CHAPTER II

A METHOD FOR FINDING A MAGIC SQUARE OF ANY ODD ORDER

We shall first begin our method by discussing a problem from a text book[1] and then find the additional conditions which added to the solution of the problem provide a method for finding a magic square of any odd order.

2.1 Consider a square divided up into n^2 equal squares. Number the columns of small squares 1, 2, ..., n from the left to right. Similarly number the rows 1, 2, ...,n from the bottom to top and let $\{c,r\}$ denote the small square in the c^{th} column and r^{th} row.

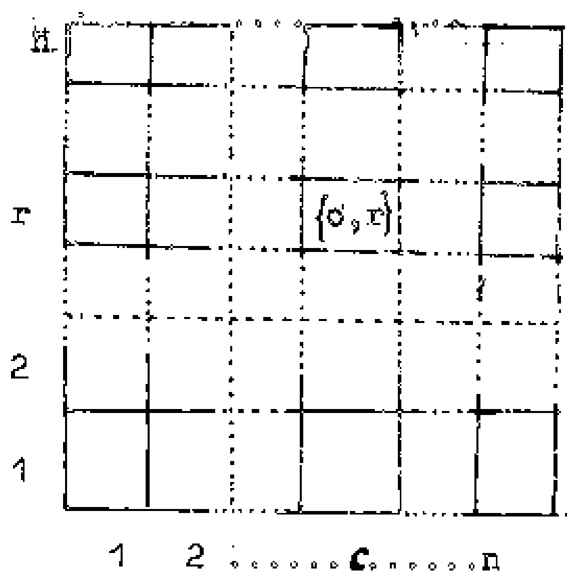

Figure 2.1 A form of a magic square.

[1] Niven, Ivan. Zuckerman, Herbert S. An Introduction to the Theory of Numbers. (New York: John Wiley & Sons, Inc., 1962), pp. 33-34.

Let a_o, b_o, α, β be positive integers less than or equal to n and such that a, b, α, β are relatively prime to n, i.e $(a,n)=(b,n)=(\alpha,n)=(\beta,n)=1$; where (x,y) denotes the greatest common divisor of x and y.

Write 1 in square $\{a_o,b_o\}$. Then count a columns to the right and b rows up from this square. If this takes you outside of the large square count as if the large square were bent into a cylider. Thus you will arrive at $\{a_o+a,b_o+b\}$ or $\{a_o+a-n,b_o+b\}$ or $\{a_o+a,b_o+b-n\}$ or $\{a_o+a-n,b_o+b-n\}$, whichever is actually one of the small squares. Write 2 in this square. Count a to the right and b up from 2 and insert 3 in that cell. Continue until you have written in 1, 2, ...,n.

We shall prove that m will have been written in $\{x_m,y_m\}$ where x_m and y_m are uniquely determined by

$$x_m \equiv a_o+a(m-1) \pmod n, \quad 1 \leq x_m \leq n$$

$$y_m \equiv b_o+b(m-1) \pmod n, \quad 1 \leq y_m \leq n, \quad 1 \leq m \leq n.$$

Proof. Consider the columns

$$x_1 = a_o$$

$$x_2 = a_o+a \text{ or } a_o+a-n \equiv a_o+a \pmod n \equiv a_o+a(2-1) \pmod n$$

$$x_3 = a_o+2a \text{ or } a_o+2a-n \equiv a_o+2a \pmod n \equiv a_o+a(3-1) \pmod n$$

$$x_4 = a_o+3a \text{ or } a_o+3a-n \equiv a_o+3a \pmod n \equiv a_o+a(4-1) \pmod n$$

....

And therefore, in general we can write

$$x_m \equiv a_o+a(m-1) \pmod n, \quad 1 \leq x_m \leq n; \quad 1 \leq m \leq n.$$

Similarly for the rows

$$y_m \equiv b_o+b(m-1) \pmod n, \quad 1 \leq y_m \leq n; \quad 1 \leq m \leq n.$$

Now suppose that we can also write x_m in the form

$$x_m \equiv a_o+ak \pmod n, \quad 1 \leq k \leq n.$$

Therefore, $a_o+a(m-1) \equiv a_o+ak \pmod n$,

or $a(m-1) \equiv ak \pmod n$.

