CHAPTER II
BASIC DEFINITIONS AND RESULTS

In this chapter, we collect definitions and results, mainly without proofs, to be used throughout the entire thesis. The first section deals with valuations and related concepts. Details and proofs can be found in McCarthy [22]. The second section deals with continued fractions and their properties. Details and proofs can be found in Lorentzen and Waadeland [20] for the classical case, and in Ruban [28], Schneider [30], Bundschuh [6], Laohakosol [13], de Weger [10], and Lianxiang [17] for the p-adic case.

2.1 Valuation

Definition 2.1. A *valuation* on a field K is a real-valued function $a \mapsto |a|$ defined on K which satisfies the following conditions:

(i) $\forall\, a \in K,\ |a| \geq 0$ and $|a| = 0 \Leftrightarrow a = 0$

(ii) $\forall\, a, b \in K,\ |ab| = |a||b|$

(iii) $\forall\, a, b \in K,\ |a + b| \leq |a| + |b|$.

There is always at least one valuation on K, namely, that given by setting $|a| = 1$ if $a \in K \backslash \{0\}$ and $|0| = 0$. This valuation is called the *trivial valuation* on K.

Definition 2.2. A valuation $|\cdot|$ on K is called *non-Archimedean* if the condition (iii), called the *triangle inequality*, is replaced by a stronger condition, called the *strong triangle inequality*

$(iii)'$ $\forall\, a, b \in K,\quad |a + b| \leq \max\{|a|, |b|\}$.

Any other valuation on K is called *Archimedean.*

A *valuated field* $(K,\ |\cdot|)$ is a field K together with a prescribed valuation $|\cdot|$. If the valuation is non-Archimedean, then K is called a *non-Archimedean valuated field.*

Examples 1) For $K = \mathbb{Q}$, the ordinary absolute value $|\cdot|$ is an Archimedean valuation on K.

2) For $K = \mathbb{Q}$ and p be a prime number. Each $a \in \mathbb{Q}\backslash\{0\}$ can be written uniquely in

the form

$$a = p^n \left(\frac{u}{v}\right),$$

where $u, \in \mathbb{Z}, v \in \mathbb{N}$, $(u,v) = 1$, $n \in \mathbb{Z}$, $p \nmid u$ and $p \nmid v$. Define

$$|a|_p = p^{-n} \text{ and } |0|_p = 0.$$

Then $|\cdot|_p$ is a non-Archimedean valuation on $\mathbb{Q}$ called the *p-adic valuation.*

3) Consider the field $\mathbb{F}_q(x)$ of rational functions over a finite field $\mathbb{F}_q$ of q elements. Let $f(x)/g(x) \in \mathbb{F}_q(x)\backslash\{0\}$. Define

$$\left|\frac{f(x)}{g(x)}\right|_\infty = q^{\deg f - \deg g} \text{ and } |0|_\infty = 0.$$

Then $|\cdot|_\infty$ is a non-Archimedean valuation on $\mathbb{F}_q(x)$.

4) Let $\pi(x)$ be an irreducible polynomial in $\mathbb{F}_q[x]$. If $f(x)/g(x) \in \mathbb{F}_q(x)\backslash\{0\}$, we can write uniquely as

$$\frac{f(x)}{g(x)} = \pi(x)^n \frac{u(x)}{v(x)},$$

where $u(x)$ and $v(x)$ are relatively prime elements of $\mathbb{F}_q[x]$, neither of which is divisible by $\pi(x)$. Define

$$\left|\frac{f(x)}{g(x)}\right|_\pi = q^{-n} \text{ and } |0|_\pi = 0.$$

Then $|\cdot|_\pi$ is a non-Archimedean valuation on $\mathbb{F}_q(x)$. We will consider mostly the case where $\pi(x) = x$, and write $|\cdot|_x$ instead of $|\cdot|_\pi$.

Since a valuation gives rise to a metric on any valuated field $(K, |\cdot|)$, the usual completion process is applicable. The valuation of K naturally extends to its completion and is still denoted by $|\cdot|$. In the case of $\mathbb{Q}$, with the usual absolute value, its completion is the field $\mathbb{R}$ of real numbers and in the case of $(\mathbb{Q}, |\cdot|_p)$, its completion is the p-adic number field $(\mathbb{Q}_p, |\cdot|_p)$, while in the cases of $(\mathbb{F}_q(x), |\cdot|_\infty)$ and $(\mathbb{F}_q(x), |\cdot|_\pi)$ the completions are $(\mathbb{F}_q((x^{-1})), |\cdot|_\infty)$ and $(\mathbb{F}_q((\pi(x))), |\cdot|_\pi)$ the fields of formal Laurent series in $1/x$ and $\pi(x)$, respectively.

Definition 2.3. i) Let $(K, |\cdot|)$ be a valuated field. The set

$$V = \{|a|; a \in K\backslash\{0\}\}$$

is easily checked to be a subgroup of the multiplicative group of nonzero real numbers and is called the *value group* of $(K, |\cdot|)$.

ii) If V is an infinite cyclic group, then $(K, \ |\cdot|)$ is called a *discrete* valuated field.

iii) A *local field* is a complete, discrete non-Archimedean valuated field.

iv) The set $\mathcal{O} = \{a \in K : |a| \leq 1\}$ is a ring, called the *valuation ring* of $(K, \ |\cdot|)$.

v) The set $\wp = \{a \in K : |a| < 1\}$ is the unique maximal ideal of $\mathcal{O}$.

vi) The field $\mathcal{O}/\wp$ is called the *residue class field* of $(K, \ |\cdot|)$.

Examples 1) $(\mathbb{Q}_p, \ |\cdot|_p)$ is a local field with $\{0, \ 1, \ 2, \ldots, \ p-1\}$ as a set of representatives of its residue class field.

2) $(\mathbb{F}_q((x^{-1})), \ |\cdot|_\infty)$ is a local field with $\mathbb{F}_q$ as a set of representatives of its residue class field.

3) $(\mathbb{F}_q((x)), \ |\cdot|_x)$ is a local field with $\mathbb{F}_q$ as a set of representatives of its residue class field.

In a local field $(K, \ |\cdot|)$ with R being the set of representatives of its residue class field, each element $\alpha \in K$ can be uniquely represented as

$$\alpha = \sum_{n=r}^{\infty} a_n \pi^n, \ a_r \neq 0$$

where $a_n \in R, \ r \in \mathbb{Z}$, and $\pi \in K$ is called a prime element which is usually normalized so that $|\pi| = q^{-1}$. Thus $|\alpha| = |\pi|^r = q^{-r}$ for any $\alpha \in K\setminus\{0\}$. Sometimes, it is convenient to use the *ordinal function* which is defined by $ord_\pi(\alpha) = r$, and so $ord_\pi(\pi) = 1$.

Examples 1) Every element $\alpha \in \mathbb{Q}_p$ can be uniquely written as the form

$$\alpha = \sum_{n=r}^{\infty} a_n p^n, \ a_r \neq 0$$

where $a_n \in \{0, \ 1, \ 2, \ldots, \ p-1\}$ and so $|\alpha|_p = p^{-r}$. The set of $\alpha \in \mathbb{Q}_p$ such that $|\alpha|_p \leq 1$ is denoted by $\mathbb{Z}_p$, that is

$$\mathbb{Z}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n \ : \ a_n \in \{0, \ 1, \ 2, \ldots, \ p-1\} \right\}.$$

2) Every element $\xi \in \mathbb{F}_q((x^{-1}))$ can be uniquely written as the form

$$\xi = \sum_{n=r}^{\infty} a_n x^{-n}, \ a_r \neq 0$$

where $a_n \in \mathbb{F}_q$ and so $|\xi|_\infty = q^{-r}$.

