# Chapter 2
# Theory of Demand for Labour Hours

Throughout this chapter, labour is considered as a factor of production and, therefore, the "theory of labour demand" is treated identically with the demand of factor inputs in general.

Section 2.1 considers the basic theory of factor input demand, where labour is considered as a variable factor of production. However, section 2.2 will relax this assumption and take the presence of fixed labour costs into account.

Section 2.3, will look more specifically into the estimation of the firm's demand for factor inputs as well as the firm's responsiveness to changes in prices of factor inputs. The procedure is the same as in section 2.1. -Deriving the expressions for the general factor input without considering any specific type of inputs does that.

Section 2.4 will concentrate on properties of the Cobb-Douglas production function, that is adopted to determine the constant output effect of changes in factor input prices.

## 2.1 Theory of Demand: The Basic Model

### Isoquants

How an individual ranks consumption ($X$) and non-labour hour ($N$) by expressing the individual's preferences in a utility function was done in section 1.1. In the production theory, a firm's production possibilities can also be expressed in a mathematical form as the firm's production function. This function, as for the case of the individual's utility function, represent the firm's preferences and therefore, the shape is determined by the firm's willingness to substitute between the inputs in question.

In theory, the production function is a mapping of the 'exact' technical relationship between a firm's factor inputs and its output, called the isoquant map.

There are two classes of production functions, which are interesting for the firm level, that is the *fixed-* and the *variable-proportions production function*.

However, the fixed-proportion production function is not treated here in depth but rather as a short passing. Meanwhile, the variable-proportion production function is treated thoroughly.

The fixed-proportion production function assumes that the production process is characterised by fixed factor-input-proportions. That is, the input-output ratio is independent of the scale of production. Therefore, 'output' requires a unique combination of inputs.

In mathematical terms:

$$q = \min\,(x_i, x_j, \cdots, x_k) \qquad (2.1)$$

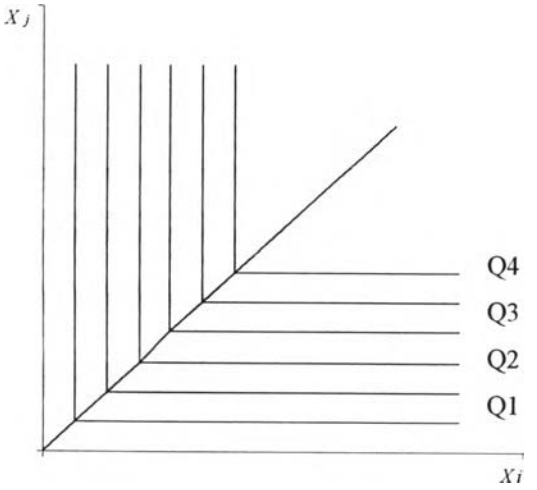
$q$ is output, and $x_i$, ...., $x_k$ are factor inputs.

Keeping all other inputs constant except $x_i$ and $x_j$ yields:

$$q_0 = x_k, \cdots, x_n \cdot \min(x_i, x_j) = B \min(x_i, x_j) \qquad (2.2)$$

The isoquant map (isoquants at different output levels) for the fixed-proportion production function is depicted in *figure 2.1*.

*Figure 2.1 Fixed proportion production function isoquant map*




The *variable-proportions production function* is one in which the same level of output may be produced by two or more combinations of inputs. For 'convenience' the function is assumed smoothly continuous and at least twice differentiable.

In mathematical terms:

$$q = f(x_i, x_j, \cdots, x_k) \qquad (2.3)$$

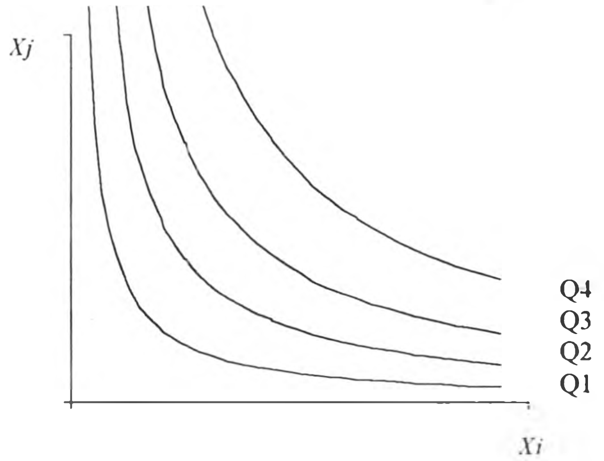
Again $q$ is output, and $x_i, \ldots, x_k$ are factor inputs.

As above, taking any variable-proportion production function and keeping all other inputs constant except $x_i$ and $x_j$ yields:

$$q_0 = x_k, \cdots, x_n \cdot f(x_i, x_j) = B\, f(x_i, x_j) \qquad (2.4)$$

The isoquant map for the variable-proportion production function is depicted in *figure 2.2* (in this case a homothetic function)

*Figure 2.2 Variable proportion production function isoquant map*




*Note:* It is obvious that illustrating more than two factors of production will need a multi-dimensional isoquant graphs.

Taking the total derivatives of *equation 2.3*, where only two inputs change and $q$ is constant, the slope of the isoquant is found:

$$dq = \frac{\partial q}{\partial x_i} dx_i + \frac{\partial q}{\partial x_j} dx_j = 0 \qquad (2.4)$$

Moreover: $\partial q / \partial x_i = MP_i$, where $MP_i$ is marginal product of the factor input in question The slope of the isoquant shows how one factor input of production can be substituted for another input while holding output constant. Examining the slope provides information about the technical possibility of substitution. That is the *marginal rate of technical substitution of* $x_i$ *for* $x_j$ (*MRTS*).

$$\frac{dx_i}{dx_j} = -\frac{\partial x_j}{\partial x_i} = MRTS = \frac{MP_j}{MP_i} \qquad (2.5)$$

*Note:* The fixed-proportion production function is not differentiable. Since they are *corners*, it makes no sense to work with “the slope” because there is no substitution possibilities between the factor inputs.

For the variable-proportion production function, we can safely assert that the curve must be negatively sloped since the opposite will not make any economic sense. No firm or industry will hire factor inputs with a negative marginal product.

In addition, quasi-concave (convex to origin) is required by both cost-minimisation and profit maximisation behaviour. This is because, as will be proved later, the assumed strict concave production function implies quasi-concavity.