And since $(a,n) = 1$, we get

$$(m-1) \equiv k \pmod n.$$

That is x_m is uniquely determined by

$$x_m \equiv a_o+a(m-1) \pmod n, \quad 1 \leq x_m \leq n.$$

And also similarly for y_m

$$y_m \equiv b_o+b(m-1) \pmod n, \quad 1 \leq y_m \leq n; \quad 1 \leq m \leq n.$$

Q. E. D.

2.2 We shall show that all these cells $\{x_m, y_m\}$ are different but that continuing the process one more step would put n+1 in $\{a_o, b_o\}$ which is already occupied by 1.

Proof. Suppose that there exists the number k in the cell $\{x_m, y_m\}$ as well as m. This implies that

$$a_o+a(k-1) \equiv a_o+a(m-1) \pmod n$$

and $b_o+b(k-1) \equiv b_o+b(m-1) \pmod n$,

which implies that $k = m$.

That is all these cells $\{x_m, y_m\}$ are different.

But when the number is n+1 we have

$$x_{n+1} \equiv a_o+a(n+1-1) \quad (\text{mod } n)$$
$$\equiv a_o+an \quad (\text{mod } n) \equiv a_o \ (\text{mod } n),$$
$$y_{n+1} \equiv b_o+b(n+1-1) \quad (\text{mod } n)$$
$$\equiv b_o+bn \quad (\text{mod } n) \equiv b_o \ (\text{mod } n),$$

which is already occupied by 1.

Q. E. D.

2.3 Now having reached $\{a_o, b_o\}$ again, count α to the right and β up and write n+1 in this cell which may or may not already be occupied. Then revert to the original process with step a, b to insert n+2, n+3, ..., 2n. Continue in this manner, using the extra step α, β just for n+1, 2n+1,......, (n-1)n+1, and stopping when n^2 has been inserted.

Then for $1 \leq m \leq n^2$ we shall prove that m is in $\{x_m, y_m\}$ where
$$x_m \equiv a_o+a(m-1)+\alpha\left[\frac{m-1}{n}\right] \quad (\text{mod } n), \ 1 \leq x_m \leq n^2,$$
$$y_m \equiv b_o+b(m-1)+\beta\left[\frac{m-1}{n}\right] \quad (\text{mod } n), \ 1 \leq y_m \leq n^2,$$
and $\left[\frac{m-1}{n}\right]$ is the quotient omitting any remainder when n is divided into m-1.

<u>Proof</u>. For the columns we have

$$x_m \equiv a_o+a(m-1) \quad (\text{mod } n), \quad 1 \leq m \leq n,$$
$$x_m \equiv a_o+a(m-1)+\alpha \quad (\text{mod } n), \quad n+1 \leq m \leq 2n$$
$$x_m \equiv a_o+a(m-1)+2\alpha \quad (\text{mod } n), \quad 2n+1 \leq m \leq 3n$$

....

Therefore in general we can write

$$x_m \equiv a_0+a(m-1)+\alpha\left[\frac{m-1}{n}\right] \qquad (\mathrm{mod}\ n), \quad 1 \leq m \leq n^2,$$

Similarly for the rows we can write

$$y_m \equiv b_0+b(m-1)+\beta\left[\frac{m-1}{n}\right] \qquad (\mathrm{mod}\ n), \quad 1 \leq m \leq n^2,$$

where $\left[\frac{m-1}{n}\right]$ is the quotient omitting any remainder when n is divided into $m-1$.

Q. E. D.

2.4 We shall now prove that if $(a\beta-b\alpha, n) = 1$ then each cell contains one and only one integer m, $1 \leq m \leq n^2$.