3) Every element $\xi \in \mathbb{F}_q((x))$ can be uniquely written as the form

$$\xi = \sum_{n=r}^{\infty} a_n x^n, \ a_r \neq 0$$

where $a_n \in \mathbb{F}_q$ and so $|\xi|_x = q^{-r}$.

2.2 Classical continued fractions

The expansion

$$b_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{\ddots}}}}}$$

is called a *continued fraction.*

It is more convenient to use the notation

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n;\ldots] \tag{2.1}$$

for the above continued fractions. The elements $a_1, a_2, a_3, \ldots$ are called its *partial numerators*; $b_0, b_1, b_2, \ldots$ its *partial denominators.* We assume that all partial denominators are not equal to zero.

The *terminating* or *finite* continued fraction

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots;\ a_n,\ b_n] := \frac{p_n}{q_n}$$

is called the n^{th} *convergent* of the continued fraction (2.1).

In $\mathbb{R}$, it is known that any real number can be represented as a continued fraction of the form

$$[b_0;\ 1,\ b_1;\ 1,\ b_2;\ldots;\ 1,\ b_n;\ldots] := [b_0,\ b_1,\ b_2,\ldots,b_n,\ldots]$$

where $b_0 \in \mathbb{Z},\ b_i \in \mathbb{N}\ \ (i \geq 1)$. This is called a *simple* continued fraction and the b_i are called its *partial quotients.* Such representation is unique for real irrationals, but for real rationals, we have the following characterization.

Theorem 2.4. *Any finite simple continued fraction represents a rational number. Conversely, any rational number can be expressed as a finite simple continued fraction, and in exactly two ways,*

$$\begin{aligned}[b_0,\ b_1,\ b_2,\ldots,\ b_n] &= [b_0,\ b_1,\ b_2,\ldots,\ b_{n-1}, b_n - 1,\ 1] \ \ \textit{if}\ \ b_n \geq 2,\\ &= [b_0,\ b_1,\ b_2,\ldots,\ b_{n-1} + 1] \ \ \textit{if}\ \ b_n = 1.\end{aligned}$$

An infinite simple continued fraction

$$[b_0,\ b_1,\ b_2,\ldots]$$

is said to be *periodic* if there is an integer r such that $b_n = b_{n+r}$ for all sufficiently large integers n. A well known theorem of Lagrange characterizing infinite, periodic, simple continued fractions states that:

Theorem 2.5. *An infinite, periodic, simple continued fraction is a quadratic irrational number, and conversely.*

2.3 Continued fractions in the field $\mathbb{Q}_p$

There are many kinds of p-adic continued fractions constructed by various authors. We shall consider only two types, namely, *Ruban continued fraction* first developed by Ruban [28] and *Schneider continued fraction* first developed by Schneider [30].

The process for the expansion of the p-adic Ruban continued fraction, denoted by p-adic **RCF**, is as follows:

Since $\xi \in \mathbb{Q}_p$ can be represented uniquely as

$$\xi = \sum_{i=r}^{\infty} a_i p^i$$

where $r \in \mathbb{Z}$, $a_r \neq 0$, $a_i \in \{0,\ 1, \ldots,\ p-1\}:=\mathbb{F}_p \quad (i \geq r)$, define

$$[\xi] := \sum_{i=r}^{0} a_i p^i \in \mathbb{F}_p\left[p^{-1}\right], \quad (\xi) := \sum_{i=1}^{\infty} a_i p^i$$

and we call $[\xi]$ and (ξ) the *head* part and the *tail* part of ξ, respectively. The head and tail parts of ξ are uniquely determined, and so uniquely $\xi = [\xi] + (\xi)$. Let $b_0 = [\xi] \in \mathbb{F}_p[p^{-1}]$. Hence $|b_0| = p^{-r} \geq 1$.

If $(\xi) = 0$, the process stops. Otherwise, write ξ in the form $\xi = b_0 + \xi_1^{-1}$, where $\xi_1^{-1} = (\xi)$ with $|\xi_1|_p > 1$. As above, we can uniquely write $\xi_1 = [\xi_1] + (\xi_1)$. Let $b_1 = [\xi_1] \in \mathbb{F}_p[p^{-1}] \backslash \{0\}$.

If $(\xi_1) = 0$, the process stops. Otherwise, write ξ_1 in the form $\xi_1 = b_1 + \xi_2^{-1}$, where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2|_p > 1$. As above, we can uniquely write $\xi_2 = [\xi_2] + (\xi_2)$. Let $b_2 = [\xi_2] \in \mathbb{F}_p[p^{-1}] \backslash \{0\}$.

Again, if $(\xi_2) = 0$, the process stops. Otherwise proceed in the same manner.

Therefore ξ has a unique p-adic **RCF** of the form $[b_0,\ b_1,\ b_2, \ldots]$ where all $b_i \in \mathbb{F}_p[p^{-1}] \backslash \{0\} \quad (i \geq 1)$.

It is quite trivial that a finite p-adic **RCF** always represents a rational number. However, there exist infinitely many rational numbers with infinite periodic p-adic **RCF**'s. Laohakosol [16] gave a characterization of rational numbers via p-adic **RCF** as follows:

Theorem 2.6. *Let $\xi \in \mathbb{Q}_p \backslash \{0\}$. Then ξ is a rational number if and only if its p-adic **RCF** is either finite or periodic from a certain fraction onwards with the shape*

$$[(p-1)p^{-1} + (p-1),\ (p-1)p^{-1} + (p-1), \ldots].$$

Schneider [30] constructed another type of p-adic continued fraction, denoted henceforth by p-adic **SCF**, as follows:

Let $\xi \in \mathbb{Q}_p \backslash \{0\}$. It can be assumed without loss of generality that $|\xi|_p = 1$. Then ξ can be represented uniquely as

$$\xi = \sum_{i=0}^{\infty} c_i p^i$$

where $c_i \in \mathbb{F}_p \ \ (i \geq 0),\ c_0 \neq 0$. Let $b_0 = c_0$ and write ξ in the form $\xi = b_0 + a_1 \xi_1^{-1}$ with $|\xi_1|_p = 1 = |b_0|_p, a_1 = p^{\alpha_1} \ \ (\alpha_1 \in \mathbb{N})$. Let

$$\xi_1 = \sum_{i=0}^{\infty} d_i p^i$$

where $d_i \in \mathbb{F}_p \ \ (i \geq 0),\ d_0 \neq 0$. Let $b_1 = d_0$ and write ξ_1 in the form $\xi_1 = b_1 + a_2 \xi_2^{-1}$ with $|\xi_2|_p = 1 = |b_1|_p,\ a_2 = p^{\alpha_2} \ \ (\alpha_2 \in \mathbb{N})$. Continuing in the same manner, we have generally

$$\xi_n = b_n + \frac{a_{n+1}}{\xi_{n+1}} \quad (n \geq 0)$$

where $b_n \in \mathbb{F}_p \backslash \{0\},\ a_{n+1} = p^{\alpha_{n+1}}$ with $|b_n|_p = 1 = |\xi_{n+1}|_p$. Therefore ξ has a unique p-adic **SCF** of the form

$$\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n; \ldots]$$

where $a_n = p^{\alpha_n},\ \alpha_n \in \mathbb{N},\ b_n \in \mathbb{F}_p \backslash \{0\}$. The expansion into p-adic **SCF** is unique. The following theorem contains a necessary and sufficient condition for rationality of p-adic numbers.

