whence $x\oplus y$ belongs to $A\oplus B$. This proves $A\oplus B\leqslant G$. ■

Theorem 3.5.8 (The second isomorphism theorem). *Let G be a gyrogroup and let $A,B\leqslant G$. If $B\trianglelefteq G$, then $A\cap B\trianglelefteq A$ and $(A\oplus B)/B\cong A/(A\cap B)$ as gyrogroups.*

Proof. As in Lemma 3.5.7, $B=\ker\phi$. Note that $A\cap B\trianglelefteq A$ since $\ker\phi\big|_A=A\cap B$. Hence, $A/(A\cap B)$ admits the quotient gyrogroup structure.

Define $\varphi\colon A\oplus B\to A/(A\cap B)$ by $\varphi(a\oplus b)=a\oplus(A\cap B)$ for $a\in A$ and $b\in B$. To see that φ is well defined, suppose that $a\oplus b=a_1\oplus b_1$, where $a,a_1\in A$ and $b,b_1\in B$. Note that $b_1=\ominus a_1\oplus(a\oplus b)=(\ominus a_1\oplus a)\oplus\text{gyr}[\ominus a_1,a]b$. Set $b_2=\ominus\text{gyr}[\ominus a_1,a]b$. Then $b_2\in B$ and $b_1=(\ominus a_1\oplus a)\ominus b_2$. The right cancellation law I gives $\ominus a_1\oplus a=b_1\boxplus b_2=b_1\oplus\text{gyr}[b_1,\ominus b_2]b_2$, which implies $\ominus a_1\oplus a\in A\cap B$. By Proposition 3.5.4, $a_1\oplus(A\cap B)=a\oplus(A\cap B)$.

As we computed in the lemma, if $a,c\in A$ and $b,d\in B$, then

$$(a\oplus b)\oplus(c\oplus d)=(a\oplus c)\oplus\text{gyr}[a,c]\tilde{b}$$

for some $\tilde{b}\in B$. Hence, $\varphi((a\oplus b)\oplus(c\oplus d))=(a\oplus c)\oplus A\cap B=\varphi(a\oplus b)\oplus\varphi(c\oplus d)$. This proves $\varphi\colon A\oplus B\to A/(A\cap B)$ is a surjective gyrogroup homomorphism whose kernel is

$$\{a\oplus b\colon a\in A,b\in B,a\in A\cap B\}=B.$$

Thus, $B\trianglelefteq A\oplus B$ and by the first isomorphism theorem, $(A\oplus B)/B\cong A/(A\cap B)$. ■

Theorem 3.5.9 (The third isomorphism theorem). *Let G be a gyrogroup and let H,K be normal subgyrogroups of G such that $H\subseteq K$. Then $K/H\trianglelefteq G/H$ and*

$$(G/H)/(K/H)\cong G/K$$

as gyrogroups.

Proof. Let ϕ and ψ be gyrogroup homomorphisms of G such that $\ker\phi=H$ and $\ker\psi=K$. Define $\varphi\colon G/H\to G/K$ by $\varphi(a\oplus H)=a\oplus K$ for $a\in G$. Note that φ is well defined since $H\subseteq K$. Furthermore, φ is a surjective gyrogroup homomorphism such that $\ker\varphi=K/H$. Hence, $K/H\trianglelefteq G/H$. By the first isomorphism theorem, $(G/H)/(K/H)\cong G/K$. ■

Theorem 3.5.10 (The lattice isomorphism theorem). *Let G be a gyrogroup and let $N\trianglelefteq G$. There is a bijection Φ from the set of subgyrogroups of G containing N onto the set of subgyrogroups of G/N. The bijection Φ has the following properties*

(1) *$A\subseteq B$ if and only if $\Phi(A)\subseteq\Phi(B)$*

(2) *$A\leqslant_L G$ if and only if $\Phi(A)\leqslant_L G/N$*

(3) *$A\trianglelefteq G$ if and only if $\Phi(A)\trianglelefteq G/N$*

for all subgyrogroups A and B of G containing N.

Proof. Set $\mathcal{S} = \{K \subseteq G\colon K \leqslant G \text{ and } N \subseteq K\}$. Let $\mathcal{T}$ denote the set of subgyrogroups of G/N. Define a map Φ by $\Phi(K) = K/N$ for $K \in \mathcal{S}$. By Proposition 3.5.2, $\Phi(K) = K/N = \Pi(K)$ is a subgyrogroup of G/N, where $\Pi\colon G \to G/N$ is the canonical projection. Hence, Φ maps $\mathcal{S}$ to $\mathcal{T}$.

Assume that $K_1/N = K_2/N$, with K_1, K_2 in $\mathcal{S}$. For $a \in K_1$, $a \oplus N \in K_2/N$ implies $a \oplus N = b \oplus N$ for some $b \in K_2$. Hence, $\ominus b \oplus a \in N$. Since $N \subseteq K_2$, $\ominus b \oplus a \in K_2$, which implies $a = b \oplus (\ominus b \oplus a)$ is in K_2. This proves $K_1 \subseteq K_2$. By interchanging the roles of K_1 and K_2, one obtains similarly that $K_2 \subseteq K_1$. Hence, $K_1 = K_2$ and Φ is injective.

Let Y be an arbitrary subgyrogroup of G/N. By Proposition 3.5.3,

$$\Pi^{-1}(Y) = \{a \in G\colon a \oplus N \in Y\}$$

is a subgyrogroup of G containing N for $a \in N$ implies $a \oplus N = 0 \oplus N \in Y$. Because $\Phi(\Pi^{-1}(Y)) = Y$, Φ is surjective. This proves Φ defines a bijection from $\mathcal{S}$ onto $\mathcal{T}$.

