CHAPTER I

FEYNMAN PATH INTEGRAL AND SOME TECHNIQUES OF PATH INTEGRATION

1.1 INTRODUCTION

It[1] is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations : the differentail equation of Schrödinger , and the matrix algebra of Heisenberg. The two , apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This thesis will describe what is essentially a third formulation of non -relativistic quantum theory. This formulation was suggested by some of Dirac ' s remarks concerning the relation of classical action to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time .

In chapter I - III , we review the theory concerning the path Integral and derivation of the Baker - Haudroff lemma for our the practical work .

For chapter IV - V , we use the Baker - Haudroff lemma for finding the coordinate operators for the several different Hamiltonians. Then , we can find the propagators of the systems , particularly the hamiltonian has been composed with quadratic and cubic potential terms. And finally , we discuss and conclude the propagator for potential with quadratic and cubic terms.

1.2 FEYNMAN' S FORMULATION

The[2] basic difference between classical mechanics and quantum mechanics should now be apparent. In classical mechanics a definite path in the xt - plane is associated with the particle motion ; in contrast, in quantum mechanics all possible paths must play roles including those which do not bear any resemblance to the classical path . Yet we must somehow be able to reproduce classical mechanics in a smooth manner in the limit $\hbar \rightarrow 0$.

As a young graduate student at Princeton University, R.P. Feynman tried to attack this problem. In looking for a possible clue, he was said to be intrigued by a mysterious remark in Dirac's book which, in our notation amounts, to the following statement :

$$\exp\left[i\int_{t_1}^{t_2} \frac{dt L_{classical}(x,\dot{x})}{\hbar} \right] \quad \textit{corresponds to} \quad \langle x_2, t_2 | x_1, t_1 \rangle .$$

Feynman attempted to make sense out of this remark. Is "corresponds" to the same thing as "is equal to" or "is proportional to"? In so doing he was led to formulate a space - time approach to quantum mechanics based on ***path integrals***.

In Feynman's formulation the classical action plays a very important role . For compactness, we introduce a new notation:

$$S(n,n-1) \equiv \int_{t_{n-1}}^{t_n} dt L_{classical}(x,\dot{x}). \qquad (1.2.1)$$

Because $L_{classical}$ is a functon of x and $\dot{x}$ S(n,n-1) is defined only after a definite path is specified along which the integration is to be carried out. So even though the path dependence is not explicit in this notation, it is understood that we are considering a particular path in evaluating the integral. Imagine now that we are following some prescribed path. We concentrate our attention on a small segment along that path, say between (x_{n-1}, t_{n-1}) and (x_n, t_n) According to Dirac, we are instructed to associate $\exp[iS(n,n-1)/\hbar]$ with that segment. Going along the definite path we are set to follow, we successively multiply expressions of this type to obtain

$$\prod_{n=2}^{N} \exp\left[\frac{iS(n,n-1)}{\hbar}\right] = \exp\left[\left(\frac{i}{\hbar}\right)\sum_{n=2}^{N} S(n,n-1)\right] = \exp\left[\frac{iS(N,1)}{\hbar}\right]. \qquad (1.2.2)$$

This does not yet give $\langle x_N, t_N | x_1, t_1 \rangle$; rather, this equation is the contribution to $\langle x_N, t_N | x_1, t_1 \rangle$ arising from the particular path we have considered we must still integrate over $x_2, x_3, \ldots, x_{n-1}$. At the same time, exploiting the composition property, we let the time interval between t_{n-1} and t_n be infinitesimally small. Thus our candidate expression for $\langle x_N, t_N | x_1, t_1 \rangle$ may be written, in some loose sense, as

$$\langle x_N, t_N | x_1, t_1 \rangle \sim \sum_{all\ paths} \exp\left[\frac{iS(N,1)}{\hbar}\right]. \tag{1.2.3}$$

where the sum is to be taken over an innumerably infinite set of paths!

Before presenting a more precise formulation, let us see whether considerations along this line make sense in the classical limit. $\hbar \to 0$, the exponential in (1.2.3) oscillates very violently, so there is a tendency for cancellation among various contributions from neighboring paths. This is because $\exp[iS/\hbar]$ for definite path and $\exp[iS/\hbar]$ for a slightly different path have very different because of the smallness of $\hbar$. So most paths do *not* contribute when $\hbar$ is regarded as a small quantity. However, there is an important exception.