Theorem 2.7. *Let $\xi \in \mathbb{Q}_p \backslash \{0\}$. Then ξ is rational if and only if its p-adic **SCF** is either finite or periodic with period length 1 and $a_n = p,\ b_n = p-1$ for all sufficiently large n.*

2.4 Ruban continued fractions in the field $\mathbb{F}_q((x^{-1}))$

In this section, our universe is the field $\mathbb{F}_q((x^{-1}))$ of formal series over a finite field $\mathbb{F}_q$. It is well-known that elements of $\mathbb{F}_q((x^{-1}))$ are formal series (in x) uniquely written as

$$\xi = a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots,$$

where the coefficients $a_m, a_{m-1}, a_{m-2}, \dots$ are in $\mathbb{F}_q$ with $a_m \neq 0$. Thus $\mathbb{F}_q(x)$, the quotient field of $\mathbb{F}_q[x]$, is a subfield of $\mathbb{F}_q((x^{-1}))$. A valuation $|\xi|$ in $\mathbb{F}_q((x^{-1}))$ is defined by putting

$$|0| = 0, \quad |\xi| = q^m \text{ if } \xi = a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots, \ a_m \neq 0.$$

The construction of the continued fraction for ξ runs as follows:
Define $\xi = [\xi] + (\xi)$, where

$$[\xi] := a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1 x + a_0, \quad (\xi) := a_{-1}x^{-1} + a_{-2}x^{-2} + \dots.$$

We call $[\xi]$ and (ξ) the *head* and the *tail* parts of ξ, respectively. Clearly, the head and tail parts of ξ are uniquely determined. Let $\beta_0 = [\xi] \in \mathbb{F}_q[x]$, so that $|\beta_0| = |\xi| \geq 1$, provided $[\xi] \neq 0$.

If $(\xi) = 0$, then the process stops. If $(\xi) \neq 0$, then write $\xi = \beta_0 + \xi_1^{-1}$, where $\xi_1^{-1} = (\xi)$ with $|\xi_1| > 1$. Next write $\xi_1 = [\xi_1] + (\xi_1)$. Let $\beta_1 = [\xi_1] \in \mathbb{F}_q[x] \backslash \mathbb{F}_q$, so that $|\beta_1| = |\xi_1| > 1$.

If $(\xi_1) = 0$, then the process stops. If $(\xi_1) \neq 0$, then write $\xi_1 = \beta_1 + \xi_2^{-1}$, where $\xi_2^{-1} = (\xi_1)$ with $|\xi_2| > 1$. Let $\beta_2 = [\xi_2] \in \mathbb{F}_q[x] \backslash \mathbb{F}_q$, then $|\beta_2| = |\xi_2| > 1$.

Again, if $(\xi_2) = 0$, then the process stops. If $(\xi_2) \neq 0$, then continue in the same manner. By so doing, we obtain the unique representation

$$\xi = [\beta_0, \ \beta_1, \ \beta_2, \dots, \ \beta_{n-1}, \ \xi_n] := \beta_0 + \frac{1}{\beta_1 +} \frac{1}{\beta_2 +} \cdots \frac{1}{\beta_{n-1} +} \frac{1}{\xi_n},$$

where $\beta_i \in \mathbb{F}_q[x] \backslash \mathbb{F}_q \ \ (i \geq 1)$, $\xi_n \in \mathbb{F}_q((x^{-1}))$, $|\xi_n| > 1$ if exists, and ξ_n is referred to as the n^{th} *complete quotient*. The sequence (β_n) is uniquely determined and the β_n are called the *partial quotients* of ξ.

In order to establish convergence, we define two sequences (C_n), (D_n) as follows:

$$C_{-1} = 1, \ C_0 = \beta_0, \ \ C_{n+1} = \beta_{n+1} C_n + C_{n-1} \ \ (n \geq 0)$$
$$D_{-1} = 0, \ D_0 = 1 \ \ , \ \ D_{n+1} = \beta_{n+1} D_n + D_{n-1} \ \ (n \geq 0).$$

The results in the following lemma are easily verified by induction.

Lemma 2.8. *For any* $n \geq 0$, $\alpha \in \mathbb{F}_q((x^{-1}))\backslash\{0\}$, *we have*
i) $\dfrac{\alpha C_n + C_{n-1}}{\alpha D_n + D_{n-1}} = [\beta_0,\ \beta_1,\ \beta_2, \ldots,\ \beta_n,\ \alpha]$,
ii) $C_n D_{n-1} - C_{n-1} D_n = (-1)^{n-1}$,
iii) $|D_n| > |D_{n-1}|$,
iv) $|D_n| = |\beta_1 \beta_2 \ldots \beta_n| \quad (n \geq 1)$,
v) $\xi - \dfrac{C_n}{D_n} = \dfrac{(-1)^n}{D_n(\xi_{n+1} D_n + D_{n-1})} \quad (n \geq 1)$.

From Lemma 2.8 (i), we have

$$\frac{C_n}{D_n} = \frac{\beta_n C_{n-1} + C_{n-2}}{\beta_n D_{n-1} + D_{n-2}} = [\beta_0,\ \beta_1,\ \beta_2, \ldots,\ \beta_n] \ \ (n \geq 1),$$

and so C_n/D_n is called the n^{th} *convergent* of the **RCF** of ξ. If $(\xi_n) = 0$ for some n, then $\xi = [\beta_0,\ \beta_1,\ \beta_2, \ldots,\ \beta_{n-1}]$, i.e. the **RCF** of ξ terminates. Otherwise, $(\xi_n) \neq 0$ for all n and the **RCF** is infinite and this is the case of interest from now on. Since $|\xi_n| = |\beta_n| \geq q$, Lemma 2.8 (iii) and (iv) give

$$|D_n(\xi_{n+1} D_n + D_{n-1})| = |D_n|^2 |\beta_{n+1}| \geq q^{2n+1}.$$

Using Lemma 2.8 (v), we get the approximation

$$\left|\xi - \frac{C_n}{D_n}\right| \leq \frac{1}{q^{2n+1}} \to 0 \quad (n \to \infty),$$

which immediately implies that $C_n/D_n \to \xi$, and enables us to write $\xi = [\beta_0,\ \beta_1,\ \beta_2,\ \beta_3, \ldots]$, where the right hand side is referred to as the **RCF** of ξ.

As in the classical case, the following characterization of rational elements in $\mathbb{F}_q((x^{-1}))$ via their **RCF** is well-known, see e.g. [29].

Theorem 2.9. *Let* $\xi \in \mathbb{F}_q((x^{-1}))$. *Then* ξ *is rational if and only if its continued fraction is finite.*

An infinite continued fraction of the shape $[\beta_0,\ \beta_1,\ \beta_2, \ldots]$ is said to be *periodic* if there are positive integers k, N such that $\beta_n = \beta_{n+k}$ for all $n \geq N$ and is denoted by

$$[\beta_0,\ \beta_1, \ldots, \beta_{N-1},\ \overline{\beta_N,\ \beta_{N+1}, \ldots, \beta_{N+k-1}}].$$

The following theorem is easily checked and we omit the proof.

