แบบจำลองอินเฟลชันภายใต้กาลอวกาศแบบไม่สลับที่

นายพิทยุทธ วงศ์จันทร์

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INFLATIONARY MODEL WITHIN NONCOMMUTATIVE SPACE-TIME

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เป็นที่รู้จักกันดีอยู่แล้วว่า แบบจำลองทางเอกภพวิทยาที่น่าเชื่อถือมากที่สุดในปัจจุบันนี้คือ แบบจำลองอินเฟลชัน ซึ่งอธิบายว่า เอกภพในยุคเริ่มแรกนั้นขยายตัวอย่างรวดเร็ว โดยแบบจำลองนี้ ถูกสมมุติให้เกิดขึ้นที่สเกลพลังงานสูงๆ ซึ่งทฤษฎีสัมพัทธภาพทั่วไปไม่สามารถใช้ได้อีกต่อไป เนื่องจากผลกระทบของทฤษฎีควอนตัมมีความสำคัญมากกว่า ทฤษฎีเส้นเชือกเป็นทฤษฎีที่มีความเป็นไปได้มากที่สุด ที่สามารถอธิบายฟิสิกส์ในสเกลนี้ โดยคุณสมบัติทั่วไปของทฤษฎีนี้คือ ความสัมพันธ์ของความไม่แน่นอนในกาลอวกาศเชิงเส้นเชือก ซึ่งความสัมพันธ์นี้ จะนำไปสู่คุณสมบัติของกาลอวกาศแบบไม่สลับที่ในที่สุด ดังนั้น จุดประสงค์ของวิทยานิพนธ์นี้คือ ศึกษาแบบจำลองอินเฟลชัน โดยคำนึงถึงผลของ กาลอวกาศแบบไม่สลับที่เข้าไปด้วย โดยทั่วไปแล้ว ปริมาณที่ใช้เปรียบเทียบกับ ข้อมูลจากการสังเกตการณ์ คือ ค่าสเปกตรัมกำลัง ของคลื่นคอสมิกไมโครเวฟแบ็กกราวน์ โดยในวิทยานิพนธ์นี้ แบ่งการเปรียบเทียบออกเป็นสองส่วน ขึ้นอยู่กับสภาวะสูญญากาศของมัน ซึ่งคือสภาวะสุญญากาศแบบ แอเดียแบติก และแบบความไม่แน่นอนน้อยสุด ผลของทั้งสองสภาวะสูญญากาศนี้ เป็นไปในทางเดียวกันคือ ผลของกาลอวกาศแบบไม่สลับที่ จะกดค่าสเปกตรัมที่มัลติโพลต่ำๆลง ซึ่งผลที่ได้นี้สอดคล้องกับผลจากการสังเกตการณ์มากกว่า ผลที่ได้จากการคำนวณใน กาลอวกาศแบบสลับที่ได้

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It is well known nowadays that the best candidate for the cosmological model is the inflationary model which describes the rapid expansion of the universe at the early time. This inflationary scenario is assumed to take place at a very high energy scale, at which classical General Relativity is no longer valid as the quantum effects become more relevant. String theory is the most promising candidate for describing the nature of physics in this scale. The universal property of this theory is the stringy space-time uncertainty relation. The noncommutative property of space and time is the important consequence of this property. The aim of this thesis is to study the inflationary model in which the effect of noncommutative space-time is taken into account. The Cosmic Microwave Background (CMB) power spectrum is the convenient quantity that one uses to compare the result of the model with observed data. The comparison can be divided into two classes according to their vacua, which are the adiabatic vacuum and the minimized uncertainty vacuum. The result of both vacua is consistent that the noncommutative effect suppresses the spectrum at the low multipole. This result is more compatible with observation than the result of the commutative space-time.

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Field of study Physics Advisor's signature ....

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Table **page**

CHAPTER I

INTRODUCTION

The standard BigBang model was constructed from the principle of cosmology, in which one assumes that the universe is both isotropic and homogeneous. The solution of the Einstein equation consistent with this principle is known as the "Friedmann-Robertson-Walker solution" [1, 2, 3, 5]. Despite the fact that this solution has been very successful due to its consistency with many observed data, it cannot explain some important puzzles in cosmology, such as the horizon and the flatness problems. In 1981, Alan Guth [4] proposed a very nice solution to these puzzles known as the "inflationary model." The idea of this model is that the universe expanded extremely fast during some period of the early time. The consequent prediction of the inflationary model is that the background temperature of the universe is nearly uniform [6]. Such a prediction is consistent with the cosmic microwave background (CMB) radiation observation [7]. Nevertheless, the most important powerfulness of this model is that it not only solves the flatness and the horizon problems, but also provides us the understanding of the mechanism of structure formations. According to the inflationary model, the quantum fluctuations generated during the inflationary period are stretched to the cosmological scale and seed the structures we observe nowadays. This effect also leads to the CMB anisotropy, and therefore the inflationary model does not predict the perfectly uniform CMB.

As the inflation is assumed in to take place at very high energy scales, the theories originally used to construct the models of cosmology, such as general relativity, are no longer valid because the quantum effects become more important. At these scales, string theory, which is the theory in which the fundamental object is a string instead of a point-like particle, is the most promising candidate for describing the physics. However, it is rather complicated to directly derive the cosmological model from string theory. In Ref. [26], Ho and Brandenberger constructed an inflationary theory in which a universal property of string theory, which is the stringy space-time uncertainty relation (SSUR) $\Delta x_p \Delta t_p \geq l_s^2$ [27], is taken into account. Such an uncertainty relation implies that space and time coordinates do not commuted with each other, and therefore the effective space-time geometry is noncommutative. This noncommutative inflationary theory is the subject of this thesis.

This thesis is organized in the following way. In Chapter 2, we review the fundamental concepts of modern cosmology, putting an emphasis on the inflationary models. In Chapter 3, we consider the cosmological perturbation theory in order to connect the theoretical results with the observational data, in particular the CMB power spectrum. We start by discussing the power spectrum of the inflaton field fluctuations [9], and then calculate the power spectrum associated with the curvature and the tensor perturbations by perturbing the Einstein equation [10, 11]. This chapter ends with the discussion of the CMB anisotropy and the calculation of the CMB power spectrum [12, 13].

In order to discuss the noncommutative inflation, the concept of the noncommutative field theory needs to be introduced first; this will be done in Chapter 4. In the first part of this chapter, the motivation for the noncommutative algebra is given by considering the generalized uncertainty principle [21, 22]. A brief discussion of field theory [15] then follows. We close this chapter with the presentation of the idea of deformation from a commutative field theory to its noncommutative counterpart [16, 17]. In Chapter 5, the concept of the noncommutative inflation is introduced, and the associated power spectra of the curvature perturbation [26, 28] based on both adiabatic and minimized uncertainty vacua [31, 32] are obtained. Finally, the comparison between the power spectra in commutative and noncommutative space-time is done in Chapter 6 [34, 35].

CHAPTER II

COSMOLOGY AND INFLATIONARY MODELS

Since the ancient time, people have wondered about the structures and the evolution of the universe. Due to the fact that the distribution of celestial objects, like stars and galaxies, looks the same to us no matter what direction we look at, and as the Earth where we all live has no right to be a special location in the universe, scientists were led to postulate what is known as the cosmological principle. This principle states that every position of the universe is in no sense preferred [1], and therefore the universe has to be homogeneous and isotropic. The recent data from the observation of the cosmic microwave background (CMB) radiation is an important evidence which confirms this principle. However, people still had no idea about the "dynamics" of the universe before the twentieth century.

In the early twentieth century, Edwin Hubble observed many galaxies and found that they are moving away from us with the velocities proportional to their distances from Earth. This led him to conclude that the universe is expanding. Thus if one were to construct a theory which describes the universe, such a theory should obey the cosmological principle and has to give a dynamics of the universe in agreement with Hubble's discovery. It turns out that the appropriate theory for modeling the universe with such characteristics is the general theory of relativity, whose main idea is that the presence of the mass-energy causes the space-time geometry to become curved, and the space-time curvature in turn causes anything with energy to move. Such an interdependence between the space-time curvature and the mass-energy is encoded mathematically in an equation known as the Einstein equation. To describe the universe using this theory, one needs to find the solution to this equation which has all the required characteristics. Such a solution, fortunately, exists and is known as the Friedmann-Robertson-Walker (FRW) solution, which describes three possible kinds of the universe (flat, closed, and open) depending on the matter contained in it. Despite their differences in many aspects, one thing that these three kinds of the universe has in common

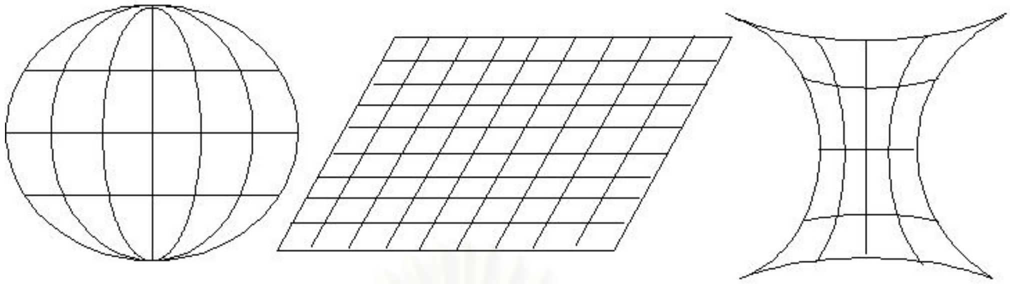

Figure 2.1: Three-dimensional spaces as the hypersurfaces of constant curvature embedded in a flat 4-dimensional Euclidean space: (a) spherical space; (b) flat space; (c) hyperbolic space.

is the prediction that the universe started from an initial singularity known as the Big Bang. This Big Bang model, however, cannot explain many puzzles in cosmology, such as the horizon and the flatness problems. A nice way to solve these problems is to postulate a model, called the inflationary model, which predicts a rapid expansion of the universe during some early period of the universe. In this chapter, we will describe the modern theory of cosmology in detail.

2.1 Friedmann-Robertson-Walker Solution

In this section, the Friedmann-Robertson-Walker solution will be discussed in detail. The convention for the notations used here is as follows. The signature of the space-time metric is chosen to be mostly plus. All the results in this thesis are expressed in the natural unit, where $\hbar = c = k_B = 1$ and the Newton's gravitational constant G is expressed in terms of the Planck mass as $G = m_{pl}^{-2}/8\pi$. The Greek indices (such as ν, μ) run from 0 to 3, where the index 0 is for the timelike component of any tensor. The roman indices (such as i, j, k) run from 1 to 3. The Einstein summation convention is also used.

To construct a 4-dimensional space-time describing the universe, we need to take into account the cosmological principle which states that the universe is both homogeneous and isotropic. In the mathematical language, this implies that the spatial part of the space-time must be a space of constant curvature. In differential geometry, the curvature of a 3-dimensional space is expressed in terms of the Ricci scalar, $R^{(3)}$, associated with it. Thus we need to have [1]

$$R^{(3)} = 6K, \tag{2.1}$$

with the sectional curvature K being a constant, for the spatial part of the space-time. This 3-dimensional curvature can be classified into three groups depending on the sign of the sectional curvature: spherical spaces ($K > 0$), flat spaces ($K = 0$), and hyperbolic spaces ($K < 0$). All these three distinct spaces are illustrated in Figure 2.1. Among these three types, whichever that turns out to describe the universe will depend on the mass-energy density of the universe. To see this, we start with the Einstein equation which relates the space-time curvature with the mass-energy density:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.2}$$

Here $g_{\mu\nu}$ is the metric tensor; $G_{\mu\nu}$, $R_{\mu\nu}$, and R are the Einstein tensor, the Ricci tensor, and the Ricci scalar, respectively; and $T_{\mu\nu}$ is the energy-momentum tensor. The left-hand side of this equation represents the space-time curvature, while the right-hand side describes the energy density and the pressure in the universe. The Ricci tensor and the Ricci scalar are proportional to the second derivatives of the space-time metric, $g_{\mu\nu}$, with respect to the space-time coordinates. For the homogeneous and isotropic universe, the matter in the universe is typically regarded as a perfect fluid with the energy-momentum tensor taking the form

$$T^{\mu}_{\ \nu} = diag(-\rho, p, p, p), \tag{2.3}$$

where ρ is the energy density and p is the pressure.

To explicitly write the left-hand side of the Einstein equation, the space-time metric is the first object that we have to determine. As mentioned earlier, there are 3 types of the spatial metric depending on the sign of the 3-dimensional curvature. We start with the case $K > 0$ in which the 3-dimensional space is a sphere embedded in a 4-dimensional Euclidean space. Thus we consider a three-dimensional sphere of radius a as a hypersurface described by an algebraic equation

$$a^2 \ = \ x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{2.4}$$

Then the line element on this hypersurface takes the form

$$\begin{aligned} dl^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \\ &= dx_1^2 + dx_2^2 + dx_3^2 + \frac{-(x_1dx_1 + x_2dx_2 + x_3dx_3)}{a^2 - x_1^2 + x_2^2 + x_3^2}. \end{aligned} \tag{2.5}$$

Using the spherical polar coordinates (r, θ, ϕ), where $r^2 = x_1^2 + x_2^2 + x_3^2$ and $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\sin\phi$, $x_3 = r\cos\theta$, the line element takes the form

$$\begin{aligned} dl^2 &= dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \frac{r^2}{a^2 - r^2}dr^2) \\ &= \frac{1}{1 - r^2/a^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= a^2\Big(\frac{1}{1 - r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big). \end{aligned} \tag{2.6}$$

Notice that in the last line of the above equation, the rescaling $r \to ar$ has been used. One should observe that the metric is independent of the value of the radius a (which turns out to be inversely proportional to the 3-dimensional curvature). For the case of $K < 0$ corresponding to the hyperbolic space, the line element can be obtained by replacing $a^2 \to -a^2$ and $dx_4^2 \to -dx_4^2$ in the analysis of the spherical hypersurface. As for the flat space with $K = 0$, we obtain the line element by simply taking $dx_4^2 = 0$ or $a \to \infty$ before rescaling. Thus, the line element for all three cases takes the form

$$dl^2 = a^2\Big(\frac{1}{1 - kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big), \tag{2.7}$$

where

$$k = \begin{cases} 1 & \text{spherical space} \\ 0 & \text{flat space} \\ -1 & \text{hyperbolic space} \end{cases} \tag{2.8}$$

The construction of the space-time metric for a homogeneous and isotropic universe goes as follows. We start with a 3-dimensional space of constant curvature obtained above, then assign a "cosmic time" t to it and multiply its line element by a time-dependent scale factor $a^2(t)$ (this will make its size to depend on t). Treating this 3-dimensional space as a space-like hypersurface at time t in a space-time, the space-time describing the universe is constructed as a collection of such hypersurfaces over all values of cosmic time t. Such a "time-slicing" of a space-time geometry is depicted in Figure 2.2. Note that these hypersurfaces cannot intersect with each other, otherwise the notion of "evolving in time" of the universe would not be consistently defined. With this construction, the space-time metric for a homogeneous and isotropic universe takes the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= g_{00}dx^{0\ 2} + dl^2 \\ &= -dt^2 + a^2(t)\Big(\frac{1}{1 - kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big). \end{aligned} \tag{2.9}$$

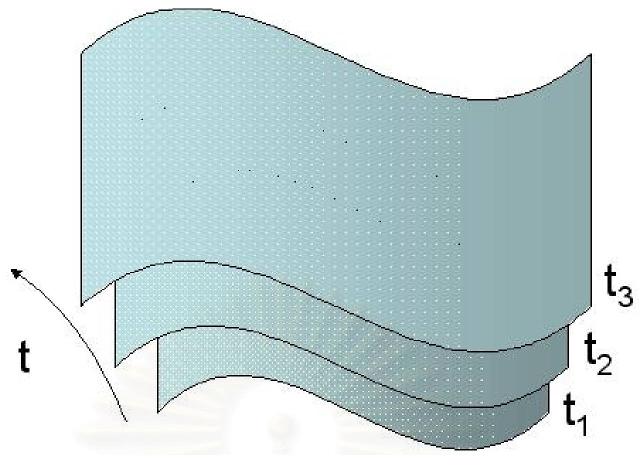

Figure 2.2: Time-slicing of space-like hypersurfaces.

This metric is known as the Friedmann-Robertson-Walker (FRW) metric, and the coordinates of the space-like hypersurface form a "comoving frame." That the above metric can explain the expansion of the universe is as follows. Imagine that all the galaxies are spread over the space-like hypersurface, and each of them is fixed at some point of the comoving frame. As time goes by, the scale factor $a(t)$ changes so that the proper distance between galaxies also changes. That it appeared to Hubble that the universe is expanding is because the scale factor $a(t)$ was increasing at the time of his observation. It should be mentioned here that another form of the FRW metric often used in cosmology is

$$ds^2 = a^2(\eta)\Big(- d\eta^2 + \frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big), \tag{2.10}$$

where the η is called the conformal time, which is defined as

$$\eta = \int \frac{1}{a(t)}dt. \tag{2.11}$$

To calculate the expansion rate of the universe, consider the coordinate difference of two points in the comoving frame, $\triangle x_{\rm como}$. Then the physical distance, $\triangle x_{\rm phys}$, is obtained by multiplying $\triangle x_{\rm como}$ by a scale factor:

$$\triangle x_{\rm phys} = a(t)\triangle x_{\rm como}. \tag{2.12}$$

The stretching rate of the distance which is observed by an observer in the physical frame is

$$\frac{1}{|\triangle x_{\rm phys}|}\frac{d|\triangle x_{\rm phys}|}{dt} = \frac{\dot{a}}{a} \equiv H = \frac{1}{a}\frac{a'}{a} = \frac{1}{a}\mathcal{H}, \tag{2.13}$$

where $\mathcal{H}$ is the Hubble parameter in the conformal time, and H is the Hubble parameter in the cosmic time. In cosmology, it is conventional to use a subscript "0" to mean "at the present time." Thus H_0 is the value of the Hubble parameter at present. Note that a dot, $(\dot{-})$, over a quantity represents the derivative with respect to the cosmic time of that quantity, while the prime, $(-')$, represents the derivative with respect to the conformal time.

Form the FRW metric, the non-zero components of the metric tensor can be read off as follows:

$$\begin{aligned} g_{00} &= -1 \;; \qquad g_{11} = \frac{a^2}{1-kr^2} \;; \\ g_{22} &= a^2r^2 \;; \qquad g_{33} = a^2r^2\sin^2\theta. \end{aligned} \tag{2.14}$$

Using the fact that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$, the non-zero components of the inverse metric can be easily found:

$$\begin{aligned} g^{00} &= -1 \;; \qquad g^{11} = \frac{1-kr^2}{a^2} \;; \\ g^{22} &= \frac{1}{a^2r^2} \;; \qquad g^{33} = \frac{1}{a^2r^2\sin^2\theta}. \end{aligned} \tag{2.15}$$

The Christoffel symbol or the affine connection which is proportional to the first-order derivative of the metric is defined by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\rho}\left(\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\beta\rho} - \partial_\rho g_{\beta\gamma}\right), \tag{2.16}$$

and, for the FRW metric, the non-zero components are

$$\begin{aligned} \Gamma^0_{ij} &= \frac{1}{2}g^{0\rho}\left(\partial_i g_{\rho j} + \partial_j g_{i\rho} - \partial_\rho g_{ij}\right) \\ &= -\frac{1}{2}g^{00}\partial_0 g_{ij} = \frac{\dot{a}}{a}g_{ij} \;; \end{aligned} \tag{2.17}$$

$$\begin{aligned} \Gamma^i_{0j} &= \frac{1}{2}g^{i\rho}\left(\partial_0 g_{j\rho} + \partial_j g_{\rho 0} - \partial_\rho g_{0j}\right) \\ &= \frac{1}{2}g^{ik}\partial_0 g_{jk} = \frac{\dot{a}}{a}g^{ik}g_{jk} = \frac{\dot{a}}{a}\delta^i_j \;; \end{aligned} \tag{2.18}$$

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2}g^{1\rho}\left(\partial_1 g_{1\rho} + \partial_1 g_{\rho 1} - \partial_\rho g_{11}\right) \\ &= \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{kr}{1-kr^2}g^{11}g_{11} = \frac{kr}{1-kr^2} \;; \end{aligned} \tag{2.19}$$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}\, g^{1\rho}\left(\partial_2 g_{2\rho} + \partial_2 g_{\rho 2} - \partial_\rho g_{22}\right) \\
&= -\frac{1}{2} g^{11}\partial_1 g_{22} \;=\; -\frac{1}{2}\frac{(1-kr^2)}{a^2} 2ra^2 \;=\; -r(1-kr^2)\;; \qquad (2.20) \\
\Gamma^1_{33} &= \frac{1}{2}\, g^{1\rho}\left(\partial_3 g_{3\rho} + \partial_3 g_{\rho 3} - \partial_\rho g_{33}\right) \\
&= -\frac{1}{2} g^{11}\partial_1 g_{33} \;=\; -\frac{1}{2}\frac{(1-kr^2)}{a^2} 2ra^2 \sin^2\theta \\
&= -r(1-kr^2)\sin^2\theta\;; \qquad (2.21) \\
\Gamma^2_{12} &= \frac{1}{2}\, g^{2\rho}\left(\partial_1 g_{2\rho} + \partial_2 g_{\rho 1} - \partial_\rho g_{12}\right) \\
&= \frac{1}{2} g^{22}\partial_1 g_{22} \;=\; \frac{1}{2}\frac{1}{a^2r^2} 2ra^2 \;=\; \frac{1}{r}\;; \qquad (2.22) \\
\Gamma^3_{13} &= \frac{1}{2}\, g^{3\rho}\left(\partial_1 g_{3\rho} + \partial_3 g_{\rho 1} - \partial_\rho g_{13}\right) \\
&= \frac{1}{2} g^{33}\partial_1 g_{33} \;=\; \frac{1}{2}\frac{1}{a^2r^2\sin^2\theta} 2ra^2\sin^2\theta \;=\; \frac{1}{r}\;; \qquad (2.23) \\
\Gamma^2_{33} &= \frac{1}{2}\, g^{2\rho}\left(\partial_3 g_{3\rho} + \partial_3 g_{\rho 3} - \partial_\rho g_{33}\right) \\
&= -\frac{1}{2} g^{22}\partial_2 g_{33} \;=\; -\frac{1}{2}\frac{1}{a^2r^2} 2r^2a^2\sin\theta\cos\theta \\
&= -\sin\theta\cos\theta\;; \qquad (2.24) \\
\Gamma^3_{23} &= \frac{1}{2}\, g^{3\rho}\left(\partial_2 g_{3\rho} + \partial_3 g_{\rho 2} - \partial_\rho g_{23}\right) \\
&= \frac{1}{2} g^{33}\partial_2 g_{33} \;=\; \frac{1}{2}\frac{1}{a^2r^2\sin^2\theta} 2r^2a^2\sin\theta\cos\theta \\
&= \cot\theta. \qquad (2.25)
\end{aligned}
$$

The Ricci tensor is proportional to the second-order derivative of the space-time metric and is defined by

$$
R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}\,. \qquad (2.26)
$$

For the FRW metric, the non-zero components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{00} - \Gamma^\alpha_{\sigma 0}\Gamma^\sigma_{0\alpha} \\
&= -\partial_0 \Gamma^i_{i0} - \Gamma^i_{0j}\Gamma^j_{i0} \\
&= -\partial_0\left(3\frac{\dot a}{a}\right) - 3\left(\frac{\dot a}{a}\right)^2 \;=\; -3\frac{\ddot a}{a} \qquad (2.27)
\end{aligned}
$$

and

$$
\begin{aligned}
R_{ij} &= \partial_\alpha \Gamma^\alpha_{ij} - \partial_i \Gamma^\alpha_{j\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{ij} - \Gamma^\alpha_{\sigma j}\Gamma^\sigma_{i\alpha} \\
&= \partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{jk} + \Gamma^k_{0k}\Gamma^0_{ij} + \Gamma^k_{lk}\Gamma^l_{ij} \\
&\quad - \Gamma^k_{0j}\Gamma^0_{ik} - \Gamma^0_{kj}\Gamma^k_{i0} - \Gamma^k_{lj}\Gamma^l_{ik} \\
&= \partial^0\Big(\frac{\dot a}{a} g_{ij}\Big) + 3\Big(\frac{\dot a}{a}\Big)^2 - \Big(\frac{\dot a}{a}\Big) g_{ij} - \Big(\frac{\dot a}{a}\Big) g_{ij} \\
&\quad + \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{jk} + \Gamma^k_{lk}\Gamma^l_{ij} - \Gamma^k_{lj}\Gamma^l_{ik} \\
&= \Big(\frac{\ddot a}{a} + 2\Big(\frac{\dot a}{a}\Big)^2\Big) g_{ij} + \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{jk} + \Gamma^k_{lk}\Gamma^l_{ij} - \Gamma^k_{lj}\Gamma^l_{ik} \\
&= \Big(\frac{\ddot a}{a} + 2\Big(\frac{\dot a}{a}\Big)^2 + 2\frac{k}{a^2}\Big) g_{ij}. \qquad (2.28)
\end{aligned}
$$

Note that the last line of (2.28) was obtained by calculating the Ricci tensor component by component. Next, the Ricci scalar for the FRW space-time can be calculated as follows:

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
&= g^{00} R_{00} + g^{ij} R_{ij} \\
&= 3\frac{\ddot a}{a} + 3\Big(\frac{\ddot a}{a} + 2\Big(\frac{\dot a}{a}\Big)^2 + 2\frac{k}{a^2}\Big) \\
&= 6\Big(\frac{\ddot a}{a} + \Big(\frac{\dot a}{a}\Big)^2 + \frac{k}{a^2}\Big), \qquad (2.29)
\end{aligned}
$$

Using the above results, we obtain the non-zero components of the Einstein tensor as

$$
\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2} R g_{00} \\
&= 3\Big(\Big(\frac{\dot a}{a}\Big)^2 + \frac{k}{a^2}\Big)\ ; \\
G^0_0 &= -3\Big(\Big(\frac{\dot a}{a}\Big)^2 + \frac{k}{a^2}\Big)\ ; \qquad (2.30) \\
G_{ij} &= R_{ij} - \frac{1}{2} R g_{ij} \\
&= -\Big(2\frac{\ddot a}{a} + \Big(\frac{\dot a}{a}\Big)^2 + \frac{k}{a^2}\Big) g_{ij} \\
G^i_j &= -\Big(2\frac{\ddot a}{a} + \Big(\frac{\dot a}{a}\Big)^2 + \frac{k}{a^2}\Big) \delta^i_j. \qquad (2.31)
\end{aligned}
$$

Having obtained the explicit form of the quantities on the left-hand side of the Einstein equation for the FRW metric, our next step is to consider the quantity on the right-hand side of the Einstein equation which is the energy-momentum tensor $T_{\mu\nu}$. For our convenience, we switch to work in the Cartesian coordinates. The first component, T_{00}, is the energy density, and the momentum density corresponds to the components T_{0i}. For the stress tensor, its components are T_{ij}. Recall that the matter in the universe is assumed to be in the form of the perfect fluid which has no heat conduction and no viscosity. Hence it looks isotropic in its rest frame [5]. The first condition (no heat conduction) makes $T_{0i} = T_{i0} = 0$, and the second one (no viscosity or no shear force) makes $T_{ij} = 0; \quad i \neq j$. Due to the isotropy of the fluid, its energy-momentum tensor must, therefore, be a diagonal matrix with the diagonal components:

$$T^0_0 \;=\; -\rho\,, \qquad T^i_j \;=\; p\,\delta^i_j \tag{2.32}$$

with ρ and p being respectively an energy density and the pressure. Since $T_{\mu\nu}$ is diagonal, its form will not change if we change the coordinates to the spherical coordinates used in the FRW metric. Thus this form of the energy-momentum tensor is ready for use in cosmology.

An important property of the energy-momentum tensor is the vanishing of its divergence, $D_\mu T^\mu_\nu = 0$, where D_μ is the covariant derivative. This property leads to the conservation of the energy, $D_\mu T^\mu_0 = 0$, and the conservation of the momentum, $D_\mu T^\mu_k = 0$. By considering the time component of the conservation equations, we obtain

$$\begin{aligned} D_\mu T^\mu_0 &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\rho} T^\rho_0 - \Gamma^\rho_{\mu 0} T^\mu_\rho \\ &= \partial_0 T^0_0 + \Gamma^k_{k0} T^0_0 - \Gamma^j_{i0} T^i_j \\ &= -\dot\rho - 3\frac{\dot a}{a}\rho - 3\frac{\dot a}{a}p \\ \Rightarrow \qquad \dot\rho &= -3\frac{\dot a}{a}(\rho + p). \end{aligned} \tag{2.33}$$

Using the equation of state, $p = \omega\rho$, we can write the energy density in terms of the scale factor as

$$\rho \;\propto\; a^{-3(1+\omega)}. \tag{2.34}$$

In order to find the parameter ω in the equation of state, we consider the number density of particles in the universe, $n_{\rm como}$. In the comoving frame, $n_{\rm como}$ is constant but, in the physical frame, the number density varies with time due to the scale factor:

$$n_{\rm phys} = \frac{n_{\rm como}}{a^3}. \tag{2.35}$$

It follows from equation (2.35) that, for massive (non-relativistic) particles, the energy density is

$$\rho_m \;=\; m n_{\rm phys} \;\propto\; \frac{1}{a^3}. \tag{2.36}$$

By comparing (2.36) with (2.34), we conclude that $\omega = 0$ for the non-relativistic matter (so-called dust). For the relativistic massless particles such as photons, it is affected by the redshift from the expansion of the universe, and therefore its frequency, ν, will decrease by a factor $1/a$. Thus, the energy density of the radiation takes the form

$$\rho_r \;\propto\; \nu n_{\rm phys} \;\propto\; \frac{1}{a^4}. \tag{2.37}$$

By comparing (2.37) with (2.34), we get $\omega = 1/3$ for the radiation. These properties will be important for the following chapters.

Let us now consider the dynamics of the expanding FRW universe by substituting the energy-momentum tensor and the Einstein tensor into the Einstein equation. We get

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \;=\; \frac{8\pi G}{3}\rho \;, \tag{2.38}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \;=\; -8\pi G p. \tag{2.39}$$

These are known as the Friedmann equations. Combining these two equations, we obtain the Raychaudhuri equation which takes the form

$$\frac{\ddot{a}}{a} \;=\; -\frac{4\pi G}{3}(\rho + 3p). \tag{2.40}$$

These three equations (2.38)-(2.40) encode the dynamics of the FRW metric. Before analyzing the Friedmann equations in order to determine the evolution of the universe, we would like to mention here that the current observational data indicate that the universe is expanding, $\dot{a} > 0$, and flat, $k \rightarrow 0$. Despite this fact, the spherical and hyperbolic spaces will also be considered here. Let us start by considering the flat and hyperbolic spaces. The first Friedmann equation is rewritten as

$$\dot{a}^2 \;=\; \frac{8\pi G}{3}\rho a^2 \;+\; |k|. \tag{2.41}$$

Since the right-hand side of this equation is always positive, then $\dot{a}$ is non-zero. Thus we can conclude that the universe expands forever in these cases, $\dot{a} > 0$. The rate of the expansion can be analyzed by considering the derivative of the ρa^3 with respect to the cosmic time, t,

$$\begin{aligned} \frac{d(\rho a^3)}{dt} \;&=\; a^3\dot{\rho} \;+\; 3a^2\dot{a}\rho \\ &=\; \frac{-3\rho a^2 \dot{a}}{\omega} \;\leq\; 0. \end{aligned} \tag{2.42}$$

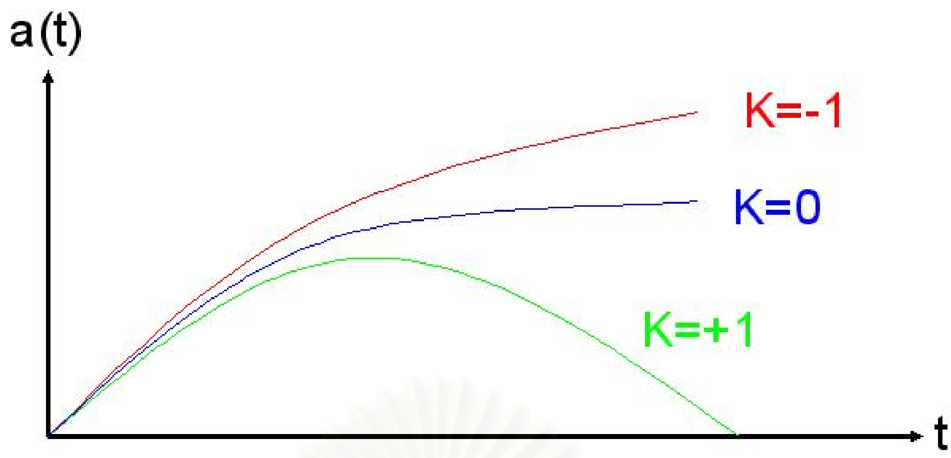

Figure 2.3: Three different evolutions of the universe: the red line ($k = -1$) corresponds to the open universe; the blue line ($k = 0$) corresponds to the flat universe; and the green line ($k = 1$) corresponds to the closed universe.

This implies that ρa^3 is a decreasing function, and so is ρa^2. Thus, if $t \to \infty$ then $\rho a^2 \to 0$, which in turn implies that $\dot{a} \to 1$ for the hyperbolic space and $\dot{a} \to 0$ for the flat space. From this analysis, we conclude that for the hyperbolic space the universe expands forever and faster than the flat space, so we call it is the "open universe." For the flat space, the universe stops expanding at the infinite time, and we call it the "flat universe" as shown in Figure 2.3. For the spherical space, the Friedmann equation can be rewritten as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1. \tag{2.43}$$

By using the similar analysis, it is not hard to see that $\dot{a}$ is a decreasing function, and therefore will vanish at some finite time, t, and then will become more and more negative. This means that the universe will collapse at some time in the future and then continue to contract to the zero size again. This is called the Big Crunch. The universe with this type of evolution is called the "closed universe."