The proof of Item 1 is straightforward. From Propositions 3.5.2 and 3.5.3, we have Item 2. To prove Item 3, suppose that $A \trianglelefteq G$. Then $A = \ker \psi$, where $\psi\colon G \to H$ is a gyrogroup homomorphism. Define $\varphi\colon G/N \to H$ by $\varphi(a \oplus N) = \psi(a)$. Since $N \subseteq A$, φ is well defined. Also, φ is a gyrogroup homomorphism. Since $\ker \varphi = A/N$, we have $A/N \trianglelefteq G/N$. Suppose conversely that $\Phi(A) \trianglelefteq G/N$. Then $A/N = \ker \phi$, where ϕ is a gyrogroup homomorphism of G/N. Set $\varphi = \phi \circ \Pi$. Thus, φ is a gyrogroup homomorphism of G with kernel A and hence $A \trianglelefteq G$. ■

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References

[1] Ferreira, M.: Factorizations of Möbius gyrogroups. *Adv. Appl. Clifford Algebras* **19**, 303–323 (2009)

[2] Ferreira, M.: *Hypercomplex Analysis and Applications*, I. Sabadini and F. Sommen (eds.), chap. Gyrogroups in projective hyperbolic Clifford analysis, pp. 61–80. Trends in Mathematics. Springer, Basel (2011)

[3] Ferreira, M.: Harmonic analysis on the Einstein gyrogroup. *J. Geom. Symmetry Phys.* **35**, 21–60 (2014)

[4] Ferreira, M., Ren, G.: Möbius gyrogroups: A Clifford algebra approach. *J. Algebra* **328**, 230–253 (2011)

[5] Kasparian, A., Ungar, A.A.: Lie gyrovector spaces. *J. Geom. Symmetry Phys.* **1**, 3–53 (2004)

[6] Kim, S., Lawson, J.: Unit balls, Lorentz boosts, and hyperbolic geometry. *Results. Math.* **63** (2013)

[7] Lawson, J.: Clifford algebras, Möbius transformations, Vahlen matrices, and B-loops. *Comment. Math. Univ. Carolin.* **51**(2), 319–331 (2010)

[8] Sabinin, L.V., Sabinina, L.L., Sbitneva, L.V.: On the notion of gyrogroup. *Aequat. Math.* **56**, 11–17 (1998)

[9] Sönmez, N., Ungar, A.A.: The Einstein relativistic velocity model of hyperbolic geometry and its plane separation axiom. *Adv. Appl. Clifford Algebras* **23**, 209–236 (2013)

[10] Ungar, A.A.: *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, *Fundamental Theories of Physics*, vol. 117. Kluwer Academic, Dordrecht (2001)

[11] Ungar, A.A.: *Analytic Hyperbolic Geometry: Mathematical Foundations and Applications*. World Scientific, Hackensack, NJ (2005)

[12] Ungar, A.A.: The proper-time Lorentz group demystified. *J. Geom. Symmetry Phys.* **4**, 69–95 (2005)

[13] Ungar, A.A.: Einstein's velocity addition law and its hyperbolic geometry. *Comput. Math. Appl.* **53**, 1228–1250 (2007)

[14] Ungar, A.A.: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*. World Scientific, Hackensack, NJ (2008)

[15] Ungar, A.A.: From Möbius to gyrogroups. *Amer. Math. Monthly* **115**(2), 138–144 (2008)

[16] Ungar, A.A.: *A Gyrovector Space Approach to Hyperbolic Geometry*. Synthesis Lectures on Mathematics and Statistics #4. Morgan & Claypool, San Rafael, CA (2009)

[17] Ungar, A.A.: Hyperbolic geometry. *J. Geom. Symmetry Phys.* **32**, 61–86 (2013)

CHAPTER IV

LAGRANGE'S THEOREM FOR GYROGROUPS AND THE CAUCHY PROPERTY*

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Abstract. We prove a version of Lagrange's theorem for gyrogroups and use this result to prove that gyrogroups of particular order have the strong Cauchy property.

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4.1 Introduction

Lagrange's theorem (that the order of any subgroup of a finite group Γ divides the order of Γ) is well known in group theory and has impact on several branches of mathematics, especially finite group theory, combinatorics, and number theory. Lagrange's theorem proves useful for unraveling mathematical structures. For instance, it is used to prove that any finite field must have prime power order. Certain classification theorems of finite groups arise as an application of Lagrange's theorem [9, 10, 17]. Further, Fermat little's theorem and Euler's theorem may be viewed as a consequence of this theorem. Also relevant are the orbit-stabilizer theorem and the Cauchy-Frobenius lemma (or Burnside's lemma). A history of Lagrange's theorem on groups can be found in [15].

In loop theory, the Lagrange property becomes a nontrivial issue. For example, whether Lagrange's theorem holds for Moufang loops was an open problem in the theory of Moufang

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loops for more than four decades [5, p. 43]. This problem was answered in the affirmative by Grishkov and Zavarnitsine [11]. In fact, not every loop satisfies the Lagrange property as one can construct a loop of order five containing a subloop of order two. Nevertheless, some loops satisfy the Lagrange property.

Baumeister and Stein [1] proved a version of Lagrange's theorem for Bruck loops by studying in detail the structure of a finite Bruck loop. Foguel et al. [7] proved that left Bol loops of *odd* order satisfy the strong Lagrange property. It is, however, still an open problem whether or not Bol loops satisfy the Lagrange property [6, p. 592]. In the same spirit, we focus on the Lagrange property for *gyrogroups* or *left Bol loops with the* A_ℓ*-property* in the loop literature. In [18], we proved that the order of an *L-subgyrogroup* of a finite gyrogroup G divides the order of G. In this paper, we extend this result by proving that the order of *any* subgyrogroup of G divides the order of G, see Theorem 4.5.7.

A gyrogroup is a group-like structure, introduced by Ungar, arising as an algebraic structure that regulates the set of relativistically admissible vectors in $\mathbb{R}^3$ with Einstein addition [19]. The origin of a gyrogroup is described in [22, Chapter 1]. There are two prime examples of gyrogroups, namely the *Einstein gyrogroup*, which consists of the relativistic ball in $\mathbb{R}^3$ with Einstein addition [19], and the *Möbius gyrogroup*, which consists of the complex unit disk with Möbius addition [21].

In this paper, we prove that Lagrange's theorem holds for gyrogroups and apply this result to show that finite gyrogroups of particular order have the Cauchy property. Our results are strongly based on results by Foguel and Ungar [8] and Baumeister and Stein [1]. For basic terminology and definitions in loop theory, we refer the reader to [2, 12, 14].

4.2 Gyrogroups

In this section, we summarize definitions and basic properties of gyrogroups. Much of this section can be found in [20].

Let $(G, \oplus)$ be a magma. Denote the group of automorphisms of G with respect to $\oplus$ by $\mathrm{Aut}(G, \oplus)$.

Definition 4.2.1 ([20]). A magma $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms:

(G1) $\exists 0 \in G \forall a \in G$, $0 \oplus a = a$; (left identity)

(G2) $\forall a \in G \exists b \in G$, $b \oplus a = 0$; (left inverse)

(G3) $\forall a, b \in G \exists \mathrm{gyr}[a,b] \in \mathrm{Aut}(G, \oplus) \forall c \in G$,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c; \qquad \text{(left gyroassociative law)}$$

(G4) $\forall a, b \in G$, $\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]$. (left loop property)

The following theorem gives a characterization of a gyrogroup.