Suppose that we consider a path that satisfies

$$\delta S(N,1) = 0, \tag{1.2.4}$$

where the change in S is due to a slight deformation of the path with the end points fixed. This is precisely the classical path by virtue of Hamilton's principle. We denote the S that satisfies (1.2.4) by S_{min}. We now attempt to deform the path a little bit from the classical path. The resulting S is still equal to S_{min} to first order in deformation. This means that the phase of $\exp[iS/\hbar]$ does not vary very much as we deviate slightly from the classical path even if $\hbar$ is small. As a result, as long as we stay near the classical path, constructive interference between neighboring path is possible. In the $\hbar \to 0$ limit, the major contributions must then arise from a very narrow strip (or a tube in higher dimensions) containing the classical path, as shown in Figure 1.1. Our (or Feynman's) guess based on Dirac's mysterious remark makes good sense because the classical path gets singled out in the $\hbar \to 0$ limit.

To formulate Feynman's conjecture more precisely, let us go back to $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$, there the time difference $t_n - t_{n-1} = \varepsilon$ is assumed to be infinitesimally small. We write

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \left[\frac{1}{w(\varepsilon)} \right] \exp\left[\frac{iS(n,n-1)}{\hbar} \right], \qquad (1.2.5)$$

where we evaluate $S(n, n-1)$ in a moment in the $\varepsilon \to 0$ limit. Notice that we have inserted a weight factor, $1/w(\varepsilon)$, which is assumed to depend only on the time interval $t_n - t_{n-1}$ and not on $V(x)$. That such a factor is needed is clear from dimensional considerations; according to the way we

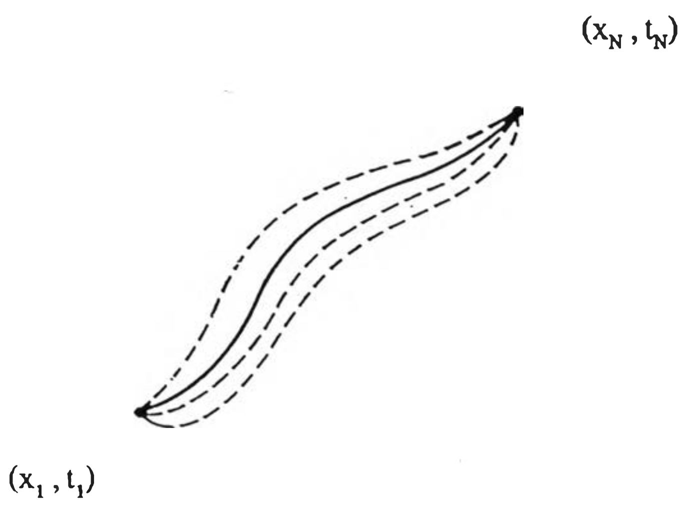

Fig 1.1 Paths important in the $\hbar \to 0$ limit.

normalized our position eigenkets, $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$ must have the dimension of 1/length.

We now look at the exponential in (1.2.5). Our task is to evaluate the $\varepsilon \to 0$ limit of S(n, n - 1). Because the time interval is so small, it is legitmate to make a straight - line approximation to the path joining (x_{n-1}, t_{n-1}) and (x_n, t_n) as follows:

$$S(n,n-1) = \int_{t_{n-1}}^{t_n} dt \left[\frac{m\dot{x}^2}{2} - V(x) \right]$$
$$= \varepsilon \left\{ \left(\frac{m}{2} \right) \left[\frac{x_n - x_{n-1}}{\varepsilon} \right]^2 - V\left(\frac{(x_n + x_{n-1})}{2} \right) \right\}. \tag{1.2.6}$$

As an example, we consider specifically the free - particle case, V = 0. Equation (1.2.5) now becomes

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \left[\frac{1}{w(\varepsilon)} \right] \exp\left[\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} \right], \tag{1.2.7}$$

We see that the exponent appearing here is completely identical to the one in the expression for the free - particle propagator. The reader may work out a similar comparison for the simple harmonic oscillator.