What we have done so far is merely the qualitative analysis. The exact solution for the scale factor can be separately calculated for the matter and radiation cases. Unfortunately, this will not be done here. Anyway, the exact solutions can be found in many textbooks on cosmology, such as [5] and [1].

2.2 Inflationary Models

In the previous section, the Big Bang model or the Friedmann model was analyzed. This model, however, has many problems, and so people have switched to use the inflationary model for explaining the evolution of the early universe. In this section, the problems of the Big Bang model will be discussed in the first part, following by the exposition of the inflationary model. To deeply understand the inflationary model, the dynamics of the inflaton, which is the scalar field responsible for driving the inflation, will be considered in the last part of this section.

2.2.1 The Flatness Problem

In order to discuss the cosmological problems, we first introduce the density parameter Ω,

$$\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G}{3H^2}\rho, \tag{2.44}$$

where ρ_c is the critical energy density which is the energy density in the flat universe, $k = 0$. The evolution of the universe in the FRW model is dictated by the energy density containing in it:

$$k = \begin{cases} +1 \Rightarrow & \Omega > 1 \;\Rightarrow \quad \text{open universe} \\ \;\;0 \Rightarrow & \Omega = 1 \;\Rightarrow \quad \text{flat universe} \\ -1 \Rightarrow & \Omega < 1 \;\Rightarrow \quad \text{closed universe.} \end{cases} \tag{2.45}$$

Next, consider the evolution of the density parameter with respect to the change of the scale factor in the logarithmic scale:

$$\begin{aligned} \frac{d\Omega}{d\ln a} &= a\frac{d\Omega}{da} \\ &= a\frac{H^2}{\rho_c}\,\frac{d}{da}\Big(\frac{\rho}{H^2}\Big) \\ &= \frac{a}{\rho_c}\,\Big(\frac{d\rho}{da} - \frac{2\rho}{H}\frac{dH}{da}\Big) \\ &= \frac{a}{\rho_c}\,\Big[\frac{-3(1+\omega)}{a}\rho - \frac{2\rho}{H\dot{a}}\Big(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\Big)\Big] \\ &= \frac{a}{\rho_c}\,\Big[\frac{-3(1+\omega)}{a}\rho - \frac{2\rho}{H\dot{a}}\Big(\frac{-4\pi G}{3}(1+3\omega)\rho - H^2\Big)\Big] \\ &= \Omega\,\Big(-3(1+\omega) + \Omega(1+3\omega) + 2\Big) \\ &= \Omega\,(\Omega - 1)(1+3\omega). \end{aligned} \tag{2.46}$$

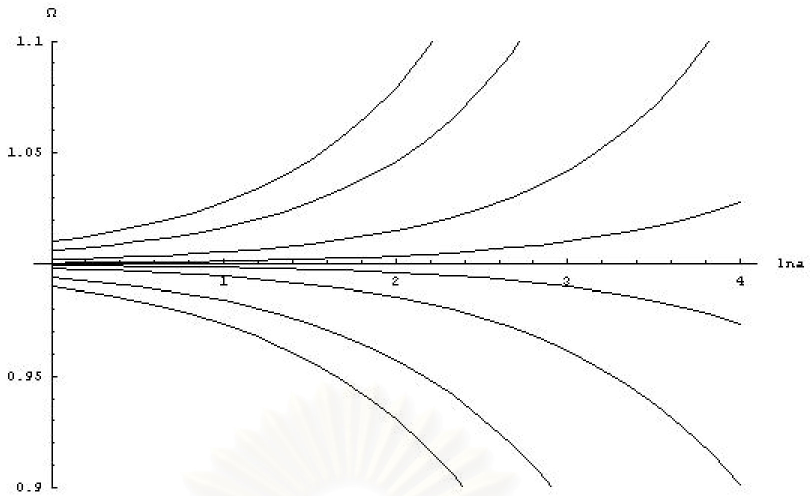

Figure 2.4: The plots of the density parameter as a function of $\ln a$.

This equation implies that the density parameter is deviated away from that of the flat universe if it slightly deviates from one at the early time. For an ordinary matter, this is a repelling behavior of the density parameter which obeys an inequality

$$\frac{d|\Omega - 1|}{d\ln a} \;\; > \;\; 0 \; ; \; 1 + 3\omega > 0. \tag{2.47}$$

This behavior of the density parameter is numerically illustrated in Figure 2.4. There is no problem if the density parameter at the present time is not equal one. But from the recent data of the observation, it indicates that $\Omega = 1.02 \pm 0.05$ [7] which makes the universe at the early time (at the nucleosynthesis time) is extremely flat, $\Omega = 1 \pm 10^{-12}$ [6]. This is a curious behavior of the universe and is referred to as the "flatness problem."

2.2.2 The Horizon Problem

In this subsection, we discuss why the standard Big Bang model cannot describe the thermal equilibrium in the CMB. First, let us consider the horizon size of the universe, d_H, which is defined as a distance that a photon travels from the moment of the Big Bang. This distance can be approximated as $d_H \;\sim\; t \;\sim\; H^{-1}$. Thus any two points, which are separated by more than the horizon size, cannot have ever been in causal contact and therefore cannot be in thermal equilibrium. Next, we consider the physical distance, d_p, between any two points in the sky. This distance is scaled by the scale factor with respect to the comoving distance, and thus can be approximated as $d_p \sim a$. Consider a constant quantity expressed in terms of the ratio of two distances and $|\Omega - 1|$,

$$\left(\frac{d_p}{d_H}\right)^2 |\Omega - 1| = \left(\frac{a}{H^{-1}}\right)^2 \frac{|k|}{a^2 H^2} = \text{const.} \tag{2.48}$$

Thus the derivative of this quantity vanishes,

$$2|\Omega - 1| \frac{d}{d\ln a}\left(\frac{d_p}{d_H}\right) + \left(\frac{d_p}{d_H}\right)^2 \frac{d|\Omega - 1|}{d\ln a} = 0. \tag{2.49}$$

Using the condition (2.47), we obtain a condition which leads to the horizon problem:

$$\frac{d}{d\ln a}\left(\frac{d_p}{d_H}\right) < 0 \ ; \ 1 + 3\omega > 0. \tag{2.50}$$

This equation implies that the physical distance stretches more slowly than the horizon size. This implies that the physical distance between any two points was larger than the horizon size at the early time. However, the photons began to travel freely about 300,000 years after the Big Bang (the decoupling time which will be discussed in the next chapter), so the photons had 300,000 years to be in causal contact. But this 300,000 years corresponds to a degree of the angular distance in the sky which is not large enough to describe the uniform temperature of the CMB radiation that is uniform over the entire sky to one part in 10^5. This shortcoming of the Big Bang model is called the horizon problem. Other shortcomings of this model such as the entropy and magnetic monopole problems can be found in general textbooks [2, 3]. In this thesis, we discuss only the flatness and horizon problems in order to introduce the inflationary model.

2.2.3 The Inflation of the Early Universe

The solution of both flatness and horizon problems can be obtained by considering the conditions (2.47) and (2.50) leading to the flatness and horizon problems, and the Raychuadhuri equation (2.40). The idea of the model, which can solve the problems, is to assume that the universe expanded with an acceleration at some period in the early time [4]. The Raychuadhuri equation tells us that this is possible only if the "matter" in the universe has a negative pressure ($\omega < -1/3$) during that period of time. People call this period the inflation period. This model causes the repelling behavior in Figure 2.4 to change to the attracting behavior during the period of inflation, and thus it is not necessary that the universe needs to be extremely flat before the inflationary period (in other words, the energy density can differ from the critical density); this solves the flatness problem. As for the horizon problem, the rapid expansion of the universe during the inflation period causes any two points that used to be in causal contact (that is, in thermal equilibrium) before the inflation to become far apart more than the horizon size when the inflation ends. This means that any two points that might appear to us that they have never been in thermal contact according to the Big Bang model were indeed in thermal contact at the early time. Thus there is no horizon problem if one assumes the inflation period at the early time.

In order to solve the problems exactly, one needs the solution of the question "how should the universe be expanded?," or in the other words, "how long must the universe be maintained in the inflation period?" Conveniently, one chooses the extreme condition in which $\omega = -1$ during the period of inflation. This condition leads to the constant Hubble parameter (from the Friedmann equations (2.38)-(2.39)) and the constant energy density (from the conservation of energy (2.33)), and the resulting model is called the "de Sitter stage." In order to determine period of the inflation one introduces a new parameter N, called the number of e-folding, which is defined as

$$\begin{aligned} \frac{\dot{a}}{a} &= H_I \\ \ln\left(\frac{a(t_e)}{a(t_i)}\right) &= H_I(t_e - t_i) \\ \frac{a(t_e)}{a(t_i)} &= e^N, \end{aligned} \tag{2.51}$$

where $N = H_I(t_e - t_i)$, and t_e , t_i and H_I denote the ending time, the initial time of inflation and the constant Hubble parameter in the inflation period respectively. The condition that must be satisfied in order to solve the horizon problem is that the physical distance d_p at the initial time of inflation must be smaller than the horizon size $d_H = H_I^{-1}$ during inflation. Using this condition, one obtains

$$\begin{aligned} d_p(t_i) &= d_H(t_0)\Big(\frac{a(t_e)}{a(t_0)}\frac{a(t_i)}{a(t_e)}\Big) < d_H(t_i) \\ &= d_H(t_0)\Big(\frac{T(t_0)}{T(t_e)}e^{-N}\Big) < d_H(t_i) \\ \Rightarrow \quad N &> \ln\Big(\frac{T(t_0)}{T(t_e)}\Big) + \ln\Big(\frac{d_H(t_0)}{d_H(t_i)}\Big) \\ &\sim \ln\Big(\frac{T(t_0)}{T(t_e)}\Big) + \ln\Big(\frac{T^2(t_e)}{T^2(t_0)}\Big) \\ &= \ln\Big(\frac{T(t_e)}{T(t_0)}\Big) \; = \; \ln\Big(\frac{10^{15}\mathrm{GeV}}{10^{-13}\mathrm{GeV}}\Big) \\ N &> 65, \end{aligned} \tag{2.52}$$

where t_0 denotes the present time and $T(t_e)$ is the temperature at the end of inflation. Above, we have used $a \propto T^{-1}$ and $d_H = H^{-1} \sim m_{pl}\rho_R^{-1/2} \sim m_{pl}a^2 \sim m_{pl}T^{-2}$. That the universe must have a negative pressure during inflation implies that inflation is the period in which the vacuum energy dominates. This statement can be understood by imagining that, during that period, the universe expands very rapidly and thus dilutes all particles in the universe, hence the vacuum energy dominates eventually.

An important thing that needs to be mentioned is that, if we assume that the universe is in the de Sitter stage with $\omega = -1$ during inflation, then the universe must rapidly expand forever because ω is a constant value. Such a situation surely cannot occur in reality, otherwise the universe would be filled with the vacuum energy forever and there would be no matter that we see around us nowadays. The way out of this difficulty is that the universe has to be approximately de Sitter during inflation and the de Sitter characteristic of the space-time dies away at later time. It turns out that, to achieve this, a scalar field called an inflation with appropriate dynamics is needed to drive inflation. This will be discussed in the next subsection.

2.2.4 Dynamics of Inflation

As we have mentioned in the previous subsection, the scalar field is the best choice as the driving source of inflation. Generally, one chooses a real scalar field, φ, which is coupled to the gravity and has the potential $V(\varphi)$. This real scalar field is called an "inflaton," and the action can be written as

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\mathcal{L} \\
&= -\int d^4x\sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\Big]. \qquad (2.53)
\end{aligned}
$$

To obtain the equation of motion, we perform a variation of this action with respect to the inflation field and set it to zero:

$$
\begin{aligned}
\delta S &= -\int d^4x\sqrt{-g}\Big[g^{\mu\nu}\partial_\mu\varphi\delta(\partial_\nu\varphi) + \frac{\delta V(\varphi)}{\delta\varphi}\delta\varphi\Big] \\
0 &= -\int d^4x\left(\partial_\nu\Big[g^{\mu\nu}\sqrt{-g}\partial_\mu\varphi\delta\varphi\Big] - \Big[\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu\varphi) - \sqrt{-g}\frac{\delta V(\varphi)}{\delta\varphi}\Big]\delta\varphi\right) \\
0 &= \frac{1}{\sqrt{-g}}\partial_\nu(g^{\mu\nu}\sqrt{-g}\partial_\mu\varphi) - \frac{\delta V(\varphi)}{\delta\varphi}. \qquad (2.54)
\end{aligned}
$$

Thus the equation of motion of the inflaton field is

$$
\ddot{\varphi} + 3H\dot{\varphi} + \partial_\varphi V(\varphi) = 0. \qquad (2.55)
$$

This equation of motion can be written in terms of the conformal time as

$$
\varphi'' + 2\mathcal{H}\varphi' + a^2\partial_\varphi V(\varphi) = 0. \qquad (2.56)
$$

In the above equations, $\nabla^2\varphi$ vanishes due to the fact that the scalar field is homogeneous and isotropic. Next, we consider the energy-momentum tensor in the universe. By assuming that the inflaton dominates at the early time, we get

$$
\begin{aligned}
T_{\mu\nu} &= \partial_\mu\varphi\,\partial_\nu\varphi + \mathcal{L}g_{\mu\nu} \\
&= \partial_\mu\varphi\,\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + V(\varphi)\right), \qquad (2.57) \\
T^\mu_\nu &= g^{\mu\rho}T_{\rho\nu} \\
&= g^{\mu\rho}\,\partial_\rho\varphi\,\partial_\nu\varphi - g^{\mu\rho}\,g_{\rho\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + V(\varphi)\right). \qquad (2.58)
\end{aligned}
$$

For the (00)-component, one obtains

$$
\begin{aligned}
T^0_0 &= g^{0\rho}T_{\rho 0} \\
-\rho &= g^{0\rho}\,\partial_\rho\varphi\,\partial_0\varphi - g^{0\rho}\,g_{\rho 0}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi + V(\varphi)\right) \\
-\rho &= -\dot{\varphi}^2 - \Big(-\frac{1}{2}\dot{\varphi} + \frac{1}{2}a^{-2}\nabla^2\varphi + V(\varphi)\Big) \\
\rho &= \frac{1}{2}\dot{\varphi} + V(\varphi), \qquad (2.59)
\end{aligned}
$$

and, by considering the (ij)-components, one gets

$$
\begin{aligned}
T^i_j &= g^{i\rho}T_{\rho j} \\
p\ \delta^i_j &= g^{i\rho}\,\partial_\rho\varphi\,\partial_j\varphi\ -g^{i\rho}\,g_{\rho j}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi\ +\ V(\varphi)\right) \\
p\ \delta^i_j &= a^{-2}\delta^i_j\nabla^2\varphi-\delta^i_j\Big(-\frac{1}{2}\dot{\varphi}+\frac{1}{2}a^{-2}\nabla^2\varphi+V(\varphi)\Big) \\
p &= \frac{1}{2}\dot{\varphi}-V(\varphi).
\end{aligned}
\tag{2.60}
$$

From the condition of inflation $\rho+3p<0$, one obtains

$$
\begin{aligned}
\rho &< -3p \\
\frac{1}{2}\dot{\varphi}+V(\varphi) &< -3\Big(\frac{1}{2}\dot{\varphi}-V(\varphi)\Big) \\
\dot{\varphi} &< V(\varphi).
\end{aligned}
\tag{2.61}
$$

This is the condition for dynamical inflation which results in the so-called "quasi-de Sitter stage" of the universe. In order to recover the de Sitter stage, one takes an extreme limit $\dot{\varphi}\to 0$ (and therefore the inflation field is approximately constant) so that $\rho=-p$. In this limit, the Friedmann equation takes the form

$$
\begin{aligned}
\Big(\frac{\dot{a}}{a}\Big)^2 = H^2 &= \frac{1}{3m_{pl}^2}\Big(\frac{1}{2}\dot{\varphi}^2+V(\varphi)\Big) \\
&= \frac{V(\varphi)}{3m_{pl}^2}=\text{constant}.
\end{aligned}
\tag{2.62}
$$

Note that we have considered the Friedmann equation in the flat space, $k=0$. In this de Sitter limit, the potential is constant and so there is no dynamics of inflation. In the real situation, the universe has to be in the quasi-de Sitter stage, and one allows the potential to depend on time with the condition that $(1/2)\dot{\varphi}^2 \ll V(\varphi)$ (so as to make the universe almost de Sitter) for a sufficiently long period of time. The physical meaning of this condition is that the inflaton slow-rolls in the flat potential and this implies the domination of the friction term in the equation of motion. Thus, one can approximate the equation of motion as

$$
3H\dot{\varphi}+\partial_\varphi V(\varphi)\ \simeq\ 0, \tag{2.63}
$$

and the Friedmann equation in the quasi-de Sitter stage can be approximated as

$$
H^2(t)\ =\ \frac{V[\varphi(t)]}{3m_{pl}^2}. \tag{2.64}
$$

This is known as the "slow-roll approximation." At this point, it is appropriate to defined some appropriate parameters, known as the slow-roll parameters, relevant to the dynamics of inflation. The first one is

$$
\epsilon=-\frac{\dot{H}}{H^2}. \tag{2.65}
$$

This slow-roll parameter can be written in terms of the potential of the inflaton by differentiating H^2 with respect to the cosmic time, and then using the slow-roll condition (2.63):

$$\begin{aligned}\epsilon(\varphi) &= -\frac{\dot{H}}{H^2} \\ &= -\frac{1}{2H^3}\frac{dH^2}{dt} \\ &= -\frac{V'(\varphi)\dot{\varphi}}{6m_{pl}^2H^3} \\ &= \frac{1}{2m_{pl}^2}\frac{\dot{\varphi}^2}{H^2}, \qquad (2.66) \\ &= \frac{m_{pl}^2}{2}\Big(\frac{V'(\varphi)}{V(\varphi)}\Big)^2. \qquad (2.67)\end{aligned}$$

The importance of this parameter can be seen if one notices that the second derivative of the scale factor can be expressed as

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon(\varphi)). \qquad (2.68)$$

This equation implies that the de Sitter stage corresponds to $\epsilon = 0$. For the quasi-de Sitter stage, $\epsilon < 1$ and the change of the slow-roll parameter depends on the shape of the potential. Generally, the inflation starts with the inflaton slowly rolling in the flat potential and this corresponds to $\epsilon \to 0$. After that the potential is no longer flat, the kinetic term of the inflaton dominates and so $\epsilon \to 1$, which corresponds to the period that inflation stops. After inflation ends, the inflaton oscillates about the minimum of the potential well. Then the inflaton decays and creates the ordinary particles; this makes the universe thermalized. This is known as the "reheating period" [8]. Another slow-roll parameters can be calculated in the similar way and are defined by

$$\eta(\varphi) = m_{pl}^2\Big(\frac{V''(\varphi)}{V(\varphi)}\Big) = \frac{1}{3}\Big(\frac{V''(\varphi)}{H^2}\Big), \qquad (2.69)$$

$$\delta(\varphi) = \eta(\varphi) - \epsilon(\varphi) = -\frac{\ddot{\varphi}}{H\dot{\varphi}}. \qquad (2.70)$$

These slow-roll parameters can be written in terms of the conformal time as

$$\epsilon(\varphi) = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{1}{2m_{pl}^2}\Big(\frac{\varphi'}{\mathcal{H}^2}\Big), \qquad (2.71)$$

$$\delta(\varphi) = \eta(\varphi) - \epsilon(\varphi) = 1 - \frac{\varphi''}{\mathcal{H}\varphi'}. \qquad (2.72)$$

Finally, by using (2.51) and (2.64), the number of e-fold can be written as

$$N = \frac{1}{m_{pl}^2}\int_{\varphi_f}^{\varphi_i}\frac{V(\varphi)}{V'(\varphi)}d\varphi. \qquad (2.73)$$

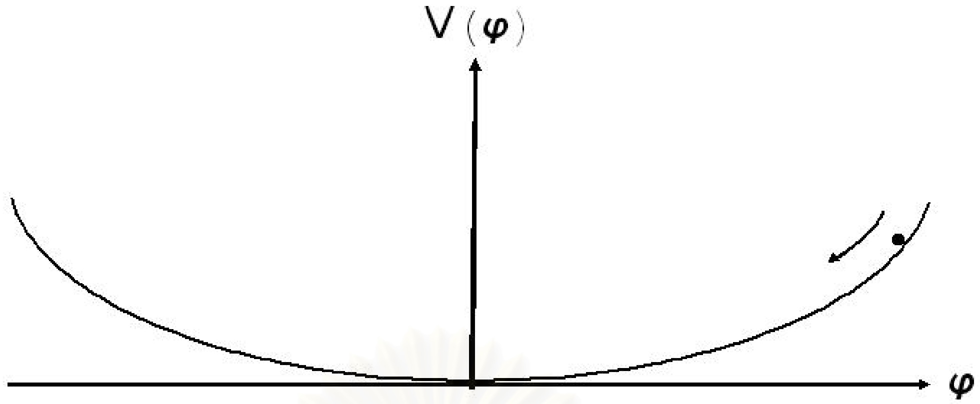

Figure 2.5: The dynamics in the large-field model.

The number of e-fold, which must be greater than 65, will constrain the potential. We will calculate it in the next subsection.

2.2.5 Classification of Inflationary Models

Since the dynamics of the inflaton field depend on the shape of the potential, $V(\phi)$, we can classify the dynamics into three classes. They are the large-field model, the small-field model and the hybrid model.

- The large-field model

In the large-field model, the initial value of the inflaton is assumed to be large. We consider the situation where the inflaton evolves from the initial large value to a small value as illustrated in Figure 2.5. The generic potential of this type is

$$V(\varphi) = \Lambda^4 \left(\frac{\varphi}{\mu}\right)^p, \tag{2.74}$$

where p is an integer number. The shape of the potential in (2.74) is controlled by two mass-dimensional parameters Λ and μ. The parameter Λ corresponds to the vacuum energy density during inflation, while the parameter μ gives us the width of the potential and corresponds to the change of the inflaton field $\Delta\varphi$. As an example, we consider the quadratic inflaton potential with $\Lambda = \mu = m/\sqrt{2}$ where m is the mass of the inflaton. Thus the potential takes the form

$$V(\varphi) = \frac{1}{2}m^2\varphi^2. \tag{2.75}$$

From this potential, equations (2.63) and (2.64) become

$$H^2 = \frac{m^2\varphi^2}{6m_{pl}^2} \tag{2.76}$$

and

$$3H\dot{\varphi} + m^2\varphi \simeq 0. \tag{2.77}$$

Using H in (2.76), we can solve (2.77) to get

$$\varphi \cong \varphi_i - \sqrt{\frac{2}{3}} m m_{pl} t. \tag{2.78}$$

By substituting this solution back into (2.76), the scale factor takes the form

$$a \cong a_i \exp\left[\frac{m}{\sqrt{6}m_{pl}}\left(\varphi_i t - \sqrt{\frac{2}{3}} m m_{pl} t^2\right)\right]. \tag{2.79}$$

We now have three parameters, φ_f, φ_i and m, to determine. First, let us consider the slow-roll parameters which, in this case, take the form

$$\epsilon = \eta = \frac{2m_{pl}^2}{\varphi^2}. \tag{2.80}$$

The inflation ends when $\epsilon = 1$, so the value of the inflaton at the end of inflation reads $\varphi_f = \sqrt{2}m_{pl}$. After this, the inflaton will oscillate and will enter the reheating period. Second, the initial inflaton field can be determined from the number of e-fold which takes the form

$$N = \frac{(\varphi_i^2 - 2m_{pl}^2)}{4m_{pl}^2} > 65. \tag{2.81}$$

This constraint constrains the initial value of the inflaton $\varphi_i > 16m_{pl}$. This means that the initial value of the inflaton field is distributed chaotically, and so people call this type of model as the "chaotic inflationary model." Unfortunately, there is no particle physics motivation for this potential [2]. However, this model is useful for studying inflation. The mass of the inflaton can be determined by analyzing the data from observation [2] which yields the value of $m = 1.8 \times 10^{13}\text{GeV} = 10^{-6}m_{pl}$.

- The small-field model

In opposite to the large-field model, the small-field model is the model in which the inflaton initially rolls down the potential from the small value of φ to the large value of φ where the potential is minimum. The shape of the potential

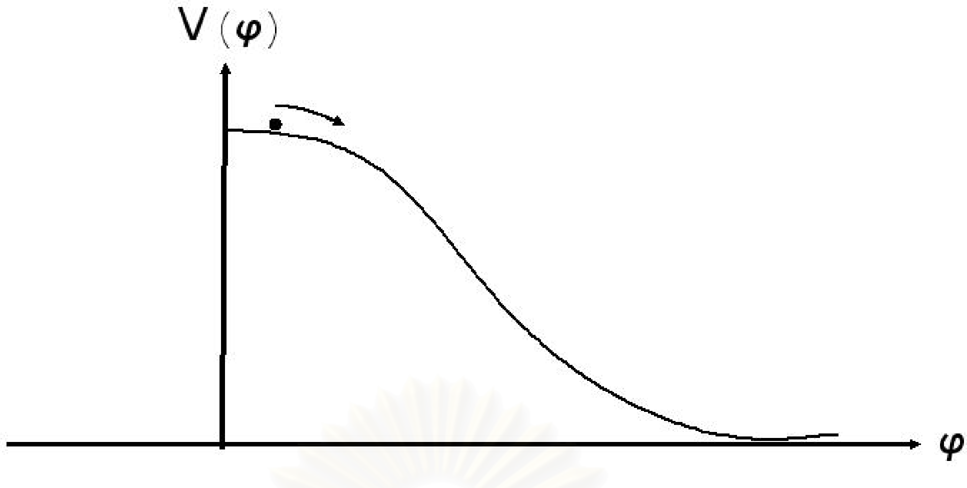

Figure 2.6: The dynamics of the small-field model.

for the small-field model is illustrated in Figure 2.6. The generic potential of this type can be written as

$$V(\varphi) \;=\; \Lambda^4\Big(1-(\varphi/\mu)^p\Big). \tag{2.82}$$

The popular potential in this case is the potential which gives the scale factor proportional to some power of the cosmic time, $a = a_0 t^p \;\; ; p > 1$. The model with this potential is called the "power-law inflation." The advantage of this potential is that the equation for the generation of the density perturbation can be solved exactly. In order to find the potential, one considers the Raychuadhuri and Friedmann equations which can be written as

$$\dot{H} \;=\; -\frac{\dot{\varphi}^2}{2m_{pl}^2} = -\frac{p}{t^2}, \tag{2.83}$$

where $H = p/t$. From this equation, one obtains the cosmic time as

$$t \;=\; \exp\Big(\frac{\varphi}{\sqrt{2p}\; m_{pl}}\Big). \tag{2.84}$$

By using the Friedmann equation and equation (2.83), one obtains

$$\begin{aligned} H^2 &= \frac{1}{3m_{pl}^2}\Big(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\Big) \\ \Rightarrow \quad V(\varphi) &= 3m_{pl}^2\frac{p^2}{t^2} - \frac{1}{2}\dot{\varphi}^2 \\ &= m_{pl}^2\Big(\frac{(3p^2-p)}{t^2}\Big) \\ &= V_0 \exp\Big(-\sqrt{\frac{2}{p}}\;\frac{\varphi}{m_{pl}}\Big), \end{aligned} \tag{2.85}$$

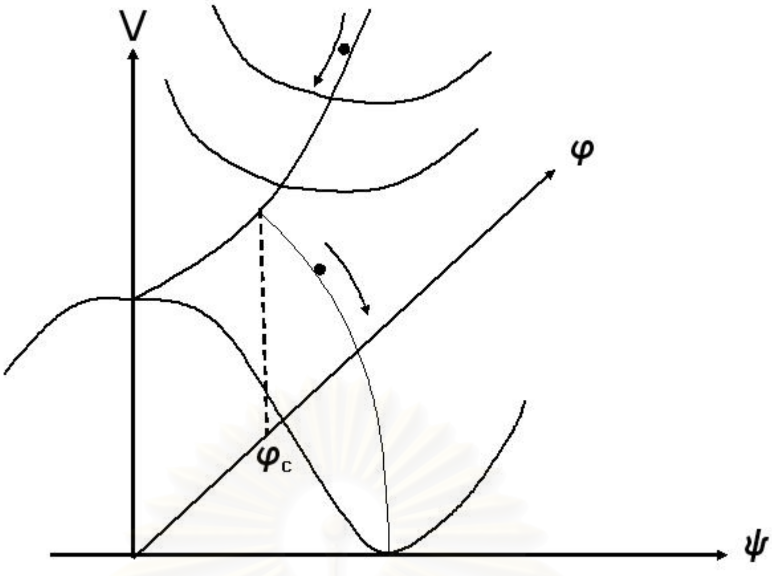

Figure 2.7: The dynamics of the hybrid model.

where $V_0 = m_{pl}^2(3p^2 - p)$. From this potential the slow-roll parameters can be determined immediately:

$$\epsilon = \frac{1}{p}; \qquad \eta = \frac{2}{p}. \tag{2.86}$$

This slow-roll parameters are constant; this implies that the inflation will never end. This is the disadvantage of the power-law inflation. However, as we have derived above, we are not concerned with the slow-roll approximation $\dot{\varphi}^2 \ll V(\varphi)$. Then one can exactly solve the equation for the generic perturbation and this is the advantage of this model.

- The hybrid model

Both two models that we discussed above are the single-field model. We now present the model with the multiple-field potential. Indeed, this model contains only two scalar fields, whose dynamics are depicted in Figure 2.7. As shown this figure, one of the fields, φ, is responsible for the inflationary stage (similar to the large-field model), but the end of inflation is not at the origin. The other one, ψ, is responsible for the end of inflation, where the inflaton rapidly rolls down to the true minimum of the potential; this stage is just like the small-field model. Therefore, people call this model the hybrid model. However, if the inflaton slowly rolls in the second step, one has two stages of inflation and the model is called the "double-inflation model." An example of the potential for the hybrid model takes the form

$$V = \frac{\lambda}{4}\Big(\psi^2 - \frac{M^2}{\lambda}\Big)^2 + \frac{1}{2}g^2\varphi^2\psi^2 + \frac{1}{2}m^2\varphi^2. \tag{2.87}$$

The transition between these two phases occurs when the sign of the mass parameter of ψ changes, which occurs when $g^2\varphi^2 - M^2 = 0$. Thus one can write this critical point as $\varphi_c = M/g$. The inflation stage corresponds to $\varphi > \varphi_c$ and stops when $\varphi < \varphi_c$. Thus the number of e-fold is determined by considering only the first part of evolution ($\varphi > \varphi_c$), and takes the form

$$N \simeq \frac{M^4}{4m^2 m_{pl}^2}\ln\frac{\varphi_i}{\varphi_c}. \tag{2.88}$$

In order to find the parameters in this model, one needs the exhaustive observation which is quite complicated because there are more parameters in the potential; this is a drawback of this model. However, some drawbacks of the single-field model are absent in this model, and moreover this model satisfies the particle theory due to the occurrence of the symmetry breaking in the end of inflation. The detail calculation is skipped due to its complication. However, a nice discussion of this calculation can be found in [2].

CHAPTER III

COSMOLOGICAL PERTURBATION

As we mentioned in the previous chapter, the inflationary model not only solves the flatness and the horizon problems, but also provides us the mechanism of structure formations. The structure formations are originated from quantum fluctuations in the microscopic scales. Then the inflation magnifies them to be in the macroscopic scales and become a seed of the structures that we observe today. In this chapter we will determine the power spectrum by using the theory of cosmological perturbation, and the result will be compared with the observation in the last part of this chapter. For simplicity, we will discuss the fluctuations of a single scalar field in the first section. Next, we will offer some ideas of the metric perturbation in the second section. In the third section, we will determine the power spectrum of the scalar perturbation and the amplitude of the tensor perturbation by perturbing the Einstein field equation.

3.1 Scalar Field Fluctuations

Our goal of this section is to find the power spectrum of the generic fluctuation fields, which actually are the inflaton fields. Thus, in the first part of this section, we will define the power spectrum of the generic fluctuations. We then calculate the power spectrum of the fluctuation fields. The basic idea of this calculation which we are going to do can be applied to determine the power spectrum of the metric perturbation amplitudes.

3.1.1 The Power Spectrum of Generic Fluctuation Fields

The power spectrum is an important quantity which characterizes the properties of the perturbations. According to quantum field theory, a scalar field, $\varphi(\vec{x}, t)$, can be quantized by replacing it with the field operator $\hat{\varphi}(\vec{x}, t)$. In analogy with quantum theory, the field operator obeys the equal-time commutation relations which take the form

$$[\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}'), \tag{3.1}$$

$$[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{x}',t)] = 0, \tag{3.2}$$

$$[\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{x}',t)] = 0, \tag{3.3}$$

where $\hat{\pi}(\vec{x},t)$ is the conjugate momentum of $\hat{\phi}(\vec{x},t)$. Moreover, the field operator can be expanded in terms of the creation and the annihilation operators, $\hat{a}^\dagger$ and $\hat{a}$, respectively:

$$\hat{\phi}(\vec{x},t) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^{3/2}}\Big(\phi_k(t)e^{ikx}\hat{a}_k + \phi_k^\dagger(t)e^{-ikx}\hat{a}_k^\dagger\Big). \tag{3.4}$$

Generally, the power spectrum of $\mathcal{P}(k)$ is defined by

$$\langle 0 \mid \phi^2(\vec{x},t) \mid 0\rangle = \int \frac{dk}{k}\mathcal{P}(k). \tag{3.5}$$

The quantity on the left-hand side can be calculated by using the properties of the creation and annihilation operators such as $\hat{a}_k \mid 0\rangle = 0$ and $\langle 0 \mid \hat{a}_k^\dagger \hat{a}_{k'} \mid 0\rangle = \delta^3(k-k')$. The result is

$$\langle 0 \mid \phi^2(\vec{x},t) \mid 0\rangle = \int \frac{dk}{k}\frac{k^3}{2\pi^2} \mid \phi_k(t) \mid^2 . \tag{3.6}$$

Therefore, the power spectrum can be written as

$$\mathcal{P}_\phi(k) = \frac{k^3}{2\pi^2} \mid \phi_k(t=t_k) \mid^2, \tag{3.7}$$

where t_k is the crossing time which we will discuss later. Moreover, the quantities relevant to the observation are the spectral index, n, and the running of the spectral index, r. The spectral index for the scalar perturbation is defined as

$$n_s = 1 + \frac{d\ln \mathcal{P}_s}{d\ln k}, \tag{3.8}$$

and the spectral index for the tensor perturbation is defined by

$$n_T = \frac{d\ln \mathcal{P}_T}{d\ln k}. \tag{3.9}$$

The running of the spectral index can be defined as

$$r_{s,T} = \frac{dn_{s,T}}{d\ln k}, \tag{3.10}$$

where the subscripts s and T denote the scalar perturbation and the tensor perturbation, respectively.