Theorem 2.10. *Let $\xi \in \mathbb{F}_q((x^{-1}))$. If the continued fraction of ξ is periodic, then ξ is an irrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x]$, $a \neq 0$.*

For the converse of Theorem 2.10 we have:

Theorem 2.11. *Let $\xi \in \mathbb{F}_q((x^{-1}))$. If ξ is an irrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x]$, $a \neq 0$, then the continued fraction of ξ is periodic.*

2.5 Schneider continued fractions in the field $\mathbb{F}_q((x^{-1}))$

Since every element $\xi \in \mathbb{F}_q((x^{-1}))\backslash\{0\}$ can be uniquely written in the form

$$\xi = \sum_{n=r}^{\infty} c_n x^{-n},$$

the construction of **SCF** for ξ runs as follows: Define $b_0 = \sum_{n=r}^{0} c_n x^{-n}$ $(r \leq 0)$, so that $|b_0| \geq 1$, provided $b_0 \neq 0$.

If $\xi = b_0$, the process stops. Otherwise, write $\xi - b_0 = \sum_{n=\alpha_1}^{\infty} c_n x^{-n}$ where $\alpha_1 \geq 1$, $c_{\alpha_1} \in \mathbb{F}_q\backslash\{0\}$. Define $a_1 = x^{-\alpha_1}$, $\xi_1^{-1} = \sum_{n=\alpha_1}^{\infty} c_n x^{n-\alpha_1}$. Then $|a_1| = q^{-\alpha_1}$, $|\xi_1^{-1}| = 1$, and

$$\xi = b_0 + \frac{a_1}{\xi_1} := [b_0;\ a_1,\ \xi_1].$$

Write $\xi_1 = \sum_{n=0}^{\infty} c_n^{(1)} x^{-n}$, $c_0^{(1)} \in \mathbb{F}_q\backslash\{0\}$. Let $b_1 = c_0^{(1)}$, so that $b_1 \in \mathbb{F}_q\backslash\{0\}$ and $|b_1| = 1$.

If $\xi_1 = b_1$, the process stops. Otherwise, write $\xi_1 - b_1 = \sum_{n=\alpha_2}^{\infty} c_n^{(1)} x^{-n}$ where $\alpha_2 \geq 1$, $c_{\alpha_2}^{(1)} \in \mathbb{F}_q\backslash\{0\}$. Define $a_2 = x^{-\alpha_2}$, $\xi_2^{-1} = \sum_{n=\alpha_2}^{\infty} c_n^{(1)} x^{n-\alpha_2}$. Then $|a_2| = q^{-\alpha_2}$, $|\xi_2^{-1}| = 1$, and

$$\xi = b_0 + \frac{a_1}{b_1+}\frac{a_2}{\xi_2} := [b_0;\ a_1,\ b_1;\ a_2,\ \xi_2].$$

Write $\xi_2 = \sum_{n=0}^{\infty} c_n^{(2)} x^{-n}$, $c_0^{(2)} \in \mathbb{F}_q\backslash\{0\}$. Let $b_2 = c_0^{(2)}$, so that $b_2 \in \mathbb{F}_q\backslash\{0\}$ and $|b_2| = 1$.

If $\xi_2 = b_2$, the process stops. Otherwise, write $\xi_2 - b_2 = \sum_{n=\alpha_3}^{\infty} c_n^{(2)} x^{-n}$ where $\alpha_3 \geq 1$, $c_{\alpha_3}^{(2)} \in \mathbb{F}_q \backslash \{0\}$.

In general if $\xi_k = b_k$, the process stops. Otherwise, write $\xi_k - b_k = \sum_{n=\alpha_{k+1}}^{\infty} c_n^{(k)} x^{-n}$ where $\alpha_{k+1} \geq 1$, $c_{\alpha_{k+1}}^{(k)} \in \mathbb{F}_q \backslash \{0\}$. Define $a_{k+1} = x^{-\alpha_{k+1}}$, $\xi_{k+1}^{-1} = \sum_{n=\alpha_{k+1}}^{\infty} c_n^{(k)} x^{n-\alpha_{k+1}}$. Then $|a_{k+1}| = q^{-\alpha_{k+1}}$, $|\xi_{k+1}^{-1}| = 1$, and

$$\xi = b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_k}{b_k+} \frac{a_{k+1}}{\xi_{k+1}} := [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_k,\ b_k;\ a_{k+1},\ \xi_{k+1}],$$

where $b_0 \in \mathbb{F}_q[x]$, $b_k \in \mathbb{F}_q \backslash \{0\}$, $a_k = x^{-\alpha_k}$, $\alpha_k \in \mathbb{N}$ $(k \geq 1)$.

We call the uniquely constructed b_n, a_n and ξ_n the n^{th} *partial denominator, the* n^{th} *partial numerators* and the n^{th} *complete quotient* of the **SCF** of ξ, respectively.

Next we define two sequences (A_n), (B_n).

$$A_{-1} = 1, A_0 = b_0, \quad A_{n+1} = b_{n+1} A_n + a_{n+1} A_{n-1} \ (n \geq 0)$$
$$B_{-1} = 0, \quad B_0 = 1, \quad B_{n+1} = b_{n+1} B_n + a_{n+1} B_{n-1} \ (n \geq 0).$$

The results in the following lemma are easily verified by induction.

Lemma 2.12. *For any* $n \geq 0$, $\alpha \in \mathbb{F}_q((x^{-1})) \backslash \{0\}$, *we have*
(i) $\dfrac{\alpha A_n + a_{n+1} A_{n-1}}{\alpha B_n + a_{n+1} B_{n-1}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n;\ a_{n+1},\ \alpha]$,
(ii) $A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} a_1 a_2 \cdots a_n$ $(n \geq 1)$,
(iii) $|B_n| = |b_n| = |\xi_n| = 1$, $|a_n| = q^{-\alpha_n}$ $(n \geq 1)$,
(iv) $|\xi_{n+1} B_n + a_{n+1} B_{n-1}| = |\xi_{n+1} B_n| = 1$,
(v) $\xi - \dfrac{A_n}{B_n} = \dfrac{(-1)^n a_1 a_2 \cdots a_{n+1}}{B_n(\xi_{n+1} B_n + a_{n+1} B_{n-1})}$ $(n \geq 1)$.

From Lemma 2.12 (i), we have

$$\frac{A_n}{B_n} = \frac{b_n A_{n-1} + a_n A_{n-2}}{b_n B_{n-1} + a_n B_{n-2}} = [b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n] \ \ (n \geq 1),$$

and so A_n/B_n is called the n^{th} *convergent* of **SCF**. If the **SCF** of ξ is finite, i.e. $\xi_n = b_n$ for some n, then the **SCF** of ξ terminates and is equal to $[b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots;\ a_n,\ b_n]$.

Assume that $\xi_n \neq b_n$ for all n. By Lemma 2.12 (iii), (iv), (v), we have

$$\left|\xi - \frac{A_n}{B_n}\right| = q^{-(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1})} \to 0 \ \ (n \to \infty),$$

so A_n/B_n converges to ξ enabling us to write $\xi = [b_0;\ a_1,\ b_1;\ a_2,\ b_2;\ldots]$, and the right-hand expression is referred to as the **SCF** of ξ.

Observe from above that $[b_0; a_1, b_1; \ldots; a_n, b_n] = A_n/B_n \quad (n \geq 1)$, so another way of representing a continued fraction is based on the following matrix representation, see e.g. van der Poorten [26],

$$\begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & a_n \end{bmatrix} \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{bmatrix}.$$

Any 2×2 matrix $\begin{bmatrix} u & v \\ w & z \end{bmatrix}$ over the field $\mathbb{F}_q((x^{-1}))$ is said to have an *admissible decomposition* if it can be written as

$$\begin{bmatrix} u & v \\ w & z \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & a_n \end{bmatrix} \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix},$$

where the a_i's and b_i's are of the form mentioned in the construction of **SCF**. Such a_i's and b_i's are referred to as *admissible partial numerators* and *denominators*, respectively.