Theorem 4.2.2 ([8]). *Suppose that $(G, \oplus)$ is a magma. Then $(G, \oplus)$ is a gyrogroup if and only if $(G, \oplus)$ satisfies the following properties:*

(g1) $\exists 0 \in G \forall a \in G, 0 \oplus a = a$ *and* $a \oplus 0 = a$; (two-sided identity)

(g2) $\forall a \in G \exists b \in G, b \oplus a = 0$ and $a \oplus b = 0$. (two-sided inverse)
For $a, b, c \in G$, define

$$\mathrm{gyr}[a,b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)), \qquad \text{(gyrator identity)}$$

then

(g3) $\mathrm{gyr}[a,b] \in \mathrm{Aut}(G, \oplus)$; (gyroautomorphism)

(g3a) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c$; (left gyroassociative law)

(g3b) $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b,a]c)$; (right gyroassociative law)

(g4a) $\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]$; (left loop property)

(g4b) $\mathrm{gyr}[a,b] = \mathrm{gyr}[a, b \oplus a]$. (right loop property)

Definition 4.2.3 ([20]). A gyrogroup G having the additional property that

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a) \qquad \text{(gyrocommutative law)}$$

for all $a, b \in G$ is called a *gyrocommutative gyrogroup*.

The *gyrogroup cooperation*, $\boxplus$, is defined by the equation

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b, \qquad a, b \in G. \tag{4.1}$$

Theorem 4.2.4 ([20]). *Let G be a gyrogroup and let $a,b \in G$. The unique solution of the equation $a \oplus x = b$ in G for the unknown x is $x = \ominus a \oplus b$, and the unique solution of the equation $x \oplus a = b$ in G for the unknown x is $x = b \boxplus (\ominus a)$.*

By Theorem 4.2.4, the following cancellation laws hold in gyrogroups.

Theorem 4.2.5 ([20]). *Let G be a gyrogroup. For all $a,b,c \in G$,*

(1) $a \oplus b = a \oplus c$ *implies* $b = c$; (general left cancellation law)

(2) $\ominus a \oplus (a \oplus b) = b$; (left cancellation law)

(3) $(b \ominus a) \boxplus a = b$; (right cancellation law I)

(4) $(b \boxplus (\ominus a)) \oplus a = b$. (right cancellation law II)

Let G be a gyrogroup. For $a \in G$, the *left gyrotranslation by* a, $L_a \colon x \mapsto a \oplus x$, and the *right gyrotranslation by* a, $R_a \colon x \mapsto x \oplus a$, are permutations of G. Further, one has the following composition law

$$L_a \circ L_b = L_{a \oplus b} \circ \mathrm{gyr}[a,b]. \tag{4.2}$$

From this it can be proved that every gyrogroup forms a left Bol loop with the A_ℓ-property, where the gyroautomorphisms correspond to *left inner mappings* or *precession maps*. In fact, gyrogroups and left Bol loops with the A_ℓ-property are equivalent, see for instance [16].

4.3 Subgyrogroups

Let G be a gyrogroup. A nonempty subset H of G is called a *subgyrogroup* if it is a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a,b]$ to H becomes an automorphism of H for all $a,b \in H$. If H is a subgyrogroup of G, we write $H \leqslant G$. We have the following subgyrogroup criterion, as in the group case.

Proposition 4.3.1 ([18]). *A nonempty subset H of G is a subgyrogroup if and only if*

(1) $a \in H$ *implies* $\ominus a \in H$ *and*

(2) $a,b \in H$ *implies* $a \oplus b \in H$.

Subgyrogroups that arise as groups under gyrogroup operation are of great importance in the study of gyrogroups.

Definition 4.3.2 ([8]). A nonempty subset X of a gyrogroup $(G, \oplus)$ is a *subgroup* if it is a group under the restriction of $\oplus$ to X.

The following proposition shows that any subgroup of a gyrogroup is simply a subgyrogroup with trivial gyroautomorphisms.

Proposition 4.3.3. *A nonempty subset X of a gyrogroup G is a subgroup if and only if it is a subgyrogroup of G and* $\operatorname{gyr}[a,b]\big|_X = \operatorname{id}_X$ *for all $a, b \in X$.*

Just as in group theory, we obtain the following results.

Proposition 4.3.4. *Let G be a gyrogroup and let $\mathcal{H}$ be a nonempty collection of subgyrogroups of G. Then the intersection $\bigcap_{H\in\mathcal{H}} H$ forms a subgyrogroup of G.*

Proof. This follows directly from the subgyrogroup criterion. ∎

Proposition 4.3.5. *Let A be a nonempty subset of a gyrogroup G. There exists a unique subgyrogroup of G, denoted by $\langle A \rangle$, such that*

(1) *$A \subseteq \langle A \rangle$ and*

(2) *if $H \leqslant G$ and $A \subseteq H$, then $\langle A \rangle \subseteq H$.*

Proof. Set $\mathcal{H} = \{H \colon H \leqslant G \text{ and } A \subseteq H\}$. Then $\langle A \rangle := \bigcap_{H\in\mathcal{H}} H$ is a subgyrogroup of G satisfying the two conditions. The uniqueness follows from condition (2). ∎

The subgyrogroup generated by one-element set $\{a\}$ is called the *cyclic subgyrogroup generated by a*, which will be denoted by $\langle a \rangle$. Next, we will give an explicit description of $\langle a \rangle$.

Let G be a gyrogroup and let $a \in G$. Define recursively the following notation:

$$0 \cdot a = 0, \quad m \cdot a = a \oplus ((m-1) \cdot a), m \geqslant 1, \quad m \cdot a = (-m) \cdot (\ominus a), m < 0. \tag{4.3}$$

We also define the right counterparts:

$$a \cdot 0 = 0, \quad a \cdot m = (a \cdot (m-1)) \oplus a, m \geqslant 1, \quad a \cdot m = (\ominus a) \cdot (-m), m < 0. \tag{4.4}$$

Lemma 4.3.6. *Let G be a gyrogroup. For any element a of G,*

$$\operatorname{gyr}[a \cdot m, a] = \operatorname{gyr}[m \cdot a, a] = \operatorname{gyr}[a, m \cdot a] = \operatorname{gyr}[a, a \cdot m] = \operatorname{id}_G$$

for all $m \in \mathbb{Z}$.