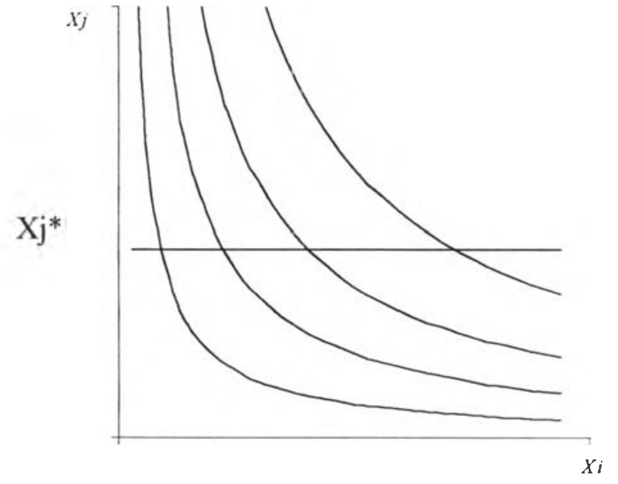
Moreover, this quasi concavity implies diminishing *MRTS*. This makes economic sense: since, when using a high $x_j/x_i$-ratio, only a small proportion of $x_i$ is required to substitute a greater number of $x_j$ to keep output at a constant level.

*Note:* We will never be able to observe an isoquant in the real world. However, there is a strong empirical case for the belief that production isoquants are not concave to the origin. The reason is that, if they are concave to the origin; we would observe firms employing only one factor of production (corner solutions), which is inconsistent and discontinuous response to factor price change.

### *Concavity of the Production Function*

For pedagogical reasons, we now consider the two factor-input function and the short run, where $x_i$ and $x_j$ as the factors of production. With fixed $x_j$, if we wish to increase output we have to increase the use of $x_i$. This is illustrated in *figure 2.3*, in which $x_j^*$ is the fixed factor of production.

*Figure 2.3 Increasing output in the short run*




As the firm moves along the $x_j^*$ -line to expand its output each isoquant the firm "passes" has a shallower slope. That the slope gets more shallow as the firm passes the isoquants means, *ceteris paribus*, the distance between any two isoquants must be increasing, and hence, marginal products ($MP_i$) is positive and decreasing and the production function is concave. In mathematical terms: $\partial q/\partial x_i > 0$, and $\partial^2 q/\partial x_i^2 < 0$.

The picture can be vanished if the production does not show constant returns to scale or if the returns to scale is varying with factor proportions or output. Then the distance between any two successive isoquants is differing not only due to diminishing $MP_i$.

Hence, the $MP_i$ depends on; first the nature of production function; second the size of the other factor input(s); and third the number of previous factor inputs employed.

### *The Marginal Revenue Product of Factor Inputs*

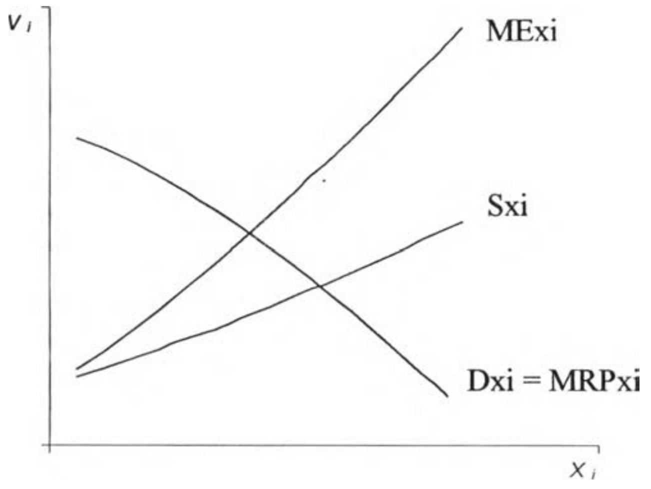
The firm is concerned not only with the output level each factor-input produces but also with the price obtained for this output when it is sold in the output market. We define the marginal revenue product of $x_i$ as $MRP_i = MP_i \cdot MR$.

If the firm is a price taker, it will face a perfectly elastic demand curve and the marginal revenue is equal to the price ($MR=P$), hence $MP_i \cdot P = MRP_i$. For a whole industry of price taker-firms, the price of output will change as output changes.

### *Pricing in the Factor Input Market*

As discussed in section 1.3, if the firm is not a price taker in the factor input market in question, the firm will face an upward sloping supply curve ($s_{x_i}$), as illustrated in *figure 2.4*.

*Figure 2.4 Equilibrium wage and Lh employment*




Given the price function of a factor input, as derived in section 1.2 (*equation 1.10.b*):

$$V_i = v_i + f(x_i) \qquad (2.6)$$

Under the assumption that the firm cannot carry out first-degree price discrimination[1] or any of higher degrees, the marginal cost of employing an additional factor inputs is above the price $v_i$, that is $f(x_i)>0$. The total cost of employing the factor input in question ($TE$) is:

$$TE_{x_i} = x_i v_i = x_i\,(v_i + f(x_i)) = x_i v_i + x_i f(x_i) \qquad (2.7)$$

The marginal expense of employing an additional worker is:

$$ME_{x_i} = \partial TE_{x_i}/\partial x_i = v_i + \partial f(x_i)/\partial x_i \qquad (2.8)$$

The marginal expense function is depicted in *figure 2.4*.

$\partial f(x_i)/\partial x_i \rightarrow 0$, in *equation 2.8*, as the factor input market moves toward perfect competition. Therefore, $\partial f(x_i)/\partial x_i=0$ under perfect competition, hence:

$$ME_{x_i} = v_I \qquad (2.9)$$

### *Hiring in the Factor Input Market*

In the long run the firm by definition no longer has to work with a particular level of $x_j$, thus increasing output only by adding other inputs than $x_j$.

We will in the following examine the hiring of factor inputs as if all inputs were variable (long run). Then we proceed to the special case of one or more fixed factor inputs (short run).

The firm's constrains on activities in the long run are two. First, the technical possibilities open to the firm, represented by the production function:

$$q = f(x_i, x_j, \cdots, x_k) \qquad (2.3)$$

Second, the financial resources of the firm summarised in the budget constraint given the output ($q_0$).

$$B = vx_i + vx_j + \cdots + vx_k \qquad (2.10)$$

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[1] First degree price discrimination involves selling each unit of the product separately and charging the highest price possible for each unit sold. -Or the firm can by its factor inputs separately and pay different price for each unit. –It will decrease the total costs of employing the input sharply.

Where $x$ is the factor input proportion and $v$ is the price of the factor input.

*Note: equation 2.10* is not to be mistaken with the fact that "firms are not considered to have a budget constraints as in the case of consumers. Firms produce as much as the demand allows"[2]. However, for a chosen output-level ($q_0$) the firm has a budget constraint.

Mathematically, the firm tries to minimise total cost, given the production possibilities $q = f(x_i, x_j, \cdots, x_k) = q_0$. Setting up the Lagrangian expression for the $k$-factor input case:

$$L^* = v_i x_i + v_j x_j + \cdots + v_k x_k + \lambda[q_0 - f(x_i \; x_j \cdots x_k] \qquad (2.11)$$

As we saw above in *(equation 2.6)* the price of a factor-input is given by:

$$v_i = v_i + f(x_i) \qquad (2.6)$$

The first order conditions, for a minimum:

$$\frac{\partial L^*}{\partial x_i} = v_i + \frac{\partial f(x_i)}{\partial x_i} - \lambda \frac{\partial f(x_i, x_j, \cdots x_k)}{\partial x_i} = 0 \quad \text{, for all } i, \ldots, k \qquad (2.12)$$

$$\frac{\partial L^*}{\partial \lambda} = q_0 - f(x_i, x_j, \cdots, x_k) = 0 \qquad (2.13)$$

We saw from *equation 2.8* that:

$$ME_{xi} = v_i + \partial f(x_i)/\partial x_I \qquad (2.8)$$

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[2] Walter Nicholson, Microeconomic Theory, (The Dryden Press, Harcourt Brace College Publishers. 1998) p. 641.