<u>Proof</u>. Suppose that the numbers k and m are inserted in the same cell $\{x_m, y_m\}$. Then we have

$$a_0+a(m-1)+\alpha\left[\frac{m-1}{n}\right] \equiv a_0+a(k-1)+\alpha\left[\frac{k-1}{n}\right] \qquad (\mathrm{mod}\ n)$$

Or $$am+\alpha\left[\frac{m-1}{n}\right] \equiv ak+\alpha\left[\frac{k-1}{n}\right] \quad (\mathrm{mod}\ n)\ldots\ldots(1)$$

and $$b_0+b(m-1)+\beta\left[\frac{m-1}{n}\right] \equiv b_0+b(k-1)+\beta\left[\frac{k-1}{n}\right] \qquad (\mathrm{mod}\ n)$$

or $$bm+\beta\left[\frac{m-1}{n}\right] \equiv bk+\beta\left[\frac{k-1}{n}\right] \quad (\mathrm{mod}\ n)\ldots\ldots(2).$$

Multiplying (1) by β we obtain:

$$a\beta m+\alpha\beta\left[\frac{m-1}{n}\right] \equiv a\beta k+\alpha\beta\left[\frac{k-1}{n}\right] \quad (\mathrm{mod}\ n)\ldots\ldots(3).$$

Multiplying (2) by α we obtain:

$$b\alpha m+\alpha\beta\left[\frac{m-1}{n}\right] \equiv b\alpha k+\alpha\beta\left[\frac{k-1}{n}\right] \quad (\mathrm{mod}\ n)\ldots\ldots(4).$$

Subtracting (4) from (3) we then have

$$a\beta m-b\alpha m \equiv a\beta k-b\alpha k \qquad (\mathrm{mod}\ n)$$

or $$(a\beta-b\alpha)m \equiv (a\beta-b\alpha)k \qquad (\mathrm{mod}\ n)$$

This implies that $m \equiv k\ (\mathrm{mod}\ n)$ if $(a\beta-b\alpha, n) = 1$.

Multiplying (1) by b we obtain:

$$abm + b\alpha\left[\frac{m-1}{n}\right] \equiv abk + b\alpha\left[\frac{k-1}{n}\right] \pmod{n} \quad \ldots(5)$$

Multiplying (2) by a we obtain:

$$abm + a\beta\left[\frac{m-1}{n}\right] \equiv abk + a\beta\left[\frac{k-1}{n}\right] \pmod{n} \quad \ldots(6)$$

Subtracting (5) from (6) we then have

$$(a\beta - b\alpha)\left[\frac{m-1}{n}\right] \equiv (a\beta - b\alpha)\left[\frac{k-1}{n}\right] \pmod{n}$$

If now $(a\beta - b\alpha, n) = 1$, this congruence shows that m and k lie between the same two multiples of n. For since the values of $\left[\frac{m-1}{n}\right]$ and $\left[\frac{k-1}{n}\right]$ range between 0 and n-1 the congruence $\left[\frac{m-1}{n}\right] \equiv \left[\frac{k-1}{n}\right] \pmod{n}$ is also an equality. But if m and k lie between the same two multiples of n, $m \equiv k \pmod{n}$ is also an equality giving $m = k$.

That is if $(a\beta - b\alpha, n) = 1$ then each cell contains not more than one integer m, $1 \leq m \leq n^2$.

Furthermore, since there are altogether n^2 cells, and since all the numbers from 1 to n^2 have been inserted, each in a different cell, every cell contains at least one number.

Q. E. D.

2.5 From now on we shall assume $(a\beta - b\alpha, n) = 1$. Write $m-1 = qn + s$, $0 \leq s \leq n-1$.

Consider the number m which we insert in the c^{th} column $m = qn + s + 1$

Since $1 \leq m \leq n^2$, we have $0 \leq q \leq n-1$.