3.1.2 Dynamics of Scalar Fluctuation Fields

We begin this subsection by considering the action of a free scalar field which takes the form

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\mathcal{L} \\ &= -\int d^4x\sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\Big]. \end{aligned} \tag{3.11}$$

It is convenient to use the conformal time as the time coordinate. To consider a small fluctuation of φ, we let $\varphi \to \varphi + \delta\varphi = \varphi + \phi$, where ϕ is called the fluctuation field. Then, the perturbed action can be written as

$$\begin{aligned} S[\varphi+\phi] &= S[\varphi] + S[\phi] \\ &= S[\varphi] + \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\partial^2 V(\varphi)}{\partial\varphi^2}\phi^2\Big). \end{aligned} \tag{3.12}$$

Actually, the second term in the integral above can be interpreted as the mass term which is of the same order as the slow-roll parameter, $V''(\varphi) = 3\eta_\phi H^2$. For convenience, one considers the massless scalar field, and so this term can be neglected. However, if one considers the massive scalar field, one can put this term back in the last step. In (3.12), we treat φ as a classical background field, and the fluctuation field ϕ as a quantum field. Thus by using the Euler-Lagrange equation, the equation of motion takes the form

$$\phi'' + \frac{2a'}{a}\phi' - \nabla^2\phi = 0. \tag{3.13}$$

Note that this equation of motion can also be obtained by directly substituting $\varphi(\vec{x},\eta) = \varphi_0(\eta) + \phi(\vec{x},\eta)$ into the equation of motion of the inflaton field. In order to find the power spectrum, we must quantize this fluctuation field. By using (3.4), we can consider ϕ as a function of the conformal time:

$$\phi_k'' + \frac{2a'}{a}\phi_k' + k^2\phi_k = 0. \tag{3.14}$$

Let us consider the qualitative behavior of the solution to this equation. Since this equation looks complicated, we therefore consider two extreme cases with respect to the wavelength of perturbation, λ.

For the first case, $k \gg aH$ and this corresponds to the perturbation that has wavelength much smaller than the horizon size, $\lambda \ll H^{-1}$. In this regime, the friction term (second term) can be neglected and so equation (3.14) is reduced to

$$\phi_k'' + k^2\phi_k = 0. \tag{3.15}$$

The solution of this equation can be easily obtained and takes the form $\phi_k \sim \exp(ik\eta)$. This means that the fluctuation modes are the oscillating modes as long as their wavelengths are smaller than the horizon size.

For the second case, $k \ll aH$ and this corresponds to the fluctuation modes with wavelengths much larger than the horizon size, $\lambda \gg H^{-1}$. Thus, one can neglect the effect of the third term, and so equation (3.14) is reduced to

$$\phi_k'' + 2aH\phi_k' = 0. \tag{3.16}$$

The solution of this equation is simply a constant. This implies that these fluctuation modes are frozen.

From this analysis, we can conclude that, as the scale factor increases faster than the horizon during inflation, the fluctuation modes, whose wavelengths are smaller than the horizon size, oscillate until their wavelengths are of the same order as the horizon size, these fluctuation modes then cease to oscillate and become frozen. After the inflation ends, the horizon expands faster than the scale factor. Thus, the fluctuations are frozen at some specific time, called the "crossing time." At some time after that, the fluctuations will reenter the horizon and oscillate again. Since one considers the fluctuation modes that reenter the horizon at the decoupling time, then these fluctuations will make the perturbations of the matter density and we observe them as a CMB anisotropies nowadays.

Next, we determine the exact solution of equation (3.14). For convenience, we let $u_k = a\phi_k$. Then the equation of motion changes to

$$u_k'' + \Big(k^2 - \frac{a''}{a}\Big)u_k = 0. \tag{3.17}$$

By using the relation from the quasi de Sitter stage (2.65),

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = 1 - \epsilon, \tag{3.18}$$

the factor a''/a in (3.17) can be written as

$$\begin{aligned} \frac{a''}{a} &= \mathcal{H}' + \mathcal{H}^2 \\ &= \mathcal{H}^2\Big(2 - \epsilon\Big). \end{aligned} \tag{3.19}$$

Using equation (3.18) again, one obtains

$$\begin{aligned} d\mathcal{H} &= (1-\epsilon)\mathcal{H}^2 d\eta, \\ \mathcal{H} &= \frac{-1}{(1-\epsilon)}\frac{1}{\eta}, \\ \mathcal{H}^2 &= (1+2\epsilon)\frac{1}{\eta^2}. \end{aligned} \tag{3.20}$$

By substituting the above form of $\mathcal{H}$ into (3.19), we find

$$\frac{a''}{a} = \frac{1}{\eta^2}(2+3\epsilon) = \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big), \tag{3.21}$$

where $\nu = 3/2 + \epsilon$. Thus, the equation of motion becomes

$$u_k'' + \Big(k^2 - \frac{1}{\eta^2}(\nu^2 - 1/4)\Big)u_k = 0. \tag{3.22}$$

By considering the differential equation of the Bessel function,

$$x^2\frac{d^2}{dx^2}J_\nu(kx) + x\frac{d}{dx}J_\nu(kx) + (x^2k^2 - \nu^2)J_\nu(kx) = 0, \tag{3.23}$$

it is not hard to verify that the solution of equation (3.22) takes the form

$$u_k = \sqrt{-k\eta}\Big(A_kJ_\nu(-k\eta) + B_kN_\nu(-k\eta)\Big), \tag{3.24}$$

where $N_\nu(-k\eta)$ is the Neumann function. The constants A_k and B_k can be found by using the boundary conditions corresponding to the two limits discussed above. For the case $k \gg aH$ which corresponds to $-k\eta \gg \nu$, the fluctuation modes take the form

$$\begin{aligned} u_k &= \sqrt{\frac{2}{\pi}}\Big(A_k\cos(-k\eta - \nu\pi/2 - \pi/4) + B_k\sin(-k\eta - \nu\pi/2 - \pi/4)\Big) \\ &\sim e^{-ik\eta}. \end{aligned} \tag{3.25}$$

This implies $B_k = iA_k$. Thus, the general form of u_k can be written as

$$\begin{aligned} u_k &= \sqrt{-k\eta}C_ke^{i\frac{\pi}{2}(\nu+1/2)}\Big(J_\nu(-k\eta) + iN_\nu(-k\eta)\Big), \\ &= \sqrt{-k\eta}C_ke^{i\frac{\pi}{2}(\nu+1/2)}\Big(H_\nu^{(1)}(-k\eta)\Big), \end{aligned} \tag{3.26}$$

where $H_\nu^{(1)}(-k\eta)$ is the first kind Hankel's function and C_k are some constants that can be found by using the normalization in (3.1). Indeed, from the action of the fluctuation field in (3.12), the conjugate momentum can be written as

$$\begin{aligned} \hat{\pi}(\vec{x},\eta) &= a^2\frac{d\hat{\phi}(\vec{x},\eta)}{d\eta} \\ &= \int\frac{dk}{(2\pi)^{3/2}}a^2\Big(\phi_k'e^{ikx}\hat{a}_k + \phi_k'^{\dagger}e^{-ikx}\hat{a}_k^{\dagger}\Big). \end{aligned} \tag{3.27}$$

By calculating the commutator in (3.1), using the commutator of annihilation and creation operators,

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = \delta^3(\vec{k} - \vec{k}'), \tag{3.28}$$

and the property of the Bessel function,

$$J_\nu(x)N'_\nu(x) - J'_\nu(x)N_\nu(x) = \frac{2}{\pi x}, \tag{3.29}$$

one finds that C_k takes the form $\sqrt{\pi/4k}$. Thus, the exact solution can be expressed as

$$\phi_k = \sqrt{\pi/4}e^{i\frac{\pi}{2}(\nu+1/2)}\frac{\sqrt{-\eta}}{a}H^{(1)}_\nu(-k\eta). \tag{3.30}$$

With this solution, one finds the power spectrum of the inflaton field as

$$\mathcal{P}_\phi(k) = \frac{k^3(-\eta_k)}{8\pi a^2(\eta_k)}\Big(J^2_\nu(-k\eta_k) + N^2_\nu(-k\eta_k)\Big). \tag{3.31}$$

In this form, one cannot find the spectral index and the running spectral index. However, these parameters can be determined by using the adiabatic approximation. By doing that, it appears that the slow-roll parameters adiabatically change with time and the crossing time can be approximated as $-k\eta_k \to 0$. Thus the fluctuation modes take the form

$$\phi_k = \frac{e^{i\frac{\pi}{2}(\nu-1/2)}}{\sqrt{2k}}\frac{2^{\nu-3/2}}{a}\frac{\Gamma(\nu)}{\Gamma(3/2)}(-k\eta)^{-\nu}, \tag{3.32}$$

where we have used the asymptotic form of Hankel's function,

$$H^{(1)}_\nu(x \ll 1) = \sqrt{2/\pi}e^{-i\frac{\pi}{2}}2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}x^{-\nu}. \tag{3.33}$$

With this result, the power spectrum takes the form

$$\mathcal{P}_\phi(k) = Ak^{-2\epsilon}, \tag{3.34}$$

where

$$A = \frac{2^{2\epsilon}}{\pi^2}\frac{\Gamma^2(3/2+\epsilon)}{\Gamma^2(3/2)}(1-\epsilon)^{2(1+\epsilon)}(aH)^{2\epsilon}H^2, \tag{3.35}$$

and the spectral index can be written as $n = 1 - 2\epsilon$. Furthermore, the power spectrum and the spectral index of the massive scalar fields can be determined by replacing ν^2 with $\nu_m^2 = \nu^2 - 3\eta_\phi$ and $\nu_m = 3/2 + \epsilon - \eta_\phi$. Thus, the power spectrum and the spectral index can be expressed as

$$\mathcal{P}_{m\phi}(k) = A_m k^{2(\eta_\phi - \epsilon)}, \tag{3.36}$$

where

$$A_m = \frac{2^{2(\epsilon-\eta_\phi)}}{\pi^2}\frac{\Gamma^2(3/2+\epsilon-\eta_\phi)}{\Gamma^2(3/2)}(1-\epsilon)^{2(1+\epsilon-\eta_\phi)}(aH)^{2(\epsilon-\eta_\phi)}H^2, \tag{3.37}$$

and $n_m = 1 - 2\epsilon + 2\eta_\phi$.

3.2 The Metric Perturbation

As we have mentioned in the previous section, the structures that we can observe nowadays are generated from the quantum fluctuations which become classical perturbations by the inflation. Moreover, we have shown that the generic scalar field can fluctuate in the inflation period. Since, the inflaton dominates in that period, then the generic scalar field is the inflaton field. Therefore, in this subsection, we will show that the inflaton field also causes the perturbation in the curvature through the Einstein equation, $\delta\varphi \Rightarrow \delta T_{\mu\nu} \Rightarrow \delta g_{\mu\nu}$. On the other hand, the perturbation of the curvature will also generate the fluctuation in the inflaton field, $\delta g_{\mu\nu} \Rightarrow \delta\varphi$. This coupling between the perturbations of curvature and scalar field, $\delta g_{\mu\nu} \Leftrightarrow \delta\varphi$, allows us to determine both the metric and inflaton field perturbations at the same time.

The metric fluctuations can be considered in the same way as the scalar field fluctuations in that one expresses the metric as a linear combination of the background metric (FRW metric) and the small fluctuation metric,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(t) + g_{\mu\nu}(\vec{x}, t). \tag{3.38}$$

Generally, the metric perturbation can be decomposed into three parts, namely, the scalar, the vector, and the tensor perturbations, according to their spins. In this thesis, we will consider only the scalar and the tensor perturbations. The vector perturbation can be neglected because it corresponds to the rotational-velocity fields which are not excited during the inflation stage.

3.2.1 Scalar Perturbation

In this subsection, we will determine the power spectrum of the scalar perturbation by perturbing the Einstein equation. Generally, the metric with the scalar perturbation takes the form

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 - 2A & \partial_i B \\ \partial_i B & (1 - 2\psi)\,\delta_{ij} + D_{ij}E \end{pmatrix}, \tag{3.39}$$

where $D_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\,\delta_{ij}\,\nabla^2\right)$, and A, B, ψ, E are the perturbation parameters which have small values. To find the inverse metric, $g^{\mu\nu}$, one writes

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} -1 + x & \partial_i y \\ \partial_i y & (1 + 2\chi)\,\delta^{ij} + D^{ij}z \end{pmatrix}, \tag{3.40}$$

where x, y, z, and χ can be calculated by using the relation

$$g^{\mu\alpha}\, g_{\alpha\nu} \;=\; \left(g^{\mu\alpha}_{(0)} \,+\, g^{\mu\alpha}\right)\,\left(g^{(0)}_{\alpha\nu} \,+\, g_{\alpha\nu}\right) \;=\; \delta^{\mu}_{\nu}\,. \tag{3.41}$$

By considering the (0,0) component, we find

$$\begin{aligned} g^{0\alpha}g_{\alpha 0} &= g^{00}g_{00} + g^{0i}g_{i0} \\ &= (-1+x)(-1-2A) + \partial_i B \partial_i y \\ &= 1 + x + 2A = 1, \\ \Rightarrow \qquad x &= 2A. \end{aligned} \tag{3.42}$$

Note that we kept only the terms of linear order in perturbations in the above calculation. Similarly, the consideration of the $(0,i)$ components gives

$$\begin{aligned} g^{0\alpha}g_{\alpha i} &= g^{00}g_{0i} + g^{0j}g_{ji} \\ &= (-1 \,+\, 2\,A)(\partial_i B) \,+\, \partial^j Y\,[(1 \,-\, 2\,\psi)\delta_{ji} \,+\, D_{ji}E] \\ &= -\partial_j B + \partial^i y \delta_{ij} = 0, \\ \Rightarrow \qquad y &= B. \end{aligned} \tag{3.43}$$

For the (i,j) components, one obtains

$$\begin{aligned} g^{i\alpha}g_{\alpha j} &= g^{i0}g_{0j} + g^{0k}g_{kj} \\ &= \partial^i B\,\partial_j B \,+\, \left((1 \,+\, 2\,\chi)\delta^{ik} \,+\, D^{ik}z\right)\,\left((1-2\,\psi)\delta_{kj} \,+\, D_{kj}E\right) \\ &= (1 \,-\, 2\,\psi \,+\, 2\,\chi)\delta^i_j \,+\, D^i_j E \,+\, D^i_j z \;=\; \delta^i_j, \\ \Rightarrow \qquad \chi &= \psi; \qquad z = -E. \end{aligned} \tag{3.44}$$

Thus, $g^{\mu\nu}$ can be expressed in the form

$$g^{\mu\nu} \;=\; \frac{1}{a^2}\begin{pmatrix} -1 \,+\, 2\,A & \partial^i B \\ \partial^i B & (1 \,+\, 2\,\psi)\delta^{ij} \,-\, D^{ij}E \end{pmatrix}. \tag{3.45}$$

We now determine the perturbed affine connection by using the above metric. The background affine connection takes the form

$$\Gamma^{\alpha}_{\beta\gamma} \;=\; \frac{1}{2}\,g^{\alpha\rho}\,(\partial_\beta g_{\rho\gamma} \,+\, \partial_\gamma g_{\beta\rho} \,-\, \partial_\rho g_{\beta\gamma}) \tag{3.46}$$

with $g_{\mu\nu}$ being the FRW metric. One of the components of this affine connection is

$$\Gamma^0_{00} = \frac{1}{2}\,g^{00}(\partial_0 g_{00} \,+\, \partial_0 g_{00} \,-\, \partial_0 g_{00}) = \frac{1}{2}\frac{1}{a^2}\partial_0 a^2 = \frac{a'}{a}. \tag{3.47}$$

Other components can also be determined in the same way. The results are

$$\Gamma^i_{0j} \;=\; \frac{a'}{a}\,\delta^i_j\,; \qquad \Gamma^0_{ij} \;=\; \frac{a'}{a}\delta_{ij}\,; \tag{3.48}$$

$$\Gamma^i_{00} \quad = \quad \Gamma^0_{0i} \quad = \quad \Gamma^i_{jk} \quad = \quad 0\,. \tag{3.49}$$

We next calculate the affine connection perturbation which takes the form

$$\begin{aligned}\delta\Gamma^\alpha_{\beta\gamma} \quad &= \quad \frac{1}{2}\,\delta g^{\alpha\rho}\left(\partial_\beta g_{\rho\gamma} \,+\, \partial_\gamma g_{\beta\rho} \,-\, \partial_\rho g_{\beta\gamma}\right) \\ &\quad +\frac{1}{2}\,g^{\alpha\rho}\left(\partial_\beta \delta g_{\rho\gamma} \,+\, \partial_\gamma \delta g_{\beta\rho} \,-\, \partial_\rho \delta g_{\beta\gamma}\right),\end{aligned} \tag{3.50}$$

where $\delta g_{\mu\nu}$ represents the metric perturbation, which is the scalar perturbation in this case. Its components are as follows:

$$\begin{aligned}\delta\Gamma^0_{00} \quad &= \quad \frac{1}{2}\Big(\delta g^{00}\partial_0 g_{00} + \delta g^{0i}\partial_0 g_{i0} + \delta g^{00}\partial_0 g_{00} \\ &\quad +\delta g^{0i}\partial_0 g_{0i} - \delta g^{00}\partial_0 g_{00} - \delta g^{0i}\partial_i g_{00}\Big) \\ &\quad +\frac{1}{2}\Big(g^{00}\partial_0\delta g_{00} + g^{0i}\partial_0\delta g_{i0} + g^{00}\partial_0\delta g_{00} \\ &\quad +g^{0i}\partial_0\delta g_{0i} - g^{00}\partial_0\delta g_{00} - g^{0i}\partial_i\delta g_{00}\Big) \\ &= \quad \frac{1}{2}\Big(\frac{-2A}{a^2}2aa' + \frac{1}{a^2}\partial_0(-2A(-a^2))\Big) \\ &= \quad \frac{1}{2}\Big(\frac{-4Aa'}{a} + \frac{4Aa'}{a} + 2A'\Big) = A',\end{aligned} \tag{3.51}$$

$$\begin{aligned}\delta\Gamma^0_{0i} \quad &= \quad \frac{1}{2}\Big(\delta g^{00}\partial_0 g_{0i} + \delta g^{0j}\partial_0 g_{ji} + \delta g^{00}\partial_i g_{00} \\ &\quad +\delta g^{0j}\partial_i g_{0j} - \delta g^{00}\partial_0 g_{0i} - \delta g^{0j}\partial_j g_{0i}\Big) \\ &\quad +\frac{1}{2}\Big(g^{00}\partial_0\delta g_{0i} + g^{0j}\partial_0\delta g_{ji} + g^{00}\partial_0\delta g_{00} \\ &\quad +g^{0j}\partial_i\delta g_{0j} - g^{00}\partial_0\delta g_{0i} - g^{0j}\partial_j\delta g_{0i}\Big) \\ &= \quad \frac{1}{2}\Big(\frac{2A}{a^2}\partial_i(-a^2(1+2A)) + \frac{(-1+2A)}{a^2}\partial_i(-a^2 2A) + \frac{\partial^j B}{a^2}\partial_0 a^2\delta_{ij}\Big) \\ &= \quad \partial_i A + \frac{a'}{a}\partial_i B,\end{aligned} \tag{3.52}$$

$$\begin{aligned}\delta\Gamma^i_{00} \quad &= \quad \frac{1}{2}\Big(2\delta g^{i0}\partial_0 g_{00} + 2\delta g^{ij}\partial_0 g_{0j} - \delta g^{i0}\partial_0 g_{00} - \delta g^{ij}\partial_j g_{00}\Big) \\ &\quad +\frac{1}{2}\Big(2g^{i0}\partial_0\delta g_{00} + 2g^{ij}\partial_0\delta g_{0j} - g^{i0}\partial_0\delta g_{00} - g^{ij}\partial_j\delta g_{00}\Big) \\ &= \quad \frac{1}{2}\Big(\frac{\partial^i B}{a^2}\partial_0(-a^2) + \frac{(2)}{a^2}\partial_0(a^2\partial^i B) - \frac{1}{a^2}\partial^i(-a^2 2A)\Big) \\ &= \quad \frac{a'}{a}\partial^i B + \partial^i B' + \partial^i A,\end{aligned} \tag{3.53}$$

$$\begin{aligned}
\delta\Gamma^0_{ij} &= \frac{1}{2}\Big(2\delta g^{00}\partial_i g_{0j} + 2\delta g^{ok}\partial_i g_{kj} - \delta g^{00}\partial_0 g_{ij} - \delta g^{0k}\partial_k g_{ij}\Big) \\
&\quad +\frac{1}{2}\left(2g^{00}\partial_i\delta g_{0j} + 2g^{0k}\partial_i\delta g_{kj} - g^{00}\partial_0\delta g_{ij} - g^{0k}\partial_k\delta g_{ij}\right) \\
&= \frac{1}{2}\Big(\frac{-2A}{a^2}\partial_0(a^2)\delta_{ij} + \frac{(-2)}{a^2}\partial_i(a^2\partial_j B) - \frac{-1}{a^2}\partial_0((-2\psi\delta_{ij} + D_{ij}E)a^2)\Big) \\
&= -2A\frac{a'}{a}\delta_{ij} - \partial_i\partial_j B - \psi\delta_{ij} + \frac{1}{2}D_{ij}E' + \frac{a'}{a}(-2\psi\delta_{ij} + D_{ij}E), \qquad (3.54)
\end{aligned}$$

$$\begin{aligned}
\delta\Gamma^i_{0j} &= \frac{1}{2}\Big(\delta g^{i0}\partial_0 g_{0j} + \delta g^{ik}\partial_0 g_{kj} + \delta g^{i0}\partial_j g_{00} \\
&\quad +\delta g^{ik}\partial_j g_{ko} - \delta g^{i0}\partial_0 g_{0j} - \delta g^{ik}\partial_k g_{0j}\Big) \\
&\quad +\frac{1}{2}\left(g^{i0}\partial_0\delta g_{0j} + g^{ik}\partial_0\delta g_{kj} + g^{i0}\partial_j\delta g_{00}\right. \\
&\quad \left.+g^{ik}\partial_j\delta g_{k0} - g^{i0}\partial_0\delta g_{0j} - g^{ik}\partial_k\delta g_{0j}\right) \\
&= \frac{1}{2}\Big(\frac{2\psi\delta^{ik} - D^{ik}E}{a^2}\partial_0 a^2\delta_{kj} + \frac{\delta^{ik}}{a^2}\partial_j(\partial_k B a^2) \\
&\quad +\frac{\delta^{ik}}{a^2}\partial_0((-2\psi\delta_{kj} + D_{kj}E)a^2) - \frac{\delta^{ik}}{a^2}\partial_k(\partial_j B a^2)\Big) \\
&= -\psi'\delta^i_j + \frac{1}{2}D^i_j E', \qquad (3.55)
\end{aligned}$$

and

$$\begin{aligned}
\delta\Gamma^i_{jk} &= \frac{1}{2}\Big(\delta g^{i0}\partial_j g_{0k} + \delta g^{il}\partial_j g_{lj} + \delta g^{i0}\partial_k g_{0j} \\
&\quad +\delta g^{il}\partial_k g_{lj} - \delta g^{i0}\partial_0 g_{jk} - \delta g^{il}\partial_l g_{jk}\Big) \\
&\quad +\frac{1}{2}\left(g^{i0}\partial_j\delta g_{0k} + g^{il}\partial_j\delta g_{lk} + g^{i0}\partial_k\delta g_{0j}\right. \\
&\quad \left.+g^{il}\partial_k\delta g_{lj} - g^{i0}\partial_0\delta g_{jk} - g^{il}\partial_l\delta g_{jk}\right) \\
&= \frac{1}{2}\Big(\frac{-\partial^i B}{a^2}\partial_0 a^2\delta_{jk} + \delta_{il}\partial_j(-2\psi\delta_{lk} + D_{lk}E) \\
&\quad +\delta_{il}\partial_k(-2\psi\delta_{lj} + D_{lj}E) - \delta_{il}\partial_l(-2\psi\delta_{jk} + D_{jk}E) \\
&= -\frac{a'}{a}\partial^i B\delta_{jk} - \delta^i_k\partial_j\psi - \delta^i_j\partial_k\psi + \delta_{jk}\partial^i\psi \\
&\quad +\frac{1}{2}D^i_k\partial_j E + \frac{1}{2}D^i_j\partial_k E - \frac{1}{2}D_{jk}\partial_i E. \qquad (3.56)
\end{aligned}$$

In order to determine the left-hand side of the perturbed Einstein equation, one must calculate the perturbed Ricci tensor and the perturbed Ricci scalar. The Ricci tensor is defined as

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}\,. \qquad (3.57)$$

Thus, the (0,0) component of the unperturbed Ricci tensor takes the form

$$\begin{aligned} R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\sigma\alpha} \Gamma^\sigma_{00} - \Gamma^\alpha_{\sigma 0} \Gamma^\sigma_{0\alpha} \\ &= -\partial_0 \Gamma^i_{0i} + \Gamma^i 0i \Gamma^0_{00} - \Gamma^i 0i \Gamma^i_{0i} \\ &= -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 . \end{aligned} \tag{3.58}$$

Other components can be calculated in the same way. The results are

$$R_{ij} = \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij}\,; \qquad R_{0i} = 0\,. \tag{3.59}$$

The first-order perturbation of the Ricci tensor takes the form

$$\begin{aligned} \delta R_{\mu\nu} &= \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\mu \delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha} \Gamma^\sigma_{\mu\nu} + \Gamma^\alpha_{\sigma\alpha} \delta\Gamma^\sigma_{\mu\nu} \\ &\quad - \delta\Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu} \delta\Gamma^\sigma_{\mu\alpha} . \end{aligned} \tag{3.60}$$

The $(0,0)$, $(0,i)$, (i,j) components can be calculated as follows:

$$\begin{aligned} \delta R_{00} &= \partial_0 \delta\Gamma^0_{00} + \partial_i \delta\Gamma^i_{00} - \partial_0 \delta\Gamma^0_{00} - \partial_0 \delta\Gamma^i_{0i} \\ &\quad + \delta\Gamma^0_{00} \Gamma^0_{00} + \delta\Gamma^i_{0i} \Gamma^0_{00} + \Gamma^0_{00} \delta\Gamma^0_{00} + \Gamma^i_{0i} \delta\Gamma^0_{00} \\ &\quad - \delta\Gamma^0_{00} \Gamma^0_{00} - \delta\Gamma^i_{0i} \Gamma^i_{0i} - \Gamma^0_{00} \delta\Gamma^0_{00} - \Gamma^i_{0i} \delta\Gamma^i_{0i} \\ &= \partial_i\Big(\frac{a'}{a}\partial^i B + \partial^i B' + \partial^i A\Big) - 3\partial_0\Big(-\psi' + \frac{1}{2}D^i_i E'\Big) + A' 3\frac{a'}{a} \\ &\quad + 3\frac{a'}{a}\Big(-\psi' + \frac{1}{2}D^i_i E'\Big) - 2\Big(3\frac{a'}{a}\Big)\Big(-\psi' + \frac{1}{2}D^i_i E'\Big) \\ &= \frac{a'}{a}\partial_i \partial^i B + \partial_i \partial^i B' + \partial_i \partial^i A + 3\psi'' + 3\frac{a'}{a}A' + 3\frac{a'}{a}\psi', \end{aligned} \tag{3.61}$$

$$\begin{aligned} \delta R_{0i} &= \partial_0 \delta\Gamma^0_{0i} + \partial_j \delta\Gamma^j_{0i} - \partial_0 \delta\Gamma^0_{i0} - \partial_0 \delta\Gamma^j_{ij} \\ &\quad + \delta\Gamma^0_{j0} \Gamma^j_{0i} + \delta\Gamma^j_{kj} \Gamma^k_{0i} + \Gamma^0_{00} \delta\Gamma^0_{0i} + \Gamma^j_{0j} \delta\Gamma^0_{0i} \\ &\quad - \delta\Gamma^j_{00} \Gamma^0_{ij} - \delta\Gamma^0_{0j} \Gamma^j_{i0} - \Gamma^0_{00} \delta\Gamma^0_{i0} - \Gamma^k_{0j} \delta\Gamma^j_{ik} \\ &= \partial_j\Big(-\psi'\delta^j_i + \frac{1}{2}D^j_i E'\Big) - \partial_0\Big(-\frac{a'}{a}\partial^i B - 3\partial_i\psi\Big) \\ &\quad + 3\frac{a'}{a}\Big(\partial_i A + \frac{a'}{a}\partial_i B\Big) - \frac{a'}{a}\Big(\frac{a'}{a}\partial_i B + \partial_i B' + \partial_i A\Big) \\ &\quad + \frac{a'}{a}\delta^k_i\Big(-\frac{a'}{a}\partial_k B - 3\partial_k \psi\Big) \\ &\quad + \frac{a'}{a}\delta^k_j\Big(-\frac{a'}{a}\partial^j B\delta_{ik} - \delta^j_k \partial_i\psi - \delta^j_i \partial_k \psi + \delta_{ik}\partial^j\psi \\ &\quad + \frac{1}{2}D^j_k \partial_i E + \frac{1}{2}D^j_i \partial_k E - \frac{1}{2}D_{ik}\partial_j E\Big) \\ &= \frac{a''}{a}\partial_i B + \left(\frac{a'}{a}\right)^2 \partial_i B + 2\frac{a'}{a}\partial_i A + 2\partial_i\psi' + \frac{1}{2}\partial_j D^j_i E', \end{aligned} \tag{3.62}$$

and

$$\begin{aligned}
\delta R_{ij} &= \partial_0 \delta\Gamma^0_{ij} + \partial_k \delta\Gamma^k_{ij} - \partial_i \delta\Gamma^0_{j0} - \partial_i \delta\Gamma^k_{kj} \\
&+ \delta\Gamma^0_{00}\,\Gamma^0_{ij} + \delta\Gamma^k_{0k}\,\Gamma^0_{ij} + \Gamma^0_{00}\,\delta\Gamma^0_{ij} + \Gamma^k_{0k}\,\delta\Gamma^0_{ij} \\
&- \delta\Gamma^k_{0i}\,\Gamma^0_{jk} - \delta\Gamma^0_{ki}\,\Gamma^k_{j0} - \Gamma^0_{ki}\,\delta\Gamma^k_{j0} - \Gamma^k_{0i}\,\delta\Gamma^0_{jk} \\
&= \partial_0\Big(-2A\frac{a'}{a}\delta_{ij} - \partial_i\partial_j B - \psi\delta_{ij} + \frac{1}{2}D_{ij}E' + \frac{a'}{a}(-2\psi\delta_{ij} + D_{ij}E)\Big) \\
&+\partial_k\Big(-\frac{a'}{a}\partial^k B\delta_{ij} - \delta^k_j\partial_i\psi - \delta^k_i\partial_j\psi + \delta_{ij}\partial^k\psi \\
&+\frac{1}{2}D^k_j\partial_i E + \frac{1}{2}D^k_i\partial_j E - \frac{1}{2}D_{ij}\partial_k E\Big) \\
&-\partial_i\Big(\partial_j A + \frac{a'}{a}\partial_j B\Big) - \partial_i\Big(-\frac{a'}{a}\partial^k B\delta_{kj} - \delta^k_k\partial_j\psi - \delta^k_j\partial_k\psi + \delta_{kj}\partial^k\psi\Big) \\
&+\frac{a'}{a}A'\delta_{ij} + \frac{a'}{a}\Big(-\psi\delta^k_k + \frac{1}{2}D^k_k E'\Big)\delta_{ij} \\
&+4\frac{a'}{a}\Big(-2A\frac{a'}{a}\delta_{ij} - \partial_i\partial_j B - \psi'\delta_{ij} + \frac{1}{2}D_{ij}E' + \frac{a'}{a}(-2\psi\delta_{ij} + D_{ij}E)\Big) \\
&-\frac{a'}{a}\Big(-\psi\delta^k_i + \frac{1}{2}D^k_i E'\Big)\delta_{jk} \\
&-\delta^k_j\frac{a'}{a}\Big(-2A\frac{a'}{a}\delta_{ki} - \partial_k\partial_i B - \psi'\delta_{ki} + \frac{1}{2}D_{ki}E' + \frac{a'}{a}(-2\psi\delta_{ki} + D_{ki}E)\Big) \\
&-\frac{a'}{a}\Big(-\psi\delta^k_j + \frac{1}{2}D^k_j E'\Big)\delta_{ki} \\
&-\delta^k_i\frac{a'}{a}\Big(-2A\frac{a'}{a}\delta_{kj} - \partial_k\partial_j B - \psi'\delta_{kj} + \frac{1}{2}D_{kj}E' + \frac{a'}{a}(-2\psi\delta_{kj} + D_{kj}E)\Big) \\
&= \delta_{ij}\Big[-5\frac{a'}{a}\psi' - \frac{a'}{a}A' - \frac{a'}{a}\partial_k\partial^k B - \psi'' - 2\frac{a''}{a}A \\
&-2A\Big(\frac{a'}{a}\Big)^2 - 2\frac{a''}{a}\psi - 2\Big(\frac{a'}{a}\Big)^2\psi + \partial_k\partial^k\psi\Big] \\
&+\frac{1}{2}D_{ij}E'' + \frac{a''}{a}D_{ij}E + \Big(\frac{a'}{a}\Big)^2 D_{ij}E + \frac{a'}{a}D_{ij}E' - \partial_i\partial_j B' - 2\frac{a'}{a}\partial_i\partial_j B \\
&-\partial_i\partial_j A + \partial_i\partial_j\psi - \frac{1}{2}\partial_k\partial^k D_{ij}E + \frac{1}{2}\partial_j\partial_k D^k_i E + \frac{1}{2}\partial_i\partial^k D_{kj}E. \qquad (3.63)
\end{aligned}$$