Hence, *equation 2.12* can be rewritten as:

$$\frac{\partial L^*}{\partial x_i} = ME_{X_i} - \lambda \frac{\partial q}{\partial x_i} = 0 \Rightarrow$$

$$\lambda \frac{\partial q}{\partial x_i} = \lambda MP_i = ME_{x_i} \qquad (2.14)$$

The Lagrangian multiplier (λ) can be interpreted as marginal cost (*MC*), because it reflects the change in the objective (*TC*) for one unit change in the constraint ($q$-$q_0$). We also know for a profit maximising firm that *MR=MC* hence:

$$ME_{x_i} = MRP_i \qquad (2.15.b)$$

For perfect competition:

$$v_i = MRP_i \qquad (2.16.b)$$

The obtained result makes it clear that the firm hires factor inputs to minimise cost and to maximise profits. Examining how firms react to input price changes require that we take account of both of these motivations.

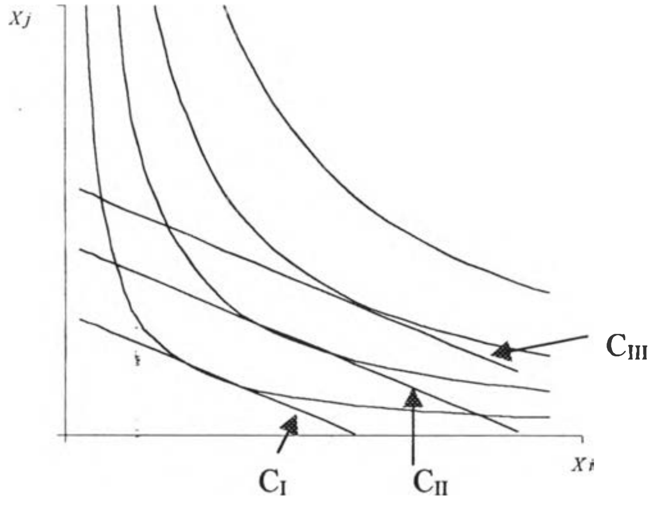
*Note:* Since, in the short run, the only constraint is at least one fixed factor input, the short run conditions for hiring of factor inputs are the same as in the long run.

### *Cost Minimisation*

Having developed the concept of the production possibilities and the budget constraint, we now proceed to look at how the firm minimises its costs. For pedagogical reasons we stick to the two factor input case $q = f(x_i, x_j)$.

The firm's budget constraint (*equation 2.10*) can be illustrated in the isoquant diagram (*figure 2.5*) as the isocost line $0 = v_i x_i + v_j x_j - B$. The slope of the budget constraint line is given as $v_j/v_i$.

*Figure 2.5 The firm's cost minimisation of production*




Still under the assumption of profit maximising and cost minimising behaviour, the firm will try to reach the lowest possible cost with given output ($q_0$). The point of tangency of the isoquants and isocost lines is the minimum cost the firm can reach.
Mathematically the conditions for cost minimisation:

$$\frac{\partial q / \partial x_i}{\partial q / \partial x_j} = \frac{\partial x_i}{\partial x_j} = -\frac{v_j}{v_i} \qquad (2.17)$$

Moreover, as already showed in *equation 2.5*:

$$\frac{MP_j}{MP_i} = \frac{MRP_J}{MRP_I} = \frac{v_i}{v_j} \qquad (2.18)$$

Therefore, in equilibrium, each factor of production is employed up to the point of equality between the ratio of its marginal product to the price of that input. "That is if $MP_i = MP_j$ but $v_i < v_j$, the firm will not maximising profit since it is getting more extra output for the dollar spent on $x_i$ than on $x_j$"[3]. Hence:

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[3] Dominick Salvatore, Managerial Economics. (McGraw-Hill International Edition, 1996), p. 244.

$$\frac{MRP_i}{v_i} = \frac{MRP_j}{v_j} = \cdots = \frac{MRP_k}{v_k} = \qquad (2.19)$$

### ***Homogeneous vs. Nonhomogeneous Production Function***

The neo-classical theory assumes the production function to be homogeneous[4]. For homogeneous production functions an equiproportional change in the factor inputs ($\psi$) increases output ($\Delta q = q_1 - q_0$) with the proportion $\psi^{\upsilon}$:

$$q_1 = f(\psi(x_i), \psi(x_j), \cdots \psi(x_k)) = \psi^{\upsilon} q_0 \qquad (2.20)$$

Hence, $\upsilon$ is the function coefficient and is defined as the elasticity of output with respect to an equiproportional variation in all inputs. "Thus the function coefficient ($\upsilon$) is the proportional change in output relative to the proportional change in the inputs for movement along any ray from the origin in input space"[5].

### ***Returns to Scale***

As it was apparent from *equation 2.20,* elasticity of scale ($\varepsilon$) is equivalent to the degree of homogeneity ($\upsilon$). Hence $\upsilon$ is equivalent to the elasticity of scale coefficient $\varepsilon$ and is constant for any homogeneous function[6].

I mathematically terms $\varepsilon$ is the sum of all factor-input's output elasticities:

$$\varepsilon = \sum[(\partial q / \partial x_i)(x_i / q)] \qquad (2.21)$$

Hence if $\varepsilon = 1$ there is constant returns to scale, if $\varepsilon < 1$ there is decreasing returns to scale, and $\varepsilon > 1$ there is increasing returns to scale.

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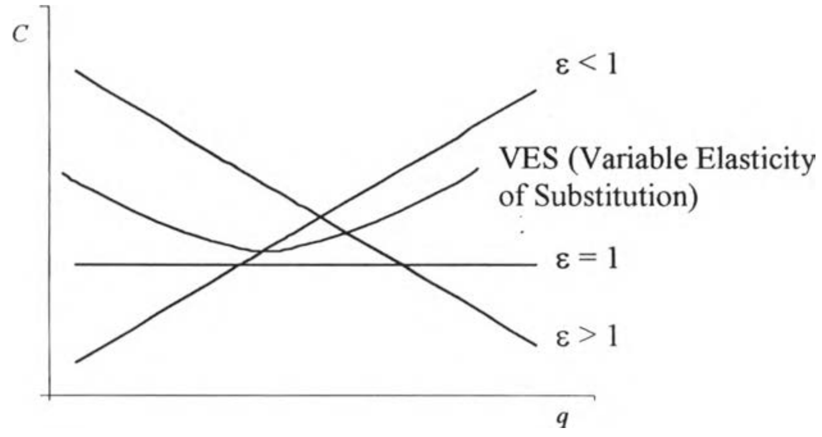
[4] Erkin I. Bairam, Homogeneous and Nonhomogeneous Production Functions: Theory and Applications. (England: Avebury Ashgate Publishing Ltd., 1994) p. 8.