The following lemma summarizes their major identities, the proof of which is easily done by induction.

Proposition 2.13. *With the above notation, we have*

(i) $$\begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & a_n \end{bmatrix} \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_n \\ B_n \end{bmatrix},$$

(ii) $$\begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_n \end{bmatrix} \cdots \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_n & B_n \\ A_{n-1} & B_{n-1} \end{bmatrix},$$

(iii) $$\begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_n \end{bmatrix} \cdots \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_n & B_n \\ A_{n-1} & B_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -b_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_n & B_n \\ A_{n-1} & B_{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} B_n \\ B_{n-1} \end{bmatrix}.$$

Theorem 2.14. *Let $\xi \in \mathbb{F}_q((x^{-1}))$. Then ξ is rational if and only if its* ***SCF*** *is finite.*

Definition 2.15. An infinite **SCF** of the shape $[b_0; a_1, b_1; a_2, b_2; \ldots]$ is said to be *periodic* if there is a positive integer r such that $a_m = a_{m+r}$ and $b_m = b_{m+r}$ for all sufficiently large integer m and such **SCF** is denoted by

$$\xi = [b_0; a_1, b_1; \ldots; a_{m-1}, b_{m-1}; \overline{a_m, b_m; \ldots; a_{m+r-1}, b_{m+r-1}}].$$

If the integers m and r are the least among all possible such values, then we call $[b_0; a_1, b_1; \ldots; a_{m-1}, b_{m-1}]$ its *preperiod* of length m, and $[a_m, b_m; \ldots; a_{m+r-1}, b_{m+r-1}]$ its *period* of length r.

The following theorem is easily checked and we omit the proof.

Theorem 2.16. *If the SCF of $\xi \in \mathbb{F}_q((x^{-1}))$ is periodic, then ξ is a root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x]$, $a \neq 0$.*

Whether the converse of Theorem 2.16 holds generally is still not known, we first give examples of quadratic irrationals with periodic **SCF**'s.

Example Let the **SCF** of $\xi \in \mathbb{F}_q((x^{-1}))$ be of the form

$$[b_0;\ a_1,\ b_1;\ a_2,\ b_2; \ldots]$$

where $a_i = x^{-\alpha_i}$ and let the sequence of positive integers (γ_i) be defined by $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_2 - \alpha_1, \ldots,\ \gamma_i = \alpha_i - \alpha_{i-1} + \cdots + (-1)^{i+1}\alpha_1 \ \ (i \geq 1)$. Assume $\gamma_i \geq 0 \ \ (i \geq 1)$. If ξ is an irrational root of a quadratic equation of the form $at^2 + bt + c = 0$ where $a, b, c \in \mathbb{F}_q[x]$, $a \neq 0$, then its **SCF** is periodic.

Since $\xi = [b_0;\ x^{-\alpha_1},\ b_1;\ x^{-\alpha_2},\ b_2; \ldots]$, then inverting the $x^{-\alpha_i}$'s, we see that

$$\begin{aligned}
\xi =&[b_0;\ 1,\ b_1x^{\alpha_1};\ x^{-(\alpha_2-\alpha_1)},\ b_2;\ x^{-\alpha_3},\ b_3; \ldots] \\
=&[b_0;\ 1,\ b_1x^{\alpha_1}; 1,\ b_2x^{\alpha_2-\alpha_1};\ x^{-(\alpha_3-(\alpha_2-\alpha_1))},\ b_3; \ldots] \\
&\vdots \\
=&b_0 + \frac{1}{b_1x^{\gamma_1}+}\ \frac{1}{b_2x^{\gamma_2}+} \cdots \frac{1}{b_ix^{\gamma_i}+} \cdots
\end{aligned}$$

which is just the **RCF** ξ and so must be periodic, see e.g. Mkaouar [23], say

$$\xi = [b_0, b_1x^{\gamma_1}, \ldots, b_ix^{\gamma_i}, \overline{b_{i+1}x^{\gamma_{i+1}},\ b_{i+2}x^{\gamma_{i+2}}, \ldots, b_{i+r}x^{\gamma_{i+r}}}].$$

Inverting this continued fraction, we get

$$\xi = [b_0; x^{-\gamma_1}, b_1; x^{-\gamma_1}, b_2x^{\gamma_2}; 1, b_3x^{\gamma_3}; \ldots; \overline{1, b_{i+1}x^{\gamma_{i+1}}; \ldots; 1, b_{i+r}x^{\gamma_{i+r}}}]$$

$$= [b_0; x^{-\gamma_1}, b_1; x^{-\gamma_1-\gamma_2}, b_2; x^{-\gamma_2}, b_3x^{\gamma_3}; 1, b_4x^{\gamma_4}; \ldots; \overline{1, b_{i+1}x^{\gamma_{i+1}}; \ldots; 1, b_{i+r}x^{\gamma_{i+r}}}]$$

$$\vdots$$

$$= [b_0; a_1, b_1; \ldots; a_{i+r}, b_{i+r}; \overline{a'_{i+r+1}, b_{i+r+1}; a'_{i+r+2}, b_{i+r+2}; \ldots; a'_{i+2r}, b_{i+2r}}],$$

where

$$a'_{i+r+1} = \frac{1}{x^{\gamma_{i+r}+\gamma_{i+1}}},\ a'_{i+r+2} = \frac{1}{x^{\gamma_{i+1}+\gamma_{i+2}}}, \ldots, a'_{i+2r} = \frac{1}{x^{\gamma_{i+r-1}+\gamma_{i+r}}}$$

which is a periodic **SCF**.

Example. Another class of quadratic irrationals with periodic **SCF**'s is provided by the following observation: if $\xi \in \mathbb{F}_q((x^{-1}))$ satisfies $a\xi^2 + b\xi + c = 0$, where $a,\ b \in \mathbb{F}_q\backslash\{0\}$, and $c = c_1x^{-\alpha}$, $c_1 \in \mathbb{F}_q\backslash\{0\}$, then

$$\xi = \frac{-b}{a} - \frac{c}{a\xi},$$

and so its **SCF** is periodic of the form

$$\left[\frac{-b}{a}; \overline{x^{-\alpha}, \frac{b}{c_1}; x^{-\alpha}, \frac{-b}{a}}\right]$$

.

Our analysis of periodic **SCF**'s follows the ideas given in van der Poorten [26]. If ξ is quadratic, and ξ' its conjugate, its discriminant is defined as $disc\ \xi = (\xi - \xi')^2$.

Theorem 2.17. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ have periodic **SCF**. If $|disc\ \xi| \geq 1$, then its preperiod is of length at most 2.*

Proof. Since ξ has periodic **SCF**, then by Theorem 2.16 it is quadratic over $\mathbb{F}_q(x)$. Assume that $|disc\ \xi| \geq 1$. Write $\xi = [b_0; a_1, b_1; \ldots; a_n, \xi_n]$ where $\xi_n = [b_n; a_{n+1}, b_{n+1}; \ldots] = b_n + a_{n+1}/\xi_{n+1}$ $(n \geq 1)$. Now ξ_n is quadratic, and so conjugating, we get $\xi'_n = b_n + a_{n+1}/\xi'_{n+1}$, where $'$ denotes conjugation.