We next consider the Ricci scalar. The unperturbed Ricci scalar is

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} \\
&= g^{00}R_{00} + g^{ij}R_{ij} \\
&= \frac{-1}{a^2}\Big(-3\frac{a''}{a} + 3\Big(\frac{a'}{a}\Big)^2\Big) + \frac{\delta^{ij}}{a^2}\Big(\frac{a''}{a} + \Big(\frac{a'}{a}\Big)^2\Big)\delta_{ij} \\
&= \frac{6a''}{a^3}, \qquad (3.64)
\end{aligned}$$

and the perturbation of the Ricci scalar can be found as

$$
\begin{aligned}
\delta R &= \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\
&= \delta g^{00} R_{00} + \delta g^{ij} R_{ij} + g^{00}\delta R_{00} + g^{ij}\delta R_{ij} \\
&= \frac{2A}{a^2}\left(-3\frac{a''}{a} + 3\Big(\frac{a'}{a}\Big)^2\right) + \frac{2\psi\delta^{ij} - D^{ij}E}{a^2}\left(\frac{a''}{a} + \Big(\frac{a'}{a}\Big)^2\right)\delta_{ij} \\
&\quad + \frac{-1}{a^2}\left(\frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}A' + 3\frac{a'}{a}\psi'\right) \\
&= \frac{\delta^{ij}}{a^2}\Big[\delta_{ij}\Big(- 5\frac{a'}{a}\psi' - \frac{a'}{a}A' - \frac{a'}{a}\partial_k\partial^k B - \psi'' - 2\frac{a''}{a}A \\
&\quad -2A\Big(\frac{a'}{a}\Big)^2 - 2\frac{a''}{a}\psi - 2\Big(\frac{a'}{a}\Big)^2\psi + \partial_k\partial^k\psi\Big) \\
&\quad +\frac{1}{2}D_{ij}E'' + \frac{a''}{a}D_{ij}E + \Big(\frac{a'}{a}\Big)^2 D_{ij}E + \frac{a'}{a}D_{ij}E' - \partial_i\partial_j B' - 2\frac{a'}{a}\partial_i\partial_j B \\
&\quad -\partial_i\partial_j A + \partial_i\partial_j\psi - \frac{1}{2}\partial_k\partial^k D_{ij}E + \frac{1}{2}\partial_j\partial_k D^k_i E + \frac{1}{2}\partial_i\partial^k D_{kj}E\Big] \\
&= \frac{1}{a^2}\Big(- 6\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i A - 6\psi'' \\
&\quad -6\frac{a'}{a}A' - 18\frac{a'}{a}\psi' - 12\frac{a''}{a}A + 4\partial_i\partial^i\psi + \partial_k\partial^i D^k_i E\Big). \qquad (3.65)
\end{aligned}
$$

Using the above results for the Ricci tensor and the Ricci scalar, the components of the Einstein tensor,

$$G_{\mu\nu} \,=\, R_{\mu\nu} \,-\, \frac{1}{2}\, g_{\mu\nu}\, R\,, \qquad (3.66)$$

can be easily found:

$$
\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2}\, g_{00}\, R \\
&= -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 - \frac{1}{2}(-a^2)\frac{6a''}{a^3} \\
&= 3\left(\frac{a'}{a}\right)^2, \qquad (3.67)
\end{aligned}
$$

$$G_{0i} \,=\, 0\,; \qquad G_{ij} \,=\, \left(-\, 2\,\frac{a''}{a} \,+\, \left(\frac{a'}{a}\right)^2\right)\delta_{ij}\,. \qquad (3.68)$$

Thus, the perturbation of the Einstein tensor takes the form

$$\delta G_{\mu\nu} \,=\, \delta R_{\mu\nu} \,-\, \frac{1}{2}\,\delta g_{\mu\nu}\, R \,-\, \frac{1}{2}\, g_{\mu\nu}\,\delta R\,, \qquad (3.69)$$

and its components read

$$
\begin{aligned}
\delta G_{00} &= \delta R_{00} - \frac{1}{2}\,\delta g_{00}\,R - \frac{1}{2}\,g_{00}\,\delta R \\
&= \frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}A' + 3\frac{a'}{a}\psi' \\
&\quad -\frac{1}{2}(-2A)a^2\frac{6a''}{a^3} - \frac{1}{2}(-a^2)\Bigg(\frac{1}{a^2}\Big(-6\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i A \\
&\quad -6\psi'' - 6\frac{a'}{a}A' - 18\frac{a'}{a}\psi' - 12\frac{a''}{a}A + 4\partial_i\partial^i\psi + \partial_k\partial^i D^k_i E\Big)\Bigg) \\
&= -2\frac{a'}{a}\,\partial_i\partial^i\,B - 6\frac{a'}{a}\,\psi' + 2\,\partial_i\partial^i\,\psi + \frac{1}{2}\,\partial_k\partial^i D^k_i E\,, \qquad (3.70)
\end{aligned}
$$

$$
\begin{aligned}
\delta G_{0i} &= \delta R_{0i} - \frac{1}{2}\,\delta g_{0i}\,R - \frac{1}{2}\,g_{0i}\,\delta R \\
&= \frac{a''}{a}\partial_i B + \left(\frac{a'}{a}\right)^2\partial_i B + 2\frac{a'}{a}\partial_i A + 2\partial_i\psi' + \frac{1}{2}\partial_j D^j_i E' - \frac{1}{2}\partial_i B a^2\frac{6a''}{a^3} \\
&= -2\frac{a''}{a}\,\partial_i B + \left(\frac{a'}{a}\right)^2\,\partial_i B + 2\partial_i\,\psi' + \frac{1}{2}\partial_k\,D^k_i E' + 2\,\frac{a'}{a}\,\partial_i A\,, \qquad (3.71)
\end{aligned}
$$

$$
\begin{aligned}
\delta G_{ij} &= \delta R_{ij} - \frac{1}{2}\,\delta g_{ij}\,R - \frac{1}{2}\,g_{ij}\,\delta R \\
&= \delta_{ij}\Big[-5\frac{a'}{a}\psi' - \frac{a'}{a}A' - \frac{a'}{a}\partial_k\partial^k B - \psi'' - 2\frac{a''}{a}A \\
&\quad -2A\Big(\frac{a'}{a}\Big)^2 - 2\frac{a''}{a}\psi - 2\Big(\frac{a'}{a}\Big)^2\psi + \partial_k\partial^k\psi\Big] \\
&\quad +\frac{1}{2}D_{ij}E'' + \frac{a''}{a}D_{ij}E + \Big(\frac{a'}{a}\Big)^2 D_{ij}E + \frac{a'}{a}D_{ij}E' - \partial_i\partial_j B' - 2\frac{a'}{a}\partial_i\partial_j B \\
&\quad -\partial_i\partial_j A + \partial_i\partial_j\psi - \frac{1}{2}\partial_k\partial^k D_{ij}E + \frac{1}{2}\partial_j\partial_k D^k_i E + \frac{1}{2}\partial_i\partial^k D_{kj}E \\
&\quad -\frac{1}{2}\frac{(-2\psi\delta_{ij} + D_{ij}E)}{a^2}\frac{6a''}{a^3} \\
&\quad -\frac{1}{2}\delta_{ij}a^2\frac{1}{a^2}\Big(-6\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i A - 6\psi'' \\
&\quad -6\frac{a'}{a}A' - 18\frac{a'}{a}\psi' - 12\frac{a''}{a}A + 4\partial_i\partial^i\psi + \partial_k\partial^i D^k_i E\Big) \\
&= \Bigg(2\frac{a'}{a}\,A' + 4\frac{a'}{a}\,\psi' + 4\frac{a''}{a}\,A - 2\left(\frac{a'}{a}\right)^2 A + 4\frac{a''}{a}\,\psi - 2\left(\frac{a'}{a}\right)^2\psi \\
&\quad +2\psi'' - \partial_k\partial^k\,\psi + 2\frac{a'}{a}\,\partial_k\partial^k B + \partial_k\partial^k B' + \partial_k\partial^k A - \frac{1}{2}\partial_k\partial^m\,D^k_m\,E\Bigg)\delta_{ij} \\
&\quad -\partial_i\partial_j B' + \partial_i\partial_j\psi - \partial_i\partial_j A + \frac{a'}{a}\,D_{ij}E' - 2\,\frac{a''}{a}\,D_{ij}E \\
&\quad + \left(\frac{a'}{a}\right)^2 D_{ij}E + \frac{1}{2}D_{ij}E'' + \frac{1}{2}\partial_k\partial_i D^k_j E \\
&\quad +\frac{1}{2}\partial^k\partial_j D_{ik}E - \frac{1}{2}\partial_k\partial^k D_{ij}E - 2\,\frac{a'}{a}\partial_i\partial_j B\,. \qquad (3.72)
\end{aligned}
$$

For our convenience, we will express the perturbed Einstein tensor which has one index up and one index down in the form

$$\begin{aligned} \delta G^{\mu}_{\nu} &= \delta(g^{\mu\alpha\, G_{\alpha\nu}}) \\ &= \delta g^{\mu\alpha}\, G_{\alpha\nu} \,+\, g^{\mu\alpha}\, \delta G_{\alpha\nu}\,. \end{aligned} \tag{3.73}$$

Its components are

$$\begin{aligned} \delta G^0_0 &= \delta g^{00} G_{00} + \delta g^{0i} G_{i0} + g^{00}\delta G_{00} + g^{0i}\delta G_{i0} \\ &= \frac{2A}{a^2} 3\Big(\frac{a'}{a}\Big)^2 + \frac{-1}{a^2}\Big(-2\frac{a'}{a}\,\partial_i\partial^i\, B - 6\frac{a'}{a}\,\psi' + 2\,\partial_i\partial^i\,\psi + \frac{1}{2}\,\partial_k\partial^i D^k_i E\Big) \\ &= \frac{1}{2}\Big(2\frac{a'}{a}\,\partial_i\partial^i\, B + 6\frac{a'}{a}\,\psi' - 2\,\partial_i\partial^i\,\psi - \frac{1}{2}\,\partial_k\partial^i D^k_i E + 6\Big(\frac{a'}{a}\Big)^2 A\Big), \end{aligned} \tag{3.74}$$

$$\begin{aligned} \delta G^0_i &= \delta g^{00} G_{0i} + \delta g^{0j} G_{ji} + g^{00}\delta G_{0i} + g^{0j}\delta G_{ji} \\ &= \frac{2\partial^j}{a^2}\Big(-2\frac{a''}{a} + \Big(\frac{a'}{a}\Big)^2\Big)\delta_{ij} \\ &\quad + \frac{-1}{a^2}\Big(-2\frac{a''}{a}\,\partial_i B + \left(\frac{a'}{a}\right)^2\,\partial_i B + 2\partial_i\,\psi' + \frac{1}{2}\partial_k\, D^k_i E' + 2\,\frac{a'}{a}\,\partial_i A\Big) \\ &= \frac{-1}{a^2}\Big(\,2\,\frac{a'}{a}\partial_i A \,+\, 2\,\partial_i\psi' \,+\, \frac{1}{2}\,\partial_k D^k_i E'\Big), \end{aligned} \tag{3.75}$$

$$\begin{aligned} \delta G^i_j &= \delta g^{i0} G_{0j} + \delta g^{ik} G_{kj} + g^{i0}\delta G_{0j} + g^{ik}\delta G_{kj} \\ &= \frac{2\psi\delta^{ik} - D^{ik}E}{a^2}\Big(-2\frac{a''}{a} + \Big(\frac{a'}{a}\Big)^2\Big)\delta_{kj} \\ &\quad + \frac{\delta^{ik}}{a^2}\left(\Big[2\frac{a'}{a}\,A' + 4\frac{a'}{a}\,\psi' + 4\frac{a''}{a}\,A - 2\left(\frac{a'}{a}\right)^2 A + 4\frac{a''}{a}\,\psi - 2\left(\frac{a'}{a}\right)^2\psi\right. \\ &\quad + 2\psi'' - \partial_k\partial^k\,\psi + 2\frac{a'}{a}\,\partial_k\partial^k B + \partial_k\partial^k B' + \partial_k\partial^k A - \frac{1}{2}\partial_k\partial^m\, D^k_m\, E\Big]\delta_{kj} \\ &\quad - \partial_k\partial_j B' + \partial_k\partial_j\psi - \partial_k\partial_j A + \frac{a'}{a}\, D_{kj}E' - 2\,\frac{a''}{a}\, D_{kj}E \\ &\quad + \left(\frac{a'}{a}\right)^2 D_{kj}E + \frac{1}{2}D_{kj}E'' + \frac{1}{2}\partial_l\partial_k D^l_j E \\ &\quad \left. + \frac{1}{2}\partial^l\partial_j D_{kl}E - \frac{1}{2}\partial_l\partial^l D_{kj}E - 2\,\frac{a'}{a}\partial_k\partial_j B\right) \\ &= \frac{1}{a^2}\left(\Big[2\,\frac{a'}{a}A' \,+\, 4\,\frac{a''}{a}A \,-\, 2\left(\frac{a'}{a}\right)^2 A \,+\, \partial_i\partial^i A \,+\, 4\,\frac{a'}{a}\psi' \,+\, 2\,\psi''\right. \\ &\quad - \partial_i\partial^i\psi \,+\, 2\,\frac{a'}{a}\partial_i\partial^i B \,+\, \partial_i\partial^i B' \,-\, \frac{1}{2}\partial_k\partial^m D^k_m E\Big]\delta^i_j \\ &\quad - \partial^i\partial_j A \,+\, \partial^i\partial_j\psi \,-\, 2\,\frac{a'}{a}\partial^i\partial_j B \,-\, \partial^i\partial_j B' \,+\, \frac{a'}{a}D^i_j E' \,+\, \frac{1}{2}\,D^i_j E'' \\ &\quad \left. + \frac{1}{2}\,\partial_k\partial^i\, D^k_j E \,+\, \frac{1}{2}\,\partial_k\partial_j\, D^{ik}E \,-\, \frac{1}{2}\,\partial_k\partial^k\, D^i_j E\,.\right) \end{aligned} \tag{3.76}$$

For the right-hand side of the Einstein equation, one needs to calculate the perturbation of the energy-momentum tensor. In the period of inflation, the universe is dominated by the inflaton field with the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right), \tag{3.77}$$

whose components take the forms

$$\begin{aligned} T_{00} &= \phi'^2 - g_{00}\left(\frac{1}{2}\frac{-\phi'^2+(\vec{\nabla}\phi)^2}{a^2} + V(\phi)\right) \\ &= \frac{1}{2}\phi'^2 + a^2V(\phi), \end{aligned} \tag{3.78}$$

$$T_{0i} = 0\,; \qquad T_{ij} = \left(\frac{1}{2}\,\phi'^2 - V(\phi)\,a^2\right)\delta_{ij}\,. \tag{3.79}$$

The perturbation of the energy-momentum tensor arises from both the metric perturbation and the inflaton field fluctuations, and takes the form

$$\begin{aligned} \delta T_{\mu\nu} &= \partial_\mu\delta\phi\,\partial_\nu\phi + \partial_\mu\phi\,\partial_\nu\delta\phi - \delta g_{\mu\nu}\left(\frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right) \\ &\quad - g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi + g^{\alpha\beta}\,\partial_\alpha\delta\phi\,\partial_\beta\phi + \partial_\phi V\,\delta\phi\right). \end{aligned} \tag{3.80}$$

Its components are calculated as follows:

$$\begin{aligned} \delta T_{00} &= 2\partial_0\delta\phi\,\partial_0\phi - \delta g_{00}\left(\frac{1}{2}\,g^{00}\,\partial_0\phi\,\partial_0\phi + \frac{1}{2}\,g^{ij}\,\partial_i\phi\,\partial_j\phi + V(\phi)\right) \\ &\quad - g_{00}\Big(\frac{1}{2}\delta g^{00}\,\partial_0\phi\,\partial_0\phi + \frac{1}{2}\delta g^{ij}\,\partial_i\phi\,\partial_j\phi + g^{00}\,\partial_0\delta\phi\,\partial_0\phi \\ &\quad + g^{ij}\,\partial_i\delta\phi\,\partial_j\phi + \partial_\phi V\,\delta\phi\Big) \\ &= 2\delta\phi'\phi' - (-2A)a^2\left(\frac{1}{2}\frac{(-1)}{a^2}\phi'^2 + V(\phi)\right) \\ &\quad -(-a^2)\left(\frac{1}{2}\frac{2A}{a^2}\phi'^2 + \frac{-1}{a^2}\delta\phi'\phi' + \partial_\phi V\delta\phi\right) \\ &= \delta\phi'\phi' + 2Aa^2V(\phi) + a^2\partial_\phi V\delta\phi, \end{aligned} \tag{3.81}$$

$$\begin{aligned} \delta T_{0i} &= \partial_0\delta\phi\,\partial_i\phi + \partial_i\delta\phi\,\partial_0\phi - \delta g_{0i}\left(\frac{1}{2}\,g^{00}\,\partial_0\phi\,\partial_0\phi + \frac{1}{2}\,g^{ij}\,\partial_i\phi\,\partial_j\phi + V(\phi)\right) \\ &= \partial_i\delta\phi\phi' - \partial_iBa^2\left(\frac{1}{2}\frac{(-1)}{a^2}\phi'^2 + V(\phi)\right) \\ &= \partial_i\delta\phi\phi' + \frac{1}{2}\partial_iB\phi'^2 - a^2\partial_iBV(\phi), \end{aligned} \tag{3.82}$$

$$\begin{aligned}
\delta T_{ij} &= \partial_i\delta\phi\,\partial_j\phi + \partial_j\delta\phi\,\partial_i\phi - \delta g_{ij}\left(\frac{1}{2}g^{00}\,\partial_0\phi\,\partial_0\phi + \frac{1}{2}g^{ij}\,\partial_i\phi\,\partial_j\phi + V(\phi)\right)\\
&\quad - g_{ij}\Big(\frac{1}{2}\delta g^{00}\,\partial_0\phi\,\partial_0\phi + g^{00}\,\partial_0\delta\phi\,\partial_0\phi + \partial_\phi V\delta\phi\Big)\\
&= -(-2\psi\delta_{ij} + D_{ij}E)a^2\left(\frac{1}{2}\frac{(-1)}{a^2}\phi'^2 + V(\phi)\right)\\
&\quad -a^2\delta_{ij}\left(\frac{1}{2}\frac{2A}{a^2}\phi'^2 + \frac{-1}{a^2}\delta\phi'\phi' + \partial_\phi V\delta\phi\right)\\
&= (-\psi\phi'^2 + 2\psi a^2 V - A\phi'^2 + \phi'\delta\phi' - a^2\partial_\phi V\delta\phi)\delta_{ij}\\
&\quad +\Big(\frac{\phi'^2}{2} - a^2 V\Big)D_{ij}E. \qquad (3.83)
\end{aligned}$$

To use the above result for the perturbed energy-momentum tensor in the Einstein equation, we need to raise the first index of the energy-momentum tensor up by using the metric $g^{\mu\nu}$ and then vary the result according to

$$\begin{aligned}
\delta T^\mu_\nu &= \delta(g^{\mu\alpha}\,T_{\alpha\nu})\\
&= \delta g^{\mu\alpha}\,T_{\alpha\nu} + g^{\mu\alpha}\,\delta T_{\alpha\nu}. \qquad (3.84)
\end{aligned}$$

Its components read

$$\begin{aligned}
\delta T^0_0 &= \delta g^{00}\,T_{00} + \delta g^{0i}\,T_{i0} + g^{00}\,\delta T_{00} + g^{0i}\,\delta T_{i0}\\
&= \frac{2A}{a^2}\left(\frac{1}{2}\phi'^2 + a^2 V(\phi)\right) + \frac{-1}{a^2}\left(\delta\phi'\phi' + 2Aa^2 V(\phi) + a^2\partial_\phi V\delta\phi\right)\\
&= \frac{A\phi'^2}{a^2} - \frac{\delta\phi'\phi'}{a^2} - \partial_\phi V\delta\phi, \qquad (3.85)
\end{aligned}$$

$$\begin{aligned}
\delta T^i_0 &= \delta g^{i0}\,T_{00} + \delta g^{ij}\,T_{j0} + g^{i0}\,\delta T_{00} + g^{ij}\,\delta T_{j0}\\
&= \frac{\partial^i B}{a^2}\left(\frac{1}{2}\phi'^2 + a^2 V(\phi)\right) + \frac{\delta^{ij}}{a^2}\left(\partial_j\delta\phi\phi' + \frac{1}{2}\partial_j B\phi'^2 + a^2\partial_j BV\right)\\
&= \frac{\partial^i\delta\phi\phi'}{a^2} + \frac{\partial^i B\phi'^2}{a^2}, \qquad (3.86)
\end{aligned}$$

$$\begin{aligned}
\delta T^0_i &= \delta g^{00}\,T_{0i} + \delta g^{0j}\,T_{ji} + g^{00}\,\delta T_{0i} + g^{0j}\,\delta T_{ji}\\
&= \frac{\partial^j B}{a^2}\left(\frac{1}{2}\phi'^2 - a^2 V(\phi)\right)\delta_{ij} + \frac{-1}{a^2}\left(\partial_i\delta\phi\phi' + \frac{1}{2}\partial_i B\phi'^2 + a^2\partial_i BV\right)\\
&= \frac{-\partial_i\delta\phi\phi'}{a^2}, \qquad (3.87)
\end{aligned}$$

$$\begin{aligned}
\delta T^i_j &= \delta g^{i0}\,T_{0j} + \delta g^{ik}\,T_{kj} + g^{i0}\,\delta T_{0j} + g^{ik}\,\delta T_{kj}\\
&= \frac{2\psi\delta_{ik} - D_{ik}E}{a^2}\left(\frac{1}{2}\phi'^2 + a^2 V(\phi)\right)\delta_{kj}\\
&\quad +\frac{\delta^{ik}}{a^2}\Big((-\psi\phi'^2 + 2\psi a^2 V - A\phi'^2 + \phi'\delta\phi' - a^2\partial_\phi V\delta\phi)\delta_{kj}\\
&\quad +\Big(\frac{\phi'^2}{2} - a^2 V\Big)D_{kj}E\Big)\\
&= \Big(\frac{-A\phi'^2}{a^2} + \frac{\delta\phi'\phi'}{a^2} - \partial_\phi V\delta\phi\Big)\delta^i_j. \qquad (3.88)
\end{aligned}$$

Before proceeding further, it should be noted that there is one thing that we need to be careful about. When we say we are perturbing the space-time metric, we really mean that we use another metric which describes the space-time geometry slightly different from the original (background) space-time. We, therefore, have to make sure that the perturbed metric, which we use in doing calculations, really describes the geometry different from the unperturbed one. This is a deep issue, since it might be that the "perturbed metric" that we use indeed describes the same geometry as that of the original one but in a different coordinate system. More generally, two perturbed metrics might correspond to the same geometry but different coordinate systems. This issue therefore concerns the choices of the coordinate system or the "gauge choices." (Thus "choosing a gauge" simply means "choosing a coordinate system.") To discuss this issue in detail, one needs a map between two geometries that enables us to compare any quantity evaluated with respect to different geometries but at the same space-time point. This means that if different numerical values of this quantity on different geometries are linked by such a map, then they are associated to the same space-time point.

There are two ways to solve this problem. The first one is to choose a coordinate system (or a gauge) to work with, which is full of dangers due to the presence of unphysical degrees of freedom. However, the physical property can be determined by using the longitudinal gauge in the computation of the curvature perturbation [10]. The other solution is to do things in a gauge invariant manner, that is, to use the gauge-invariant quantities, such as the Bardeen's potentials defined by

$$\Phi = -A + \frac{1}{a}\left[\left(-B + \frac{E'}{2}\right)a\right]', \tag{3.89}$$

$$\Psi = -\psi - \frac{1}{6}\nabla^2 E + \frac{a'}{a}\left(B - \frac{E'}{2}\right). \tag{3.90}$$

This implies that the gauge invariant counterparts of the perturbation of the inflaton field and the energy-density perturbation take the forms

$$\tilde{\delta\phi} = -\delta\phi + \phi'\left(\frac{E'}{2} - B\right) \tag{3.91}$$

$$\tilde{\delta\rho} = -\delta\rho + \rho'\left(\frac{E'}{2} - B\right). \tag{3.92}$$

To work out things in a gauge invariant way, we first express all quantities in the perturbed Einstein equation in terms of the gauge-invariant quantities. From the

Bardeen's potential, the perturbation parameters take the forms

$$A = -\Phi + D' + \mathcal{H}D, \tag{3.93}$$
$$\psi = -\Psi - \frac{1}{6}\nabla^2 E - \mathcal{H}D, \tag{3.94}$$

where $D = E'/2 - B$. By substituting the above results into the Einstein tensor, one obtains

$$\begin{aligned}
\delta G^0_0 &= \frac{1}{a^2}\Big(2\mathcal{H}\nabla^2 B + 6\mathcal{H}\psi' - 2\nabla^2\psi - \frac{1}{2}\partial_k\partial^i D^i_k E + 6\mathcal{H}A\Big) \\
&= \frac{1}{a^2}\Big(2\mathcal{H}\nabla^2 B + 6\mathcal{H}(-\Psi' - \frac{1}{6}\nabla^2 E' - \mathcal{H}'D - \mathcal{H}D') \\
&\quad -2\nabla^2(-\Psi - \frac{1}{6}\nabla^2 E - \mathcal{H}D) - \frac{1}{3}\nabla^2\nabla^2 E + 6\mathcal{H}(-\Phi + D' + \mathcal{H}D)\Big) \\
&= \frac{2}{a^2}\Big(- 3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi + 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}')D\Big),
\end{aligned} \tag{3.95}$$

$$\begin{aligned}
\delta G^0_i &= \frac{-1}{a^2}\Big(2\mathcal{H}\partial_i A + 2\partial_i\psi' + \frac{1}{3}\partial_i\nabla^2 E'\Big) \\
&= \frac{-1}{a^2}\partial_i\Big(2\mathcal{H}(-\Phi + D' + \mathcal{H}D) + 2(-\Psi' - \frac{1}{6}\nabla^2 E' - \mathcal{H}'D - \mathcal{H}D') + \frac{\nabla^2 E'}{3}\Big) \\
&= \frac{2}{a^2}\partial_i\Big(\mathcal{H}\Phi + \Psi' - (\mathcal{H}^2 - \mathcal{H}')D\Big),
\end{aligned} \tag{3.96}$$

$$
\begin{aligned}
\delta G^i_j &= \frac{1}{a^2}\Bigg(\delta^i_j\Big(4\mathcal{H}\psi' + 2\mathcal{H}A' + 2\mathcal{H}\nabla^2 B + 2\psi'' + 4\mathcal{H}'A + 2\mathcal{H}^2 A \\
&\quad -\nabla^2\psi + \nabla^2 B' + \nabla^2 A - \frac{1}{3}\nabla^2\nabla^2 E\Big) + \frac{1}{2}D^i_j E'' + \mathcal{H}D^i_j E' \\
&\quad -\partial^i\partial_j B' - 2\mathcal{H}\partial^i\partial_j B - \partial^i\partial_j A + \partial^i\partial_j\psi - \frac{1}{2}\nabla^2 D^i_j E - \frac{2}{3}\nabla^2\partial^i\partial_j E\Bigg) \\
&= \frac{1}{a^2}\Bigg(\delta^i_j\Big(4\mathcal{H}(-\Psi' - \frac{1}{6}\nabla^2 E' - \mathcal{H}'D - \mathcal{H}D') \\
&\quad +2\mathcal{H}(-\Psi' + D'' + \mathcal{H}'D + D'\mathcal{H}) + 2\mathcal{H}\nabla^2 B \\
&\quad +2(-\Psi'' - \frac{1}{6}\nabla^2 E'' - \mathcal{H}''D - 2\mathcal{H}'D' - \mathcal{H}D'') \\
&\quad +4\mathcal{H}'(-\Phi + D' + \mathcal{H}D) + 2\mathcal{H}^2(-\Phi + D' + \mathcal{H}D) \\
&\quad -\nabla^2(-\Psi - \frac{1}{6}\nabla^2 E - \mathcal{H}D) + \nabla^2 B' \\
&\quad +\nabla^2(-\Phi + D' + \mathcal{H}D) - \frac{1}{3}\nabla^2\nabla^2 E\Big) \\
&\quad +\frac{1}{2}D^i_j E'' + \mathcal{H}D^i_j E' - \partial^i\partial_j B' - 2\mathcal{H}\partial^i\partial_j B \\
&\quad -\partial^i\partial_j(-\Phi + D' + \mathcal{H}D) + \partial^i\partial_j(-\Psi - \frac{1}{6}\nabla^2 E - \mathcal{H}D) \\
&\quad -\frac{1}{2}\nabla^2 D^i_j E - \frac{2}{3}\nabla^2\partial^i\partial_j E\Bigg) \\
&= \frac{-2}{a^2}\Bigg(\delta^i_j\Big(\frac{1}{2}\nabla^2(\Phi - \Psi) + 2\mathcal{H}\Psi' + \Psi'' + \mathcal{H}\Phi' \\
&\quad +(2\mathcal{H}' + \mathcal{H}^2)\Phi + (\mathcal{H}'' - \mathcal{H}\mathcal{H}' - \mathcal{H}^3)D\Big) - \partial^i\partial_j(\Phi - \Psi)\Bigg). \qquad (3.97)
\end{aligned}
$$

Observe that the above result is not gauge invariant. Thus we replace it with its gauge-invariant counterpart. For the (0,0) component,

$$
\begin{aligned}
\delta\tilde{G}^0_0 &= -\delta G^0_0 + (G^0_0)'D \\
&= \frac{2}{a^2}\Big(-3\mathcal{H}(\mathcal{H}\Phi + \Psi') - \nabla^2\Psi + 3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}')D\Big) \\
&\quad +\Big(\frac{-6\mathcal{H}'\mathcal{H}}{a^2} + \frac{6\mathcal{H}^3}{a^2}\Big)D \\
&= \frac{2}{a^2}\Big(3\mathcal{H}(\mathcal{H}\Phi + \Psi') - \nabla^2\Psi\Big), \qquad (3.98)
\end{aligned}
$$

The corresponding gauge-invariant component of the energy-momentum tensor is

$$\begin{aligned}
\widetilde{\delta T}^0_0 &= -\delta T^0_0 + (T^0_0)'D \\
&= \Big(\frac{A\phi'^2}{a^2} - \frac{\delta\phi'\phi'}{a^2} - \partial_\phi V \delta\phi\Big) + \Big(\frac{-1}{a^2}(1/2\phi'^2 + a^2 V(\phi))\Big)' D \\
&= -(-\Phi + D' + \mathcal{H}D)\frac{\phi'^2}{a^2} - (-\tilde{\delta\phi} + \phi'' D + \phi' D')\frac{\phi'}{a^2} \\
&\quad + (-\tilde{\delta\phi} + \phi' D)\partial_\phi V + \frac{\mathcal{H}\phi'^2}{a^2}D - \frac{\phi'\phi''}{a^2}D - \partial_\phi V \phi' D \\
&= \Phi\frac{\phi'^2}{a^2} - \tilde{\delta\phi}\frac{\phi'}{a^2} - \tilde{\delta\phi}\partial_\phi V. \qquad (3.99)
\end{aligned}$$

By substituting (3.98) and (3.99) into the Einstein equation, one obtains

$$3\mathcal{H}(\mathcal{H}\Phi + \Psi') - \nabla^2\Psi = 4\pi G\Big(\Phi\phi'^2 - \tilde{\delta\phi}\,\phi' - a^2\tilde{\delta\phi}\partial_\phi V\Big). \qquad (3.100)$$

For the $(0, i)$ component, we find

$$\begin{aligned}
\widetilde{\delta G}^0_i &= -\delta G^0_i + (G^0_i)'D \\
&= \frac{-2}{a^2}\partial_i\Big(\mathcal{H}\Phi + \Psi' - (\mathcal{H}^2 - \mathcal{H}')D\Big), \qquad (3.101)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\delta T}^0_i &= -\delta T^0_i + (T^0_i)'D \\
&= \frac{\partial_i\delta\phi\phi'}{a^2} + \frac{\partial_j B\phi'^2}{a^2} \\
&= \frac{\phi'}{a^2}\partial_i(-\tilde{\delta\phi} + \phi' D). \qquad (3.102)
\end{aligned}$$

The Einstein equation for this component takes the form

$$\partial_i\Big(\mathcal{H}\Phi + \Psi'\Big) = 4\pi G\phi'\partial_i\tilde{\delta\phi}. \qquad (3.103)$$