[5] Ferguson, C. E., The Neo-classical Theory of Production and Distribution. (Cambridge: the University Press , 1969) p. 79.

[6] Ibid. p. 81.

That the elasticity of scale is constant for homogeneous production functions restricts the possibility of obtaining the 'U'-shaped cost curve as so often assumed, unless factor prices change as factor demand grows (that is imperfect input market).

*Figure 2.6 Cost curves for homogeneous and non-homogeneous functions*




It must however, be remembered that the point where average cost starts to increase may be larger than the largest existing company in the industry. Hence, the assumption of homogenous production function does not necessarily spoil the picture of the real world behaviour.

### *The Elasticity of Substitution*

Another important characteristic of the production process is how easy it is to substitute one factor-input for another, holding output constant. This is the elasticity of substitution ($\sigma$).

For the two factor input case, $q_0=f(x_i, x_j)$, the shape of the isoquant determines the substitution effect. A sharp curve will have very little substitution effect whereas a shallow curve will have large substitution effect.

The elasticity of substitution is defined as: "the proportionate change in factor-inputs ratio as a result of proportionate change in the marginal rate of technical substitution".

$$\sigma = \frac{\partial(x_j / x_i)/(x_j / x_i)}{\partial(\partial x_j / \partial x_i)/(\partial x_j / \partial x_i)} \qquad (2.22)$$

Due to the cost minimisation condition (*equation 2.17*), the slope of the isoquant ($\partial x_i/\partial x_j$) is equal to the relative factor prices ($-v_j/v_i$)[7] hence *equation 2.22* can be rewritten as:

$$\sigma = \frac{\partial(x_j / x_i)/(x_j / x_i)}{\partial(v_i / v_j)/(v_i / v_j)} \qquad (2.23)$$

The relationship between $\sigma$ and the shape of the isoquant is as follows:

(1) If the isoquants are linear, which is the two factor inputs in question are perfect substitutes, then $\sigma=\infty$. That is, the firm will employ only one factor of production (corner solutions), which is inconsistent and discontinuous response to factor price change.

(2) If the isoquant are $\angle 90°$ (fixed-proportion production function), there is no substitution possibility between the two products and $\sigma= 0$. Therefore, factor proportions are independent of relative prices (and output for homogeneous production functions).

(3) For $0<\sigma>\infty$, it can be said that as the isoquant get more curved the factor input proportion are becoming less sensitive to relative price changes and less substitutable.

In addition to the above discussion, the elasticity of substitution is a useful toll in analysing the behaviour of factor shares. This is because from *equation 2.23*[8] we have:

$$\sigma = \frac{\textit{percent change in } (x_i / x_j)}{\textit{percent change in } (v_j / v_i)} \qquad (2.24)$$

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[7] David F. Heathfield and Soeren Wibe, An Introduction to Cost and Production Functions, (Houndmills, Basingstoke, Hampshire RG21 2XS and London: Macmillan Education Ltd., 1987) p. 59.

[8] Nicholson, Microeconomic Theory p. 652.

If $\sigma=1$: $v_j/v_i$ will change in exactly the same proportion as $x_j/x_i$ does. Therefore, factor share income will stay constant, regardless of output level.
Is $\sigma>1$: $v_j/v_i$ will change less than $x_j/x_i$ does. Therefore, factor income of $x_j$ will rise as the $x_j/x_i$ increase.
Is $\sigma<1$: $v_j/v_i$ will change more than $x_j/x_i$ does. Therefore, factor income of $x_j$ will fall as the $x_j/x_i$ increase.

The above factor income shares is considered an important issue in labour economics, especially for labour markets associates (trade unions, employers federation, and the government).

Extending the elasticity of substitution to the general *k*-factor of production raises different problems. That is because we cannot speak of factor proportions and slope of isoquant curves. If we use the same definition as for the two factor input case, it will be necessary to require that other inputs are held constant. This is a rather artificial restriction: "in the "real world production processes", it is likely that any change in the ratio of two inputs will be accompanied by changes in the level of other inputs. Some of these may be complementary with the ones changed, whereas others may be substitutes"[9].

R.G.D. Allen (1932) developed the concept of *Allen partial elasticities of substitution* (*AES*). Having the unconstrained production function:

$$q = f(x_i, x_j, \cdots, x_k) \tag{2.3}$$

The *AES* is defined as:

$$\sigma_{ij} = (x_i \frac{\partial q}{\partial x_i} + x_j \frac{\partial q}{\partial x_j} + \cdots + x_k \frac{\partial q}{\partial x_k}) / (x_i \cdot x_j) \cdot (F_{ij} / F) \tag{2.25}$$

*Note:* that $\sigma_{ij} = \sigma_{ji}$ for all $i, j$.

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[9] Nicholson, Microeconomic Theory p. 304.

$|F|$ is the determinant of the bordered Hessian to $q = f(x_i, x_j, \cdots, x_k)$.

$$|F| = \begin{vmatrix} 0 & f_i & f_j & \cdots & f_k \\ f_i & f_{ii} & f_{ij} & \cdots & f_{ik} \\ f_j & f_{ji} & f_{jj} & \cdots & f_{jk} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_k & f_{ki} & f_{kj} & \cdots & f_{kk} \end{vmatrix}$$

Where $|F_{ij}|$ is defined as the cofactor of $f_{ij}$ in $|F|$.

"The concept cofactor is closely related to the minor denoted by $M_{ij}$. -The cofactor is a minor with a prescribed algebraic sign attached to it. The rule of sign is as follows. If the sum of the two subscripts (in this case $i=1$ and $j=2$, $i+j=3$) in the minor is odd, then the cofactor takes the opposite sign of the minor. If the sum is even, the cofactor takes the same sign as the minor".

The minor $|M_{ij}|$ is taking the Hessian matrix and deleting the $i$'th row and the $j$'th column:

$$M_{ij} = \begin{vmatrix} 0 & f_i & \cdots & f_k \\ f_j & f_{ji} & \cdots & f_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ f_k & f_{ki} & \cdots & f_{kk} \end{vmatrix}$$

Hence the cofactor $|F_{ij}| = -|M_{ij}|$.

## 2.2 Response in Input Demand to Changes in Input Prices

In order to examine how the firm reacts to changes in the wage and how we obtain the function for the firm's long run responsiveness, we will proceed in the same fashion as above. Deriving the expressions for the general $k$-factor inputs without considering the type of inputs does that.

To answer the question of: how the firm reacts to changes in the wage (if labour is in question) and how we obtain the firm's long run responsiveness, we must recognise that a change in price does two things.
First it changes the relative price and rotates the isocost line, causing the point of minimum cost to change. Second it changes the total cost of producing the output and therefore provides the firm with an incentive to alter its output level. The first is *substitution effect* and the second is *scale effect*.