We now show that for all $h \in \mathbb{N} \cup \{0\}$, if $|\xi'_n| < 1$, then

$$|\xi'_{n+h}| < 1. \tag{2.2}$$

Clearly, this holds for $h = 0$. Assume that $|\xi'_{n+j-1}| < 1,\ j = 1,\ 2, \ldots, h$. Consider

$$\left|\frac{a_{n+h}}{\xi'_{n+h}}\right| = |\xi'_{n+h-1} - b_{n+h-1}| = |b_{n+h-1}| = 1 \text{ because } |\xi'_{n+h-1}| < 1,$$

i.e. $|\xi'_{n+h}| = |a_{n+h}| = q^{-\alpha_{n+h}} < 1$.

Next, we show that if $|\xi'_1| \geq 1$, then

$$|disc\ \xi| < 1. \tag{2.3}$$

If $|\xi'_1| \geq 1$, then

$$|\xi' - b_0| = \frac{|a_1|}{|\xi'_1|} \leq |a_1| = q^{-\alpha_1},$$

while

$$|\xi - b_0| = \frac{|a_1|}{|\xi_1|} = |a_1| = q^{-\alpha_1}.$$

Then

$$|disc\ \xi| = |\xi - \xi'|^2 = |(\xi - b_0) - (\xi' - b_0)|^2 \leq \max\{|\xi - b_0|^2, |\xi' - b_0|^2\} = q^{-2\alpha_1} < 1.$$

From (2.2), (2.3) and the hypothesis, we deduce that $|\xi'_{n+h}| < 1$ for integers $h \geq 0,\ n \geq 1$.

Let $[b_0; a_1, b_1; \ldots; a_{m-1}, b_{m-1}]$ be the preperiod of ξ of minimal length m with period of length r. For each $h = 0, 1, 2, \ldots$, we have

$\xi_{m+r+h} = \xi_{m+h},\ b_{m+r+h} = b_{m+h},\ a_{m+r+h} = a_{m+h}$.

Thus

$$b_{m-1} - \xi'_{m-1} = -\frac{a_m}{\xi'_m} = -\frac{a_{m+r}}{\xi'_{m+r}} = b_{m+r-1} - \xi'_{m+r-1}. \tag{2.4}$$

If $m > 1$, then (2.4) is the construction of **SCF** of $-a_m/\xi'_m$ which is unique. Thus $b_{m-1} = b_{m+r-1}$ and $\xi'_{m-1} = \xi'_{m+r-1}$. Consider

$$\frac{a_{m-1}}{\xi'_{m-2} - b_{m-2}} = \xi'_{m-1} = \xi'_{m+r-1} = \frac{a_{m+r-1}}{\xi'_{m+r-2} - b_{m+r-2}}.$$

If $m \geq 3$, then $|\xi'_{m-2} - b_{m-2}| = |b_{m-2}| = 1 = |b_{m+r-2}| = |\xi'_{m+r-2} - b_{m+r-2}|$. By the uniqueness of expansion of ξ'_{m-1}, $a_{m-1} = a_{m+r-1}$, contradicting the minimality of m. Hence $m \leq 2$. □

From the proof of Theorem 2.17 we have the following proposition.

Proposition 2.18. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ have periodic **SCF**. If $|\xi'_1| < 1$, where ξ'_1 is the quadratic conjugate of ξ_1, then its preperiod is of length at most 2.*

Proposition 2.19. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ be quadratic irrational with $|\xi| = 1$ satisfying $A\xi^2 + B\xi + C = 0$ and $|A| < |B|$. If ξ has periodic **SCF**, then its preperiod is of length at most 2.*

Proof. For all $n \geq 1$, we can write

$$\xi = \frac{\xi_n A_{n-1} + a_n A_{n-2}}{\xi_n B_{n-1} + a_n B_{n-2}}.$$

Substituting to the quadratic equation $A\xi^2 + B\xi + C = 0$. Then

$$\begin{aligned}0 &= A(\xi_n^2 A_{n-1}^2 + 2\xi_n a_n A_{n-1}A_{n-2} + a_n^2 A_{n-2}^2)\\ &+B(\xi_n^2 A_{n-1}B_{n-1} + \xi_n a_n A_{n-1}B_{n-2} + \xi_n a_n B_{n-1}A_{n-2} + a_n^2 A_{n-2}B_{n-2})\\ &+C(\xi_n^2 B_{n-1}^2 + 2\xi_n a_n B_{n-1}A_{n-2} + a_n^2 B_{n-2}^2).\end{aligned}$$

Thus

$$\begin{aligned}0 &= \xi_n^2(AA_{n-1}^2 + BA_{n-1}B_{n-1} + CB_{n-1}^2)\\ &+\xi_n(a_n A_{n-1}B_{n-2}B + a_n B_{n-1}A_{n-2}B + 2a_n AA_{n-1}A_{n-2} + 2a_n CB_{n-1}B_{n-2})\\ &+Aa_n^2A_{n-2}^2 + Ba_n^2A_{n-2}B_{n-2} + Ca_n^2B_{n-2}^2.\end{aligned}$$

We obtain

$$\begin{aligned}|\xi_1'| = |\xi_1\xi_1'| &= \left|\frac{Aa_1^2A_{-1}^2 + Ba_1^2A_{-1}B_{-1} + Ca_1^2B_{-1}^2}{AA_0^2 + BA_0B_0 + CB_0^2}\right| = \left|\frac{Aa_1^2}{Ab_0^2 + Bb_0 + C}\right|\\ &= \left|\frac{Aa_1^2}{A(\xi - \frac{a_1}{\xi_1})^2 + B(\xi - \frac{a_1}{\xi_1}) + C}\right| = \left|\frac{Aa_1^2}{-2\xi\frac{a_1}{\xi_1}A - B\frac{a_1}{\xi_1} + A\frac{a_1^2}{\xi_1^2}}\right| = \left|\frac{Aa_1^2}{Ba_1}\right| < 1.\end{aligned}$$

By Proposition 2.18, **SCF** of ξ has its preperiod of length at most 2. □

Theorem 2.20. *Let $\xi \in \mathbb{F}_q((x^{-1}))$.*

*(i) If ξ has a periodic **SCF** of the form $[b_0; \overline{a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0}]$, then ξ is quadratic and $A_r = -\xi\xi' B_{r-1}$, $B_r = A_{r-1} - (\xi + \xi')B_{r-1}$, where ξ' is the quadratic conjugate of ξ. Moreover, we have the admissible decomposition*

$$\begin{bmatrix} -\xi\xi' B_{r-1} & A_{r-1} \\ A_{r-1} - (\xi + \xi')B_{r-1} & B_{r-1} \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} b_1 & 1 \\ a_1 & 0 \end{bmatrix}\cdots\begin{bmatrix} b_r & 1 \\ a_r & 0 \end{bmatrix}.$$

*(ii) If ξ is quadratic and there exists $r \geq 1$ such that $A_r = -\xi\xi' B_{r-1}$, $B_r = A_{r-1} - (\xi + \xi')B_{r-1}$, where ξ' is the quadratic conjugate of ξ, then ξ or ξ' has a periodic **SCF** of the form $[b_0; \overline{a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0}]$.*

Proof. (i) Assume that ξ has a periodic **SCF** of the form

$$\xi = [b_0; \overline{a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0}] = [b_0; a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0 + (\xi - b_0)].$$

Then in the matrix representation form

$$\begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & a_r \end{bmatrix} \begin{bmatrix} b_r + b_0 + (\xi - b_0) & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_r & A_{r-1} \\ B_r & B_{r-1} \end{bmatrix} \begin{bmatrix} b_r & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_r + b_0 + (\xi - b_0) & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_r & A_{r-1} \\ B_r & B_{r-1} \end{bmatrix} \begin{bmatrix} b_r & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_r & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}$$
$$= \begin{bmatrix} A_r + A_{r-1}\xi & A_{r-1} \\ B_r + B_{r-1}\xi & B_{r-1} \end{bmatrix},$$

i.e.