Proof. By induction, $\operatorname{gyr}[a, a \cdot m] = \mathrm{id}_G$ and $\operatorname{gyr}[a \cdot m, a] = \mathrm{id}_G$ for all $a \in G$ and all $m \geqslant 0$. By the right gyroassociative law, $a \cdot m = m \cdot a$ for all $m \in \mathbb{Z}$. If $m < 0$, the left and right loop properties and the left cancellation law together imply $\operatorname{gyr}[a, a \cdot m] = \mathrm{id}_G$. ■

By induction,

$$(m \cdot a) \oplus (k \cdot a) = (m + k) \cdot a \tag{4.5}$$

for all $m, k \geqslant 0$. In fact, we have the following proposition.

Proposition 4.3.7. *Let a be an element of a gyrogroup. For all $m, k \in \mathbb{Z}$,*

$$(m \cdot a) \oplus (k \cdot a) = (m + k) \cdot a.$$

Proof. The proof is routine, using (4.5) and induction. ■

Theorem 4.3.8. *Let G be a gyrogroup and let $a \in G$. Then $\langle a \rangle = \{m \cdot a \colon m \in \mathbb{Z}\}$. In particular, $\langle a \rangle$ forms a subgroup of G.*

Proof. Set $H = \{m \cdot a \colon m \in \mathbb{Z}\}$. For all $m, n \in \mathbb{Z}$, Proposition 4.3.7 implies that $\ominus(m \cdot a) = (-m) \cdot a \in H$ and $(m \cdot a) \oplus (k \cdot a) = (m + k) \cdot a \in H$. This proves $H \leqslant G$. Since $a \in H$, we have $\langle a \rangle \subseteq H$ by the minimality of $\langle a \rangle$. By the closure property of subgyrogroups, $H \subseteq \langle a \rangle$ and so equality holds.

Note that $(m \cdot a) \oplus [(n \cdot a) \oplus (k \cdot a)] = (m + n + k) \cdot a = [(m \cdot a) \oplus (n \cdot a)] \oplus (k \cdot a)$ for all $m, n, k \in \mathbb{Z}$. Thus, $\operatorname{gyr}[m \cdot a, n \cdot a]\big|_{\langle a \rangle} = \mathrm{id}_{\langle a \rangle}$ for all $m, n \in \mathbb{Z}$ and hence $\langle a \rangle$ forms a subgroup of G by Proposition 4.3.3. ■

Theorem 4.3.8 suggests us to define the *order* of an element in a gyrogroup as follows.

Definition 4.3.9. Let G be a gyrogroup and let $a \in G$. The *order* of a, denoted by $|a|$, is defined to be the cardinality of $\langle a \rangle$ if $\langle a \rangle$ is finite. In this case, we will write $|a| < \infty$. If $\langle a \rangle$ is infinite, the order of a is defined to be infinity, and we will write $|a| = \infty$.

Proposition 4.3.10. *Let G be a gyrogroup and let $a \in G$. For all $m, n \in \mathbb{Z}$,*

$$\operatorname{gyr}[m \cdot a, n \cdot a] = \mathrm{id}_G.$$

Proof. By induction, $L_{m \cdot a} = L_a^m$ for all $a \in G$ and all $m \in \mathbb{Z}$. Since $L_a^{-1} = L_{\ominus a}$, we have from (4.2) that

$$\operatorname{gyr}[m \cdot a, n \cdot a] = L_{-(m+n) \cdot a} \circ L_{m \cdot a} \circ L_{n \cdot a} = L_a^{-(m+n)} \circ L_a^m \circ L_a^n = \mathrm{id}_G$$

for all $m, n \in \mathbb{Z}$. ■

In light of the proof of Proposition 4.3.10, gyrogroups are *left power alternative*. Further, the following proposition implies that gyrogroups are *power associative*.

Proposition 4.3.11. *If a is an element of a gyrogroup, then $\langle a\rangle$ forms a cyclic group with generator a under gyrogroup operation.*

Proof. By Theorem 4.3.8, $\langle a\rangle$ is a group under gyrogroup operation. By induction, $m\cdot a = a^m$ for all $m \geqslant 0$, where the notation a^m is used as in group theory. If $m < 0$, one obtains similarly that $m\cdot a = a^m$. Hence, $\langle a\rangle$ forms a cyclic group with generator a. ■

Corollary 4.3.12. *Any gyrogroup generated by one element is a cyclic group.*

Because the *group* order of a and the *gyrogroup* order of a are the same, we obtain the following results.

Proposition 4.3.13. *Let G be a gyrogroup and let $a \in G$.*

(1) *If $|a| < \infty$, then $|a|$ is the smallest positive integer such that $|a|\cdot a = 0$.*

(2) *If $|a| = \infty$, then $m\cdot a \neq 0$ for all $m \neq 0$ and $m\cdot a \neq k\cdot a$ for all $m \neq k$ in $\mathbb{Z}$.*

Corollary 4.3.14. *Let a be an element of a gyrogroup. If $|a| = n < \infty$, then*

$$\langle a\rangle = \{0\cdot a, 1\cdot a, \ldots, (n-1)\cdot a\}.$$

Corollary 4.3.15. *Let a be an element of a gyrogroup and let $m \in \mathbb{Z}\setminus\{0\}$.*

(1) *If $|a| = \infty$, then $|m\cdot a| = \infty$.*

(2) *If $|a| < \infty$, then $|m\cdot a| = \dfrac{|a|}{\gcd(|a|, m)}$.*

4.4 Gyrogroup homomorphisms

A *gyrogroup homomorphism* is a map between gyrogroups that preserves the gyrogroup operations. A bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. We say that gyrogroups G and H are *isomorphic*, written $G \cong H$, if there exists a gyrogroup isomorphism from G to H.

Suppose that $\varphi\colon G \to H$ is a gyrogroup homomorphism. The kernel of φ is defined to be the inverse image of the trivial subgyrogroup $\{0\}$ under φ. Since $\ker\varphi$ is invariant under all the gyroautomorphisms of G, the operation

$$(a\oplus\ker\varphi)\oplus(b\oplus\ker\varphi) := (a\oplus b)\oplus\ker\varphi, \qquad a, b \in G, \tag{4.6}$$

is independent of the choice of representatives for the left cosets. The system $G/\ker\varphi$ forms a gyrogroup, called a *quotient gyrogroup*. This results in the first isomorphism theorem for gyrogroups.

Theorem 4.4.1 (The first isomorphism theorem, [18]). *If φ is a gyrogroup homomorphism of G, then $G/\ker\varphi \cong \varphi(G)$ as gyrogroups.*

A subgyrogroup N of a gyrogroup G is *normal in* G, denoted by $N \trianglelefteq G$, if it is the kernel of a gyrogroup homomorphism of G. By Theorem 4.4.1, every normal subgyrogroup N gives rise to the quotient gyrogroup G/N, along with the *canonical projection* $\Pi\colon a \mapsto a \oplus N$.