For above reasons, investigating the responsiveness, both the substitutions effect ("own-price elasticity for constant output"), and the scale effect ("own-price elasticity for variable output") must be taking into consideration.

The "own-price elasticity for constant output" ($\tau$) is derived from the production function and the "own-price elasticity for variable output" ($\eta$) is derived by examine how increased labour-hours cost influence the price of the output and how the price increase affect the output demand. Hence, the total effect from $\tau$ and $\eta$ is added to express the "own price elasticity of factor input demand" ($\gamma$) –in the following called the *price elasticity production function factor input demand*.

$$\gamma = \tau + \eta \qquad (2.26)$$

## 2.2.1 Own-Price Elasticity for Constant Output

The substitution away from labour-hours, due to the increases in price of labour-hours will depend in the theory on the individual firm's technical properties of the production function.
Some firms may find it easy to substitute machines for workers and for those firms the quantity of labour demanded will decrease substantially. Other firms may produce with a 'quasi-fixed proportion technology', and for them substitution will be less substantially.

The firm's own-price elasticity for constant output can be determined using the neo-classical production theory. First, the neo-classical production theory represents ways in which labour, capital, and land can be combined to produce goods. Second, it assumes that capital is a separate, independent input directly comparable with labour and lands, which is contrary to the classical theory. Third, it focus attention on the production possibilities and decisions within the firm.

However, the existence of a production function is predicated on assertions concerning the firm's behaviour. Because, even without random behaviour there are multiple ways a company could combine its factor inputs. Each of these different input combinations will produce at different level of costs, even at same output level. Hence, the production function always depends on the objective of the firm, which the neo-classical production theory assumes is profit maximising.

The production theory for variable-proportion-production functions start with defining a mathematical model, as already specified in *equation* 2.3, which is assumed to exist for, any good or service[10].

$$q = f(x_i, x_j, \cdots, x_k) \qquad (2.3)$$

As learned from *equation 2.11* the firm will minimise cost of production given the output ($q_0$):

$$L^* = v_i x_i + v_j x_j + \cdots + v_k x_k + \lambda[q_0 - f(x_i \; x_j \cdots x_k] \qquad (2.11)$$

From this the factor input demand function ($x_i$) can be determined. This is done for the Cobb-Douglas *k*-factor input production function, in section 2.4.

The firm's own-price elasticity for constant output is defined as:

$$\tau_{x_i} = \frac{\partial x_i}{\partial v_i}\frac{v_i}{x_i} = \frac{\partial \ln(x_i)}{\partial \ln(v_i)} \qquad (2.27)$$

[10] Bairam, Homogeneous and Nonhomogeneous Production Functions, Theory and Applications, p. 7.

### *2.2.2 Own-Price Elasticity for Variable Output*

An increase in labour costs will also rise the firm's costs and normally cause the price of the output to rise and hence reduce demand. -The scale effect will in this case reinforces the substitution effect. The size depends on the firm's elasticity of costs and the price elasticity of the product.

However, assuming the firm has the U-shaped (depicted in *figure 2.6*) average cost curve as so often assumed for in the theory of the firm, the sign of the scale effect is ambiguous. Higher labour costs may move the minimum average costs to a higher output and the firm may expand in response to this. If the firm becomes sufficiently larger after the factor price increase the scale effect might outweigh the substitution effect. Thus, for the individual firm the negatively sloping demand curve is theoretically speaking not necessarily implied in the long run. Therefore, quantifying the scale effect requires examining the chain of events that cause the output to change when labour cost changes.

The firm's sensitivity of factor input demand to changes in output is expressed as[11]:

$$\frac{\partial L}{\partial v_L} = \frac{\partial L}{\partial q} \cdot \frac{\partial q}{\partial P} \cdot \frac{\partial P}{\partial MC} \cdot \frac{\partial MC}{\partial v_L} \qquad (2.28)$$

In terms of elasticities:

$$\eta = e_{x_i,q} \cdot e_{q,P} \cdot e_{P,MC} \cdot e_{MC,v_i} \qquad (2.29)$$

$e_{x_i,q}$ : The output elasticity of factor input demand. And is the inverse of the output elasticity of the factor-input in question.

$e_{q,P}$ : The price elasticity of demand for the goods produced.

$e_{P,MC}$ : Elasticity of product price with respect to *MC*.

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[11] Nicholson, Microeconomic Theory, p. 644.

$e_{MC,v_i}$ : The elasticity of total costs on labour costs: is the share of the factor input in total cost[12].

Hence the final model expressing the firm' response to factor price changes is:

$$\gamma = \tau + e_{x_i,q} \cdot e_{q,P} \cdot e_{P,MC} \cdot e_{MC,v_i} \qquad (2.26.a)$$

## 2.3 Labour as Quasi-Fixed Factor of Production

A Firm changes its demand for inputs more slowly than the shocks to input demand warrant. The explanation for this slow adjustment is that, because the firm must incur adjustment costs that are inherent in the act of changing the amount of the input used, the response to shocks will not be instantaneous.

"Besides direct remuneration the firm face what is called "non-wage labour costs" (NWLC), which include fringe benefit payment, obligatory social welfare contributions, severance pays, recruitment and training expenditures as well as other special cost items. Not only are such costs quantitatively important, but they also influence the firms labour market behaviour in ways that are not captured by studies that concentrate primarily on the role of direct wages"[13].

The fixed costs of employment and labour hour adjustment, related to and as a consequence of NWLC, then include:

(i) Training expenditures

(ii) Recruitment expenditures

(iii) Severance pays (mandated and otherwise).

(iv) Overhead cost of maintaining the personnel function dealing with recruitment and worker outflows.

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[12] Nicholson, Microeconomic Theory, p. 649.

[13] Robert A. Hart, The Economics of Non-Wage Labour Costs, (London: Goerge Allen & Urwin Ltd.,1984), preface.

(v) Social welfare contributions

The fixed costs of employment and labour hour adjustment, not related to NWLC, include:

(vi) Disruptions to production occurring when changing employment causes workers' assignments to be rearranged.

The above cost items are substantial, as workers must be hired and trained to replace those who depart. Moreover, payments for days not worked and social welfare costs account for 85 percent of total NWLC (in the case of USA, UK, and Germany). NWLC accounted for 38 percent of total labour cost in France in 1981 and for 34 percent in Germany[14].

Hiring labour-hours until $ME_{xi} = MRP_L$ is appropriate for labour costs as variable costs. But with substantial component of fixed total labour cost we must look at hiring labour as a investment with most of the cost encountering in the initial period of employment and the greatest returns at the later period[15].

"Now the decision rule is to take employment up to the point at which the NPV (net present value) of the $MRP_L$ is equal to the NPV of the investment and wage costs of the marginal employee". Hence, *equation 2.15* is rewritten to consider NWLC.

$$NPVMLC = NPVMRP_L \qquad (2.15.b)$$

$NPVMLC$ is the $NPV$ of labour costs (investment and wage costs) for the period. $NPVMRP_L$ is the $NPV$ of the revenue of the last employee for the period.