$$\frac{A_r + A_{r-1}\xi}{B_r + B_{r-1}\xi} = [b_0; a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0 + (\xi - b_0)] = \xi,$$

and so

$$B_{r-1}\xi^2 - (A_{r-1} - B_r)\xi - A_r = 0.$$

So $A_r = -\xi\xi' B_{r-1}$, $B_r = A_{r-1} - (\xi + \xi')B_{r-1}$. Assume that ξ is quadratic and there exists $r \geq 1$ such that $A_r = -\xi\xi' B_{r-1}$, $B_r = A_{r-1} - (\xi + \xi')B_{r-1}$, where ξ' is the quadratic conjugate of ξ. Then we have the admissible decomposition

$$\begin{bmatrix} -\xi\xi' B_{r-1} & A_{r-1} \\ A_{r-1} - (\xi + \xi')B_{r-1} & B_{r-1} \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ a_1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_r & 1 \\ a_r & 0 \end{bmatrix}.$$

(ii) Let $\zeta = [b_0; \overline{a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0}]$. By the same reasoning as above, $B_{r-1}(\zeta^2 - (\xi + \xi')\zeta + \xi\xi') = 0$. Then ζ is a quadratic root of the $t^2 - (\xi + \xi')t + \xi\xi' = 0$, and so $\zeta = \xi$ (or ξ'). □

Corollary 2.21. *Let $\xi \in \mathbb{F}_q((x^{-1}))$. If ξ has a periodic **SCF** of the form $[b_0; \overline{a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, \delta_r + b_0}]$, where $b_r = \delta_r + b_0$, then we have the admissible decomposition*

$$\begin{bmatrix} -\xi\xi' B_{r-1} & A_{r-1} \\ A_{r-1} - (\xi + \xi')B_{r-1} & B_{r-1} \end{bmatrix} = \begin{bmatrix} A_r & A_{r-1} \\ B_r & B_{r-1} \end{bmatrix} \begin{bmatrix} b_r & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \delta_r & 1 \\ 1 & 0 \end{bmatrix}.$$

Theorem 2.22. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ have a periodic* ***SCF*** *of the form* $[b_0; \overline{a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + b_0}]$, *with ξ' being its quadratic conjugate. If $\xi + \xi' + b_r \in \mathbb{F}_q\backslash\{0\}$, then $\xi + \xi' + b_r = b_0$ and $a_1, b_1, \ldots, a_{r-1}, b_{r-1}, a_r$ are palindrome.*

Proof. Assume that $\xi + \xi' + b_r \in \mathbb{F}_q\backslash\{0\}$. By Theorem 2.20, we have the admissible decomposition

$$\begin{bmatrix} -\xi\xi' B_{r-1} & A_{r-1} \\ A_{r-1} - (\xi + \xi')B_{r-1} & B_{r-1} \end{bmatrix} = \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ a_1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_r & 1 \\ a_r & 0 \end{bmatrix}.$$

Multiplying both sides of the equation on the right by the matrix $\begin{bmatrix} 1 & 0 \\ \xi + \xi' & 1 \end{bmatrix}$, we have

$$\begin{bmatrix} A_{r-1}(\xi + \xi') - \xi\xi' B_{r-1} & A_{r-1} \\ A_{r-1} & B_{r-1} \end{bmatrix}$$
$$= \begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ a_1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_{r-1} & 1 \\ a_{r-1} & 0 \end{bmatrix} \begin{bmatrix} b_r + (\xi + \xi') & 1 \\ a_r & 0 \end{bmatrix}.$$

Then

$$\frac{A_{r-1}(\xi + \xi') - \xi\xi' B_{r-1}}{A_{r-1}} = [b_0; a_1, b_1; \ldots; a_{r-1}, b_{r-1}; a_r, b_r + (\xi + \xi')]. \tag{2.5}$$

The left hand side of (2.5) is

$$\xi + \xi' - \frac{\xi\xi' B_{r-1}}{A_{r-1}} = \xi + \xi' + \frac{A_r}{A_{r-1}} = [\xi + \xi' + b_r; a_r, b_{r-1}; \ldots; a_1, b_0], \tag{2.6}$$

noting that

$$\frac{A_r}{A_{r-1}} = [b_r; a_r, b_{r-1}; \ldots; a_1, b_0].$$

Since $\xi + \xi' + b_r \in \mathbb{F}_q\backslash\{0\}$, then by the uniqueness of **SCF** and (2.5) and (2.6), we get $\xi + \xi' + b_r = b_0$, $a_1 = a_r$, $b_1 = b_{r-1}, \ldots, a_r = a_1$, $\xi + \xi' + b_r = b_0$ i.e. $a_1, b_1, \ldots, a_{r-1}, b_{r-1}, a_r$ are palindrome. □

Proposition 2.23. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ be a quadratic irrational with $|disc\ \xi| \geq 1$ and $b_0 \neq 0$ and whose defining equation is $At^2 + Bt + C = 0$, where $A, B, C \in \mathbb{F}_q[x]$, $A \neq 0$. If ξ has a periodic* ***SCF*** *of preperiod length 1, then $|\xi\xi'| \leq |b_0|^2$, and $|\xi + \xi'| \leq |b_0|$. Moreover, If $|b_0| > 1$, then $|\xi\xi'| = |b_0|^2$.*

Proof. Assume that ξ has its preperiod of length 1. Write

$$\xi = [b_0; \overline{a_1, b_1; \dots ; a_r, b_r}] = [b_0; a_1, b_1; \dots ; a_r, b_r + (\xi - b_0)].$$

Then in the matrix representation form

$$\begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & a_r \end{bmatrix} \begin{bmatrix} b_r + (\xi - b_0) & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_r & A_{r-1} \\ B_r & B_{r-1} \end{bmatrix} \begin{bmatrix} b_r & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_r & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \xi - b_0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} A_r + A_{r-1}(\xi - b_0) & A_{r-1} \\ B_r + B_{r-1}(\xi - b_0) & B_{r-1} \end{bmatrix},$$

i.e.

$$\frac{A_r + A_{r-1}(\xi - b_0)}{B_r + B_{r-1}(\xi - b_0)} = [b_0; a_1, b_1; \dots ; a_{r-1}, b_{r-1}; a_r, b_r + b_0 + (\xi - b_0)] = \xi,$$

and so

$$B_{r-1}\xi^2 + (B_r - b_0 B_{r-1} - A_{r-1})\xi - A_r + b_0 A_{r-1} = 0.$$

Clearly, $B_{r-1} \neq 0$. Hence

$$|\xi\xi'| = \frac{|-A_r + b_0 A_{r-1}|}{|B_{r-1}|} \leq |b_0|^2,$$

and

$$|\xi + \xi'| = \frac{|B_r - b_0 B_{r-1} - A_{r-1}|}{|B_{r-1}|} \leq |b_0|.$$

□

Lemma 2.24. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ be a quadratic irrational with $|disc\ \xi| \geq 1$ and $b_0 \neq 0$. Assume ξ has a periodic* ***SCF****.*
(i) If ξ has its preperiod of length 1 with period length r, then $|a_r| = |(\xi' - b_0)|$.
(ii) If ξ has its preperiod of length 2 with period length r, then $|a_{r+1}| = |\xi_1'|$.