We state the second isomorphism theorem for gyrogroups for easy reference; its proof can be found in [18].

Theorem 4.4.2 (The second isomorphism theorem). *Let G be a gyrogroup and let $A, B \leqslant G$. If $B \trianglelefteq G$, then $A \oplus B \leqslant G$, $A \cap B \trianglelefteq A$, and $(A \oplus B)/B \cong A/(A \cap B)$ as gyrogroups.*

4.5 The Lagrange property

Throughout this section, all gyrogroups are finite. A version of the Lagrange property for loops can be found in [5]. In terms of gyrogroups, the Lagrange property can be restated as follows.

Definition 4.5.1. A gyrogroup G is said to have the *Lagrange property* if for each subgyrogroup H of G, the order of H divides the order of G.

A version of the following proposition for loops was proved by Bruck in [2]. As the first isomorphism theorem and the second isomorphism theorem hold for gyrogroups, we also have the following proposition:

Proposition 4.5.2. *Let H be a subgyrogroup of a gyrogroup G and let B be a normal subgyrogroup of H. If B and H/B have the Lagrange property, then so has H.*

Corollary 4.5.3. *Let N be a normal subgyrogroup of a gyrogroup G. If N and G/N have the Lagrange property, then so has G.*

Proof. Taking $H = G$ in the proposition, the corollary follows. ■

Proposition 4.5.4. *Let X be a subgroup of a gyrogroup G. If $H \leqslant X$, then $|H|$ divides $|X|$. In other words, every subgroup of G has the Lagrange property.*

Proof. Suppose that $H \leqslant X$. Since $\operatorname{gyr}[a,b]\big|_H = \mathrm{id}_H$ for all $a,b \in H$, H forms a subgroup of G. By definition, X forms a group and H becomes a subgroup of X. By Lagrange's theorem for groups, $|H|$ divides $|X|$. ■

Lagrange's theorem holds for *all* gyrocommutative gyrogroups, as shown by Baumeister and Stein in [1, Theorem 3] in the language of Bruck loops.

Theorem 4.5.5. *In a gyrocommutative gyrogroup G, if $H \leqslant G$, then $|H|$ divides $|G|$. In other words, every gyrocommutative gyrogroup has the Lagrange property.*

Proof. Let G be a gyrocommutative gyrogroup and let $H \leqslant G$. Then G is a Bruck loop and H becomes a subloop of G. By Theorem 3 of [1], $|H|$ divides $|G|$, which completes the proof. ■

The next theorem, due to Foguel and Ungar, enables us to extend Lagrange's theorem to all finite gyrogroups.

Theorem 4.5.6 (Theorem 4.11, [8]). *If G is a gyrogroup, then G has a normal subgroup N such that G/N is a gyrocommutative gyrogroup.*

Theorem 4.5.7 (Lagrange's theorem). *If H is a subgyrogroup of a gyrogroup G, then $|H|$ divides $|G|$. That is, every gyrogroup has the Lagrange property.*

Proof. Let G be a gyrogroup. By Theorem 4.5.6, G has a normal subgroup N such that G/N is gyrocommutative. Because $N = \ker \Pi$, where $\Pi\colon G \to G/N$ is the canonical projection, N is a normal subgyrogroup of G. By Proposition 4.5.4 and Theorem 4.5.5, N and G/N have the Lagrange property. By Corollary 4.5.3, G has the Lagrange property. ■

4.6 Applications

In this section, we provide some applications of Lagrange's theorem. Throughout this section, all gyrogroups are finite.

Proposition 4.6.1. *Let G be a gyrogroup and let $a \in G$. Then $|a|$ divides $|G|$. In particular, $|G| \cdot a = 0$.*

Proof. By definition, $|a| = |\langle a \rangle|$. By Lagrange's theorem, $|a|$ divides $|G|$. Write $|G| = |a|k$ with $k \in \mathbb{N}$, so $|G| \cdot a = (|a|k) \cdot a = \underbrace{|a| \cdot a \oplus \cdots \oplus |a| \cdot a}_{k \text{ copies}} = 0$. ■

Although we know that a left Bol loop of prime order is a cyclic group by a result of Burn [3, Corollary 2], we present the following theorem as an application of Lagrange's theorem.

Theorem 4.6.2. *If G is a gyrogroup of prime order p, then G is a cyclic group of order p under gyrogroup operation.*

Proof. Let a be a nonidentity element of G. Then $|a| \neq 1$ and $|a|$ divides p. It follows that $|a| = p$, which implies $G = \langle a \rangle$ since G is finite. By Proposition 4.3.11, $\langle a \rangle$ is a cyclic group of order p, which completes the proof. ■

The Cauchy property

In the loop literature, it is known that left Bol loops of odd order satisfy the Cauchy property [7, Theorem 6.2]. However, Bol loops fail to satisfy the Cauchy property as Nagy proves the existence of a simple right Bol loop of exponent 2 and of order 96 [13, Corollary 3.7]. This also implies that gyrogroups fail to satisfy the Cauchy property since any Bol loop of exponent 2 is necessarily a Bruck loop, hence is a gyrocommutative gyrogroup.

In this subsection, we apply Lagrange's theorem and results from loop theory to establish that some finite gyrogroups satisfy the Cauchy property.

Definition 4.6.3 (The weak Cauchy property, WCP). A finite gyrogroup G is said to have the *weak Cauchy property* if for every prime p dividing $|G|$, G has an element of order p.

Definition 4.6.4 (The strong Cauchy property, SCP). A finite gyrogroup G is said to have the *strong Cauchy property* if every subgyrogroup of G has the weak Cauchy property.

The Cauchy property is an invariant property of gyrogroups, as shown in the following proposition.

Proposition 4.6.5. *Let G and H be gyrogroups and let $\phi\colon G \to H$ be a gyrogroup isomorphism.*

(1) *If G has the weak Cauchy property, then so has H.*

(2) *If G has the strong Cauchy property, then so has H.*

Proof. (1) It suffices to prove that $|\phi(a)| = |a|$ for all $a \in G$. By induction, $\phi(n \cdot a) = n \cdot \phi(a)$ for all $a \in G$ and all $n \in \mathbb{N}$. Let $a \in G$. Since $|a| \cdot a = 0$, we have $|a| \cdot \phi(a) = \phi(|a| \cdot a) = \phi(0) = 0$. If there were a positive integer $m < |a|$ for which $m \cdot \phi(a) = 0$, then we would have $\phi(m \cdot a) = 0$ and would have $m \cdot a = 0$, contradicting the minimality of $|a|$. Hence, $|a|$ is the smallest positive integer such that $|a| \cdot \phi(a) = 0$, which implies $|\phi(a)| = |a|$ by Proposition 4.3.13 (1).