Now for this c^{th} column we have

$$c \equiv a_0 + a(qn+s) + \alpha\left[\frac{qn+s}{n}\right] \pmod{n}$$

$$\equiv a_0 + aqn + as + \alpha\left[q+\frac{s}{n}\right] \pmod{n}$$

$$\equiv a_0 + as + \alpha q \pmod{n}.$$

From which we have

$$as \equiv c - a_0 - \alpha q \pmod{n}$$

That is the entries in the c^{th} column are just the $m = qn + s + 1$ for which $0 \leq q \leq n-1$, $0 \leq s \leq n-1$, and $as \equiv c - a_0 - \alpha q \pmod{n}$.

Similarly for the r^{th} row we have:

$$bs \equiv r - b'_0 - \beta q \pmod{n}.$$

2.6 We shall prove that in each column and in each row there is one and only one s for each q and that each s is distinct from all the others.

Proof. Consider the relation $as \equiv c - a_0 - \alpha q \pmod{n}$. The values of a, a_0, α, c are given or can be fixed. If the value of q is also given, the right hand side of this relation is therefore like an arbitrary constant. Hence the relation reduces to the form of congruences of degree 1 as: $as \equiv K \pmod{n}$, where K is an arbitrary constant.

Now since a is relatively prime to n, s has exactly one value[2].

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[2] Ibid p.29.

Similarly for each given value of s we can show that q has exactly one value.

That is there is one and only one s for each q.

Now suppose there exist two different numbers m, say m_1, m_2 in any row or any column whose corresponding values of s are the same. This implies that the values of q corresponding to m_1 and m_2 also are the same, for there is one and only one value of s for each value of q in any row or column.

Therefore, we can write

$$m_1 = qn+s+1$$

$$m_2 = qn+s+1$$

This implies $m_1 = m_2$ which contradicts our assumption that m_1 and m_2 are different.

Therefore each s must be distinct from all the others, in each row and in each column.

Q. E. D.

2.7 We shall show that the sum of the numbers m in the c^{th} column $= \sum_{q=0}^{n-1} qn + \sum_{s=0}^{n-1} s + n = \frac{n(n^2+1)}{2}$ and that this is equal to the sum of the numbers m in a row.

<u>Proof</u>. Since in each column q and s vary from 0 to n-1, and there is one and only one s for each q, and each s is distinct from all the others, it follows that the sum of the numbers m in the c^{th} column is

$$\therefore \sum (qn+s+1) = \sum_{q=0}^{n-1} qn + \sum_{s=0}^{n-1} s + n$$

$$= \left[0+n+2n+\ldots+(n-1)n\right] + \left[0+1+2+\ldots+(n-1)\right] + n$$

$$= n\left[1+2+\ldots+(n-1)\right] + \left[1+2+\ldots+(n-1)+n\right]$$

$$= \frac{n(n-1)n}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n(n^2+1)}{2}$$

Similarly for the sum of the numbers m in a row we get $\frac{n(n^2+1)}{2}$.

Q. E. D.

Now since $\frac{n(n^2+1)}{2}$ is independent of c and r, the sums in each row and in each column are the same, and the square array of integers has the magic square property in both columns and rows.

2.8 At this point we shall call the square array of integers whose sums in each row and in each column are equal a QUASI-MAGIC SQUARE. We have seen that the initial square $\{a_0, b_0\}$ is subject to no conditione. The only essential conditions for making the square array of integers a quasi-magic square are that a, b, α, β, $a\beta - b\alpha$ are relatively prime to n, where a, b, α, β, n are positive integers as in 2.1.

2.9 We shall now show that the conditions in 2.8 can not be fulfilled if n is even.

<u>Proof</u>. If n is even, a, b, α, β must be odd, for

$$(a,n) = (b,n) = (\alpha,n) = (\beta,n) = 1.$$

Then $a\beta$ must also be odd

$b\alpha$ must also be odd

and therefore $a\beta - b\alpha$ must be even.

Hence the condition $(a\beta - b\alpha, n) = 1$ does not hold.

So that the conditions in 2.8 can not be fulfilled if n is even.