Due to NWLC's quantitatively importance NWLC is considered a quasi-fixed cost and therefore, labour-hours costs are also considered quasi fixed. In other words, studies of

[14] Hart, The Economics of Non-Wage Labour Costs, table 1.2.

the firm's adjustment of labour hours in the **short to medium run** must differentiate between number of employees and hours per labour. Therefore we cannot speak about wage only but about cost of labour-hours. Hence, when turning attention into labour-hours in chapter 7, $v_L$ changes to cost of labour-hours ($C_{Lh}$) to reflect the additional cost.

## *2.4 The Cobb-Douglas Production Function Properties*

### ***Homogeneous Function***

Given the Cobb-Douglas production function:

$$q = A(t)\, x_i^{\alpha}\, x_j^{\beta} \cdots x_k^{\Omega} \qquad (2.30)$$

Changing the factor inputs proportional by ($\psi$) gives the degree of homogeneity:

$$q_1 = A(t)\,((\psi x_i^{\alpha}), (\psi x_j^{\beta}), \cdots (\psi x_k^{\Omega})) = \psi^{(\alpha+\beta+\cdots+\Omega)} = \psi^{\upsilon}\, q_0 \qquad (2.31)$$

As we immediately see the production function is homogeneous of degree, $\upsilon$, because $\upsilon=\alpha+\beta+ \ \dots \ +\Omega$[16], which is in conformity with the neo-classical theory.

### ***Homothetic Function***

All isoquants of the Cobb-Douglas or any variable homogeneous production function are asymptotic to the axes no matter what level of output is chosen (the isoquants never touch the axes). Since neither inputs can go to zero, it is necessary in a Cobb-Douglas world to have some of both inputs ($x_i>0$, for $i= 1,2,\dots,k$).

The slope of an isoquant (*MRTS*) is, as proven in *equation 2.32*, proportional to factor proportions ($x_i/x_j$) and is irrespective of the level of output. Thus, the slope is the same along any factor-ray through the origin.

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[15] This is because there is a great incentive for the firm to share the return to specific investment (special skills) with the worker in the form of work premium. The wage paid increases but so do the tenure and the total investments in selection and initial training will fall.

[16] Bairam, Homogeneous and Nonhomogeneous Production Functions. Theory and Applications. p. 8.

For that reason, all isoquants are radial expansions ("blow ups") and the Cobb-Douglas function or any homogeneous function is therefore homothetic. Determining *MRTS* using *equation 2.5* yields:

$$MRTS = \frac{\partial q / \partial x_i}{\partial q / \partial x_j} = \frac{\beta \ A(t) \ x_i^{\alpha} \ x_j^{\beta-1}}{\alpha \ A(t) \ x_i^{\alpha-1} \ x_j^{\beta}} = \frac{\beta}{\alpha} \frac{x_i}{x_j} \qquad (2.32)$$

### *Returns to Scale*

The Cobb-Douglas production function can exhibit any degree of scale. As already given in *equation 2.21* the elasticity of scale is defined as:

$$\varepsilon = \sum [(\partial q / \partial x_i)(x_i / q)] \qquad (2.21)$$

The output elasticity for factor-input $x_i$:

$$(\partial q / \partial x_i)(x_i / q) = \frac{\alpha}{x_i} A(t) \ x_i^{\alpha} \ x_j^{\beta} \cdots x_k^{\Omega} \frac{x_i}{q} = \alpha \qquad (2.33)$$

Hence, output elasticity for any factor-input is equal to its own exponents. Therefore, the returns to scale of the Cobb-Douglas production function is the sum of the factor input exponents.

$$\varepsilon = \alpha + \beta + \cdots + \Omega \qquad (2.34)$$

Nowhere does the returns to scale mention anything about output or factor proportions, hence the returns to scale in the Cobb-Douglas function is independent of output and factor proportions.

### *Perfect Competition in all Markets*

The neo-classical production function assumes perfect competition in all markets. That is, prices of output, capital, and labour are predetermined – the firm is a price taker.

In a competitive output market, all firms will produce at minimum cost of operation. If the firms have the so often assumed 'U'-shaped cost curve, the minimum cost of operation is at constant returns to scale. If the company wish to expand its production it will multiply all its factors of production and then still operate on constant returns to scale.

Hence, the competitive behaviour sets constant returns to scale as the norm. However, as mentioned above, the Cobb-Douglas production function does allow the elasticity of scale to take other values than unity ($\varepsilon \neq 1$).

### *Homogeneous Factor Inputs*

"In the neo-classical production theory the facto inputs remain unchanged in character and is assumed homogeneous within them selves"[17]

All factor inputs are homogeneous: adding more of any factor input simply means more of the same. It makes use of existing factor inputs to produce more of the same output and there is only one type of each factor input.

In mathematically terms (taking the two factor input case): the production function is given as: $q=f(x_i^*, x_j^*)$, where $x_i^*$ and $x_j^*$ is the aggregate of the different $x_i$ and $x_j$ respectively.

Hence, $x_i^*=f(x_{i1}, x_{i2}, \ldots, x_{in})$ and $x_j^*=f(x_{j1}, x_{j2}, \ldots, x_{jn})$. If $x_{i1}= x_{i2}= \ldots =x_{in} \Rightarrow x_i^*=x_i$ and if $x_{j1}=x_{j2}= \ldots =x_{jn} \Rightarrow x_j^*=x_j$.

"It should be noted that $x_i$ and $x_j$ is are separable in the sense that the ratio of marginal physical products of e.g. $x_i$ independent of the quantity of $x_j$. If not, a rise in $x_j$ may change the marginal products of one or more types of $x_i$'s and thus change the aggregate $x_i$ measure ($x_i^*$), even though the number of $x_i$ has not changed"[18].

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[17] Ferguson, The Neo-classical Theory of Production and Distribution, p. 60.

[18] Peter Fallon and Donald Verry, The Economics of Labour Markets, (Oxford: Phillip Allan Publishers Limited, 1998), section 3.4.

As just proved, different (inhomogeneous) labour hours can be combined into, the Cobb-Douglas functions if the total labour hours can be aggregated. The aggregation can be done in the following ways (among others)[19]:

(1) Weight the $x_i$'s by some index, $x_i^* = (x_{i1}/m_1) + (x_{i2}/m_2) + \dots + (x_{in}/m_n)$, there are perfect substitution between all $x_i$'s.
(2) Relaxing the perfect substitution and introduce constant elasticity of substitution (CES-function), $x_i^* = [a_1 x_{i1} + a_2 x_{i2} + \dots + a_3 x_{in}]$ $a_1 + a_2 + \dots + a_n = 1$.