(2) Let $B \leqslant H$. Set $A = \phi^{-1}(B)$. Then $A \leqslant G$ and A has the WCP. Since $\phi\big|_A$ is a gyrogroup isomorphism from A onto B, B has the WCP by Item 1. ■

Corollary 4.6.6. *Let G and H be gyrogroups. If $G \cong H$, then G has the weak (resp. strong) Cauchy property if and only if H has the weak (resp. strong) Cauchy property.*

Theorem 4.6.7. *Let H be a subgyrogroup of a gyrogroup G and let B be a normal subgyrogroup of H.*

(1) *If B and H/B have the weak Cauchy property, then so has H.*

(2) *If B and H/B have the strong Cauchy property, then so has H.*

Proof. (1) Suppose that p is a prime dividing $|H|$. Since $|H| = [H\colon B]|B|$, p divides $|H/B|$ or $|B|$. If p divides $|B|$, then B has an element of order p and we are done. We may therefore assume that $p \nmid |B|$. Hence, p divides $|H/B|$. By assumption, H/B has an element of order p, say $a \oplus B$. By induction, $n \cdot (a \oplus B) = (n \cdot a) \oplus B$ for all $n \geqslant 0$. Hence, by Proposition 4.3.13 (1), p is the smallest positive integer such that $p \cdot a \in B$. In particular, $a \notin B$. Note that $\gcd(|a|, p) = 1$ or p. If $\gcd(|a|, p) = 1$ were true, we would have $|p \cdot a| = \dfrac{|a|}{\gcd(|a|, p)} = |a|$, and would have $a \in \langle a \rangle = \langle p \cdot a \rangle \leqslant B$, a contradiction. Hence, $\gcd(|a|, p) = p$, which implies p divides $|a|$. Write $|a| = mp$. Then $|m \cdot a| = \dfrac{|a|}{\gcd(|a|, m)} = p$, which finishes the proof of (1).

(2) Suppose that B and H/B have the SCP. Let $A \leqslant H$. By assumption, $A \cap B$ has the WCP. Since $A \oplus B/B \leqslant H/B$, $A \oplus B/B$ has the WCP. Since $A/A \cap B \cong A \oplus B/B$, $A/A \cap B$ has the WCP. By Item 1, A has the WCP. ■

Corollary 4.6.8. *Let N be a normal subgyrogroup of a gyrogroup G. If N and G/N have the weak (strong) Cauchy property, then so has G.*

Consider a gyrogroup G of order pq, where p and q are primes. If pq is odd, by a result of Foguel, Kinyon, and Phillips [7, Theorem 6.2], G has the weak Cauchy property. Since any subgyrogroup of G is of order $1, p, q$ or pq, every subgyrogroup of G has the weak Cauchy property as well. This implies that G has the strong Cauchy property. If pq is even, at least one of p or q must be 2. Hence, G is of order $2\tilde{p}$, where $\tilde{p}$ is a prime. By a result of Burn [3, Theorem 4], G is a group, hence has the strong Cauchy property. This proves the following theorem.

Theorem 4.6.9 (Cauchy's theorem). *Let p and q be primes. Every gyrogroup of order pq has the strong Cauchy property.*

Theorem 4.6.10. *Let p and q be primes and let G be a gyrogroup of order pq. If $p = q$, then G is a group. If $p \neq q$, then G is generated by two elements; one has order p and the other has order q.*

Proof. In the case $p = q$, G is a left Bol loop of order p^2, hence must be a group by Burn's result [3, Theorem 5].

Suppose that $p \neq q$. Let a and b be elements of order p and q, respectively. By Lagrange's theorem, $\langle a \rangle \cap \langle b \rangle = \{0\}$. For all $m, n, s, t \in \mathbb{Z}$, if

$$(m \cdot a) \oplus (n \cdot b) = (s \cdot a) \oplus (t \cdot b),$$

then $\ominus(s \oplus a) \oplus (m \cdot a) = (t \cdot b) \boxplus (\ominus(n \cdot b)) = (t \cdot b) \ominus (n \cdot b)$ belongs to $\langle a \rangle \cap \langle b \rangle$. Hence, $\ominus(s \oplus a) \oplus (m \cdot a) = 0$ and $(t \cdot b) \ominus (n \cdot b) = 0$ and so $m \cdot a = s \cdot a$ and $n \cdot b = t \cdot b$. This proves $\{(m \cdot a) \oplus (n \cdot b) \colon 0 \leq m < p, 0 \leq n < q\}$ contains pq distinct elements of G. Since G is finite, it follows that $G = \{(m \cdot a) \oplus (n \cdot b) \colon 0 \leq m < p, 0 \leq n < q\} = \langle a, b \rangle$. ■

In general, gyrogroups of order pq, where p and q are distinct primes not equal to 2, need not be groups. This is a situation where gyrogroups are different from Moufang loops. As Moufang loops are *diassociative*, every Moufang loop generated by two elements must be a group. This implies that Moufang loops of order pq are groups [4, Proposition 3].

Let G be a finite *nongyrocommutative* gyrogroup. By Theorem 4.5.6, G has a normal subgroup N such that G/N is gyrocommutative. Because G is nongyrocommutative, we have N is nontrivial, since otherwise $\Pi \colon G \to G/N$ would be a gyrogroup isomorphism and G and G/N would be isomorphic gyrogroups. From this we can deduce the following results.

Theorem 4.6.11. *Let p be a prime. Every nongyrocommutative gyrogroup of order p^3 has the strong Cauchy property.*

Proof. Let G be a nongyrocommutative gyrogroup of order p^3. As noted above, G has a nontrivial normal subgroup N. By Lagrange's theorem, $|N| = p, p^2$ or p^3. If $|N| = p^3$, then $G = N$ is a group, hence has the SCP. If $|N| \in \{p, p^2\}$, then $|N| \in \{p, p^2\}$. In any case, N and G/N form groups. Hence, N and G/N have the SCP and by Corollary 4.6.8, G has the SCP. ■

Theorem 4.6.12. *Let p, q and r be primes. Every nongyrocommutative gyrogroup of order pqr has the strong Cauchy property.*

Proof. The proof follows the same steps as in the proof of Theorem 4.6.11. ■

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References

[1] Baumeister, B., Stein, A.: The finite Bruck loops. *J. Algebra* **330**, 206–220 (2011)

[2] Bruck, R.H.: *A Survey of Binary Systems*. Springer, Berlin Heidelberg (1971)