Q. E. D.

We can note that for odd n the values $a = b = \alpha = 1$, $\beta = 2$ always give a quasi-magic square.

(This completes the solution of the problem referred to at the beginning of this chapter.).

2.10 We shall now find the conditions that make both diagonals of the square have the same sum as each row and each column, which is equal to $\frac{n(n^2+1)}{2}$.

Consider the relations in 2.5 which give

$$as \equiv c - a_0 - \alpha q \quad (\text{mod } n) \ldots\ldots\ldots\ldots\ldots(1)$$

$$bs \equiv r - b_0 - \beta q \quad (\text{mod } n) \ldots\ldots\ldots\ldots\ldots(2).$$

Since in the ascending diagonal (bottom left to top right), c is equal to r in each cell. By subtracting (2) from (1) we get

$$(a-b)s \equiv b_0-a_0-(\alpha-\beta)q \quad (\text{mod } n) \ldots\ldots(3).$$

And in the descending diagonal (top left to bottom right), the sum of c and r is equal to n+1 in each cell.

Therefore by adding (1) and (2) we get

$$(a+b)s \equiv 1-a_0-b_0-(\alpha+\beta)q \quad (\text{mod } n) \ldots\ldots(4).$$

The sum of the n numbers $m = nq+s+1$ in both diagonals will be equal to $\frac{n(n^2+1)}{2}$ if $\sum q$ and $\sum s$ are equal to $\frac{n(n-1)}{2}$ as in 2.7.

There are two possible ways in which $\sum q$ and $\sum s$ can be equal to $\frac{n(n-1)}{2}$. Either q and s run through the integers from 0 to n-1 as in 2.7, or q and s remain constant. The later case is possible if q and s are equal to $\frac{n-1}{2}$.

We may consider the values of q and s in the central cell, which is the common cell of both diagonals, for these two possible ways of choosing values for q and s in both diagonals. There are four possible cases for the values of q and s in the central cell.

These four possible cases are as follows:

2.11 Case I. Both values of q and s are constant equal to $\frac{n-1}{2}$ in the central cell. This is possible for either one of the following:

(i). In one diagonal we hold the value of q constant equal to $\frac{n-1}{2}$ while the value of s runs through the integers from 0 to n-1, and in the other diagonal we hold the value of s constant equal to $\frac{n-1}{2}$ while the value of q runs through the integers from 0 to n-1.

(ii). We hold the value of q constant while the value of s runs through the integers from 0 to n-1 in one diagonal; and q and s both run through the integers from 0 to n-1 in the other diagonal.

(iii). We hold the value of s constant while the value of q runs through the integars from 0 to n-1 in one diagonal; and q and s both run through the integers from 0 to n-1 in the other diagonal.

(iv). Both values of q and s run through the integers from 0 to n-1 in both diagonals.

Possibilities (ii), (iii) and (iv) will be considered in the other cases. In (i) we can find the conditions for the starting cell by substituting the value $\frac{n-1}{2}$ for q and s in (3) and (4). We get

$$2a_0-2b_0 \equiv a-b+\alpha-\beta \quad (\text{mod } n)\ldots\ldots\ldots(5)$$

$$2a_0+2b_0 \equiv a+b+\alpha+\beta+2 \quad (\text{mod } n)\ldots\ldots\ldots(6).$$

Adding (5) and (6) we get

$$2a_0 \equiv a+\alpha+1 \quad (\text{mod } n)\ldots\ldots(A).$$

Subtracting (5) from (6) we have

$$2b_0 \equiv b+\beta+1 \quad (\text{mod } n)\ \ldots\ldots(B).$$

So we have (A) and (B) as the essential conditions for making both diagonals magic. These ere the conditions for the starting cell in case I.