We remarked earlier that the weight factor $1/w(\varepsilon)$ appearing in (1.2.5) is assumed to be independent of V(x), so we may as well evaluate it for the free particle. Noting the orthonormality, in the sense of δ - function, of Heisenberg - picture position eigenkets at equal times,

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \Big|_{t_n = t_{n-1}} = \delta(x_n - x_{n-1}), \quad (1.2.8)$$

we obtain

$$\frac{1}{w(\varepsilon)} = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}. \quad (1.2.9)$$

where we have used

$$\int_{-\infty}^{\infty} d\xi \exp\left(\frac{im\xi^2}{2\hbar\varepsilon}\right) = \sqrt{\frac{2\pi i\hbar\varepsilon}{m}} \quad (1.2.10)$$

and

$$\lim_{\varepsilon \to 0} \sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \exp\left(\frac{im\xi^2}{2\hbar\varepsilon}\right) = \delta(\xi). \quad (1.2.11)$$

This weight factor is, of course, anticipated from the expression for the free - particle propagator

To summarize, as $\varepsilon \to 0$ we are led to

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \exp\left(\frac{iS(n,n-1)}{\hbar}\right) = \delta(\xi). \quad (1.2.12)$$

The final expression for the transition amplitude with $t_N - t_1$ finite is

$$\langle x_N, t_N | x_1, t_1 \rangle = \lim_{N \to \infty} \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{(n-1)/2}$$

$$\times \int dx_{N-1} \int dx_{N-2} \cdots \int dx_2 \prod_{n=2}^{N} \exp\left[\frac{iS(n,n-1)}{\hbar} \right],$$

(1.2.13)

where the $N \to \infty$ limit is taken with x_N and t_N fixed. It is customary here to define a new kind of multidimensional (in fact, infinite - dimensional) integral operator

$$\int_{x_1}^{x_N} D[x(t)] \equiv \lim_{N \to \infty} \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{(N-1)/2} \int dx_{N-1} \int dx_{N-2} \cdots \int dx_2$$

(1.2.14)

and write (1.2.13) as

$$\langle x_N, t_N | x_1, t_1 \rangle = \int_{x_1}^{x_N} D[x(t)] \exp\left[i \int_{t_1}^{t_N} \frac{dt\ L_{classical}(x, \dot{x})}{\hbar} \right].$$

(1.2.15)

This expression is known as **Feynman's path integral.** Its meaning as the sum over all possible paths should be apparent from (1.2.13). The formulation is mathematically equivalent to the more usual formulations. There are, therfore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problem for which the new point of view offers a distinct advantage.

1.3 SOME TECHNIQUES OF PATH INTEGRATION

The Feynman path integral provides an elegant approach towards solving a quantum mechanical problem once the classical Lagrangian is know. The concept of a path integral is natural generalization of the Young's double slit experiment. This experiment involves the principle of superposition of two states corresponding to the option an electron has of going through either of the two slits which we have presented and shown up the substance in the chapter II In Feynman approach the basic quantity is the propagator which represents physically the probability amplitude for a particle to travel from one space-time point to another. In quantum mechanics a particle moving from one point to another may follow different paths. The propagator is thus given by the sum of the amplitudes corresponding to all the paths connecting the two points. The contribution of each path has a phase proportional to the classical action for that path. The propagator is thus a "sum over paths" or a path integral. This geometrical way of expressing the quantum superposition principle is intuitively appealing since it allows us to visualize the constructive and destructive interference arising from many different paths.[3]

The last five decades It has been shown the growth of path integral as a powerful tool for solving problems in diverse areas of physics. Apart from its aesthetic appeal the paths approach easily lends itself to new approximation techniques. The well - know approximation schemes like the semi - classical or WKB method and perturbation expansion may be naturally incorporated within the frame- work of a path integral with added advantages of great formal simplicity and physical insight.

Despite the great success of the path integral as a methematical tool in physical applications, the summability of paths into a closed analytical form for the propagator has ofter been a difficult task. Feynman's original time - slicing scheme whereby the propagator is defined as the limit (as $N \rightarrow \infty$) of an N - dimensional Riemann integral provides exact answers only for the cases of the free particle, harmonic oscillator and the more general quadratic potentials. We may well agrue that the harmonic oscillator and other related quadratic potentials are the only cases of interest in most applications. Also a rigorous mathematical justification of the path integral exists only for quadratic actions. Extending the path integral approach beyond this though difficult is a worthwhile exercise. An example in the hydrogen atom problem which can be treated easily in the Schrödinger approach yielding the energy eigenvalues and eigenfunctions. But Feynman's path integral failed to provide solutions to these standard problems in quantum mechanics. Breakthroughs have been made and the difficulties posed by nontrivial (non - quadratic) path integrals are now slowly begin surmounted. New techniques in path integral calculus have been discovered and are being developed.

In this thesis we have developed and extended the new techniques which we have called that the commutator techniques for finding the propagator for potential of an oscillator with quadratic and cubic terms. We expound proudly this new techniques in the chapter IV. For chapter III we indicate to derivation of Baker - Hausdorff Rule which has been played important role in this new techniques.