Finally, for the (i, j) components, we find

$$\begin{aligned}
\widetilde{\delta G}^i_j &= -\delta G^i_j + (G^i_j)'D \\
&= \frac{2}{a^2}\Bigg(\delta^i_j\Big(\frac{1}{2}\nabla^2(\Phi - \Psi) + 2\mathcal{H}\Psi' + \Psi'' + \mathcal{H}\Phi' \\
&\quad + (2\mathcal{H}' + \mathcal{H}^2)\Phi + (\mathcal{H}'' - \mathcal{H}\mathcal{H}' - \mathcal{H}^3)D\Big) - \partial^i\partial_j(\Phi - \Psi)\Bigg) \\
&\quad + \Big(\frac{2\mathcal{H}}{a^2}(2\mathcal{H}' + \mathcal{H}^2) + \frac{1}{a^2}(-2\mathcal{H}'' - 2\mathcal{H}\mathcal{H}')\Big)D\delta^i_j \\
&= \frac{2}{a^2}\delta^i_j\Big(\frac{1}{2}\nabla^2(\Phi - \Psi) + 2\mathcal{H}\Psi' + \Psi'' + \mathcal{H}\Phi' \\
&\quad + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \Big) - \partial^i\partial_j(\Phi - \Psi), \qquad (3.104)
\end{aligned}$$

$$\begin{aligned}
\delta\tilde{T}^i_j &= -\delta T^i_j + (T^i_j)'D \\
&= \Big((-\Phi + D' + \mathcal{H}D)\frac{\phi'^2}{a^2} - (-\tilde{\delta\phi}' + \phi''D + \phi'D')\frac{\phi'}{a^2} + (-\tilde{\delta\phi} + \phi'D)\partial_\phi V\Big)\delta^i_j \\
&\quad +(-\tilde{\delta\phi} + \phi'D)\partial_\phi V - \Big(\frac{\mathcal{H}\phi'^2}{a^2}D + \frac{\phi'\phi''}{a^2}D - \partial_\phi V\phi'D\Big)\delta^i_j \\
&= \Big(-\Phi\frac{\phi'^2}{a^2} + \tilde{\delta\phi}'\frac{\phi'}{a^2} - \tilde{\delta\phi}\partial_\phi V\Big)\delta^i_j, \qquad (3.105)
\end{aligned}$$

and

$$\frac{2}{a^2}\delta^i_j\Big(\frac{1}{2}\nabla^2(\Phi-\Psi) + 2\mathcal{H}\Psi' + \Psi'' + \mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \Big) \\ -\partial^i\partial_j(\Phi-\Psi) = \Big(-\Phi\frac{\phi'^2}{a^2} + \tilde{\delta\phi}'\frac{\phi'}{a^2} - \tilde{\delta\phi}\partial_\phi V\Big)\delta^i_j. \qquad (3.106)$$

From equation (3.106), one obtains $\partial^i\partial_j(\Phi-\Psi) = 0$ for $i \neq j$. Choosing a simple solution $\Phi = \Psi$, then equations (3.100), (3.103), and (3.106) can be rewritten as

$$\begin{aligned}
\nabla^2\Phi - 3\mathcal{H}\Phi' - \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi &= 4\pi G\left(\tilde{\delta\phi}\,\phi' + \tilde{\delta\phi}\,\frac{\partial V}{\partial\phi}\,a^2\right); \qquad (3.107) \\
\Phi' + \mathcal{H}\Phi &= 4\pi G\left(\tilde{\delta\phi}\,\phi'\right); \qquad (3.108) \\
\Phi'' + 3\mathcal{H}\Phi' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi &= 4\pi G\left(\tilde{\delta\phi}\,\phi' - \tilde{\delta\phi}\,\frac{\partial V}{\partial\phi}\,a^2\right). \qquad (3.109)
\end{aligned}$$

The first term on the right-hand side of (3.107) and (3.109) can be eliminated. This leads to

$$\Phi'' + 6\mathcal{H}\Phi' + 2\left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi = 8\pi G\tilde{\delta\phi}\,\frac{\partial V}{\partial\phi}\,a^2. \qquad (3.110)$$

Using the action in (3.11), the equation of motion for the inflaton field can be expressed as

$$\phi'' + 2\mathcal{H}\phi' = -\frac{\partial V}{\partial\phi}\,a^2\,. \qquad (3.111)$$

By substituting $\frac{\partial V}{\partial\phi}\,a^2$ from (3.111) and $\pi G\tilde{\delta\phi}$ from (3.108) into (3.110), equation (3.110) can be rewritten as

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' - \nabla^2\Phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi = 0\,. \qquad (3.112)$$

For convenience, we define a new quantity $u \equiv a\,\tilde{\delta\phi} + z\,\Phi$. Then the above equation becomes

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0, \qquad (3.113)$$

where

$$z = \frac{a\phi'}{\mathcal{H}}. \qquad (3.114)$$

In order to determine the power spectrum, we need to quantize the gauge-invariant variable, u, by repeating the steps in the calculation that we have done in the previous section. Thus, the amplitude coefficient for quantization expansion, u_k, satisfies

$$u_k'' + \Big(k^2 - \frac{z''}{z}\Big)u_k = 0. \tag{3.115}$$

The difference between (3.115) and (3.17) is only the factor $\frac{z''}{z}$ and $\frac{a''}{a}$. Therefore, one can write down the power spectrum of the curvature perturbation as

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\frac{\mid u_k(\eta = \eta_k)\mid^2}{z_k^2}, \tag{3.116}$$

where the curvature perturbation $\mathcal{R} = -u/z$. We will solve this equation by expressing the factor $\frac{z''}{z}$ in terms of the slow-roll parameters and later on set it equal to $(\nu^2 - 1/4)/\eta^2$. Let us consider

$$\begin{aligned} \frac{z'}{\mathcal{H}z} &= 1 + \frac{\phi''}{\mathcal{H}\phi'} - \frac{\mathcal{H}'}{\mathcal{H}^2} \\ &= 1 + \epsilon - \delta. \end{aligned} \tag{3.117}$$

According to the adiabatic expansion, the derivative of this object is equal to zero,

$$\Big(\frac{z'}{\mathcal{H}z}\Big)' = \frac{z''}{\mathcal{H}z} - \frac{1}{\mathcal{H}}\Big(\frac{z'}{z}\Big)^2 - \frac{\mathcal{H}'z'}{\mathcal{H}^2 z} \cong 0. \tag{3.118}$$

So we get

$$\begin{aligned} \frac{z''}{z} &= \Big(\frac{z'}{z}\Big)^2 + \frac{\mathcal{H}'}{\mathcal{H}}\frac{z'}{z} \\ &= (1+\epsilon-\delta)^2\mathcal{H}^2 + (1+\epsilon-\delta)(1-\epsilon)\mathcal{H}^2 \\ &\cong \mathcal{H}^2(2+2\epsilon-3\delta) \\ &= \frac{1}{\eta^2}(1+2\epsilon)(2+2\epsilon-3\delta) \\ &= \frac{1}{\eta^2}(2+6\epsilon-3\delta) \\ &= \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big), \end{aligned} \tag{3.119}$$

where

$$\begin{aligned} \nu &= (\frac{9}{4} + 6\epsilon - 3\delta)^{1/2} \\ &= \frac{3}{2} + 2\epsilon - \delta. \end{aligned} \tag{3.120}$$

Similar to the previous section, the solution u_k takes the form

$$u_k = \sqrt{\pi/4}e^{i\frac{\pi}{2}(\nu+1/2)}\sqrt{-\eta}H_\nu^{(1)}(-k\eta), \tag{3.121}$$

and so the power spectrum of the curvature perturbation takes the form

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3(-\eta_k)}{8\pi z^2(\eta_k)}\Big(J_\nu^2(-k\eta) + N_\nu^2(-k\eta)\Big). \tag{3.122}$$

This power spectrum can be reduced to the power spectrum of the generic scalar field by replacing z with a. Using the asymptotic form of Hankel's function in (3.33) and $z = a\phi'/\mathcal{H} = a\sqrt{2\epsilon}m_{pl}$, the power spectrum can be expressed in the form

$$\mathcal{P}_{\mathcal{R}}(k) = A_{\mathcal{R}}k^{6\epsilon-2\eta_{\mathcal{R}}}, \tag{3.123}$$

where

$$A_{\mathcal{R}} = \frac{2^{2\nu-3}}{8\pi^2 m_{pl}^2}\frac{\Gamma^2(\nu)}{\Gamma^2(3/2)}\frac{(1+\epsilon)^{1-2\nu}}{\epsilon}H^2(aH)^{2\nu-3}, \tag{3.124}$$

and the running spectral index is

$$n_{\mathcal{R}} = 1 + 6\epsilon - 2\eta_{\mathcal{R}}. \tag{3.125}$$

Next, we consider another way to derive the equation of motion in (3.113) using the action of u. To do so, one needs to find the action of the scalar perturbation field u. The idea of doing this is to expand the general action up to the second-order in perturbation variables, since the first-order terms correspond to the action of the background field. The calculation is rather complicated and lengthy. Therefore, we only quote the result taken from [11],

$$S = \frac{1}{2}\int d^4x\Big(u'^2 - \partial_i u\partial^i u + \frac{z''}{z}u^2\Big). \tag{3.126}$$

By using the Euler-Lagrange equation, this action yields the same equation of motion in (3.111). Thus, for our convenience, we will use this action in Chapter 5 in order to derive the scalar perturbation field.

3.2.2 Tensor Perturbation

As we have mentioned above, the result of the massless scalar field fluctuation can be applied to the tensor perturbation. The fluctuation of the inflaton will also make the fluctuation in the space-time metric, which we can think of as a ripple of the space-time. Generally, this ripple is called the gravitational waves described by the metric

$$g_{\mu\nu} = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})\,dx^i dx^j\right], \tag{3.127}$$

where h_{ij} has a small value. Considering the degrees of freedom of h_{ij}, this tensor can be decomposed into two parts according to their polarizations,

$$h_{ij} = h_+ \, e^+_{ij} + h_\times \, e^\times_{ij}, \tag{3.128}$$

where e^+_{ij} and $e^\times_{ij}$ respectively represent the longitudinal and transverse polarizations. The key point of this issue is that the amplitudes of this tensor act as massless scalar fields. To see this explicitly, consider the action of this tensor,

$$S_T = \frac{m_{pl}^2}{2} \int d^4x \, \sqrt{-g} \, \frac{1}{2} \partial_\mu h_{ij} \, \partial^\mu h_{ij}. \tag{3.129}$$

This action leads to the action of the massless scalar fields which takes the form

$$S_s = \frac{1}{2} \int d^4x \, a^2 \partial_\mu \phi_l \, \partial^\mu \phi_l, \tag{3.130}$$

where $\phi_l = m_{pl} h_l / \sqrt{2}$, and the equations of motion can be expressed as

$$\phi_l'' + \frac{2a'}{a}\phi_l' - \nabla^2 \phi_l = 0. \tag{3.131}$$

This is of the same form as the equation of motion for the generic massless scalar field in (3.13). Therefore, the power spectrum of the tensor perturbation can be written as

$$\begin{aligned} \mathcal{P}_T(k) &= \frac{k^3}{2\pi^2} \sum_l |h_l|^2 = 4 \times 2 \frac{k^3}{2\pi^2} \frac{|\phi_k|^2}{a^2(\eta = \eta_k)} \\ &= A_T k^{-2\epsilon}, \end{aligned} \tag{3.132}$$

where

$$A_T = \frac{8}{m_{pl}^2} \frac{2^{2\epsilon}}{\pi^2} \frac{\Gamma^2(3/2 + \epsilon)}{\Gamma^2(3/2)} (1 - \epsilon)^{2(1+\epsilon)} (aH)^{2\epsilon} H^2. \tag{3.133}$$

Notice that the multiplicative factors “2” and “4” in (3.131) correspond to the number of the polarization and number of the component for each polarization, respectively. From this power spectrum, the spectral index of the tensor perturbation can be deduced as

$$n_T = -2\epsilon. \tag{3.134}$$

3.3 From Inflation to CMB Anisotropy

Depending on the gravity sources, the evolution of the universe can be divided into two phases. The first one is the inflation era, in which the inflaton field is the main source of gravity. Therefore, the curvature perturbation is generated from the inflaton fluctuations; this was discussed in the previous section. As we have mentioned in the previous section, the amplitudes of the fluctuation modes in the inflation period are frozen on the scales larger than the Hubble radius. The main role of these fluctuations is to serve as the seed of all structures that we observe nowadays. The significant mechanism which yields the structures that we observe is the re-entry of the fluctuation modes after the end of inflation. This is the second phase which will be discussed in this section. The energy-momentum tensor will correspond to the energy density and the pressure of the perfect fluid matter or radiation. The CMB anisotropy which is generated during the recombination time will also be discussed.

3.3.1 The Hydrodynamical Perturbation

At the end of inflation, the inflaton decayed and the ordinary objects such as the radiation and the matter were created. In that period the universe reheated itself and was then dominated by the radiation. At later time, the universe was dominated by the matter because the density of the radiation deceases faster than that of the matter due to the expansion of the universe. Therefore, after the end of inflation, the evolution of the universe can be divided into the radiation and the matter periods. The perturbed energy-momentum tensor can be written in terms of the fluctuations of the energy density and pressure of the perfect fluid and takes the form

$$\begin{aligned} \delta\tilde{T}^0_0 &= -\tilde{\delta\rho}\,, \\ \delta\tilde{T}^0_i &= (\rho+p)\,a^{-1}\tilde{\delta v}_i\,, \\ \delta\tilde{T}^i_j &= \tilde{\delta p}\;\delta^i_j. \end{aligned} \tag{3.135}$$

The perturbed Einstein equation can be calculated in the same way as in the previous section, but instead using the energy-momentum tensor in (3.135). Equations (3.107), (3.108) and (3.109) can be rewritten as

$$\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = 4\pi G a^2\,\tilde{\delta\rho}\,; \tag{3.136}$$

$$\partial_i(\Phi\, a)' = 4\pi G a^2 (\rho+p)\,\tilde{\delta v}_i\,; \tag{3.137}$$

$$\Phi'' + 3\mathcal{H}\Phi' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Phi = 4\pi G a^2 \tilde{\delta p}\,. \tag{3.138}$$

In order to eliminate the right-hand side of (3.136) and (3.138), we use the definition of the adiabatic perturbation,

$$\tilde{\delta p} = c_s^2\tilde{\delta\rho}. \tag{3.139}$$

As a result,

$$\Phi'' + 3\mathcal{H}\Big(1+c_s^2\Big)\Phi' - c_s^2\nabla^2\Phi + \Big[2\mathcal{H}' + \Big(1+3c_s^2\Big)\mathcal{H}^2\Big]\Phi = 0, \tag{3.140}$$

where $c_s^2 = \dot{p}/\dot{\rho}$. This equation can be solved by eliminating the friction term Φ'. Let us define a new quantity u by

$$\Phi = 4\pi G\sqrt{\rho+p}\; u = \sqrt{4\pi G}\left(\frac{\mathcal{H}^2 - \mathcal{H}'}{a^2}\right)^{1/2} u. \tag{3.141}$$

This quantity then satisfies

$$u'' - c_s^2\nabla^2 u - \frac{z''}{z}u = 0, \tag{3.142}$$

where

$$z = \frac{1}{a}\left(\frac{\rho}{\rho+p}\right)^{1/2} = \frac{\mathcal{H}}{a}\left(\frac{2}{3}(\mathcal{H}^2 - \mathcal{H}')\right)^{-1/2}. \tag{3.143}$$

From (3.136) and (3.137), the density contrast can be written as

$$\delta = \frac{\tilde{\delta\rho}}{\rho} = \frac{2}{3\mathcal{H}^2}\Big(\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Big). \tag{3.144}$$

The goal of this part is to find the density contrast and the potential Φ which play the important role in the determination of the temperature fluctuations in the CMB anisotropy. These quantities can be obtained once u is determined. Our consideration can be divided into the matter-dominated and the radiation dominated parts.

For the matter-dominated universe, one uses the conditions

$$p = 0; \qquad c_s^2 = 0; \qquad a(\eta) = a_m\frac{\eta^2}{2}. \tag{3.145}$$

The general solutions for u, Φ and δ take the form

$$u(\vec{x},\eta) = C_1'(\vec{x})z(\eta) + C_2'(\vec{x})z(\eta)\int\frac{d\eta}{z^2}, \tag{3.146}$$

$$\Phi(\vec{x},\eta) = C_1(\vec{x}) + C_2(\vec{x},\eta)\eta^{-5}, \tag{3.147}$$

$$\delta(\vec{x},\eta) = \frac{1}{6}\Big(\nabla^2 C_1\eta^2 - 12C_1 + (\nabla^2 C_2\eta^2 - 18C_2)\eta^{-5}\Big), \tag{3.148}$$

where $C_1'(\vec{x})$, $C_2'(\vec{x})$, $C_1(\vec{x})$ and $C_2(\vec{x})$ are arbitrary functions of the spatial coordinates. From these equations (neglecting the effects of the decaying modes), one can conclude that the potential will be constant on both small and large scales. On the contrary, the density contrast will be constant only on the large scales, where the spatial derivatives can be neglected (small k), and will be proportional to η^2 or $t^{2/3}$ on the small scales. This conclusion implies that the amplitude of the curvature perturbation is constant outside the horizon in agreement with the results in the previous section. As a result, the perturbed density of the matter grows larger when it reenters the horizon. This perturbed density then causes the temperature fluctuations on the large scales of the CMB anisotropy that we will discuss later.

For the radiation-dominated universe, the conditions

$$p = \rho/3; \qquad c_s^2 = 1/3; \qquad a(\eta) = a_r\eta, \tag{3.149}$$

are applied and equation (3.142) can be rewritten as

$$u'' - \frac{1}{3}\nabla^2 u - \frac{2a_r^2}{a^2}u = 0. \tag{3.150}$$

This equation can be solved by introducing a new function which satisfies a wave equation [11]. This technical calculation leads to the solutions which take the form

$$\begin{aligned}\Phi(\vec{x},\eta) &= \frac{1}{\eta^3}\Big[\Big(\omega\eta\cos(\omega\eta) - \sin(\omega\eta)\Big)C_1 \\ &\quad + \Big(\omega\eta\sin(\omega\eta) + \cos(\omega\eta)\Big)C_2\Big]e^{i\vec{k}\cdot\vec{x}}, \qquad (3.151)\\ \delta(\vec{x},\eta) &= \frac{4}{\eta^3}\Big[\Big((\omega^2\eta^2 - 1)\sin(\omega\eta) + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2)\cos(\omega\eta)\Big)C_1 \\ &\quad + \Big((\omega^2\eta^2 - 1)\cos(\omega\eta) + \omega\eta(1 - \frac{1}{2}\omega^2\eta^2)\sin(\omega\eta)\Big)C_2\Big]e^{i\vec{k}\cdot\vec{x}}, \qquad (3.152)\end{aligned}$$

where $\omega = k/\sqrt{3}$. Both general solutions are the oscillation modes and become constant when we consider the long-wavelength limit, $\omega\eta \ll 1$. This yields the same result as the solution for the matter-dominate universe. The solution which oscillates on the short-wavelength scales will produce the temperature fluctuations on the small scale of the CMB anisotropy. The effect of these temperature fluctuations is called "the acoustic oscillation."

3.3.2 The CMB Anisotropy

As the universe expands, its temperature decreases continuously. This leads to the formation of the nuclei and atoms. The history of the universe began with the high temperature and primordial soup of the fundamental particles. As the temperature decreases, the elementary particles such as photons and neutrons were formed. Until the temperature is about 3,000 K ($t \sim 300,000$ years), the formation of atom could take place. At this time, electrons will be bounded in the atoms and decoupled from the photons. This is the first time that photons can propagate freely and it is called "the recombination time or decoupling time." There is also the furthest horizon of the light that one can observe, called "the last scattering surface." Therefore, photons from this period can be observed in all directions as a background of the universe and is called "the cosmic microwave background (CMB) radiation." The perturbation that reenters at the decoupling time will produce the fluctuations of the density of the photons in the CMB radiation and leads to the CMB anisotropy that one observes nowadays.

The temperature perturbations in the CMB arise from five physical effects. First, the peculiar velocity of our position with respect to the cosmic rest frame. This effect corresponds to the dipole moment and can be eliminated by using some technical data analysis. Second, the intrinsic perturbation in the radiation fluid, δ_r. This effect arises from the fluctuations that reentered the horizon in the radiation-dominated period which produced the temperature fluctuations on the small angular scales. This is the main effect of the small angular scales in the CMB anisotropy. One calls this effect is the "acoustic oscillation" in the last scattering surface as we have mentioned in the previous subsection. Third, the peculiar velocity of the photon on the last scattering surface which leads to the bulk motion of the baryons. This effect can be neglected in comparison with the second effect. Fourth, the Sachs-Wolfe effect which is the effect that the photons climb up the potential at the decoupling time and cause the photon red-shift that we observe. Finally, the last effect is the integrated Sachs-Wolfe effect. It is caused by photons passing through the gravitational potential which depends on time before we observe. This effect has a small effect in comparison with the Sachs-Wolfe effect. So we will not consider this effect here. The temperature perturbations of the intrinsic perturbation in the radiation fluid and the Sachs-Wolfe effect will be considered, and can be written as

$$\frac{\delta T}{T} = \frac{1}{4}\delta_r(t_{dec}, x_{dec}) + (\Phi_{in} - \Phi)(t_{dec}, x_{dec}), \tag{3.153}$$

where $\Phi_{in} = \frac{\delta T}{T}|_{in}$ is the initial temperature fluctuation which already existed before the decoupling time. The first term comes from the effect of the intrinsic perturbation in the radiation and the factor $1/4$ comes from $\rho \propto a^{-4} \propto T^4$. The second term comes from the effect of the Sachs-Wolfe effect. Φ_{in} can be determined in several ways. The easy but mathematically rigorous calculation is considered in [12] by considering the geodesic equation for the photon propagation in the metric that is perturbed by the gravitational potential Φ. One obtains

$$ds = \sqrt{1 - 2\Phi}\; dt \sim \; (1 - \Phi)dt. \tag{3.154}$$

By using the gauge transformation ($t \rightarrow t+dt$) and the conservation of the energy density equation, $a(t) = a_0 t^{2/3(1+\omega)}$, one obtains

$$\frac{\delta a}{a} = \frac{-2}{3(1+\omega)}\Phi. \tag{3.155}$$

From statistical mechanics, the energy density of the radiation is proportional to the fourth order of the temperature. Thus, one obtains

$$\frac{\delta T}{T}|_{in} \;=\; \frac{-\delta a}{a} \;=\; \frac{2}{3(1+\omega)}\Phi. \tag{3.156}$$

Finally, (3.152) reduces to

$$\frac{\delta T}{T} = \frac{1}{4}\delta_r(t_{dec}, x_{dec}) - \frac{1}{3}\Phi(t_{dec}, x_{dec}). \tag{3.157}$$

Next, we will compare this result with the observation. From the view point of the observer, the CMB anisotropies are generated at the last scattering surface which is spherically symmetric. So one can expand the temperature fluctuations in the spherical harmonics,

$$\frac{\delta T}{T}(\hat{n}, x_0) = \sum_{\ell,m} a_{\ell m}(x_0) Y_{\ell m}(\hat{n}), \tag{3.158}$$

where a_{lm} are the multipoles moments. The fluctuations are created by the homogeneous and isotropic random processes, then the result depends neither on x_0 nor on m [13]. Thus, there is no coupling between different scales and orientations which can be written as

$$\langle a^*_{\ell m}\; a_{\ell' m'}\rangle = \delta_{\ell'\ell}\delta_{m'm}C_\ell, \tag{3.159}$$

where C_ℓ is the CMB power spectrum. The two-point correlation function of the temperature fluctuations takes the form

$$\begin{aligned}
\xi(\hat{n}', \hat{n}) &= \langle \frac{\delta T}{T}(\hat{n}, x_0)\frac{\delta T}{T}(\hat{n}', x_0)\rangle \\
&= \sum_{\ell\ell'\; mm'} \langle a^*_{\ell' m'}\; a_{\ell m}\rangle Y^*_{\ell' m'}(\hat{n}')\;\; Y_{\ell m}(\hat{n}) \\
&= \sum_{\ell m} C_\ell Y^*_{\ell m}(\hat{n}')\;\; Y_{\ell m}(\hat{n}) \\
&= \frac{1}{4\pi}\sum_{\ell}(2\ell+1)C_\ell P_\ell(\hat{n}'\cdot\hat{n}).
\end{aligned} \tag{3.160}$$

Notice that the identity,

$$P_\ell(\hat{n}'\cdot\hat{n}) = \frac{4\pi}{2\ell+1}\sum_{m}\;\; Y^*_{\ell m}(\hat{n}')\;\; Y_{\ell m}(\hat{n}), \tag{3.161}$$

was used in the last step. Now, we will return to the two physical effects that generate anisotropy, These two physical effects are independent from each other. One can calculate them separately. First, let us consider the Sachs-Wolfe effect, the temperature perturbation can be expanded in the Fourier expansion and takes the form

$$\frac{\delta T}{T}(k,\hat{n},t_0) = \frac{1}{3}\Phi(k,t_{dec})e^{i\hat{k}\cdot\hat{n}(t_0-t_{dec})}. \tag{3.162}$$

By using the mathematical identity

$$\exp(i\,\hat{k}\cdot\hat{n}(t_0-t_{dec})) = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}j_{\ell}(k(t_0-t_{dec}))P_{\ell}(\hat{k}\cdot\hat{n}), \tag{3.163}$$

the two-point correlation function of the temperature fluctuations can be written as

$$\begin{aligned}
\xi(\hat{n}',\hat{n}) &= \langle\frac{\delta T}{T}(\hat{n},x_0,t_0)\frac{\delta T}{T}(\hat{n}',x_0,t_0)\rangle \\
&= \frac{1}{V}\int d^3x_0\frac{\delta T}{T}(\hat{n},x_0,t_0)\frac{\delta T}{T}(\hat{n}',x_0,t_0) \\
&= \frac{1}{(2\pi)^3}\int d^3k\frac{\delta T}{T}(\hat{n},k,t_0)\frac{\delta T}{T}(\hat{n}',k,t_0) \\
&= \frac{1}{9}\frac{1}{(2\pi)^3}\int d^3k\langle|\Phi|^2\rangle\sum_{\ell\ell'}(2\ell+1)(2\ell'+1)i^{\ell}(-i)^{\ell'} \\
&\quad j_{\ell}(k(t_0-t_{dec}))j'_{\ell}(k(t_0-t_{dec}))P_{\ell}(\hat{k}\cdot\hat{n})P_{\ell'}(\hat{k}\cdot\hat{n}') \\
&= \frac{1}{9}\int\frac{d^3k}{2\pi^3}\langle|\Phi|^2\rangle\sum_{\ell\ell'}A_{\ell'\ell}P_{\ell}(\hat{k}\cdot\hat{n})P_{\ell'}(\hat{k}\cdot\hat{n}') \\
&= \frac{1}{9}\int\frac{d^3k}{2\pi^3}\langle|\Phi|^2\rangle\sum_{\ell\ell',m'm}\frac{(4\pi)^2A_{\ell'\ell}}{(2\ell'+1)(2\ell+1)}Y^*_{\ell'm'}(\hat{k})Y_{\ell'm'}(\hat{n}')Y^*_{\ell m}(\hat{k})Y_{\ell m}(\hat{n}) \\
&= \frac{1}{9}\int\frac{dkk^2}{2\pi^3}\langle|\Phi|^2\rangle\sum_{\ell\ell',m'm}\frac{(4\pi)^2A_{\ell'\ell}}{(2\ell'+1)(2\ell+1)} \\
&\quad \int d\Omega_k Y^*_{\ell'm'}(\hat{k})Y^*_{\ell m}(\hat{k})Y_{\ell'm'}(\hat{n}')Y_{\ell m}(\hat{n}) \\
&= \frac{1}{9}\int\frac{dkk^2}{2\pi^3}\langle|\Phi|^2\rangle\sum_{\ell\ell',m'm}\frac{(4\pi)^2A_{\ell'\ell}}{(2\ell'+1)(2\ell+1)}\delta_{\ell'\ell}\delta_{m'm}Y_{\ell'm'}(\hat{n}')Y_{\ell m}(\hat{n}) \\
&= \frac{1}{9}\int\frac{dkk^2}{2\pi^3}\langle|\Phi|^2\rangle\sum_{\ell,m}(4\pi)^2j^2_{\ell}(k(t_0-t_{dec}))Y_{\ell m}(\hat{n}')Y_{\ell m}(\hat{n}) \\
&= \sum_{\ell,m}4\pi(2\ell+1)P_{\ell}(\hat{n}'\cdot\hat{n})\int\frac{dkk^2}{2\pi^3}\frac{1}{9}\langle|\Phi|^2\rangle j^2_{\ell}(k(t_0-t_{dec})).
\end{aligned} \tag{3.164}$$

Comparing with (3.160), one obtains

$$C_{\ell} = \frac{2}{\pi}\int\frac{dk}{k}\langle|\frac{1}{3}\Phi|^2\rangle k^3 j^2_{\ell}(k(t_0-t_{dec})). \tag{3.165}$$

Notice that this equation is valid for the adiabatic perturbation which $2 \leq \ell \ll (t_o - t_{dec})/t_{dec} \sim 100$. This equation will be solved if we know the exact solution of the potential Φ. It means that we must solve for the exact solution of the differential equation (3.148) which is very complicated. However, one can solve this equation numerically. Nevertheless, this equation can be solved analytically by assuming that the potential is the power-law of k,

$$\langle|\Phi|^2\rangle k^3 = Ak^{n-1}t_0^{n-1}. \tag{3.166}$$

One obtains

$$C_\ell^{(SW)} = \frac{A}{9}\frac{\Gamma(3-n)\Gamma(\ell-\frac{1}{2}+\frac{n}{2})}{2^{3-n}\Gamma(2-\frac{n}{2})\Gamma(\ell\frac{5}{2}-\frac{n}{2})}. \tag{3.167}$$

The significant point of this assumption is the scale invariant spectrum where $n = 1$.

On the small scales, $\ell \geq 100$, the intrinsic perturbations in the radiation fluid are dominated. The CMB power spectrum is

$$C_\ell = \frac{2}{\pi}\int \frac{dk}{k}\langle|\frac{1}{4}\delta_r(k,t_{dec})|^2\rangle k^3 j_\ell^2(k(t_0-t_{dec})). \tag{3.168}$$

According to the previous section, the solution of the density contrast is the oscillating solution that will induce the CMB power spectrum to oscillate as show in Figure 3.1. This figure is obtained by using the CMBFAST program [14] which is evaluated at $\Omega_b = 0.05$, $\Omega_c = 0.2$, $\Omega_\nu = 0.00$, $\Omega_\Lambda = 0.75$ and $H_0 = 65$ km/sec/Mpc.

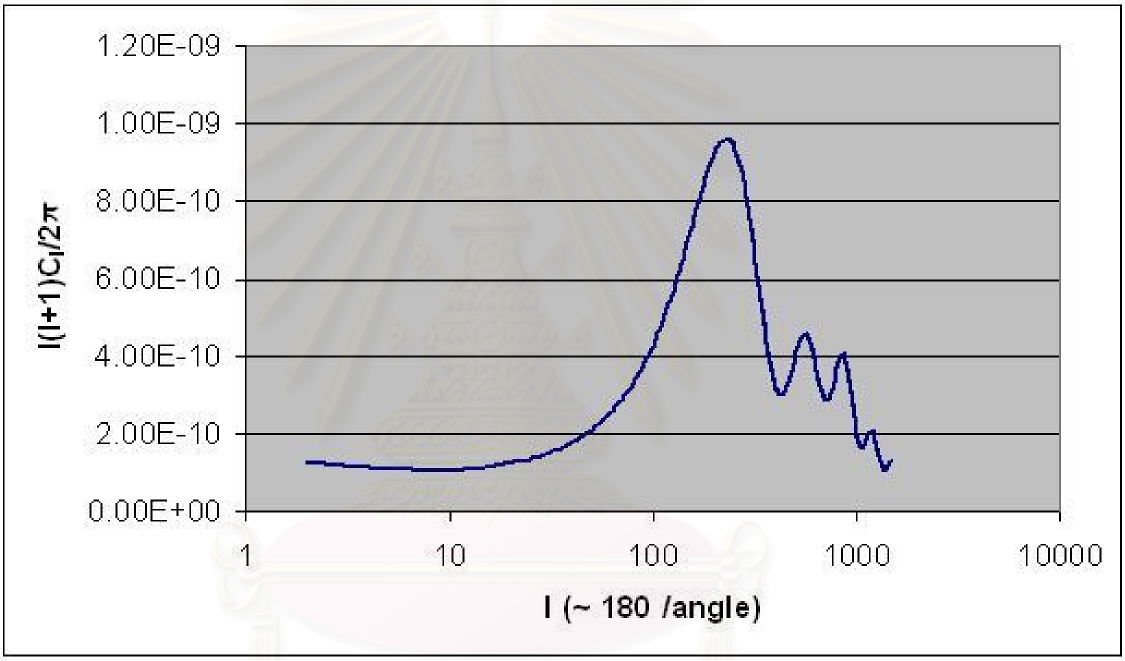

Figure 3.1: The CMB power spectrum.

CHAPTER IV

INTRODUCTION TO NONCOMMUTATIVE SPACE-TIME

The idea of noncommutative space-time was first introduced by Heisenberg in his attempt to remove the Ultraviolet divergences in quantum field theory [16]. Then, this idea was applied to explain the systems of an electron in strong external magnetic field by Rudolf Peierls [18]. However, it was Snyder who first formalized this idea in a systematic way and published it in the paper with the title "Quantized space-time" [19]. Unfortunately, his idea was ignored by the physics community because during that time the renormalization technique in quantum field theory was more attractive and was successful in predicting the data of observation. However, the idea of noncommutative space is more popular among the mathematicians. It was von Neumann who first used the idea of the noncommutative phase space in the quantum mechanics to introduce the noncommutative algebra. This is the well known aspects of "noncommutative geometry". After that, the idea of noncommutative space was revived in 1980's by three mathematicians, Connes, Woronowich and Drinfel'd who applied the notion of the differential formalism in noncommutative geometry. This formalism is the fundamental mathematical tool for studying several modern theories in physics such as the Yang-Mills gauge theory, Kaluza-Klein mechanism, standard model for particle physics, etc.