In constructing a magic square of any odd order, we therefore choose a, b, α, β such that $(a,n) = (b,n) = (\alpha,n) = (\beta,n) = (a\beta - b\alpha, n) = 1$ and start at $\{a_0, b_0\}$, where $2a_0 = a+\alpha+1 \pmod n$ and $2b_0 = b+\beta+1 \pmod n$, putting 1 there. Then following the procedure given in 2.1, we get a magic square.

Example 1. In constructing a 7^{th} order magic square we may choose $a = 1$, $b = 1$, $\alpha = 2$, $\beta = 5$, these satisfy the conditions $(a,n) = (b,n) = (\alpha,n) = (\beta,n) = (a\beta-b\alpha,n) = 1$.

We then determine the starting cell as follows:

$$2a_0 \equiv 1+2+1 \equiv 4 \pmod 7$$

or $$a_0 = 2$$

and $$2b_0 \equiv 1+5+1 \equiv 0 \pmod 7$$

or $$b_0 = 7$$

Therefore by starting with 1 in the cell $\{2,7\}$ and following the process in 2.1 we have the magic square below.

39	1	19	30	48	10	28
7	18	29	47	9	27	38
17	35	46	8	26	37	6
34	45	14	25	36	5	16
44	13	24	42	4	15	33
12	23	41	3	21	32	43
22	40	2	20	31	49	11

Figure 2.2 A magic square of order 7.

2.12 Case II. The value of q is constant equal to $\frac{n-1}{2}$ and the value of s may be any one of the integers from 0 to n-1 in the central cell. This is possible if we assign the values of q and s as in (ii), (iii) and (iv) of case I. We will consider (iii) and (iv) in later cases. Here we shall discuss (ii).

Consider the relations (3) and (4). With q constant in one diagonal the value of q will run through the integers 0 to n-1 along the other diagonal if $(\alpha \pm \beta)$ are relatively prime to n. For if $(\alpha \pm \beta)$ are relatively prime to n, there will be exactly one value[3] of q for one value of s. And since s has n different values, therefore q also has n different values. So the conditions that $(\alpha \pm \beta)$ be relatively prime to n are necessary in this case.

Now consider the central cell. By substituting the value of $q = \frac{n-1}{2}$ in (3) and (4) we get

$$2(a-b)s \equiv 2b_0-2a_0+(\alpha-\beta) \quad (\text{mod } n) \;\ldots\ldots(\text{I})$$

$$2(a+b)s \equiv 2-2a_0-2b_0+(\alpha+\beta) \quad (\text{mod } n) \;\ldots\ldots(\text{II}).$$

Adding (I) and (II) we have

$$2a_0 \equiv 1+\alpha-2as \quad (\text{mod } n)\ldots\ldots(\text{C}).$$

Subtracting (I) from (II) we get

$$2b_0 \equiv 1+\beta-2bs \quad (\text{mod } n)\ldots\ldots(\text{D}).$$

We now have (C) and (D) as the conditions for the starting cell.

Now since the value of s can take n different values, we have n starting cells. Therefore if we choose α, β in such

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[3]Dickson, L.E. Modern Elementary Theory of Numbers. (Chicago: The University of Chicago Press, 1950), p.15.

a way that $\alpha \pm \beta$ are also relatively prime to n, we can find n different magic squares according to the n different starting cells from conditions (C) and (D).

Notice that every set of values for a, b, α, β that satisfies case II also satisfies case I. The difference between case II and case I is that there is only one starting cell for case I but there are n starting cells for case II. So, there is one magic square for each set of values for a, b, α, β of case II that satisfies case I. This occurs when the starting cell of case II is the same as that of case I, which is when:

$$1+\alpha-2as \equiv a+\alpha+1 \pmod{n}$$
$$1+\beta-2bs \equiv b+\beta+1 \pmod{n}$$
$$\text{or when} \quad 2s \equiv -1 \pmod{n}.$$

<u>Example 2</u>. If we choose a=1, b=1, α=6, β=5 in constructing a magic square of order 7. We can see that not only $(a,n)=(b,n)=(\alpha,n)=(\beta,n)=(a\beta-b\alpha,n)=1$ but also $(\alpha\pm\beta,n)=1$. Therefore these choices satisfy the conditions of case II. So we can determine the starting cells from (C) and (D) and get 7 starting cells which are $\{7,3\}$, $\{6,2\}$, $\{5,1\}$, $\{4,7\}$, $\{3,6\}$, $\{2,5\}$, $\{1,4\}$, according as the the values of s are 0, 1, 2, 3, 4, 5, 6 respectively.