Proof. (i) Since

$$\xi = b_0 + \frac{a_1}{\xi_1},\ \xi_1 = b_1 + \frac{a_2}{\xi_2}, \dots,\ \xi_{r-1} = b_{r-1} + \frac{a_r}{\xi_r},\ \xi_r = b_r + \frac{a_1}{\xi_1}, \dots,$$

we have

$$\xi' = b_0 + \frac{a_1}{\xi_1'},\ \xi_1' = b_1 + \frac{a_2}{\xi_2'}, \dots,\ \xi_{r-1}' = b_{r-1} + \frac{a_r}{\xi_r'},\ \xi_r' = b_r + \frac{a_1}{\xi_1'}, \dots.$$

By the proof of Theorem 2.17, $|disc\ \xi| \geq 1 \Rightarrow |\xi_1'| < 1 \Rightarrow |\xi_n'| < 1$ $(n \geq 2)$. Then the **SCF** for $-a_1/\xi_1'$ is of the form

$$\begin{aligned}
\frac{-a_1}{\xi_1'} &= b_r - \xi_r' \text{ where } \xi_r' = (\tfrac{-a_1}{\xi_1'}),\\
&= [b_r; a_r, -\tfrac{a_r}{\xi_r'}] = [b_r; a_r, b_{r-1} - \xi_{r-1}'] \text{ where } \xi_{r-1}' = (\tfrac{-a_r}{\xi_r'}),\\
&= [b_r; a_r, b_{r-1}; a_{r-1}, -\tfrac{a_{r-1}}{\xi_{r-1}'}]\\
&= \dots\\
&= [b_r; a_r, b_{r-1}; a_{r-1}, b_{r-2}; \dots; a_2, b_1; a_1, -\tfrac{a_1}{\xi_1'}]\\
&= [b_r; \overline{a_r, b_{r-1}; a_{r-1}, b_{r-2}; \dots; a_2, b_1; a_1, b_r}].
\end{aligned}$$

Since

$$|a_r| = \left|\frac{-a_1}{\xi_1'} - b_r\right| = |\xi_r'| < 1,$$

we have

$$|a_r| = \left|\left(\frac{-a_1}{\xi_1'}\right)\right| = |(b_0 - \xi')| = |(\xi' - b_0)|.$$

(ii) Write $\xi = [b_0; a_1, b_1; \overline{a_2, b_2; \dots; a_r, b_r}]$. By (i), $|a_{r+1}| = |(\xi_1' - b_1)| = |\xi_1'|$ since $|\xi_1'| < 1$.

□

Lemma 2.25. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ be a quadratic irrational with $|disc\ \xi| \geq 1$ and $b_0 \neq 0$ and whose defining equation is $At^2 + Bt + C = 0$, where $A, B, C \in \mathbb{F}_q[x]$, $A \neq 0$. Then $|\xi_1'| = |Aa_1|/|2b_0A + B|$.*

Proof. Write $\xi = b_0 + a_1/\xi_1$. Then

$$0 = A\xi^2 + B\xi + C = A\left(b_0 + \frac{a_1}{\xi_1}\right)^2 + B\left(b_0 + \frac{a_1}{\xi_1}\right) + C.$$

Thus ξ_1 is a quadratic irrational satisfying

$$(Ab_0^2 + Bb_0 + C)\xi_1^2 + (2a_1b_0A + a_1B)\xi_1 + Aa_1^2 = 0.$$

Since $|disc\ \xi| \geq 1$, $|\xi_1'| < 1$, so that

$$1 = |\xi_1| = |\xi_1 + \xi_1'| = \left|\frac{a_1(2b_0A + B)}{Ab_0^2 + Bb_0 + C}\right|.$$

Hence

$$|\xi_1'| = |\xi_1\xi_1'| = \left|\frac{Aa_1^2}{Ab_0^2 + Bb_0 + C}\right| = \left|\frac{Aa_1}{2b_0A + B}\right|.$$

□

Proposition 2.26. *Let $\xi \in \mathbb{F}_q((x^{-1}))$ be a quadratic irrational with $|disc\ \xi| \geq 1$ and $b_0 \neq 0$ and whose defining equation is $At^2 + Bt + C = 0$, where $A, B, C \in \mathbb{F}_q[x]$, $A \neq 0$. If ξ has a periodic **SCF** of preperiod length 2, then* $|\xi\xi'| \leq \max\{|b_0^2|, |b_0||B/A|\}$.

Proof. Assume ξ has its preperiod length 2. Let $\xi = [b_0; a_1, b_1; \overline{a_2, b_2; \ldots; a_{r+1}, b_{r+1}}]$. Then $\xi = [b_0; a_1, b_1; a_2, b_2; \ldots; a_{r+1}, b_{r+1} + \theta]$, where $\theta = a_1(\xi - b_0)^{-1} - b_1$. We have

$$\begin{bmatrix} A_{r+1} & A_r \\ B_{r+1} & B_r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \theta & 1 \end{bmatrix} = \begin{bmatrix} A_{r+1} + A_r\theta & A_r \\ B_{r+1} + B_r\theta & B_r \end{bmatrix}.$$

Thus $A_{r+1} + A_r\theta = \xi(B_{r+1} + B_r\theta)$. Substituting $\theta = a_1(\xi - b_0)^{-1} - b_1$ to the above equation, we have
$(B_{r+1} - b_1B_r)\xi^2 + (B_ra_1 - B_{r+1}b_0 + B_rb_0b_1 - A_{r+1} + A_rb_1)\xi + (A_{r+1}b_0 - A_ra_1 - A_rb_1b_0) = 0$.
Then

$$\xi\xi' = \frac{A_{r+1}b_0 - A_ra_1 - A_rb_1b_0}{B_{r+1} - b_1B_r}. \tag{2.7}$$

If $b_{r+1} - b_1 \neq 0$, then $|B_{r+1} - b_1B_r| = |(b_{r+1} - b_1)B_r + a_{r+1}B_{r-1}| = |(b_{r+1} - b_1)B_r| = 1$ and $|A_{r+1}b_0 - A_ra_1 - A_rb_1b_0| = |b_0(b_{r+1} - b_1)A_r + b_0a_{r+1}Ar - 1 - A_ra_1| = |b_0|^2$. Substituting it to (2.7),

$$|\xi\xi'| = \frac{|A_{r+1}b_0 - A_ra_1 - A_rb_1b_0|}{|B_{r+1} - b_1B_r|} = |b_0|^2 \leq \max\left\{|b_0^2|, \left|\frac{b_0B}{A}\right|\right\}.$$

If $b_{r+1} - b_1 = 0$, then $|B_{r+1} - b_1B_r| = |a_{r+1}B_{r-1}| = |a_{r+1}|$.

We have

$$|a_{r+1}| = |\xi_1'| = \frac{|Aa_1|}{|2b_0A + B|}.$$

Then

$$\frac{|a_1|}{|a_{r+1}|} = \frac{|2b_0A + B|}{|A|} \leq \max\left\{|b_0|, \left|\frac{B}{A}\right|\right\}.$$

Substituting it to (2.7),

$$|\xi\xi'| = \left|\frac{A_{r+1}b_0 - A_ra_1 - A_rb_1b_0}{a_{r+1}}\right| = \left|\frac{b_0(b_{r+1} - b_0)A_r + b_0a_{r+1}A_{r-1} - A_ra_1}{a_{r+1}}\right|$$
$$= \left|b_0A_{r-1} - \frac{a_1}{a_{r+1}}A_r\right| \leq \max\left\{|b_0^2|, \left|\frac{b_0B}{A}\right|\right\}.$$

□