[3] Burn, R.P.: Finite Bol loops. *Math. Proc. Cambridge Philos. Soc.* **84**(3), 377–386 (1978)

[4] Chein, O.: Moufang loops of small order I. *Trans. Amer. Math. Soc.* **188**, 31–51 (1974)

[5] Chein, O., Kinyon, M.K., Rajah, A., Vojtěchovský, P.: Loops and the Lagrange property. *Results. Math.* **43**, 74–78 (2003)

[6] Foguel, T., Kinyon, M.K.: Uniquely 2-divisible Bol loops. *J. Algebra Appl.* **9**(4), 591–601 (2010)

[7] Foguel, T., Kinyon, M.K., Phillips, J.: On twisted subgroups and Bol loops of odd order. *Rocky Mountain J. Math.* **36**, 183–212 (2006)

[8] Foguel, T., Ungar, A.A.: Involutory decomposition of groups into twisted subgroups and subgroups. *J. Group Theory* **3**, 27–46 (2000)

[9] Gallian, J.A.: The classification of groups of order $2p$. *Math. Mag.* **74**(1), 60–61 (2001)

[10] Gallian, J.A., Moulton, D.: On groups of order pq. *Math. Mag.* **68**(4), 287–288 (1995)

[11] Grishkov, A.N., Zavarnitsine, A.V.: Lagrange's theorem for Moufang loops. *Math. Proc. Cambridge Philos. Soc.* **139**, 41–57 (2005)

[12] Kiechle, H.: *Theory of K-loops*, *Lecture notes in Mathematics*, vol. 1778. Springer, Berlin (2002)

[13] Nagy, G.P.: A class of finite simple Bol loops of exponent 2. *Trans. Amer. Math. Soc.* **361**(10), 5331–5343 (2009)

[14] Pflugfelder, H.O.: *Quasigroups and Loops: An Introduction*. Sigma Series in Pure Mathematics 7. Heldermann Verlag, Berlin (1991)

[15] Roth, R.L.: A history of Lagrange's theorem on groups. *Math. Mag.* **74**(2), 99–108 (2001)

[16] Sabinin, L.V., Sabinina, L.L., Sbitneva, L.V.: On the notion of gyrogroup. *Aequat. Math.* **56**, 11–17 (1998)

[17] Sinefakopoulos, A.: On groups of order p^2. *Math. Mag.* **70**(3), 212–213 (1997)

[18] Suksumran, T., Wiboonton, K.: Isomorphism theorems for gyrogroups and L-subgyrogroups. *J. Geom. Symmetry Phys.* **37**, 67–83 (2015)

[19] Ungar, A.A.: Einstein's velocity addition law and its hyperbolic geometry. *Comput. Math. Appl.* **53**, 1228–1250 (2007)

[20] Ungar, A.A.: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*. World Scientific, Hackensack, NJ (2008)

[21] Ungar, A.A.: From Möbius to gyrogroups. *Amer. Math. Monthly* **115**(2), 138–144 (2008)

[22] Ungar, A.A.: *A Gyrovector Space Approach to Hyperbolic Geometry*. Synthesis Lectures on Mathematics and Statistics #4. Morgan & Claypool, San Rafael, CA (2009)

CHAPTER V

THESIS CONCLUSION

5.1 Conclusion

The Möbius and Einstein gyrogroups, which are prime examples of an abstract gyrogroup, have a strong connection by means of Clifford algebra operations via Equation (2.8) given in Chapter II. Using this connection, the algebraic proof that the open unit ball of $\mathbb{R}^n$, together with Einstein addition, forms a gyrocommutative gyrogroup with the uniquely 2-divisible property is provided. Further, some algebraic properties of the Einstein gyrogroup such as criteria for associativity and commutativity are obtained from that of the Möbius gyrogroup.

In the second part of the dissertation, algebraic properties of an arbitrary gyrogroup are investigated. Some of group-theoretic theorems in abstract algebra are extended to the case of gyrogroups in a natural way, including

- **Cayley's theorem.** Every gyrogroup can be embedded into its symmetric group as a subgyrogroup.
- **The first isomorphism theorem.** If $\varphi\colon G \to H$ is a gyrogroup homomorphism, then the quoteint of G by $\ker\varphi$ and the image of φ are isomorphic as gyrogroups:

$$G/\ker\varphi \cong \varphi(G).$$

- **Lagrange's theorem.** If H is a subgyrogroup of a finite gyrogroup G, then the order of G is divisible by the order of H.
- **Cauchy's theorem.** Every gyrogroup of order pq, where p and q are primes, contains an element of order p and an element of order q.

5.2 Delimitation and limitation

None.

5.3 Suggestion for future work

The results in the dissertation indicate that gyrogroups are a natural generalization of groups. From this point of view, one direction of research in gyrogroup theory is to extend group-theoretic theorems to the case of gyrogroups. In addition, as noted in Chapter IV, not every finite gyrogroup satisfies the Cauchy property. Therefore, it is worth studying which class of finite gyrogroups satisfies the Cauchy property.

REFERENCES

[1] Baumeister, B., Stein, A.: The finite Bruck loops. *J. Algebra* **330**, 206–220 (2011)

[2] Bruck, R.H.: *A Survey of Binary Systems*. Springer, Berlin Heidelberg (1971)

[3] Burn, R.P.: Finite Bol loops. *Math. Proc. Cambridge Philos. Soc.* **84**(3), 377–386 (1978)

[4] Chein, O.: Moufang loops of small order I. *Trans. Amer. Math. Soc.* **188**, 31–51 (1974)

[5] Chein, O., Kinyon, M.K., Rajah, A., Vojtěchovský, P.: Loops and the Lagrange property. *Results. Math.* **43**, 74–78 (2003)

[6] Ferreira, M.: Factorizations of Möbius gyrogroups. *Adv. Appl. Clifford Algebras* **19**, 303–323 (2009)

[7] Ferreira, M.: *Hypercomplex Analysis and Applications*, I. Sabadini and F. Sommen (eds.), chap. Gyrogroups in projective hyperbolic Clifford analysis, pp. 61–80. Trends in Mathematics. Springer, Basel (2011)

[8] Ferreira, M.: Harmonic analysis on the Einstein gyrogroup. *J. Geom. Symmetry Phys.* **35**, 21–60 (2014)

[9] Ferreira, M., Ren, G.: Möbius gyrogroups: A Clifford algebra approach. *J. Algebra* **328**, 230–253 (2011)

[10] Foguel, T., Kinyon, M.K.: Uniquely 2-divisible Bol loops. *J. Algebra Appl.* **9**(4), 591–601 (2010)