### *Homogeneous Output*

Viewing the firm as a multi-product firm takes us due to C. E. Ferguson[20] into the concept of "pricing processes in a multi product firm".
Either with (1) Pigou's approach of price discrimination of the third degree". (2) Hick's approach of conventional marginal analysis of profit maximising of a firm that produces a variety of products by means of a variety of variable products. (3) Pfout's approach of multi-products firm, permitting switching of fixed inputs among various outputs.

The Cobb-Douglas world concentrates on a single product firm and production of a single commodity.

### *Profit Maximisation Behaviour*

The neo-classical production theory assumes that profit is the firm's only relevant goal. This implies that the firm has sufficient information about its cost structure and the output market in order to make profit maximising decisions.
This is obviously a simplification of reality because it ignores other possible goals such as, obtaining market power or prestige, etc.

Given the goal of profit maximisation: the problem for the firm is to choose (i) an output level (isoquant) and (ii) a particular technology (a point on an isoquant), such as to maximise profits.

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[19] Fallon and Verry, The Economics of Labour Markets, section 3.4.
[20] Ferguson, The Neo-classical Theory of Production and Distribution, p. 201.

(i) Since we are interested in choosing a output level ($q_0$) then $TR = q_0 \cdot MR$ is also fixed, and all maximising of profits is equivalent to minimising costs.
As stated before the slope of the isocost line must equal the slope of the isoquant. Hence, relative prices determine the input combination of factor inputs due to the cost minimisation behaviour.

(ii) The output level is determined by (or accordingly depend on) the firm's market for factors and on the output market.

*Note:* As we shall see below, the Cobb-Douglas function does not allow for variable factor-income-shares, this is because, as will be proved in *equation 2.41*, $\sigma=1$.

As explained in most production literature and as we will see in the following: the profit maximising behaviour in perfect competitive markets implies a decreasing or constant returns to scale for homogeneous production functions.

The profit function is expressed as:

$$\pi = TR - TC = P \cdot q - C \cdot q \qquad (2.35)$$

$P$ is the average price, and $C$ is average cost of operation.

The first order conditions for a maximum:

$$\frac{\partial \pi}{\partial q} = \frac{\partial P}{\partial q} q + P - \frac{\partial C}{\partial q} q - C = 0 \qquad (2.36.a)$$

The second order conditions for a maximum:

$$\frac{\partial^2 \pi}{\partial q^2} = \frac{\partial^2 P}{\partial q^2} q + 2\frac{\partial P}{\partial q} - \frac{\partial^2 C}{\partial q^2} - 2\frac{\partial C}{\partial q} < 0 \qquad (2.36.b)$$

The Cobb-Douglas function is homogeneous (elasticity of scale does not vary with output or factor proportions), hence $(\partial^2 C/\partial q^2)=0$.
The term $(\partial^2 P/\partial q^2)$ is the curvature of the output demand curve, which we can say nothing about except that: $(\partial^2 P/\partial q^2)>0$ for convex to origin; $(\partial^2 P/\partial q^2)<0$ for concave to origin; $(\partial^2 P/\partial q^2)=0$ for linear curve.

(1) For increasing returns to scale: $(\partial C/\partial q)<0$, and for perfect competition in the output market: $(\partial P/\partial q)=0$ and $P=C$. Hence $(\partial \pi/\partial q)>0$ and there is no profit maximising point of production.
Instead increasing returns to scale requires imperfect factor input market, $(\partial v_i/\partial x_i)>0$ and/or imperfect output market $(\partial P/\partial q)<0$ and $P>C$. However, the slope of the demand curve must be steeper than the slope of the average cost curve.
(2) For constant returns to scale: $(\partial C/\partial q)=0$, and perfect competition in the output market: $(\partial P/\partial q)=0$, and $P=C$ (no profit). Hence $(\partial \pi/\partial q)=0$, but $(\partial^2 \pi/\partial q^2)=0$ and there is no profit maximising point of production.
(3) For decreasing returns to scale: $(\partial C/\partial q)>0$, and perfect competition in the output market: $(\partial P/\partial q)=0$, and $P<C$ (since by assumption all other firms in the industry is operating on the lowest possible cost of production).
Hence $(\partial \pi/\partial q)$ can only be zero if $P<C$ outbalance $(\partial C/\partial q)>0$. Hence, there is a point of profit maximisation.
(4) For constant returns to scale, $(\partial C/\partial q)=0$ and imperfect competition in the output market, $(\partial P/\partial q)<0$ and $P>C$: $(\partial \pi/\partial q)$ is zero if $(\partial P/\partial q)<0$ outbalance $P>C$. Hence, there is a profit maximising point of operation.
(5) For decreasing returns to scale, $(\partial C/\partial q)<0$, and for imperfect competition in the output market, $(\partial P/\partial q)<0$ and $P>C$: $(\partial \pi/\partial q)$ is zero only if $(\partial P/\partial q)<0$ outbalance $(\partial C/\partial q)>0$. Hence, there is a profit maximising point of production.

### *Strict Concave Production Function*

"The production function is assumed to be continuos and at least twice differentiable. The neo-classical production function requires that marginal products of factor inputs are positive and decreasing (strict concave). That is:

$\partial q / \partial x_i > 0$, and $\partial q / \partial x_j > 0$

$\partial^2 q / \partial x_i^2 < 0$, and $\partial^2 q / \partial x_j^2 < 0$

If $\alpha$ and $\beta > 0$, and $q$, $x_i$, $x_j > 0$ The marginal products are positive. And if $\alpha$ and $\beta < 1$ marginal products of factor inputs diminish"[21].

For profit maximisation, the production function must be strictly concave. Strict concavity, while sufficient for cost minimisation, is not necessary. The second order conditions for cost minimisation requires only quasi concavity.
The Cobb-Douglas production function is strict concave, which also imply quasi-concavity.

Quasi concavity implies diminishing *MRTS*. This makes economic sense since when using a high $x_j/x_i$-ratio only a small proportion of $x_i$ is necessary to substitute a greater number of $x_j$ to keep output at a constant level.

***Technical Progress***

Introducing technical progress into the Cobb-Douglas production function, as is discussed in section 5.4, can be done in many sound ways, taking into consideration that technical progress is considered a function of time. Two interesting introductions are here mentioned:

$$q = Be^{mt} x_i^{\alpha} x_j^{\beta} \qquad (2.37)$$

$$q = Bt^{m} x_i^{\alpha} x_j^{\beta} \qquad (2.38)$$

Differentiating *equation 2.37* with respect to time yields:

[21] Bairam, Homogeneous and Nonhomogeneous Production Functions. Theory and Applications, p. 7.

$$\partial q / \partial t = m\, Bt^{m-1}\; x_i^{\alpha}\; x_j^{\beta} = \frac{m \cdot q}{t} \qquad (2.39)$$

Differentiating *equation 2.38* with respect to time yields:

$$\partial q / \partial t = m\, Be^{mt}\; x_i^{\alpha}\; x_j^{\beta} = m \cdot q \qquad (2.40)$$

Holding factor inputs constant, the introduction of $A(t) = Be^{mt}$, as in *equation 2.37* output is growing with constant proportion rate of *m* percent a year, regardless of factor proportions. The introduction of $A(t) = Bt^{m}$, as in (*equation 2.38*), output is growing with *m/t* percent a year, regardless of factor proportions.