In the past twenty years, there has been a lot of progress in quantum gravity in particular in string theory. One of the most important developments is the discovery of D-brane which is the multi-dimensional object in string theory [20]. The studies of D-brane show the connection between string theory and gauge theory on noncommutative space. This connection pushes the idea of noncommutative space back into main stream physics again.

In fact String theory and non-commutative space seem to have very deep connection. Already in 1980s, Yonaya discovered the stringy space-time uncertainty relation [27]. In analogy with Quantum mechanics, one can conclude that string theory implies noncommutativity between space and time coordinates. Recently, Brandenberger and Ho apply the idea of string non-commutative space-time to cosmology [26]. We will review this idea in detail in Chapter 5.

In this chapter, we start by giving the idea of the noncommutative space-time and deriving the generalized uncertainty principle. Next, the fundamental concept of scalar field theory will be reviewed in order to provide us the understanding in the concept of field theory in the noncommutative space-time. Finally, the noncommutative algebra is applied to the field theory in order to use in a cosmological regime.

4.1 Generalized Uncertainty Principle

Quantum gravity is the theory which takes an account for the effects of quantum theory and general relativity. The significant property of the quantum gravity is the existence of the minimum length scale. This length scale is known as the Planck length. The Planck length can be determined by comparing the quantum length scale, $\lambda_c = \hbar/mc$, with the classical length scale, $R_s = Gm/c^2$. Note that λ_c is the Compton wavelength and R_s is the Shcwarzschild radius. This leads to the Planck mass

$$m_{pl} = \sqrt{\frac{\hbar c}{G}} \cong 2.2 \times 10^{-8}\text{kg}, \tag{4.1}$$

Other Planck quantities can be determined by using the dimensional analysis, and take the form

$$\begin{aligned} l_{pl} &= \sqrt{\frac{\hbar G}{c^3}} \cong 1.6 \times 10^{-35}\text{m}, \\ t_{pl} &= \sqrt{\frac{\hbar G}{c^4}} \cong 0.54 \times 10^{-43}\text{sec}, \\ E_{pl} &= \sqrt{\frac{\hbar G}{c^3}} \cong 1.2 \times 10^{19}\text{GeV}, \end{aligned} \tag{4.2}$$

where l_{pl}, t_{pl} and E_{pl} are the Planck length, time and energy, respectively.

We are interested in the effect of Planck scale on the principle of Quantum Mechanics. It turns out that the Heisenberg uncertainty principle needs to be modified. This generalized version of Heisenberg uncertainty principle occurs in many models of quantum gravity for example in string theory [21]. In this section, we will derive the generalized uncertainty principle by following the work of Adler

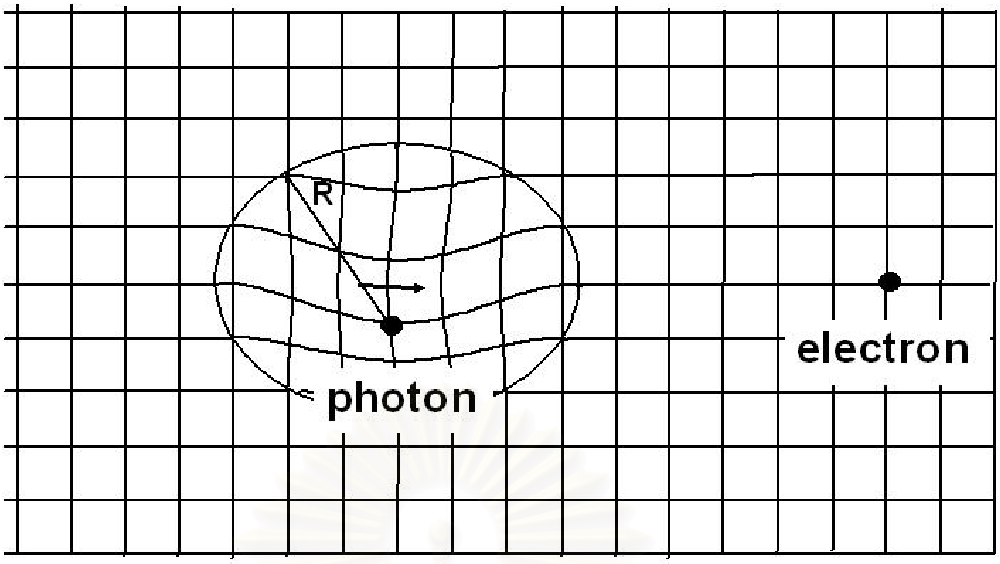

Figure 4.1: The curvature of the space-time which is curved due to the photon.

in [22]. Note that only in this section we reintroduce the speed of light c and will return to the natural unit in the next section.

Let us consider the classical scattering process of the photon and electron. We assume that at the beginning the electron is at rest in flat space-time. Suppose the photon energy is high enough to curve the space-time around it. As the photon propagating toward the electron, the electron starts to feel the curvature of the space-time as illustrate in Figure 4.1.

We assume that electromagnetic and gravitational interactions can be calculated independently. The effect of the electromagnetic interaction is responsible for the Heisenberg uncertainty principle which obeys the relation,

$$\Delta x \geq \frac{\hbar}{\Delta p}. \tag{4.3}$$

This relation implies that one cannot find the exact position and momentum of the electron at the same time. The exact position of the electron can be determined by using the very large photon momentum and this leads to the very large uncertainty for the electron momentum eventually.

In order to determine the gravitational interaction from the in-coming photon on the electron, we will consider the Einstein equation,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{4.4}$$

For simplicity, one can consider the dimensional estimate in the first place. The left-hand side (LHS) of the Einstein equation can be viewed as the second derivative of the space-time metric with respect to the space-time coordinate. Thus, one can estimate $LHS \approx \delta g_{\mu\nu}/L^2$. L is the characteristic size of the model which is the approximation radius of the region that the gravitational interaction from the photon can affect the electron. $\delta g_{\mu\nu}$ is the deviation of the flat space metric due to the photon energy and it contributes fractional uncertainty to the electron position at any points in the characteristic region. Dimensionally, one can write $\delta g_{\mu\nu} = \Delta x/L$. Thus, the left-hand side of (4.4) can be written as

$$LHS \approx \frac{\Delta x}{L^3}. \tag{4.5}$$

For the right-hand side (RHS), the energy-momentum tensor represents the energy density of the photon. By using the dimensional estimate, the right-hand side can be written as $RHS \approx GE/c^4L^3 = Gp/c^3L^3$. This photon momentum will contribute to the uncertainty of the electron momentum in the same order of magnitude, and then the right-hand side of (4.4) takes the form

$$RHS \approx \frac{G\Delta p}{c^3 L^3}. \tag{4.6}$$

From equations (4.5) and (4.6), the uncertainty of the electron position from gravitational effect can be expressed in the form

$$\Delta x \;=\; \frac{G\Delta p}{c^3} \;=\; l_{pl}^2\frac{\Delta p}{\hbar}. \tag{4.7}$$

By combining equation (4.7) with the Heisenberg uncertainty principle (4.3), one obtains the generalized uncertianty principle:

$$\Delta x \;\geq\; \frac{\hbar}{\Delta p} + l_{pl}^2\frac{\Delta p}{\hbar}. \tag{4.8}$$

This equation implies the maximum accuracy in determining the electron positions which has the limit at the Planck length. This means that one cannot probe any object which is smaller than the Planck length. However, the above consideration is only the rough approximation. More rigorous calculation can be done by using the linearized perturbation theory in general relativity. The components of the energy momentum tensor are assumed to be small. In this context, one can perturb the space-time metric by adding the small perturbation metric, $h_{\mu\nu}$, to the flat Minkowski metric, $\eta_{\mu\nu}$, such that

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{4.9}$$

From the straightforward calculation with the Lorentz gauge condition [5], one obtains the linearized Einstein equation which takes the form

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h \;=\; -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{4.10}$$

where the D'Alembertian operator $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. By considering the energy-momentum tensor of the radiation in the flat Minkowski space, one obtains

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} + \frac{1}{4}F_{\sigma\rho}F^{\sigma\rho}\eta_{\mu\nu}. \tag{4.11}$$

where $F_{\mu\rho}$ is the electromagnetic field strength tensor. By assuming the radiation propagates along x direction, the energy-momentum tensor has only four non-zero components and takes the form

$$T_{\mu\nu} = -\rho \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.12}$$

where ρ is the energy density of the radiation which takes the form $\rho = (c^2E^2 + B^2)/2$. The electromagnetic wave propagating in the x direction can be explained in terms of the wave function $f(x - ct, y, z)$. Therefore, the perturbed metric can be written as

$$h_{\mu\nu} = f(x - ct, y, z) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.13}$$

For our convenience, we are only interested in the inhomogeneous solution. Then, the linearized Einstein equation (4.10) can be reduced to the inhomogeneous wave equation of the unknown function, f, and takes the form

$$\Box f(x - ct, y, z) = \frac{16\pi G}{c^4}\rho(x - ct, y, z). \tag{4.14}$$

The energy density ρ and the unknown function f can be treated separately. Let us appling the separation of variable method such that

$$\begin{aligned} f(x - ct, y, z) &= f_{\parallel}(x - ct)f_{\perp}(y, z), \\ \rho(x - ct, y, z) &= \rho_{\parallel}(x - ct)\rho_{\perp}(y, z). \end{aligned} \tag{4.15}$$

Substituting these into equation (4.14), one obtains

$$\begin{aligned} \Box f_{\parallel}(x - ct)f_{\perp}(y, z) &= \frac{16\pi G}{c^4}\rho_{\parallel}(x - ct)\rho_{\perp}(y, z) \\ f_{\parallel}(x - ct)\Box f_{\perp}(y, z) + f_{\perp}(y, z)\Box f_{\parallel}(x - ct) &= \frac{16\pi G}{c^4}\rho_{\parallel}(x - ct)\rho_{\perp}(y, z) \\ f_{\parallel}(x - ct)(\partial_y^2 + \partial_z^2)f_{\perp}(y, z) &= \frac{16\pi G}{c^4}\rho_{\parallel}(x - ct)\rho_{\perp}(y, z) \\ \Rightarrow \qquad f_{\parallel}(x - ct) &= \frac{16\pi G}{c^4}\rho_{\parallel}(x - ct), \qquad (4.16) \\ \Rightarrow \qquad (\partial_y^2 + \partial_z^2)f_{\perp}(y, z) &= \rho_{\perp}(y, z). \qquad (4.17) \end{aligned}$$

For simplicity, one chooses the cylindrical coordinate. We assume that the radiation is enveloped by the moving cylinder of length L and radius R. We also assume that R is in the same order magnitude of the characteristic length, L. Furthermore, it is assumed that the energy density has a constant value ρ_0 inside the moving cylinder and vanishes outside the cylinder. Therefore, the equations (4.16) and (4.17) can be reduced to

$$f_{\parallel}(x-ct) = \frac{16\pi G}{c^4}\theta(x-ct)\theta(L-x+ct), \tag{4.18}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f_{\perp}}{\partial r}\right) = \rho_0\theta(r-R), \tag{4.19}$$

where θ is step function. Note that from the above equation, we assume that the moving cylinder is an azimuthally symmetric cylinder. In order to solve (4.19), we will determine the solutions for inside and outside the cylinder separately. There are four boundary conditions that we need to consider. The first is the non-vanishing of the function at the origin $f(r=0) \neq 0$. The second is the vanishing of the solution at the infinity $f(r\to\infty) = 0$. The requirement that the solutions must be smooth at the surface of the cylinder gives the last two boundary conditions: $f(r<R)|_R = f(r>R)|_R$ and $\partial_r f(r<R)|_R = \partial_r f(r>R)|_R$. Thus, after some calculations, we have

$$f = \frac{4\pi G}{c^4}\theta(x-ct)\theta(L-x+ct)\rho_0 R^2 \begin{cases} \frac{r^2}{R^2} & r<R \\ 1+\ln\frac{r^2}{R^2} & r>R \end{cases} \tag{4.20}$$

Recall that in our model, the gravitational interaction can only affect the electron if and only if the electron is in the characteristic range L from the photon. While the electromagnetic interaction is assumed to have instantaneous effect. As the gravitational interaction begins to affect the electron at the surface of the cylinder ($r \sim R$). The electromagnetic interaction will affect the electron at the same time. By replacing $r \sim R$ and $\rho_0 = E/\pi R^2 L = p/c\pi R^2 L$ into the above equation (4.20), one can approximate f function as

$$f \approx \frac{4Gp}{Lc^3}. \tag{4.21}$$

The function f is the amplitude of the perturbed metric which has the same order magnitude as $\delta g_{\mu\nu}$, then the f function can be approximated as $f \sim \Delta x/L$. Thus, the uncertainty of the electron position can be written as

$$\Delta x \approx \frac{4Gp}{c^3} = \frac{4G\Delta p}{c^3} = 4l_s^2\frac{\Delta p}{\hbar}. \tag{4.22}$$

By combining this equation with the Heisenberg uncertainty principle, one obtains the same result with the dimensional estimate analysis. As we mentioned at the beginning of this section, the calculation in string theory also yields the same generalized uncertainty principle but, in that case, replacing the Planck length with the string length. Moreover, both Planck and string lengths are the minimum length that one can observe in each theory. This consistency makes string theory one of the most promising candidates for quantum gravity. It implies that the string theory can be a good choice to describe physics at the high-energy scale such as in the early universe. We will come back to discuss this issue in the next chapter.

Another application of the generalized uncertainty principle is the prediction of black hole remnants [23], [24]. Black hole remnants are the object left over after Hawking radiation of the small black hole. The idea of the black hole remnants is in analogous with the stability problem of the hydrogen atom in quantum mechanics. In the latter, the uncertainty principle prevents the collapse of the hydrogen atom. While, in the former case, the generalized uncertainty principle prevents a small black hole to evaporate and vanish. These black hole remnants are one of the candidates for cold dark matter.

4.2 Review of Scalar Field Theory

Field theory plays the important roles in many areas of cosmology in particular the inflationary model. We will review some necessary concepts of the classical and quantum field theory. In this thesis, we consider only the real scalar field theory.

4.2.1 Scalar Field Theory

Quantum theory of the discrete system is not well defined in the relativistic limit while the propagation amplitude of free particle violates the causality [25]. However, the causality problem can be solved in the continuous system which is described by the field variable. By using the Lagrangian formalism, we can construct the relativistic field theory whose action is invariant under the Lorentz transformation in four-dimensional space-time. Let us consider a real scalar field $\phi(x) = \phi(x^0, x^1, x^2, x^3)$. In this section, we denote the 4-vector, x^μ, by using the Greek indices μ, ν, σ and the 3-dimensional vector will be denote by the arrow e.g. $\vec{x}$. In general, the action for real scalar field in 4-dimensional Minkowski space-time is

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x, \tag{4.23}$$

where $\mathcal{L}$ is the Lagrangian density. The equation of motion is the Euler-Lagrange equation:

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Big) - \frac{\partial \mathcal{L}}{\partial \phi} \;=\; 0. \tag{4.24}$$

In order to consider quantum field theory of the scalar field, we need the Hamiltonian which is defined by

$$H \;=\; \int \Big(\pi(x)\dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi) \Big) d^3x = \int \mathcal{H} d^3x, \tag{4.25}$$

where $\mathcal{H}$ is the Hamiltonian density. The conjugate momentum is defined by

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}. \tag{4.26}$$

4.2.2 Field Quantization

In this subsection, we review the canonical quantization of the scalar field. Let us consider the free scalar field, ϕ, with the mass "m". The Lagrangian density can be written as

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{4.27}$$

The Hamiltonian density is

$$\mathcal{H} \;=\; \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2. \tag{4.28}$$

In analogy with quantum mechanics, we replace the field variable and the conjugate momentum (ϕ, π) by their associated operators $(\hat{\phi}, \hat{\pi})$. These operators must obey the equal-time canonical commutation relations

$$[\hat{\phi}(t, \vec{x}) \;,\; \hat{\phi}(t, \vec{x}')] = [\hat{\pi}(t, \vec{x}) \;,\; \hat{\pi}(t, \vec{x}')] = 0, \tag{4.29}$$

and

$$[\hat{\phi}(t, \vec{x}) \;,\; \hat{\pi}(t, \vec{x}')] = \imath \delta^3(\vec{x} - \vec{x}'). \tag{4.30}$$

Moreover, the field operators also obey the Heisenberg equations which take the form

$$\dot{\hat{\phi}}(x) \;=\; \frac{1}{i}\Big[\hat{\phi}(x), \hat{H}\Big] = \hat{\pi}(x), \tag{4.31}$$

$$\dot{\hat{\pi}}(x) \;=\; \frac{1}{i}\Big[\hat{\pi}(x), \hat{H}\Big] = (\nabla^2 - m^2)\hat{\phi}(x), \tag{4.32}$$

where $\hat{H}$ is the Hamiltonian operator which is determined by replacing the classical field by the field operator. Combining these two equations of motion, one obtains

$$(\Box + m^2)\hat{\phi}(x) = 0, \tag{4.33}$$

which is the Klien-Gordon equation of the field operator. In analogy with the quantum theory, the operator can be written as the superposition of the creation and annihilation operators. One can expand the field operators as

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{1}{2E_k}}(\hat{a}_k e^{-ikx} + \hat{a}_k^\dagger e^{ikx}) \tag{4.34}$$

$$\hat{\pi}(x) = \dot{\hat{\phi}}(x) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{E_k}{2}}(\hat{a}_k e^{-ikx} - \hat{a}_k^\dagger e^{ikx}). \tag{4.35}$$

The creation and the annihilation operators have to obey the commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \tag{4.36}$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = (2\pi)^3\delta^3(k - k'). \tag{4.37}$$

From above discussions, one can state that quantum field theory is the quantum theory with infinite degrees of freedom which is referred from the infinite numbers of the creation and annihilation operators. As we mentioned in the first part of this section, the causality problem can be solved by multiple-particle regime. Therefore, quantum field theory for the scalar field cannot violate the causality by interpreting that "a measurement which is performed at one point cannot affect a measurement at another point which is separated by space-like interval" [25]. Due to this interpretation, one uses the continuous Lorentz transformation which exists only in the continuous system. This is the elegant theory to explain the nature of particle physics. However, in this work, we consider only in the free theory in order to see the difference between noncommutative and commutative field theory. Although, the most important result of the quantum field theory is to explain the interaction of the particles. This result can be determined by adding the interaction term in Lagrangian.

4.3 Noncommutative Field Theory

As we mentioned in the first part of this chapter, the idea of noncommutative space-time is inspired by quantum theory. In quantum mechanics, the phase-space variables, x_i and p_j, are replaced by the Hermitian operators, $\hat{x}_i$ and $\hat{p}_j$ which obey the commutation relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. By using the simillar concept, the noncommutivity of space-time can be defined by replacing the space-time coordinates with the space-time Hermitian operators which obey the commutation relation,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{4.38}$$

where $\theta^{\mu\nu}$ is constant real-valued antisymmetric 4×4 matrix with the dimension of length squared. In analogy with the quantum mechanics, this commutation relation implies the space-time uncertainty relation

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|. \tag{4.39}$$

This relation implies that the space-time point will be replaced by the Planck cell. One cannot probe anything inside the Planck cell. Moreover, it refers to non-local time coordinate which leads to the causality problem eventually. Since we consider the dynamics of the scalar field in noncommutative space-time, we need the multiplication product of such fields, $\phi(\hat{x})\phi(\hat{x})$. Note that, we put the "hat notation $(\hat{\ })$" over all parameters that represent the operators for this section. Thus, the mapping between the multiplication of some given functions in noncommutative and commutative space-time is defined, and well known as the Groenewold-Moyal product or "star product".

4.3.1 The Star Product

It is convenient to define the operator which maps the field in noncommutative space-time into the function in commutative space-time. The Weyl operator is the significant operator for this mapping because it is the mapping between the algebra of coordinate in classical phase space and the algebra of operator in Hilbert space which is the noncommutative structure. Therefore, in this section, we first consider the Weyl mapping in order to define the star product. And then, the property of the star product will be discussed.

The Weyl operator of any function in commutative space-time can be defined in similar way with Fourier transformation which can be written as

$$\hat{W}[f(x)] = \int \frac{d^4k}{(2\pi)^2}\tilde{f}(k)e^{ik\hat{x}}. \tag{4.40}$$

Here $\tilde{f}(k)$ are the Fourier coefficients which take the form

$$\tilde{f}(k) \;=\; \int \frac{d^4x}{(2\pi)^2} e^{ikx} f(x). \tag{4.41}$$

Substituting the Fourier coefficient into equation (4.40), one can rewrite the Weyl operator in term of a given function, $f(x)$, and the mapping operator $\hat{M}(x)$,

$$\hat{W}[f(x)] \;=\; \int d^D x \hat{M}(x) f(x), \tag{4.42}$$

where

$$\hat{M}(x) \;=\; \int \frac{d^4k}{(2\pi)^4} e^{-ikx} e^{ik\hat{x}}. \tag{4.43}$$

From this context, if one considers in the commutative limit by taking $\theta^{\mu\nu} = 0$, the mapping operator will be reduced to the delta-function, $\delta^4(\hat{x} - x)$. This yields the Weyl operator taking the form $\hat{W}[f(x)] = f(\hat{x})$. Note that, in the noncommutative case ($\theta \neq 0$), the mapping operator is no longer written as the delta-function because it is affected by the Baker-Campbell-Hausdorff formular. Now, we already have the definition of the algebraic operator. Therefore, in the next step, the derivative of the operator will be defined. Generally, the derivation can be defined as the anti-Hermitian linear derivation obeying the commutation relations,

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu, \qquad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \tag{4.44}$$

Considering the commutation relation for the derivative and the mapping operator, one obtains

$$\begin{aligned} [\hat{\partial}_\mu, \hat{M}(x)] &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} [\hat{\partial}_\mu, e^{ik\hat{x}}] \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} (ik_\mu) e^{ik\hat{x}} \\ &= \int \frac{d^4k}{(2\pi)^4} (-\partial_\mu e^{-ikx}) e^{ik\hat{x}} \\ &= -\partial_\mu \hat{M}(x). \end{aligned} \tag{4.45}$$

The second line of this equation is obtained by expanding the exponential as a power series. By using this relation, the translation of the mapping operator can be expressed as

$$\begin{aligned} e^{v\hat{\partial}}\hat{M}(x)e^{-v\hat{\partial}} &= (1+v\hat{\partial}+\frac{1}{2!}(v\hat{\partial})^2+....)\hat{M}(x)e^{-v\hat{\partial}} \\ &= (\hat{M}+\hat{M}v\hat{\partial}-v\partial\hat{M}+\frac{1}{2!}v\hat{\partial}(\hat{M}v\hat{\partial}-v\partial\hat{M})+....)e^{-v\hat{\partial}} \\ &= (\hat{M}e^{v\hat{\partial}}-v\partial\hat{M}-\frac{1}{2!}v\partial\hat{M}v\hat{\partial}-\frac{1}{2!}v\hat{\partial}v\partial\hat{M}+....)e^{-v\hat{\partial}} \\ &= (\hat{M}e^{v\hat{\partial}}-v\partial\hat{M}e^{v\hat{\partial}}+\frac{1}{2!}(v\partial)^2e^{v\hat{\partial}}+....)e^{-v\hat{\partial}} \\ &= (\hat{M}-v\partial\hat{M}+\frac{1}{2!}(v\partial)^2+....)e^{v\hat{\partial}}e^{-v\hat{\partial}} \\ &= \hat{M}(x-v). \end{aligned} \quad (4.46)$$

From this relation, it appears that the trace of the mapping operator is independent of the space-time coordinate (x^μ), $\mathrm{Tr}\Big(e^{v\hat{\partial}}\hat{M}(x)e^{-v\hat{\partial}}\Big) = \mathrm{Tr}\hat{M}(x) = \mathrm{Tr}\hat{M}(x-v)$. One can choose the mapping operator which has trace normalization, $\mathrm{Tr}\hat{M}(x)=1$. Therefore, the trace of the Weyl operator can be writen as

$$\mathrm{Tr}\hat{W}[f(x)] = \int d^4x f(x). \quad (4.47)$$

This property is helpful for us in order to determine the dynamics of scalar field whose action can be expressed in the term of Weyl operator trace. The discussion of this property will be analyzed later in the subject of the property of the star product. Now, we move to the goal of this subsection, the definition of the star product.

The star product is the mapping between the product of the operators that live on the noncommutative space-time and any functions which are the multiplication product of the fields that live on the commutative space-time. Therefore, the star product can be defined in such the way that:

$$\begin{aligned} \hat{W}[f*g] &= \hat{W}[f]\hat{W}[g] \\ &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4k'}{(2\pi)^2}\tilde{f}(k)\tilde{g}(k')\; e^{\,ik\hat{x}}\; e^{\,ik'\hat{x}} \\ &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4k'}{(2\pi)^2}\tilde{f}(k)\tilde{g}(k')\; e^{\,i\hat{x}(k+k')}\; e^{\,-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu} \\ &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4k'}{(2\pi)^2}\tilde{f}(k)\tilde{g}(k'-k)\; e^{\,-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\; e^{\,i\hat{x}k'}. \end{aligned} \quad (4.48)$$

From the third line of this equation, one uses the Baker-Campbell-Hausdorff formula for the anti-symmetric constant matrix which takes the form

$$e^{\,ik_\mu\hat{x}^\mu}\; e^{\,ik_\nu\hat{x}^\nu} = e^{\,i\hat{x}^\mu(k+k')_\mu}\; e^{\,-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}. \quad (4.49)$$

Using the relation $\hat{W}[e^{ikx}] = e^{ik\hat{x}}$ from (4.40) and (4.41), one obtains

$$\begin{aligned} f(x) * g(x) &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4k'}{(2\pi)^2}\tilde{f}(k)\tilde{g}(k'-k)\; e^{\,-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\; e^{\,ixk'} \\ &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4k'}{(2\pi)^2}\tilde{f}(k)\tilde{g}(k')\; e^{\,-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\; e^{\,ix(k'+k)} \\ &= \int \Big(\frac{d^4k}{(2\pi)^2}\tilde{f}(k)\; e^{\,ikx}\Big)\; e^{\,-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\Big(\frac{d^4k'}{(2\pi)^2}\tilde{g}(k')\; e^{\,ixk'}\Big) \\ &= f(x)e^{\,\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}g(x). \end{aligned} \tag{4.50}$$

where $\overleftarrow{\partial}_\mu$ and $\overrightarrow{\partial}_\nu$ are the derivative from right and the derivative from left, respectively. From this definition, the star product is the ordinary product added by the infinite space-time derivatives of the functions. Comparing the trace of Weyl operator in the star product function in commutative space-time, $\mathrm{Tr}\hat{W}[f * g] = \int d^4x f(x) * g(x)$, with the integration of the multiplication product function in the noncommutative space-time, $\int d^4x f(\hat{x})g(\hat{x})$, one obtains

$$f(\hat{x})g(\hat{x}) \;=\; f(x) * g(x). \tag{4.51}$$

This allows us to directly replace the multiplication product in noncommutative space-time by the star product in ordinary space-time. From this context, it promotes us to calculate the object in noncommutative space-time by the object in commutative space-time which is rather easier. From this definition of the star product, it yields some different calculations from the commutative space-time which one can summarize in the context of the properties for the star product in the next subsection.

4.3.2 The Properties of the Star Product

In this subsection, we will investigate the properties of the star product by using its definition and the properties of the trace of Weyl operaters. First of all, one considers the star product of two exponential functions which can be written as

$$\begin{aligned} e^{ikx} * e^{ipx} &= e^{ikx}\; e^{\,\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}\; e^{ipx} \\ &= e^{ikx}\Big(1 + \frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu + \frac{1}{2!}\Big(\frac{i}{2}\Big)^2\overleftarrow{\partial}_\mu\overleftarrow{\partial}_\rho\theta^{\mu\nu}\theta^{\rho\sigma}\overrightarrow{\partial}_\nu\overrightarrow{\partial}_\sigma + ...\Big)e^{ipx} \\ &= e^{ix(k+p)}\Big(1 + \frac{i}{2}k\wedge p + \frac{1}{2!}\Big(\frac{i}{2}\Big)^2(k\wedge p)^2 + ...\Big) \\ &= e^{ix(k+p)}\; e^{-\frac{i}{2}k\wedge p}, \end{aligned} \tag{4.52}$$

where $\wedge$ is the wedge product which is defined as $k\wedge p = k_\mu\theta^{\mu\nu}p_\nu$. This property is similar to the Baker-Campbell-Hausdorff formula. Using the Fourier transformation and property of the exponential star product, one obtains

$$\begin{aligned}
\Big(f(x)*g(x)\Big)*h(x) &= \int \frac{d^4k}{(2\pi)^2}\frac{d^4p}{(2\pi)^2}\frac{d^4q}{(2\pi)^2}\tilde{f}(k)\tilde{g}(p)\tilde{h}(q)\Big(e^{ikx}*e^{ipx}\Big)*e^{iqx} \\
&= \int \frac{d^4K}{(2\pi)^2}\tilde{F}(k)\ e^{-\frac{i}{2}k\wedge p}e^{ix(k+p)}*e^{iqx} \\
&= \int \frac{d^4K}{(2\pi)^2}\tilde{F}(k)\ e^{-\frac{i}{2}k\wedge p}\ e^{-\frac{i}{2}q\wedge(k+p)}\ e^{ix(k+p+q)} \\
&= \int \frac{d^4K}{(2\pi)^2}\tilde{F}(k)\ e^{-\frac{i}{2}p\wedge q}\ e^{-\frac{i}{2}k\wedge(p+q)}\ e^{ix(k+p+q)} \\
&= \int \frac{d^4K}{(2\pi)^2}\tilde{F}(k)\ e^{-\frac{i}{2}p\wedge q}e^{ikx}*e^{ix(p+q)} \\
&= \int \frac{d^4k}{(2\pi)^2}\frac{d^4p}{(2\pi)^2}\frac{d^4q}{(2\pi)^2}\tilde{f}(k)\tilde{g}(p)\tilde{h}(q)e^{ikx}*\Big(e^{ipx}*e^{iqx}\Big) \\
\Big(f(x)*g(x)\Big)*h(x) &= f(x)*\Big(g(x)*h(x)\Big). \qquad (4.53)
\end{aligned}$$

This is the associative property of the star product. Note that, we use the short notation, $\tilde{F}(k)=\tilde{f}(k)\tilde{g}(p)\tilde{h}(q)$ and $\int\frac{d^4K}{(2\pi)^2}=\int\frac{d^4k}{(2\pi)^2}\frac{d^4p}{(2\pi)^2}\frac{d^4q}{(2\pi)^2}$, while we derive this property. Next, one considers the trace of the Weyl operator for n-function which takes the form

$$\begin{aligned}
\mathrm{Tr}\Big(\hat{W}[f_1]\ \hat{W}[f_2]...\hat{W}[f_n]\Big) &= \mathrm{Tr}\hat{W}[f_1*f_2*...f_n] \\
&= \int d^4x(\ f_1(x)*f_2(x)*...f_n(x)\ ) \\
\mathrm{Tr}\Big(\hat{W}[f_n]\ \hat{W}[f_1]...\hat{W}[f_{n-1}]\Big) &= \int d^4x(\ f_n(x)*f_1(x)*...f_{n-1}(x)\ ), \qquad (4.54)
\end{aligned}$$

This implies that the cyclic property of operator trace yields the cyclic property of the star product under integration over space-time coordinate. For the special case of this property, one considers only two functions,

$$\begin{aligned}
\mathrm{Tr}\hat{W}[f*g] &= \mathrm{Tr}\hat{W}[g*f] \\
\int d^4x(\ f(x)*g(x)\ ) &= \int d^4x(\ g(x)*f(x)\ ) \\
&= \int d^4x\frac{d^4k}{(2\pi)^2}\frac{d^4p}{(2\pi)^2}\tilde{f}(k)\tilde{g}(p)\ e^{-\frac{i}{2}k\wedge p}e^{ix(k+p)} \\
&= \int d^4kd^4p\tilde{f}(k)\tilde{g}(p)\ e^{-\frac{i}{2}k\wedge p}\delta^4(k+p) \\
&= \int d^4k\tilde{f}(k)\tilde{g}(-k) \\
&= \int d^4x\ f(x)\ g(x). \qquad (4.55)
\end{aligned}$$

This means that the star product and the ordinary product are the same under integration over space-time coordinate. It is very important property of the star product because it yields the same result between the free theories in the commutative and that in noncommutative space-time. Therefore, for the free theory, the quantization in noncommutative space-time can be done by replacing with the commutative quantization. Moreover, by using the same action of free theory, it yields the same result for conservation law in order to take into account the Noether theorem. Finally, one can define the commutator which takes into account the star product,

$$\begin{aligned} [f,g]_{MB} &= f(x)*g(x)-g(x)*f(x) \\ &= f(x)e^{\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}g(x)-g(x)e^{\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}f(x) \\ &= 2if(x)\sin\Big(\frac{1}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu\Big)g(x). \end{aligned} \quad (4.56)$$

From above equation the even-order terms of $\theta^{\mu\nu}$ vanish because it is the anti-symmetric tensor. This commutator is well known as the Moyal bracket. From the property in (4.55), it yields that the integral of this commutator over the space-time coordinate vanishes.