Hence we can find 7 different magic squares by starting with these 7 various starting cells and a=1, b=1, α=6, β=5 the same way as in the previous example.

Note that when $2s \equiv -1 \pmod{7}$ or s = 3, the magic square which starts at $\{4,7\}$ also satisfies case I.

2.13 Case III. The value of s is constant equal to $\frac{n-1}{2}$ and the value of q may be any one of the integers from 0 to n-1 in the central cell. This is possible if we assign the values of q and s as in (ii), (iii) and (iv) of case I. We will not consider (iv) now but in the next case.

Possibility (iii) is similar to (ii) in case II. We find that this case occurs if the selected values of a and b such that $a \pm b$ are also relatively prime to n. By substituting the fixed value of s, which is equal to $\frac{n-1}{2}$, in (3) and (4) we get the following conditions for the starting cells.

$$2a_0 \equiv 1+a-2\alpha q \qquad (\text{mod } n) \ldots\ldots\ldots (E)$$

$$2b_0 \equiv 1+b-2\beta q \qquad (\text{mod } n) \ldots\ldots\ldots (F).$$

Because the value of q has n different values from 0 to n-1, we get n different starting cells from (E) and (F). Therefore when we choose a, b in such a way that $a \pm b$ are also relatively prime to n, we can find n different magic squares according to the n starting cells obtained from the relations (E) and (F).

This is similar to case II in that every set of values for a, b, α, β which satisfies case III also satisfies case I. This occurs when $2q \equiv -1$ (mod n).

Example 3. We may construct a 7^{th} order magic square by choosing a=2, b=1, $\alpha=1$, $\beta=1$. These satisfy the conditions $(a,n) = (b,n) = (\alpha,n) = (\beta,n) = (a\beta-b\alpha,n) = 1$ and also $(a \pm b,n) = 1$. So that we can obtain 7 starting cells from (E) and (F), which are $\{5,1\}$, $\{4,7\}$, $\{3,6\}$, $\{2,5\}$, $\{1,4\}$,

$\{7,3\}$, $\{6,2\}$ according as the values of **q** are 0, 1, 2, 3, 4, 5, 6 respectively.

Thersfore we can find 7 different magic squares from this set of values for a, b, α and β and ths magic square that satisfies case I is the one that has $\{2,5\}$ as the starting cell.

Note that as we choose the set of values for a, b, α, β in case III such that a, b, α, β, $a\beta - b\alpha$ and $a \pm b$ are relatively prime to n there will be some sets of valuea for a, b, α, β sucb that $\alpha \pm \beta$ are relatively prime to n too. These sets of values for a, b, α, β will also satisfy case II. Similarly, as we choose the set of values for a, b, α, β in case II such that a, b, α, β, $a\beta - b\alpha$, $\alpha \pm \beta$ are relatively prime to n there will be some scts of velues for a, b, α, β such that $a \pm b$ are relatively prime to **n** too. These sets of values for a, b, α, β of case II will also satisfy case III. These sets of values belong to case IV which we shall consider next.