[11] Foguel, T., Kinyon, M.K., Phillips, J.: On twisted subgroups and Bol loops of odd order. *Rocky Mountain J. Math.* **36**, 183–212 (2006)

[12] Foguel, T., Ungar, A.A.: Involutory decomposition of groups into twisted subgroups and subgroups. *J. Group Theory* **3**, 27–46 (2000)

[13] Friedman, Y., Scarr, T.: *Physical Applications of Homogeneous Balls*, *Progress in Mathematical Physics*, vol. 40. Birkhäuser, Boston (2005)

[14] Gallian, J.A.: The classification of groups of order $2p$. *Math. Mag.* **74**(1), 60–61 (2001)

[15] Gallian, J.A., Moulton, D.: On groups of order pq. *Math. Mag.* **68**(4), 287–288 (1995)

[16] Grishkov, A.N., Zavarnitsine, A.V.: Lagrange's theorem for Moufang loops. *Math. Proc. Cambridge Philos. Soc.* **139**, 41–57 (2005)

[17] Grove, L.C.: *Classical Groups and Geometric Algebra*, *Graduate Studies in Mathematics*, vol. 39. AMS, Providence, RI (2001)

[18] Issa, A.N.: Gyrogroups and homogeneous loops. *Rep. Math. Phys.* **44**(3), 345–358 (1999)

[19] Kasparian, A., Ungar, A.A.: Lie gyrovector spaces. *J. Geom. Symmetry Phys.* **1**, 3–53 (2004)

[20] Kiechle, H.: *Theory of K-loops*, *Lecture notes in Mathematics*, vol. 1778. Springer, Berlin (2002)

[21] Kim, S.: Distances of qubit density matrices on Bloch sphere. *J. Math. Phys.* **52**(10), 1–8 (2011)

[22] Kim, S., Lawson, J.: Unit balls, Lorentz boosts, and hyperbolic geometry. *Results. Math.* **63** (2013)

[23] Lawson, J.: Clifford algebras, Möbius transformations, Vahlen matrices, and B-loops. *Comment. Math. Univ. Carolin.* **51**(2), 319–331 (2010)

[24] Nagy, G.P.: A class of finite simple Bol loops of exponent 2. *Trans. Amer. Math. Soc.* **361**(10), 5331–5343 (2009)

[25] Pflugfelder, H.O.: *Quasigroups and Loops: An Introduction*. Sigma Series in Pure Mathematics 7. Heldermann Verlag, Berlin (1991)

[26] Roth, R.L.: A history of Lagrange's theorem on groups. *Math. Mag.* **74**(2), 99–108 (2001)

[27] Sabinin, L.V., Sabinina, L.L., Sbitneva, L.V.: On the notion of gyrogroup. *Aequat. Math.* **56**, 11–17 (1998)

[28] Sinefakopoulos, A.: On groups of order p^2. *Math. Mag.* **70**(3), 212–213 (1997)

[29] Sönmez, N., Ungar, A.A.: The Einstein relativistic velocity model of hyperbolic geometry and its plane separation axiom. *Adv. Appl. Clifford Algebras* **23**, 209–236 (2013)

[30] Suksumran, T.: *Essays in mathematics and its applications: In honor of Vladimir Arnold*, P. M. Pardalos and T. M. Rassias (eds.), chap. The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems, Springer, in press.

[31] Suksumran, T., Wiboonton, K.: Isomorphism theorems for gyrogroups and L-subgyrogroups. *J. Geom. Symmetry Phys.* **37**, 67–83 (2015)

[32] Ungar, A.A.: Thomas rotation and parametrization of the Lorentz transformation group. *Found. Phys. Lett.* **1**(1), 57–89 (1988)

[33] Ungar, A.A.: The holomorphic automorphism group of the complex disk. *Aequat. Math.* **47**, 240–254 (1994)

[34] Ungar, A.A.: *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, *Fundamental Theories of Physics*, vol. 117. Kluwer Academic, Dordrecht (2001)

[35] Ungar, A.A.: The hyperbolic geometric structure of the density matrix for mixed state qubits. *Found. Phys.* **32**(11), 1671–1699 (2002)

[36] Ungar, A.A.: *Analytic Hyperbolic Geometry: Mathematical Foundations and Applications*. World Scientific, Hackensack, NJ (2005)

[37] Ungar, A.A.: The proper-time Lorentz group demystified. *J. Geom. Symmetry Phys.* **4**, 69–95 (2005)

[38] Ungar, A.A.: Einstein's velocity addition law and its hyperbolic geometry. *Comput. Math. Appl.* **53**, 1228–1250 (2007)

[39] Ungar, A.A.: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*. World Scientific, Hackensack, NJ (2008)

[40] Ungar, A.A.: From Möbius to gyrogroups. *Amer. Math. Monthly* **115**(2), 138–144 (2008)

[41] Ungar, A.A.: Einstein's Special Relativity: The hyperbolic geometric viewpoint. In: PIRT Conference Proceedings, 1–35 (2009).

[42] Ungar, A.A.: *A Gyrovector Space Approach to Hyperbolic Geometry*. Synthesis Lectures on Mathematics and Statistics #4. Morgan & Claypool, San Rafael, CA (2009)

[43] Ungar, A.A.: *Essays in Mathematics and its Applications*, P. M. Pardalos and T. M. Rassias (eds.), chap. Gyrations: The missing link between classical mechanics with its underlying Euclidean geometry and relativistic mechanics with its underlying hyperbolic geometry, pp. 463–504. Springer, Berlin Heidelberg (2012)

[44] Ungar, A.A.: Hyperbolic geometry. *J. Geom. Symmetry Phys.* **32**, 61–86 (2013)

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Published articles:

(1) T. Suksumran and S. Panma, *On connected Cayley graphs of semigroups*, Thai J. Math **13**, no. 3 (2015), 641–652.

(2) T. Suksumran and K. Wiboonton, *Lagrange's theorem for gyrogroups and the Cauchy property*, Quasigroups Related Systems **22** no. 2 (2014), 283–294.

(3) T. Suksumran, K. Wiboonton, *Isomorphism theorems for gyrogroups and L-subgyrogroups*, J. Geom. Symmetry Phys. **37** (2015), 67–83.

(4) T. Suksumran and K. Wiboonton, *Einstein gyrogroup as a B-Loop*, Rep. Math. Phys. **76** (2015), 63–74.

Article in press:

(1) T. Suksumran, *Essays in mathematics and its applications: In honor of Vladimir Arnold*, P. M. Pardalos and T. M. Rassias (eds.), chap. The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems, Springer.