4.3.3 Noncommutative Perturbation Theory

In the previous subsection, we conclude that dynamics of the free scalar field is the same for both the noncommutative and commutative space-time. Therefore, in this subsection, we will discuss the different results of the noncommutative and commutative space-time when adding the interaction term in Lagrangian. After using the properties derived in the previous subsection, the action of scalar field in noncommutative space-time can be written as

$$S = \int\Big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi*\phi*\phi*\phi\Big)d^4x. \quad (4.57)$$

This leads to the equation of motion which takes the form

$$\partial_\mu\partial^\mu\phi+m^2\phi = -\frac{\lambda}{3!}\phi*\phi*\phi. \quad (4.58)$$

In order to see how the noncommutative quantization differs from the commutative case, one considers the conjugate momentum which can be rewritten as

$$\pi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\dot{\phi}-\frac{\lambda}{3!}\frac{\partial(\phi*\phi*\phi)}{\partial\dot{\phi}}. \quad (4.59)$$

From this conjugate momentum, the difference occurs due to the second term which contains the infinite number of time derivatives. This infinite number of derivative suggests that the theory is non-local in time. Therefore, the causality may be violated and the unitarity is not preserved at the quantum level [17]. However, in the special case which time and space are commute ($\theta^{0i} = 0$), the conjugate momentum in noncommutative space-time is as same as that in commutative space-time because the time derivatives in the interaction term are vanish. Thus, we can roughly state that the quantization of the interacting scalar field in noncommutative space-time can be done only in the context of space-space noncommutivity.

We should also discuss the Noether theorem and see how the noncommutative space-time affects the conservation law. Generally, the energy momentum tensor can be written in noncommutative space-time as

$$\begin{aligned} T^{\mu}_{\ \nu} &= \frac{1}{2}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu)} * \partial_\nu\phi + \partial_\nu\phi * \frac{\partial\mathcal{L}}{\partial(\partial_\mu)}\Big) - \eta^{\mu}_{\ \nu}\mathcal{L} \\ &= \frac{1}{2}\Big(\partial^\mu\phi * \partial_\nu\phi + \partial_\nu\phi * \partial^\mu\phi\Big) + \eta^\mu_\nu\Big(\frac{1}{2}\partial_\rho\phi * \partial^\rho\phi - \frac{1}{2}m^2\phi^{*2} - \frac{\lambda}{4!}\phi^{*4}\Big), \end{aligned} \tag{4.60}$$

where ϕ^{*n} is the number, n, of ϕ with the star product. Considering the divergence of this tensor, one obtains

$$\begin{aligned} \partial_\mu T^{\mu}_{\ \nu} &= \frac{1}{2}\Big(\partial_\mu\partial^\mu\phi * \partial_\nu\phi + \mu\phi * \partial_\mu\partial_\nu\phi + \partial_\mu\partial_\nu\phi * \partial^\mu\phi + \partial_\nu\phi * \partial_\mu\partial^\mu\phi\Big) \\ &\quad + \Big(\frac{1}{2}\partial_\nu\partial_\mu\phi * \partial^\mu\phi + \partial_\mu\phi * \partial_\nu\partial^\mu\phi\Big) + \frac{1}{2}m^2\Big(\partial_\nu\phi * \phi + \phi * \partial_\nu\phi\Big) \\ &\quad - \frac{\lambda}{4!}\Big(\partial_\nu * \phi^{*3} + \phi * \partial_\nu\phi * \phi^{*2} + \phi^{*2} * \partial_\nu\phi * \phi + \phi^{*3} * \partial_\nu\phi\Big) \\ &= \frac{\lambda}{4!}\Big(- \partial_\nu * \phi^{*3} + \phi * \partial_\nu\phi * \phi^{*2} + \phi^{*2} * \partial_\nu\phi * \phi - \phi^{*3} * \partial_\nu\phi\Big) \\ &= \frac{\lambda}{4!}\Big([\phi, \partial_\nu\phi]_{MB} * \phi^{*2} - \phi^{*2} * [\phi, \partial_\nu\phi]_{MB}\Big) \\ &= \frac{\lambda}{4!}\Big([\ [\phi, \partial_\nu\phi]_{MB}, \phi^{*2}\ ]_{MB}\Big). \end{aligned} \tag{4.61}$$

Note that in the second step one uses the equation of motion. The conserved charge can be rewritten as $Q_\nu = \int T^0{}_\nu d^3x$, which is the integration over only spatial coordinate. Thus, it is conserved only in the special case, $\theta^{0i} = 0$. This is consistent with conjugate momentum analysis which the problem occurs in the noncommutivity of the space and time. However, in the next chapter we will consider the effect of noncommutative space-time in a cosmological regime.

As we mentioned, quantum field theory can be constructed in the regime of the noncommutative space. In this regime, the consequence of the noncommutative effect will exist in the Feymann diagram. From the cyclic property of the star product, it yields the difference in the vertex of Feymann diagram which is invariant only under cyclic permutation of four-incoming momentum. This leads to the loop diagram which cannot be written in the plane. This diagram is known as the "non-planar diagram". Moreover, these non-planar diagrams are responsible for the UV/IR mixing effect. The UV/IR mixing is the effect of the ultraviolet divergence on the infrared behavior which has no analog in the conventional quantum field theory. For the detail of these topics please look at [16] and [17] for a review.

CHAPTER V

NONCOMMUTATIVE INFLATION

In Chapter 3, we describe the evolution of the early universe by using the inflationary model. We also assume that inflation took place at the time when the size of the early universe is very small near the Planck scales. In such a small length scale, the effect of quantum gravity cannot be neglected. String theory is the most promising candidate for the quantum gravity. This theory states that the elementary particle is a string which has the minimum length called string length l_s. As a result of this theory, there is the uncertainty relation in space and time $\Delta x_p \Delta t_p \geq l_s^2$, where t_p and x_p denote the physical time and spatial coordinate, respectively [27]. This relation can be realized by non-commutative space-time $[x_p, t_p] = il_s$ [26]. However, in string theory, this commutation relation cannot be simply written down because this relation is neither covariant nor meaningful when applied to any degree of freedom [28]. Thus, we must keep in mind that it is just a model that Brandenberger and Ho rose up, and we will follow them in the first part of this chapter.

5.1 Dynamics of Fluctuation Scalar Field in Noncommutative Space-Time

The main object of this section is to find the action for the fluctuation field in noncommutative space-time. For convenience, we will consider the action of the scalar perturbation field which is deduced from the action of the tensor perturbation field. Usually, we will consider the FRW metric in the context of the conformal time but the stringy space-time uncertainty relation (SSUR) for the conformal time is not well defined due to the uncertainty relation, $\Delta\eta\Delta x \geq \frac{l_s^2}{a^2(\eta)}$. It is not clear when $\Delta\eta$ is large because the scale factor in the right hand side can be changed. Thus, we will modify the FRW metric by defining the new time coordinate τ which relates to the comoving time by $dt = a^{-1}d\tau$, and the metric can be written in the form

$$ds^2 = a^{-2}(\tau)d\tau^2 - a^2(\tau)d\vec{x}^2. \tag{5.1}$$

Note that we consider in spatially flat universe. So the SSUR is written by

$$\Delta t\Delta x_p = \Delta\tau\Delta x \geq l_s^2. \tag{5.2}$$

The algebra of the noncommutative space-time associating to this relation can be written as

$$[\tau, x] = 2il_s^2. \tag{5.3}$$

In order to avoid the breakdown of cosmological principle by the noncommutative algebra, we will use the tricky technique for this calculation. This trick is that we first calculate the action through the star product in two dimensional space-time. This calculation no longer spoils the cosmological principle. And then we will extend this action to four dimensional space-time later. Furthermore, this trick provides FRW metric preserving both the rotational and translational symmetries which correspond to the cosmological principle as we will see later in the final part of this section.

First, we consider two dimensional space-time. The metric in (5.1) can be written as

$$g_{\mu\nu} = \begin{pmatrix} a^{-2}(\tau) & 0 \\ 0 & -a^2(\tau) \end{pmatrix} \tag{5.4}$$

$$g^{\mu\nu} = \begin{pmatrix} a^{2}(\tau) & 0 \\ 0 & -a^{-2}(\tau) \end{pmatrix}. \tag{5.5}$$

In the case of commutative space-time, one obtains the action

$$S = \int d\tau dx \frac{1}{2}(\partial_\mu\phi^\dagger g^{\mu\nu}\partial_\nu\phi). \tag{5.6}$$

For noncommutative space-time, we replace the multiplication product of the scalar field by the star product.

$$\begin{aligned} \tilde{S} &= \int d\tau dx \frac{1}{2}(\partial_\mu\phi^\dagger * g^{\mu\nu} * \partial_\nu\phi) \\ &= \int d\tau dx \frac{1}{2}\Big(\partial_\tau\phi^\dagger * a^2 * \partial_\tau\phi - \big(\partial_x\phi\big)^\dagger * a^{-2} * \partial_x\phi\Big). \end{aligned} \tag{5.7}$$

The fourier transform of ϕ can be written as

$$\phi(x,\tau) = \frac{V^{\frac{1}{2}}}{2}\int_{|k|<k_0}\frac{dk}{\sqrt{2\pi}}\Big(\phi_k(\tau)e^{ikx} + \phi_k^\dagger(\tau)e^{-ikx}\Big), \tag{5.8a}$$

$$\phi^\dagger(x,\tau) = \frac{V^{\frac{1}{2}}}{2}\int_{|k|<k_0}\frac{dk}{\sqrt{2\pi}}\Big(\phi_k^\dagger(\tau)e^{-ikx} + \phi_k(\tau)e^{ikx}\Big), \tag{5.8b}$$

where V is the volume in spatial coordinates. Let us consider the first term in (5.7), we obtain

$$\begin{aligned}
\tilde{S}_1 &= \frac{V}{8}\int\frac{d\tau dx}{2\pi}\Big(\int_{|k|<k_0}dk(\partial_\tau\phi_k^\dagger e^{-ikx} + \partial_\tau\phi_k e^{ikx}) * a^2 \\
&\quad * \int_{|k|<k_0}dp(\partial_\tau\phi_p e^{ipx} + \partial_\tau\phi_p^\dagger e^{-ipx})\Big) \\
&= \frac{V}{8}\int_{|k|<k_0}\frac{d\tau dxdkdp}{2\pi}\Big((\partial_\tau\phi_k^\dagger e^{-ikx} + \partial_\tau\phi_k e^{ikx})e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}}a^2\Big) \\
&\quad *(\partial_\tau\phi_p e^{ipx} + \partial_\tau\phi_p^\dagger e^{-ipx}) \\
&= \frac{V}{8}\int_{|k|<k_0}\frac{d\tau dxdkdp}{2\pi}\Big((\partial_\tau\phi_k^\dagger e^{-ikx} + \partial_\tau\phi_k e^{ikx})e^{il_s^2(\overleftarrow{\partial_\tau}\overrightarrow{\partial_x} - \overleftarrow{\partial_x}\overrightarrow{\partial_\tau})}a^2\Big) \\
&\quad *(\partial_\tau\phi_p e^{ipx} + \partial_\tau\phi_p^\dagger e^{-ipx}) \\
&= \frac{V}{8}\int_{|k|<k_0}\frac{d\tau dxdkdp}{2\pi}\Big((\partial_\tau\phi_k^\dagger e^{-ikx}e^{-l_s^2k\partial_\tau} + \partial_\tau\phi_k e^{ikx}e^{l_s^2k\partial_\tau})a^2\Big) \\
&\quad *(\partial_\tau\phi_p e^{ipx} + \partial_\tau\phi_p^\dagger e^{-ipx}) \\
&= \frac{V}{8}\int_{|k|<k_0}\frac{d\tau dxdkdp}{2\pi}\Big((\partial_\tau\phi_k^\dagger\partial_\tau\phi_p e^{-ix(k-p)} + \partial_\tau\phi_k^\dagger\partial_\tau\phi_p^\dagger e^{-ix(k+p)})e^{-l_s^2k\partial_\tau}a^2 \\
&\quad + (\partial_\tau\phi_k\partial_\tau\phi_p e^{ix(k+p)} + \partial_\tau\phi_k\partial_\tau\phi_p^\dagger e^{ix(k-p)})e^{l_s^2k\partial_\tau}a^2\Big) \\
&= \frac{V}{8}\int_{|k|<k_0}d\tau dk\Big((\partial_\tau\phi_k^\dagger\partial_\tau\phi_k + \partial_\tau\phi_k^\dagger\partial_\tau\phi_{-k}^\dagger)e^{-l_s^2k\partial_\tau}a^2 \\
&\quad +(\partial_\tau\phi_k\partial_\tau\phi_{-k} + \partial_\tau\phi_k\partial_\tau\phi_k^\dagger)e^{l_s^2k\partial_\tau}a^2\Big).
\end{aligned} \tag{5.9}$$

We assume that ϕ is the real scalar field. Due to the reality condition of the field $\phi(x,\tau)$, we have $\phi_k(\tau) = \phi_{-k}^\dagger(\tau)$. Thus, equation (5.9) become

$$\begin{aligned}
\tilde{S}_1 &= \frac{V}{4}\int_{|k|<k_0}d\tau dk\Big(\partial_\tau\phi_{-k}\partial_\tau\phi_k e^{-l_s^2k\partial_\tau}a^2(\tau) + \partial_\tau\phi_{-k}\partial_\tau\phi_k e^{l_s^2k\partial_\tau}a^2(\tau)\Big) \\
&\simeq \frac{V}{4}\int_{|k|<k_0}d\tau dk\Big(a^2(\tau - kl_s^2) + a^2(\tau + kl_s^2)\Big)\partial_\tau\phi_{-k}\partial_\tau\phi_k.
\end{aligned} \tag{5.10}$$

The second term in action (5.7) can be calculated in a similar way as in (5.9) and takes the form

$$\tilde{S}_2 \simeq \frac{V}{4}\int_{|k|<k_0}d\tau dk\Big(a^{-2}(\tau - kl_s^2) + a^{-2}(\tau + kl_s^2)\Big)k^2\phi_{-k}\phi_k. \tag{5.11}$$

Combining the first and the second terms, the action (5.7) reduces to

$$\tilde{S} \;\simeq\; V\int_{|k|<k_0} d\tau dk \frac{1}{2}\Big(\beta_k^+ \partial_\tau \phi_{-k}\partial_\tau \phi_k - \beta_k^- k^2 \phi_{-k}\phi_k\Big), \tag{5.12}$$

where

$$\beta_k^\pm = \frac{1}{2}(a^{\pm 2}(\tau - kl_s^2) + a^{\pm 2}(\tau + kl_s^2)). \tag{5.13}$$

From this action, we neglect the effect of SSUR by taking the commutative limit, $l_s = 0$, and then obtain $\beta_k^\pm(\tau) = a^\pm(\tau)$. So, the action in (5.12) becomes the action in commutative space-time as we expect. Note that the action is non-local in time through the $\beta_k^\pm(\tau)$ function which is the result of the realization of SSUR by noncommutative algebra. Now we already have obtained the action in the context of noncommutative algebra. Generally, the SSUR is not realized by noncommutative algebra. In this context, $\beta_k^\pm(\tau)$ can be expanded in the order of $\frac{H}{M_s}$ [29] which take the form

$$\beta_k^\pm(\tau) = \Big(1 + C_1^\pm \frac{Hp}{M_s^2} + C_2^\pm \frac{(\dot{H} + H^2)p^2}{M_s^4} + ...\Big)a^\pm(\tau). \tag{5.14}$$

Where $C^\pm$ are constants of order 1, $p = \frac{k}{a}$ is physical momentum and $M_s = \frac{1}{l_s}$ is energy scale of string theory. In this case, when we take $l_s = 0$, it yields the same result as the commutative case.

From the action above, we have an upper bound of integration of k at k_0 because we need the fluctuation modes ϕ_k which satisfy SSUR. Therefore, it means that we have an initial time at which the fluctuation modes begin. However, the upper bound of momentum k_0 is easily calculated by defining the energy in τ coordinate as $E_k = ka^{-2}(\tau)$. But in the context of non-local time, we need the effective scale factor a_e which can be defined as

$$a_e^2(\tau) = \Big(\frac{\beta_k^+}{\beta_k^-}\Big)^{1/2}. \tag{5.15}$$

Then the energy can be written as $E_k = ka_e^{-2}$. By using the relations of uncertainty principle $\Delta x \sim \frac{1}{k}$, $\Delta\tau \sim \frac{1}{E_k}$ and SSUR, we have

$$\Big(\frac{a_e(\tau)}{k}\Big)^2 \sim \Delta t \Delta x_p = \Delta\tau\Delta x \geq l_s^2. \tag{5.16}$$

Finally, the upper bound of energy scale or momentum scale can be defined as

$$k_0 \equiv \frac{a_e(\tau)}{l_s}. \tag{5.17}$$

In order to calculate the power spectrum, we have to write the action in the term of the conformal time. Since we have seen the metric above, the relation of conformal time and τ can be written as $d\eta = a^{-2}d\tau$. In the context of non-local time we replace a by a_e. Note that we put $(\tilde{\ })$ over the noncommutative parameter in order to be not confused. Then the noncommutative conformal time takes the form $d\tilde{\eta} = a_e^{-2}d\tau$. Thus, the action can be written as

$$\begin{aligned}
\tilde{S} &= V\int_{|k|<k_0} a_e^2 d\tilde{\eta}dk\frac{1}{2}\Big(\beta_k^+ a_e^{-4}\partial_{\tilde{\eta}}\phi_{-k}\partial_{\tilde{\eta}}\phi_k - k^2\beta_k^-\phi_{-k}\phi_k\Big)\\
&= V\int_{|k|<k_0} d\tilde{\eta}dk\frac{1}{2}\Big((\beta_k^+\beta_k^-)^{1/2}\phi'_{-k}\phi'_k - k^2(\beta_k^+\beta_k^-)^{1/2}\phi_{-k}\phi_k\Big)\\
&= V\int_{|k|<k_0} d\tilde{\eta}dk\frac{1}{2}y_k^2(\tilde{\eta})\Big(\phi'_{-k}\phi'_k - k^2\phi_{-k}\phi_k\Big),
\end{aligned} \tag{5.18}$$

where $y_k^2(\tilde{\eta}) = (\beta_k^+\beta_k^-)^{1/2}$, and prime denotes the derivative with respect to noncommutative conformal time $\tilde{\eta}$. To extend the action from two dimensional space-time to four dimensional space-time, we have to compare this action with the action of tensor perturbation in $d+1$ dimensions of commutative space-time (3.126) which can be rewritten as

$$S = V\int d\eta d^dk\frac{1}{2}a^{d-1}\Big(\phi'_{-k}\phi'_k - k^2\phi_{-k}\phi_k\Big). \tag{5.19}$$

From this action, we will see that the difference between the action in noncommutative and commutative space-time is $y_k^2(\tilde{\eta})$ and a^{d-1}. Therefore, we can extend the action in noncommutative space-time from $1+1$ to $d+1$ dimensions by expressing the smeared version of a^{d-1} as $\tilde{a}_k^{d-1} = y_k^2(\tilde{\eta})a^{d-1}$ for the tensor perturbation. Furthermore, we can extend to the scalar perturbation by writing the smeared version of z^{d-1} in the form of $\tilde{z}_k^{d-1} = y_k^2(\tilde{\eta})z^{d-1}$. So, the action of scalar perturbation in four dimensional noncommutative space-time can be written as

$$\tilde{S} = V\int_{|k|<k_0} d\tilde{\eta}d^3k\frac{1}{2}\tilde{z}_k^2\Big(\phi'_{-k}\phi'_k - k^2\phi_{-k}\phi_k\Big), \tag{5.20}$$

where $\tilde{z}_k^2 = y_k^2 z^2$. As we have mentioned above, the advantage of this action is that it preserves both translational and rotational symmetries of flat FRW metric. Before we move to the next section to calculate the power spectrum of the fluctuation field, we will emphasize here that first we have smeared version of scale factor or z which depends on k. Second we have an upper bound of k which depends on the time when the SSUR is saturated, denoting by $\tilde{\eta}_0$.

5.2 Power Spectrum in Noncommutative Space-Time

In this section, we will find the power spectrum of the curvature perturbation in the noncommutative space-time and compare this power spectrum to the commutative case. The quantization is a significant mechanism to yield the power spectrum. Therefore, the first part of this section, we will quantize the scalar perturbation field (normally, this field is called the gauge invariant potential). For convenience, the Heisenberg picture in which the operator depends on time is preferred. In the next subsection, we calculate the power spectrum by using the initial vacuum which minimizes the uncertainty relation at the time $\tilde{\eta} = \tilde{\eta}_0$. However, in the last subsection, the result can be compared to another initial vacuum. Furthermore, we will compare the result with the commutative case.

5.2.1 Quantization of Scalar Perturbation Field

The metric perturbation can be classified in three components as tensor, vector and scalar perturbations. The vector perturbation can be neglected because there is no rotational velocity field during inflation stage [10]. In this subsection, we will consider only the scalar perturbation because the tensor perturbation can be deduced from scalar perturbation by replacing a_k with z_k. From Chapter 3, the action for the scalar perturbation can be written as

$$S = \frac{1}{2}\int d^4x\Big(v'^2 - \partial_i v\partial^i v + \frac{z''}{z}v^2\Big). \tag{5.21}$$

To compare with the action in the previous section, we must transform this action by using the Fourier transformation,

$$v(\vec{x},\eta) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^{3/2}}\Big(v_k(\eta)e^{i\vec{k}\cdot\vec{x}} + v_k^*(\eta)e^{-i\vec{k}\cdot\vec{x}}\Big), \tag{5.22a}$$

$$v^*(\vec{x},\eta) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^{3/2}}\Big(v_k^*(\eta)e^{-i\vec{k}\cdot\vec{x}} + v_k(\eta)e^{i\vec{k}\cdot\vec{x}}\Big). \tag{5.22b}$$

The action in (5.21) can be written as

$$S = \frac{1}{2}\int d^3k d\eta\Big(v'_{-k}v'_k - (k^2 - \frac{z''}{z})v_{-k}v_k\Big). \tag{5.23}$$

By using the Euler-Lagrange equation, the equation of motion can be written as

$$v''_k + (k^2 - \frac{z''}{z})v_k = 0. \tag{5.24}$$

This equation of motion is equivalent to the equation of motion for the field $\mathcal{R}_k = -\phi_k/\tilde{z}_k$ from the noncommutative action in (5.20) by replacing $\frac{z''}{z}$ by $\frac{\tilde{z}''_k}{\tilde{z}_k}$. For the canonical quantization, the Hamiltonian formalism is a significant mechanism, and then we will find the Hamiltonian. Conveniently, the action in (5.21) can be rewritten as

$$S = \frac{1}{2}\int d^4x\Big(v'^2 - \partial_i v\partial^i v - 2\frac{z'}{z}vv' + \Big(\frac{z'}{z}\Big)^2 v^2\Big). \tag{5.25}$$

Thus, the Lagrangian density takes the form:

$$\mathcal{L} = \frac{1}{2}\Big(v'^2 - \partial_i v\partial^i v - 2\frac{z'}{z}vv' + \Big(\frac{z'}{z}\Big)^2 v^2\Big). \tag{5.26}$$

The conjugate momentum can be defined by

$$\pi = \frac{\partial\mathcal{L}}{\partial v'} = v' - \frac{z'}{z}v. \tag{5.27}$$

The Lagrangian density (5.26) can be written in terms of the conjugate momentum as

$$\mathcal{L} = \frac{1}{2}\Big(v'\pi - \partial_i v\partial^i v - \frac{z'}{z}v\pi\Big). \tag{5.28}$$

We can write the Hamiltonian density as

$$\begin{aligned}
\mathcal{H} = v'\pi - \mathcal{L} &= \frac{1}{2}\Big(v'\pi + \partial_i v\partial^i v + \frac{z'}{z}v\pi\Big). \\
&= \frac{1}{2}\Big(v'\pi - \frac{z'}{z}v\pi + \partial_i v\partial^i v + 2\frac{z'}{z}v\pi\Big) \\
&= \frac{1}{2}\Big(\pi^2 + \partial_i v\partial^i v + 2\frac{z'}{z}v\pi\Big).
\end{aligned} \tag{5.29}$$

These lead to Hamiltonian in momentum space as

$$\begin{aligned}
H &= \int d^3x\mathcal{H} = \frac{1}{2}\int d^3x\Big(v'\pi + \partial_i v\partial^i v + \frac{z'}{z}v\pi\Big) \\
&= \int d^3k\Big(\frac{1}{2}\pi_k\pi_{-k} + \frac{1}{2}k^2 v_k v_{-k} + \frac{z'}{z}(v_k\pi_{-k} + v_{-k}\pi_k)\Big).
\end{aligned} \tag{5.30}$$

Using the canonical quantization, the fields $v(\vec{x},\eta)$ and $\pi(\vec{x},\eta)$ are replaced by the operators $\hat{v}(\vec{x},\eta)$ and $\hat{\pi}(\vec{x},\eta)$ which satisfy the standard commutation relations on the η= constant hypersurface:

$$\Big[\hat{v}(\vec{x},\eta), \hat{v}(\vec{x}',\eta)\Big] = \Big[\hat{\pi}(\vec{x},\eta), \hat{\pi}(\vec{x}',\eta)\Big] = 0, \tag{5.31a}$$

$$\Big[\hat{v}(\vec{x},\eta), \hat{\pi}(\vec{x}',\eta)\Big] = i\delta^3(\vec{x} - \vec{x}'). \tag{5.31b}$$

In Fourier modes, the commutation relations above are

$$\Big[\hat{v}_k(\eta), \hat{v}_{k'}(\eta)\Big] = \Big[\hat{\pi}_k(\eta), \hat{\pi}_{k'}(\eta)\Big] = 0, \tag{5.32a}$$

$$\Big[\hat{v}_k(\eta), \hat{\pi}^\dagger_{k'}(\eta)\Big] = i. \tag{5.32b}$$

Generally, the operators can be written in the term of composition of the annihilation and creation operators as

$$\hat{\pi}_k = -i\sqrt{\frac{k}{2}}(\hat{a}_k - \hat{a}^\dagger_{-k}), \qquad \hat{\pi}_{-k} = \hat{\pi}^\dagger_k = i\sqrt{\frac{k}{2}}(\hat{a}^\dagger_k - \hat{a}_{-k}), \tag{5.33}$$

$$\hat{v}_k = \sqrt{\frac{1}{2k}}(\hat{a}_k + \hat{a}^\dagger_{-k}), \qquad \hat{v}_{-k} = \hat{v}^\dagger_k = \sqrt{\frac{1}{2k}}(\hat{a}^\dagger_k + \hat{a}_{-k}). \tag{5.34}$$

After substituting (5.33) and (5.34) into (5.30), the Hamiltonian operator takes the form

$$\hat{H} = \int d^3k\Big(\frac{k}{2}(\hat{a}_k\hat{a}^\dagger_k + \hat{a}^\dagger_{-k}\hat{a}_{-k}) + i\frac{z'}{z}(\hat{a}^\dagger_{-k}\hat{a}^\dagger_k - \hat{a}_k\hat{a}_{-k})\Big). \tag{5.35}$$

The last piece of this integrand is responsible for the squeezing and effects the time evolution of the system [30].

According to the canonical quantization in quantum field theory, the Heisenberg picture is more useful. From this picture, there is a unique vacuum state $\mid 0\rangle$. The vacuum state can be defined in the Fock representation of Hilbert space of the state that the operators act on. This vacuum state can be expressed as

$$\hat{a}_k \mid 0\rangle = 0, \quad \forall k. \tag{5.36}$$

However, this vacuum state can be defined only in the flat space-time with the constant mass. Thus, the issue of a vacuum state in the time dependent background such as the cosmological background cannot be well defined [11] because the vacuum state in the later time will no longer be a vacuum state. Nevertheless, we can pick an initial vacuum state $\mid 0\rangle_{in}$ which satisfy $\hat{a}_k(\eta = \eta^0) \mid 0\rangle_{in} = 0, \;\; \forall k$. We then transform the operators to what satisfy $\hat{a}_k(\eta) \mid 0\rangle = 0, \;\; \forall k$ which corresponds to commutation relation of the annihilation and creation operators. This transformation is called the Bogoliubov transformation which can be defined as

$$\hat{a}_k(\eta) = \alpha_k(\eta)\hat{a}_k(\eta^0) + \beta_k(\eta)\hat{a}^\dagger_{-k}(\eta^0), \tag{5.37}$$

$$\hat{a}^\dagger_k(\eta) = \beta^*_k(\eta)\hat{a}_{-k}(\eta^0) + \alpha^*_k(\eta)\hat{a}^\dagger_k(\eta^0). \tag{5.38}$$

The parameters α_k and β_k are called the Bogoliubov coefficients which obey the relation

$$\alpha_k\alpha^*_k - \beta_k\beta^*_k = 1, \quad \forall k. \tag{5.39}$$

The significant treatment is the initial vacuum state that we pick. If we pick the unreasonable vacuum, it may yield the unphysical vacuum state at the later time. However, we have a choice which corresponds to physical states. It is the adiabatic vacuum, which the initial state is assumed to be an empty vacuum in the infinite past [31]. Thus, it would rather not make sense when we talk about the SSUR which has a finite time. We will follow the idea of Danielsson [31](and also [32]) by choosing the vacuum state which minimizes the space-time uncertainty relation. This vacuum has a condition which takes the form

$$\hat{\pi}_k(\eta^0) \mid 0\rangle_{in} = ik\hat{v}_k(\eta^0) \mid 0\rangle_{in}, \tag{5.40}$$

and the physical interpretation of this vacuum has been discussed in [30]. Using (5.37), (5.38), (5.33) and (5.34) one obtains

$$\begin{aligned} \hat{v}_k(\eta) &= f_k(\eta)\hat{a}_k(\eta^0) + f_k^*(\eta)\hat{a}_{-k}^\dagger(\eta^0), & (5.41)\\ \hat{\pi}_k(\eta) &= -i\Big(g_k(\eta)\hat{a}_k(\eta^0) - g_k^*(\eta)\hat{a}_{-k}^\dagger(\eta^0)\Big), & (5.42)\end{aligned}$$

where

$$\begin{aligned} f_k(\eta) &= \sqrt{\frac{1}{2k}}\Big(\alpha_k(\eta) + \beta_k^*(\eta)\Big), \\ g_k(\eta) &= \sqrt{\frac{k}{2}}\Big(\alpha_k(\eta) - \beta_k^*(\eta)\Big). & (5.43)\end{aligned}$$

From (5.41), the equation of motion for f_k can be written in same form as the equation of motion for v_k represented in (5.24):

$$f_k'' + (k^2 - \frac{z''}{z})f_k. \tag{5.44}$$

As we explained in the end of the previous section, we can express the equation of motion in the context of noncommutative space-time by replacing $\tilde{z}_k$ by z , $\tilde{\eta}$ by η and keep in mind that the initial time depends on k. Thus, the equation of motion of f_k in noncommutative space-time takes the form

$$f_k'' + (k^2 - \frac{\tilde{z}_k''}{\tilde{z}_k})f_k. \tag{5.45}$$

From Chapter 3, the power spectrum of the curvature perturbation in noncommutative space-time can be written as

$$\begin{aligned} \mathcal{P}_{\mathcal{R}}(k) &= \frac{k^3}{2\pi^2}\frac{\langle 0 \mid \hat{v}_k^\dagger\hat{v}_k \mid 0\rangle}{\tilde{z}_k^2(\tilde{\eta}_k)} \\ &= \frac{k^3}{2\pi^2}\frac{|f_k|^2}{\tilde{z}_k^2(\tilde{\eta}_k)}, & (5.46)\end{aligned}$$

where $\tilde{\eta}_k$ is time when the mode k crosses the Hubble radius. From this power spectrum, we will write it in terms of noncommutative parameter and inflation parameters by finding solution for f_k.