2.14 Case IV. The values of q and s in the central cell occur in pairs such that there is one and only one value of s for each value of q and such that each s is distinct from all the others. This is possible only if the values of q and s both run through the integers from 0 to n-1 in both diagonals as (iv) of case I. Here we know nothing in advance about the values of q and s in the central cell, so in this case we cannot proceed in the same way as in the three previous cases. However, we can go back to consider the relations (3) and (4) from 2.10:

$$(a-b)s \equiv b_0-a_0-(\alpha-\beta)q \pmod n$$
$$(a+b)s \equiv 1-a_0-b_0-(\alpha+\beta)q \pmod n.$$

As in the proof of cases II and III, possibility (iv) will occur if and only if $a\pm b$ and $\alpha\pm\beta$ are relatively prime to n. So the conditions $a\pm b$ and $\alpha\pm\beta$ be relatively prime to n are neccessary in this case. If it is so, we can see that the values of a_0 and b_0 are subject to no conditions. Hence the only additional conditions for case IV other than $(a,n)=(b,n)=(\alpha,n)=(\beta,n)=(a\beta-b\alpha,n)=1$ are that $(a\pm b,n) = (\alpha\pm\beta,n) = 1$ too.

Notice that the initial cell $\{a_0,b_0\}$ is subject to no conditions when we choose a, b, α, β in such a way that $(a,n)=(b,n)=(\alpha,n)=(\beta,n)=(a\beta-b\alpha,n)=(a\pm b,n)=(\alpha\pm\beta,n)=1$ as in case IV. So we can find n^2 different magic squares from one such set of values for a, b, α, β by starting at any of the n^2 cells.

Note that every set of values for a, b, α, β of case IV also satisfies cases II and III and of course of case I too. Therefore of the n^2 magic squares in case IV n satisfy case II and n satisfy case III, and there is one magic square which in addition to satisfies cases II and III also satisfies case I.

Example 4. If we choose $a=2$, $b=1$, $\alpha=4$, $\beta=5$ for constructing a magic square of order 7. We can see that this set of values for a, b, α, β satisfies all the conditions of case IV. So we may start at any cell and the square will always be magic. The magic squares which start at $\{6,3\}$, $\{4,2\}$, $\{2,1\}$, $\{0,0\}$, $\{5,6\}$, $\{3,5\}$, $\{1,4\}$ also belongs to case II, and those start at $\{5,1\}$, $\{1,3\}$, $\{4,5\}$, $\{7,7\}$, $\{3,2\}$, $\{6,4\}$, $\{2,6\}$ also belongs to case III. One of which starts at $\{7,7\}$ belongs to every case.

2.15 We can summarize all of the four cases as follows:

Case I. If we choose a set of values for a, b, α, β in such a way that $(a,n) = (b,n) = (\alpha,n) = (\beta,n) = (a\beta-b\alpha,n) = 1$, we can find one magic square by starting at $\{a_0, b_0\}$, where

$2a_0 \equiv a+\alpha+1 \pmod{n}$

$2b_0 \equiv b+\beta+1 \pmod{n}$.

Case II. If a set of values for a, b, α, β also satisfies the conditions $(\alpha \pm \beta, n) = 1$ in addition to those of case I, we can find n magic squares by starting at $\{a_0, b_0\}$, where

$2a_0 \equiv 1+\alpha-2as \pmod{n}$

$2b_0 \equiv 1+\beta-2bs \pmod{n}$, s varying from 0 to n-1.

One of these magic squares belongs to case I.

Case III. If we choose a set of values for a, b, α, β in such a way that $(a \pm b, n) = 1$ in addition to the conditions of case I, we can find n magic squares by starting at $\{a_0, b_0\}$ where $2a_0 \equiv 1+a-2\alpha q \pmod{n}$

$2b_0 \equiv 1+b-2\beta q \pmod{n}$, q varying from 0 to n-1.

One of these magic squares belongs to case I.

Case IV. If a set of values for a, b, α, β also satisfies the conditions $(a \pm b, n) = (\alpha \pm \beta) = 1$ in addition to those of case I, we can find n^2 magic squares by starting at any cell. Of these squares n belongs to case II, n belongs to case III, and there is one which belongs to cases I, II and III.

Note that the conditions in cases II, III, IV can never be fulfilled if the odd n is 3 or when the odd n contains a factor 3. For if a and b are relatively prime to 3, a+b and a-b cannot simultaneously be relatively prime to 3.