5.2.2 Power Spectrum of Curvature Perturbation

In this subsection, we will find the solution for f_k from the differential equation in (5.45). We will expand $\tilde{z}_k''/\tilde{z}_k$ in the term of slow-roll parameter and introduce the noncommutative parameter μ in this expansion. Conveniently, we will find the relation between the conformal time η and the noncommutative conformal time $\tilde{\eta}$

$$\begin{aligned} d\eta &= a^{-2}d\tau \\ &= \frac{a_e^2}{a^2}d\tilde{\eta} \\ &= \frac{a(\tau+l_s^2k)a(\tau-l_s^2k)}{a^2}d\tilde{\eta} \\ &\simeq \frac{(a+l_s^2k\partial_\tau a)(a-l_s^2k\partial_\tau a)}{a^2}d\tilde{\eta} \\ &= \frac{a^2-(l_s^2k\partial_\tau a)^2}{a^2}d\tilde{\eta} \\ &= \Big(1-\Big(\frac{l_s^2k\partial_\tau a}{a^2}\Big)^2\Big)d\tilde{\eta} \\ &= \Big(1-\frac{k^2H^2}{a^2M_s^4}\Big)d\tilde{\eta} = (1-\mu)d\tilde{\eta}, \end{aligned} \tag{5.47}$$

where $\mu = \frac{k^2H^2}{a^2M_s^4}$ is a noncommutative parameter. From $\tilde{z}_k = y_k z_k$, one obtains

$$\begin{aligned} \frac{\tilde{z}_k'}{\mathcal{H}\tilde{z}_k} &= \Big(\frac{a_e}{a}\Big)^2\frac{z'}{\mathcal{H}z} \\ &= (1-\mu)(1+\epsilon-\delta) = (1+\epsilon-\delta-\mu). \end{aligned} \tag{5.48}$$

Note that, the primes over the noncommutative variables represent the derivative respect to $\tilde{\eta}$ and the primes over another variable represent the derivative with respect to η. From (5.48), we assume that μ is independent on time then y_k is also independent on time. y_k can be calculated in similar way as in the calculation of the relation of $\tilde{\eta}$ and η which takes the form

$$y_k = 1 + \mu/2. \tag{5.49}$$

Moreover, we assume that the slow-roll parameters are independent on time. Then the derivative of $\frac{\tilde{z}_k'}{\mathcal{H}\tilde{z}_k}$ with respect to $\tilde{\eta}$ is zero. This leads to

$$\frac{\tilde{z}_k''}{\tilde{z}_k} = \Big(\frac{\tilde{z}_k'}{\tilde{z}_k}\Big)^2 + \Big(\frac{a_e}{a}\Big)^2\frac{\tilde{z}_k'}{\tilde{z}_k}\frac{\mathcal{H}'}{\mathcal{H}}. \tag{5.50}$$

By using $\frac{\tilde{z}_k'}{\tilde{z}_k}$ in (5.48), $\left(\frac{a_e}{a}\right)^2$ in (5.47) and definition of slow-roll parameter $\frac{\mathcal{H}'}{\mathcal{H}^2} = 1-\epsilon$, the equation (5.50) can be written as

$$\begin{aligned} \frac{\tilde{z}_k''}{\tilde{z}_k} &= (1+\epsilon-\delta-\mu)^2\mathcal{H}^2 + (1-\mu)(1+\epsilon-\delta-\mu)(1-\epsilon)\mathcal{H}^2 \\ &= 2(Ha)^2(1+\epsilon-\frac{3}{2}\delta-2\mu). \end{aligned} \tag{5.51}$$

By using the definition of slow-roll parameters in the commutative case, the Hubble parameter for conformal time can be written as

$$\begin{aligned}\mathcal{H} = Ha &= -\frac{1}{\eta}(1-\epsilon)^{-1} \\ &= -\frac{1}{\tilde{\eta}}\Big(\frac{a_e}{a}\Big)^2(1+\epsilon) \\ &= -\frac{1}{\tilde{\eta}}(1+\epsilon+\mu). \qquad (5.52)\end{aligned}$$

Conveniently, we will set

$$\frac{\tilde{z}_k''}{\tilde{z}_k} = \frac{1}{\tilde{\eta}^2}\Big(\nu^2 - \frac{1}{4}\Big). \qquad (5.53)$$

By using (5.51), ν can be written as a function of slow-roll parameters and takes the form

$$\nu = (\frac{3}{2} + 2\epsilon - \delta). \qquad (5.54)$$

Thus, the equation of motion for f_k takes the form

$$f_k'' + \Big(k^2 - \frac{1}{\tilde{\eta}^2}(\nu^2 - \frac{1}{4})\Big)f_k. \qquad (5.55)$$

Generally, the solution of this equation can be written in the term of a combination of Bessel and Neumann functions and take the form of

$$f_k = \sqrt{-\tilde{\eta}}\Big(A_k J_\nu(-k\tilde{\eta}) + B_k N_\nu(-k\tilde{\eta})\Big). \qquad (5.56)$$

Thus the power spectrum can be written as

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}\frac{(-\tilde{\eta}_k)\Big(|A_k|^2 J_\nu J_\nu + |B_k|^2 N_\nu N_\nu + (A_k\bar{B}_k + \bar{A}_k B_k)J_\nu N_\nu\Big)}{\tilde{z}_k^2(\tilde{\eta}_k)}. \qquad (5.57)$$

Now we have the general solution of f_k, but it is not specific for arbitrary constants A_k and B_k. However, the specific solution can be expressed by determining the constants A_k and B_k. Actually, we must find their multiplication. The constants can be obtained by using the relation for Bogoliubov coefficients. Therefore, in the first step we must find g_k. Generally, g_k is proportional to derivative of f_k. For simplicity, we choose to multiply it by $-i$. By using the identity of the Bessel function, g_k takes the form

$$g_k = -ik\sqrt{-\tilde{\eta}}\Big(A_k J_{\nu-1}(-k\tilde{\eta}) + B_k N_{\nu-1}(-k\tilde{\eta})\Big). \qquad (5.58)$$

Thus, the Bogoliubov coefficients take the form

$$\begin{aligned}
\alpha_k &= \sqrt{\frac{k}{2}}(f_k + \frac{g_k}{k}) \\
&= \sqrt{\frac{-k\tilde{\eta}}{2}}\Big((A_kJ_\nu + B_kN_\nu) - i(A_kJ_{\nu-1} + B_kN_{\nu-1})\Big) \\
&= \sqrt{\frac{-k\tilde{\eta}}{2}}(C_k - iD_k), \qquad (5.59)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\beta}_k &= \sqrt{\frac{k}{2}}(f_k - \frac{g_k}{k}) \\
&= \sqrt{\frac{-k\tilde{\eta}}{2}}\Big((A_kJ_\nu + B_kN_\nu) + i(A_kJ_{\nu-1} + B_kN_{\nu-1})\Big) \\
&= \sqrt{\frac{-k\tilde{\eta}}{2}}(C_k + iD_k), \qquad (5.60)
\end{aligned}$$

where $C_k = \Big((A_kJ_\nu + B_kN_\nu) - i(A_kJ_{\nu-1} + B_kN_{\nu-1})\Big)$ and $D_k = \Big((A_kJ_\nu + B_kN_\nu) + i(A_kJ_{\nu-1} + B_kN_{\nu-1})\Big)$. From the relation of the Bogoliuibov coefficients in (5.39), we obtain

$$\begin{aligned}
\alpha_k\bar{\alpha}_k - \beta_k\bar{\beta}_k &= 1 \\
(\frac{-k\tilde{\eta}}{2})(C_k\bar{D}_k - \bar{C}_kD_k) &= 1 \\
(\frac{-k\tilde{\eta}}{2})(A_k\bar{B}_k - \bar{A}_kB_k)(J_\nu N_{\nu-1} - J_{\nu-1}N_\nu) &= 1 \\
A_k\bar{B}_k - \bar{A}_kB_k &= -i\frac{\pi}{2}. \qquad (5.61)
\end{aligned}$$

Note that, we use the recurrence relation of Bessel and Neumann functions,

$$J_\nu(x)N_{\nu-1}(x) - J_{\nu-1}(x)N_\nu(x) = -\frac{2}{\pi x}. \qquad (5.62)$$

Using the initial condition of Bogoliubov transformations, $\beta_k(\tilde{\eta}_0) = 0$, we obtain the second condition of the constants as

$$\begin{aligned}
C_{0,k} &= -iD_{0,k} \\
(A_kJ_{0,\nu} + B_kN_{0,\nu}) &= -i(A_kJ_{0,\nu-1} + B_kN_{0,\nu-1}) \\
A_k &= -\frac{(N_{0,\nu} + iN_{0,\nu-1})}{(J_{0,\nu} + iJ_{0,\nu-1})}B_k. \qquad (5.63)
\end{aligned}$$

Note that, for short notation, we replace $X_\nu(\tilde{\eta}_0)$ by $X_{0,\nu}$. By using these two conditions and recurrence relation in (5.62), one acquires

$$|A_k|^2 = -\frac{\pi^2\tilde{\eta}_0 k}{8}(N_{0,\nu}^2 + N_{0,\nu-1}^2), \tag{5.64}$$

$$|B_k|^2 = -\frac{\pi^2\tilde{\eta}_0 k}{8}(J_{0,\nu}^2 + J_{0,\nu-1}^2), \tag{5.65}$$

$$A_k\bar{B}_k = -i\frac{\pi}{4} + \frac{\pi^2\tilde{\eta}_0 k}{8}(J_{0,\nu}N_{0,\nu} + J_{0,\nu-1}N_{0,\nu-1}), \tag{5.66}$$

$$\bar{A}_k B_k = i\frac{\pi}{4} + \frac{\pi^2\tilde{\eta}_0 k}{8}(J_{0,\nu}N_{0,\nu} + J_{0,\nu-1}N_{0,\nu-1}). \tag{5.67}$$

Substitute these into (5.57), we obtain the power spectrum can be written as

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) &= \frac{k^4\tilde{\eta}_k\tilde{\eta}_0}{16\tilde{z}_k^2(\tilde{\eta}_k)}\Big((N_{0,\nu}^2 + N_{0,\nu-1}^2)J_\nu J_\nu + (J_{0,\nu}^2 + J_{0,\nu-1}^2)N_\nu N_\nu \\ &\quad -2(J_{0,\nu}N_{0,\nu} + J_{0,\nu-1}N_{0,\nu-1})J_\nu N_\nu\Big) \\ &= \frac{k^4\tilde{\eta}_k\tilde{\eta}_0}{16\tilde{z}_k^2(\tilde{\eta}_k)}F_\nu(\tilde{\eta}_0, \tilde{\eta}_k), \end{aligned} \tag{5.68}$$

where

$$\begin{aligned}F_\nu(\tilde{\eta}_0, \tilde{\eta}_k) &= \Big((N_{0,\nu}^2 + N_{0,\nu-1}^2)J_\nu J_\nu + (J_{0,\nu}^2 + J_{0,\nu-1}^2)N_\nu N_\nu \\ &\quad -2(J_{0,\nu}N_{0,\nu} + J_{0,\nu-1}N_{0,\nu-1})J_\nu N_\nu\Big). \end{aligned} \tag{5.69}$$

Now, we have already had the power spectrum in noncommutative space-time by using the vacuum which satisfies the minimum uncertainty relation. Therefore, this power spectrum can be compared with another power spectrum in two ways. First, we will compare with the power spectrum in the commutative space-time. According to [33], this power spectrum can be reduced to the power spectrum in commutative space-time by replacing $\tilde{\eta}$ by η and $\tilde{z}_k$ by z. This analysis agrees with the statement that we mentioned in the previous section. Second, we must compare with the power spectrum in another initial vacuum. Indeed there is an adiabatic vacuum. The power spectrum can be reduced to the adiabatic power spectrum by taking the limit $\tilde{\eta}_0 \to -\infty$ and $\tilde{\eta}_k \to 0^-$. From the approximation of Bessel and Neumann functions,

$$J_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu; 0 \le x \ll \nu, \tag{5.70}$$

$$J_\nu(x) \simeq \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right); x \gg \nu, \tag{5.71}$$

$$N_\nu(x) \simeq -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}; 0 \le x \ll \nu, \tag{5.72}$$

$$N_\nu(x) \simeq \left(\frac{2}{\pi x}\right)^{1/2}\sin\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right); x \gg \nu. \tag{5.73}$$

$F_\nu(\tilde{\eta}_0, \tilde{\eta}_k)$ can be approximated as

$$F_\nu(\tilde{\eta}_0, \tilde{\eta}_k) \simeq \frac{2}{\pi^2(-k\tilde{\eta}_0)}, \tag{5.74}$$

and the power spectrum takes the form

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^2}{4\pi^2 \tilde{z}_k^2(\tilde{\eta}_k)}, \tag{5.75}$$

which is in the same form of the power spectrum in adiabatic vacuum.

5.2.3 Power-Law Inflation in Noncommutative Space-Time

In order to compare the results, we must know the exact power spectrum of the curvature perturbation which depends on the slow-roll parameters and the noncommutative parameter. For our convenience, we use the power-law inflation to determine the exact power spectrum. The power-law inflation can be defined in the way that the scale factor is proportional to the power of time, $a(t) = a_0 t^p$, where p is the power-law parameter which relates to the slow-roll parameters by $p = 1/\epsilon$. So, the purpose of this subsection is to find the power spectrum which depends on the slow-roll, power-law and noncommutative parameters. First, we will calculate a $\tilde{\eta}_0$, which is the time when the fluctuation modes begin to occur, in the term of the parameters. The $\tilde{\eta}_0$ can be calculated by using the saturated uncertainty relation in (5.16). In the first step, we will write the power-law scale factor in the term of noncommutative time, τ. From $d\tau = adt$, we have

$$t = \Big(\frac{p+1}{a_0}\tau\Big)^{\frac{1}{p+1}}, \tag{5.76}$$

and from $\frac{da}{d\tau} = a^{-1}\frac{da}{dt}$, the scale factor can be written as

$$a(\tau) = \alpha_0 \tau^{\frac{p}{p+1}}, \tag{5.77}$$

where $\alpha_0 = (p+1)\Big(\frac{a_0}{p+1}\Big)^{\frac{1}{p+1}}$. By using the definition $d\tilde{\eta} = a_e^{-2}d\tau$, $\tilde{\eta}_0$ can be written in term of τ_0 as

$$\begin{aligned}
\tilde{\eta} &= \alpha_0^{-2}\int(\tau^2 - l_s^4k^2)^{\frac{-2p}{p+1}}d\tau \\
&= \alpha_0^{-2}\int\tau^{\frac{-2p}{p+1}}\Big(1-\frac{l_s^4k^2}{\tau^2}\Big)^{\frac{-p}{p+1}}d\tau \\
&= \alpha_0^{-2}\int\tau^{\frac{-2p}{p+1}}\Big(1+\frac{p}{p+1}\frac{l_s^4k^2}{\tau^2}\Big)d\tau \\
&= \alpha_0^{-2}\int\tau^{\frac{-2p}{p+1}}\Big(1+\frac{p+1}{p}\mu\Big)d\tau \\
&= \alpha_0^{-2}\Big(1+\frac{p+1}{p}\mu\Big)\Big(\frac{1+p}{1-p}\Big)\tau^{\frac{1-p}{1+p}} \\
\tilde{\eta}_0 &= \alpha_0^{-2}\Big(1+\frac{p+1}{p}\mu\Big)\Big(\frac{1+p}{1-p}\Big)\tau_0^{\frac{1-p}{1+p}} \\
&= A_0\tau_0^{\frac{1-p}{1+p}}, \qquad (5.78)
\end{aligned}$$

where

$$A_0 = \alpha_0^{-2}\Big(1+\frac{p+1}{p}\mu_0\Big)\Big(\frac{1+p}{1-p}\Big). \qquad (5.79)$$

Substituting (5.77) into (5.78), we get

$$\mu = \frac{p^2}{(p+1)^2}\frac{l_s^4k^2}{\tau^2}. \qquad (5.80)$$

Next, we will determine the horizon crossing time which occurs when $\frac{\tilde{z}_k''}{\tilde{z}_k} = k^2$. From (5.53), the crossing time takes the form

$$\tilde{\eta}_k = \frac{\sqrt{\nu^2-\frac{1}{4}}}{k}. \qquad (5.81)$$

Finally, we will consider $\tilde{z}_k(\tilde{\eta}_k)$ which takes the form

$$\begin{aligned}
\tilde{z}_k(\tilde{\eta}_k) &= z(\eta)y_k = a\frac{\dot{\phi}}{H}y_k = a\sqrt{2\epsilon}m_{pl}(1+\mu_k/2), \\
\tilde{z}_k^2(\tilde{\eta}_k) &= \frac{2m_{pl}^2}{p}(1+\mu_k)a^2(\tilde{\eta}_k) \\
&= \frac{2m_{pl}^2}{p}(1+\mu_k)\alpha_0^2A_k^{\frac{2p}{p-1}}\tilde{\eta}_k^{\frac{2p}{1-p}} \\
&= \frac{2m_{pl}^2}{p}(1+\mu_k)\alpha_0^2A_k^{\frac{2p}{p-1}}(\nu^2-1/4)^{\frac{p}{1-p}}k^{\frac{2p}{p-1}}. \qquad (5.82)
\end{aligned}$$

Substituting these into the power spectrum in (5.68), we obtain

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}}(k) &= \frac{p(\nu^2-1/4)^{\frac{1-3p}{2(1-p)}} A_k^{\frac{2p}{1-p}} A_0}{32 m_{pl}^2 \alpha_0^2 (1+\mu_k)} k^{3-\frac{2p}{p-1}} \tau_0^{\frac{1-p}{1+p}} F_\nu(\tilde{\eta}_0, \tilde{\eta}_k) \\
&= \frac{p(\nu^2-1/4)^{\frac{1-3p}{2(1-p)}}}{32 m_{pl}^2 \alpha_0^{\frac{4}{1-p}}} \Big(\frac{1+p}{1-p}\Big)^{\frac{1+p}{1-p}} \Big(1+\frac{p+1}{p}\mu_0 - \frac{3p+1}{p-1}\mu_k\Big) \\
&\quad k^{3-\frac{2p}{p-1}} \tau_0^{\frac{1-p}{1+p}} F_\nu(\tilde{\eta}_0, \tilde{\eta}_k) \\
&= B_1 \Big(1+\frac{p+1}{p}\mu_0 - \frac{3p-1}{p-1}\mu_k\Big) k^{3-\frac{2p}{p-1}} \tau_0^{\frac{1-p}{1+p}} F_\nu(\tilde{\eta}_0, \tilde{\eta}_k), \qquad (5.83)
\end{aligned}$$

where

$$B_1 = \frac{p(\nu^2-1/4)^{\frac{1-3p}{2(1-p)}}}{32 m_{pl}^2 \alpha_0^{\frac{4}{1-p}}} \Big(\frac{1+p}{1-p}\Big)^{\frac{1+p}{1-p}}.$$

Using the definition of the energy upper bound in (5.17), we obtain the effective scale factor at the beginning time, τ_0 as

$$\begin{aligned}
l_s^4 k^4 = a_e^4(\tau_0) &= a^2(\tau_0 + l_s^2 k) a^2(\tau_0 - l_s^2 k) \\
&= \alpha_0^4 (\tau_0 + l_s^2 k)^{\frac{2p}{p+1}} (\tau_0 - l_s^2 k)^{\frac{2p}{p+1}} \\
&= \alpha_0^4 \Big(\tau_0^2 - l_s^4 k^2\Big)^{\frac{2p}{p+1}}, \\
\tau_0 &= \Big[\Big(\frac{l_s k}{\alpha_0}\Big)^{\frac{2(p+1)}{p}} + l_s^4 k^2\Big]^{\frac{1}{2}}. \qquad (5.84)
\end{aligned}$$

In order to compare the result with the observation, we need to consider two regions, UV and IR regions, separately [26]. The UV region is the region in which the fluctuation modes are generated within the Hubble radius. These correspond to the modes which satisfy the relation $k \gg \alpha_0^{p+1} l_s^{p-1}$. On the other hand, the region in which the modes are generated outside the Hubble radius are the IR region corresponding to $k \ll \alpha_0^{p+1} l_s^{p-1}$. Therefore, the first and the second terms in the right-hand side of (5.84) are dominated for the UV and IR regions, respectively. In order to neglect the effect of the second term, the initial time $\tilde{\eta}_0$ can be written as

$$\tilde{\eta}_0 = -\alpha_0^{-2} \Big(\frac{p+1}{p-1}\Big)\Big(1+\frac{p+1}{p}\mu_0\Big)\Big(\frac{l_s}{\alpha_0}\Big)^{\frac{1-p}{p}} k^{\frac{1-p}{p}}, \qquad (5.85)$$

and the power spectrum in the UV region takes the form

$$\mathcal{P}_{\mathcal{R}}(k) = B_2 k^{\frac{-2}{p-1}+\frac{1}{p}} (1 - x k^{\frac{-4}{p-1}} + y k^{\frac{-1}{p}}) F_\nu(\tilde{\eta}_0, \tilde{\eta}_k), \qquad (5.86)$$

where

$$
\begin{aligned}
B_2 &= \frac{p(\nu^2-1/4)^{\frac{1-3p}{2(1-p)}}}{32m_{pl}^2\alpha_0^{\frac{4}{1-p}}}\Big(\frac{1+p}{1-p}\Big)^{\frac{1+p}{1-p}}\Big(\frac{l_s}{\alpha_0}\Big)^{\frac{1-p}{p}},\\
x &= (\nu^2-1/4)^{\frac{p+1}{p-1}}l_s^4\alpha_0^{\frac{2(p+1)}{p-1}}\ \Big(\frac{1+p}{1-p}\Big)^{\frac{2(1+p)}{1-p}}\Big(\frac{p}{p+1}\Big)^2\Big(\frac{3p-1}{p-1}\Big),\\
y &= \Big(\frac{p}{p+1}\Big)\Big(\frac{\alpha_0}{l_s}\Big)^{\frac{1-2p}{p}}. \qquad (5.87)
\end{aligned}
$$

From this power spectrum, the factor $1/p$ in the power of k will make the spectral index larger than that in adiabatic vacuum and enhances the trend from the red tilt to the blue tilt spectrum. This result can be estimated to the adiabatic vacuum by using the approximation of $F_\nu(\tilde{\eta}_0,\tilde{\eta}_k)$ in (5.74) which takes the form

$$
F_\nu(\tilde{\eta}_0,\tilde{\eta}_k) = \frac{2\alpha_0^2}{\pi^2}\Big(\frac{l_s}{\alpha_0}\Big)^{\frac{p-1}{p}}\Big(\frac{p-1}{p+1}\Big)(1-yk^{\frac{-1}{p}})k^{\frac{-1}{p}}. \qquad (5.88)
$$

Form this approximation, it leads to the power spectrum which takes the form

$$
\begin{aligned}
\mathcal{P}_{\mathcal{R}}(k) &= B_3k^{\frac{-2}{p-1}+\frac{1}{p}}(1-xk^{\frac{-4}{p-1}}+yk^{\frac{-1}{p}})(1-yk^{\frac{-1}{p}})k^{\frac{-1}{p}},\\
&= B_3k^{\frac{-2}{p-1}}(1-xk^{\frac{-4}{p-1}}), \qquad (5.89)
\end{aligned}
$$

where

$$
B_3 = \frac{p(\nu^2-1/4)^{\frac{1-3p}{2(1-p)}}\alpha_0^2}{16\pi^2m_{pl}^2\alpha_0^{\frac{4}{1-p}}}\Big(\frac{p-1}{p+1}\Big)\Big(\frac{1+p}{1-p}\Big)^{\frac{1+p}{1-p}}. \qquad (5.90)
$$

This power spectrum is equivalent to the power spectrum of adiabatic vacuum in [34]. The differences occur only from the technical calculation in the constants B_3 and x. From this power spectrum, the spectral index takes the form

$$
n-1 = \frac{-2}{p-1}+\frac{1}{p}-\Big(\frac{4}{p-1}xk^{\frac{-4}{p-1}}+\frac{y}{p}k^{\frac{-1}{p}}\Big)+\frac{d\ln F_\nu(\tilde{\eta}_0,\tilde{\eta}_k)}{d\ln k}. \qquad (5.91)
$$

Next, we will determine the power spectrum in the IR region which the second term in (5.84) dominates over the first term. The initial time $\tilde{\eta}_0$ can be written as

$$
\tilde{\eta}_0 = -\alpha_0^{-2}\Big(\frac{p+1}{p-1}\Big)\Big(1+\frac{p+1}{p}\mu_0\Big)l_s^2k, \qquad (5.92)
$$

and the power spectrum takes the form

$$
\mathcal{P}_{\mathcal{R}}(k) = B_4k^{\frac{2}{p+1}-\frac{2}{p-1}}\Big(1-\frac{2p+1}{p+1}xk^{\frac{-4}{p-1}}\Big)F_\nu(\tilde{\eta}_0,\tilde{\eta}_k), \qquad (5.93)
$$

where

$$
B_4 = \frac{p(\nu^2-1/4)^{\frac{1-3p}{2(1-p)}}}{32m_{pl}^2\alpha_0^{\frac{4}{1-p}}}\frac{p+1}{2p+1}\Big(\frac{1+p}{1-p}\Big)^{\frac{1+p}{1-p}}l_s^{\frac{2(1-p)}{1+p}}. \qquad (5.94)
$$

This power spectrum provides red tilt spectrum which is very different from the result in adiabatic vacuum that has blue tilt spectrum. However, the result is the same as the adiabatic vacuum when we think of the fluctuation modes are generated and then cross the horizon suddenly, $\tilde{\eta}_k \to \tilde{\eta}_0$. In this context, in order to compare with the adiabatic vacuum, one needs the new approximation of y_k which takes the form

$$y_k^2 \;=\; 2^{\frac{p-1}{p+1}}(l_s\alpha_0)^{\frac{2(p-1)}{p+1}}k^{\frac{-2}{p+1}}\Big(1+\frac{1}{2}\frac{p}{p+1}(l_s\alpha_0)^{\frac{2(1-p)}{p}}k^{\frac{2}{p}}\Big). \tag{5.95}$$

Furthermore, by taking the limit $\tilde{\eta}_k \to \tilde{\eta}_0$, one finds that

$$F_\nu(\tilde{\eta}_0) \;=\; \frac{4}{\pi^2 k^2 \tilde{\eta}_0^2}. \tag{5.96}$$

Substituting these into (5.68), we obtain the power spectrum

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) &= \frac{k^2}{4\pi^2 \tilde{z}_k^2}\\ &= B_5 k^{\frac{4}{p+1}}\Big(1 - x' k^{\frac{2}{p}}\Big),\end{aligned} \tag{5.97}$$

where

$$B_5 \;=\; \frac{p2^{\frac{1-p}{1+p}}}{8\pi^2 m_{pl}^2\alpha_0^2}(\alpha_0 l_s)^{\frac{2(1-p)}{1+p}}, \tag{5.98}$$

$$x' \;=\; \frac{3}{2}\frac{p}{p+1}(l_s\alpha_0)^{\frac{2(1-p)}{p}}. \tag{5.99}$$

This power spectrum is in the same form as in the adiabatic vacuum which we have mentioned above. Now, we already have analyzed the comparison between the minimized uncertainty and the adiabatic vacuum. Next, we will compare the results with the observation. Since the comparison is rather complicated, the numerical method is the significant tool for this process. In order to avoid this complicity, we deduce the result from [34] and [35] for our analysis. Therefore, the comparison between the observation and the model will be analyzed here only in the adiabatic vacuum. From these two papers, we can conclude that the noncommutative space-time affects the CMB power spectrum only for the low multipoles. In order to obtain the result only for the cosmologically relevant scale, $10^{-4}Mpc^{-1} < k < 10^{-1}Mpc^{-1}$, they use only the power spectrum in the UV region (5.89) to determine the CMB power spectrum. In [34], by using the result of [36], they pick $p = 12$ and then determine the CMB power spectrum which is illustrated in Figure 5.1. The result yields the string length is $l_s \cong 2.5\times10^{-29}$ cm. The result from [35]is more accurate than [34] because they calculate the exponent, p, by the

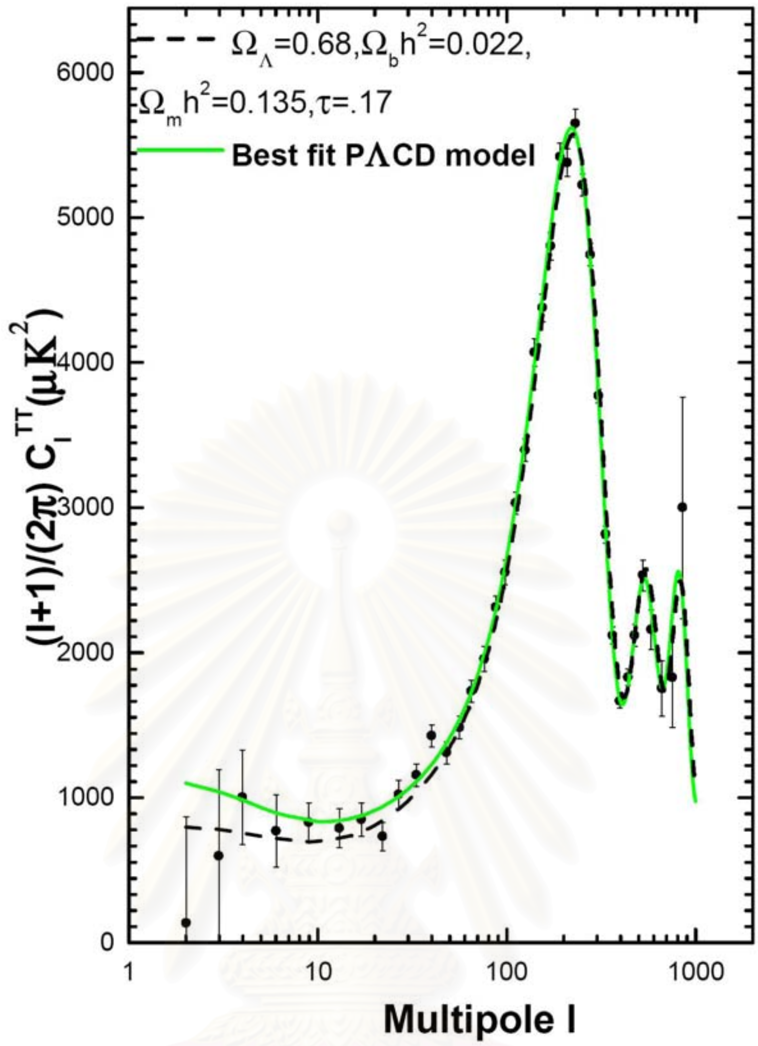

Figure 5.1: The CMB angular power spectrum from [34]. The dashing line represents the noncommutative model for the adiabatic vacuum and the green solid line represents the best fit ΛCDM without noncommutative effect.

numerical method and use another approximation for the power spectrum in the intermediate region. This leads to the result which is slightly different from [34] namely $l_s \sim 10^{-28}$ cm. However, The result from both papers yields the similar conclusion: the low multipoles is suppressed by the noncommutative effect.

CHAPTER VI

DISCUSSION AND SUMMARY

In this thesis, we have reviewed the idea of noncommutativity of space-time, arisen in the context of string theory, and its applications to cosmological models. We particularly studied the noncommutative inflation model introduced by R. Brandenberger and P. M. Ho [26].

By encoding the stringy space-time uncertainty relation in the star products of the inflaton field, we were able to determine the dynamics of inflaton fluctuations. The effects of space-time noncommutivity on the cosmological observations have been investigated. The latter was discussed in Chapter 5 by following the works by M. Li and Qing Guo Huang [34].

We found that the noncommutative nature of space-time affects the Cosmic Microwave Background power spectrum mostly in the IR region, while the effect in the UV region seems to be very small. The advantage of noncommutative inflation is that it allows us to determine not only the power spectrum, the spectral index, but also the running spectral index of the CMB. We were able to use the CMB observational data from WMAP satellite at one specific length scale to constrain the parameters in our model, and used that constraint to predict the other set of observational data in the different length scale. However, in order to see how SSUR affects the full CMB power spectrum, one needs to calculate the CMB power spectrum numerically by using the package program, such as the CMBFAST. This also involves the modification of the FORTRAN codes which is not in the scope of this thesis. The results are presented in references [34] and [35]. The major difference between the commutative and non-commutative inflation CMB power spectrum is that the low multipoles seem to be suppressed in the latter case as illustrated in Figure 5.1. The authors of [34] and [35] constrained the string length to $l_s \sim 10^{-29}$ cm and $l_s \sim 10^{-28}$ cm, respectively. Note that the difference between these results is due to the calculation techniques, and both values of sting length are roughly of the same order of magnitude.

Table 6.1: The power spectrum of curvature perturbation for the commutative and noncommutative (NC) space-time.

	Power spectrum in UV region	Power spectrum in IR region
Commutative space-time	$\mathcal{P}_{\mathcal{R}}(k) \sim k^{-\frac{2}{p-1}}$	$\mathcal{P}_{\mathcal{R}}(k) \sim k^{-\frac{2}{p-1}}$
NC space-time for adiabatic vacuum	$\mathcal{P}_{\mathcal{R}}(k) \sim k^{-\frac{2}{p-1}}\left(1 - xk^{-\frac{4}{p-1}}\right)$	$\mathcal{P}_{\mathcal{R}}(k) \sim k^{\frac{4}{p+1}}\left(1 + x'k^{\frac{2}{p}}\right)$
NC space-time for minimized uncertainty vacuum	$\mathcal{P}_{\mathcal{R}}(k) \sim k^{-\frac{2}{p-1}+\frac{1}{p}}$ $\left(1 - xk^{-\frac{4}{p-1}} + yk^{-\frac{1}{p}}\right)F_\nu(\tilde{\eta}_0, \tilde{\eta}_k)$	$\mathcal{P}_{\mathcal{R}}(k) \sim k^{-\frac{2}{p-1}+\frac{2}{p+1}}$ $\left(1 - x''k^{-\frac{4}{p-1}}\right)F_\nu(\tilde{\eta}_0, \tilde{\eta}_k)$

However, all calculations that we have mentioned above are determined in the context of the adiabatic vacuum. The calculation in the other vacuum, the minimized uncertainty relation vacuum, is applied to this approach in order to take into account the effect of the finite time. This vacuum is introduced in [30] and then Danielsson applies it to the commutative space-time cosmology [31]. For the noncommutative regime, it was Rong-Gen Cai who uses this vacuum to calculate the power spectrum [32]. From this regime, one can summarize that the noncommutative effect enhances the trend of the power spectrum from the red tilt to the blue tilt spectrum for UV region and the blue tilt to the red tilt spectrum for the IR region comparing the result from using the adiabatic vacuum. Comparing to the commutative case, the noncommutative effect enhances the trend from the red tilt to the blue tilt for both UV and IR regions. In order to see how the power spectrum is different explicitly, one can see in the Table 6.1. Because F_ν is in the term of the Bessel and Nuemann functions and it is in a very complicate term, we are able to analyze only the trend of power spectrum for the minimized uncertainty relation vacuum. However, one can summarize that the effect of the string theory such as the SSUR provides the explanation for the observed lake of

CMB power spectrum at the low multipoles. The other meaning of this summary is that the effect of string theory at the high-energy scale yields the modification for the large-scale perturbation rather than for the small scale. Nevertheless, this phenomenal effect can be explained by the fact that the large-scale modes are generated outside the horizon and then experience growth due to squeezing for less time than the modes which are generated inside.

In this thesis, we consider only the power-law inflation. The slow-roll parameters and the power of inflaton field in the potential term, p, are assumed to be constant. In general case, these parameters should not be constant especially if we want the inflation to end. Some attempts on other models of noncommutative inflation has been done, for example the author in [37]. However, they got the similar results as the results of the power-law noncommutative inflation. For more accurate calculation, the power spectrum can be calculated up to second order of the slow-roll parameters and the noncommutative parameter is not constant [38], [39], [40], [41]. The results of these calculations yield the consistency with our conclusion.

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