As we saw in *equation 2.17*, cost minimisation:

$$\frac{\partial q / \partial x_i}{\partial q / \partial x_j} = \frac{\partial x_i}{\partial x_j} = -\frac{v_j}{v_i} \qquad (2.17)$$

*Equation 2.32* showed that *MRTS* for the Cobb-Douglas production function:

$$MRTS = \frac{\partial q / \partial x_i}{\partial q / \partial x_j} = \frac{\beta\, A(t)\, x_i^{\alpha}\, x_j^{\beta-1}}{\alpha\, A(t)\, x_i^{\alpha-1}\, x_j^{\beta}} = \frac{\beta}{\alpha} \frac{x_i}{x_j} \qquad (2.32)$$

As discussed above, since $\alpha$ and $\beta$ are constants, any movement of the isoquant toward the origin preserves: $\partial x_i / \partial x_j = H_1^{-1}\ (x_i/x_j)$. Therefore, technical progress is "output augmenting"

$x_i/x_j$ is constant if $v_j/v_i$ is constant. Hence, the introduction of technical progress into the Cobb-Douglas-function leaves factor ratios unchanged if relative factor prices remain constant. That is it is Hicks neutral[22].

[22] Heathfield and Wibe. An Introduction to Cost and Production Functions. p. 122.

Taking the case of capital and labour-hours as the primary inputs, $q_0=f(\text{Lh}, \text{K})$, technical progress will neither cause a change in $q/K$ if the price of capital ($v_K$) is constant (hence, it is Harrod neutral), nor will $q/Lh$ change if $v_{Lh}$ is constant (hence, it is Solow neutral).

### *Elasticity of Substitution*

The expression for elasticity of substitution was defined in *equation 2.22*:

$$\sigma = \frac{\partial(x_j / x_i)/(x_j / x_i)}{\partial(\partial x_j / \partial x_i)/(\partial x_j / \partial x_i)} \qquad (2.22)$$

Where,

$$(\partial x_j / \partial x_i) = \frac{\partial q / \partial x_j}{\partial q / \partial x_i} = \frac{\beta}{\alpha} \frac{x_i}{x_j}$$

and hence:

$$\partial(\partial x_j / \partial x_i) = \frac{\partial^2 q / \partial x_j^2}{\partial^2 q / \partial x_i^2} = \frac{\beta}{\alpha}$$

$$\partial(x_j / x_i)/(x_j / x_i) = 1/(x_j / x_i)$$

Hence the elasticity of substitution equal unity:

$$\sigma = 1 \qquad (2.41)$$

### *Own-Price Elasticity for Constant Output*

Mathematically, the firm tries to minimise total cost, given the production possibilities:

$$q = A(t)\, x_i^{\alpha}\, x_j^{\beta} \cdots x_k^{\Omega} = q_0 \qquad (2.30)$$

Setting up the Lagrangian expression for the *k-factor* input case as done in *equation 2.11*:

$$L^* = v_i x_i + v_j x_j + \cdots + v_k x_k + \lambda[q_0 - A(t)\, x_i^{\alpha}\, x_j^{\beta} \cdots x_k^{\Omega}] \qquad (2.11)$$

The conditions for minimum are:

$$\frac{\partial L^*}{\partial x_i} = v_i - \lambda\, \alpha\, x_i^{-1}\, A(t)\, x_i^{\alpha}\, x_j^{\beta} \cdots x_k^{\Omega} = v_i - q\, \lambda \frac{\alpha}{x_i} = 0 \quad \Rightarrow$$

$$q = \frac{v_i\, x_i}{\lambda\, \alpha}$$

$$\frac{\partial L^*}{\partial x_j} = v_j - q\, \lambda \frac{\beta}{x_j} = 0 \qquad \text{,which is} \qquad q = \frac{v_j\, x_j}{\lambda\, \beta}$$

$$\frac{\partial L^*}{\partial x_k} = v_k - q\, \lambda \frac{\Omega}{x_k} = 0 \qquad \text{,which is} \qquad q = \frac{v_k\, x_k}{\lambda\, \Omega}$$

$$\frac{\partial L^*}{\partial \lambda} = q_0 - x_i^{\alpha}\, x_j^{\beta} \cdots x_k^{\Omega}$$

Taking the $(\partial L^*/\partial x_i)$ to $(\partial L^*/\partial x_k)$ yields:

$$\frac{v_i\, x_i}{\alpha} = \frac{v_j\, x_j}{\beta} = \cdots = \frac{v_k\, x_k}{\Omega}$$

Hence:

$$x_j = \frac{v_i\, x_i\, \beta}{v_j \alpha} \quad , \ \ldots\ldots\ldots , \ \textit{and} \quad x_k = \frac{v_i\, x_i\, \Omega}{v_k \alpha}$$

Substituting into the production function yields:

$$q = A(t)\, x_i^{\alpha} \left( \frac{v_i\, x_i\, \beta}{v_j\, \alpha} \right)^{\beta} \cdots \left( \frac{v_i\, x_i\, \Omega}{v_k\, \alpha} \right)^{\Omega} \quad \Rightarrow$$

$$q = A(t)\, x_i^{\alpha}\, (v_j^{\beta} \cdots v_k^{\Omega})\, (\alpha^{-\beta} \cdots \alpha^{-\Omega})\, [(\beta / v_j)^{\beta} \cdots (\Omega / v_k)^{\Omega}]$$
$$(x_i^{\beta} \cdots x_i^{\Omega})$$

Multiplying with $\alpha^{\alpha}$ and $v_i^{\alpha}$ on both sides yields:

$$q = A(t)\frac{x_i^s\, v_i^s}{\alpha^s}\left(\frac{\alpha}{v_i}\right)^{\alpha}\left(\frac{\beta}{v_j}\right)^{\beta}\cdots\left(\frac{\Omega}{v_k}\right)^{\Omega} \qquad ,s=\alpha+\beta+ \ldots +\Omega$$

Deriving the factor input demand function:

$$x_i = \left(\frac{q}{A(t)}\right)^{1/s}\frac{\alpha}{v_i}\left(\frac{v_i}{\alpha}\right)^{\alpha/s}\left(\frac{v_j}{\beta}\right)^{\beta/s}\cdots\left(\frac{v_k}{\Omega}\right)^{\Omega/s} \qquad (2.42)$$

The constant price elasticity of factor input demand ($\tau$) is already defined in *equation 2.27*:

$$\tau_{x_i} = \frac{\partial x_i}{\partial v_i}\frac{v_i}{x_i} = \frac{\partial \ln(x_i)}{\partial \ln(v_i)} \qquad (2.27)$$

Transforming *equation 2.42* into log-linear form:

$$\ln(x_i) = 1\frac{1}{s}\ln\left(\frac{q}{A(t)}\right) + \ln(\alpha) - \ln(v_i) + \frac{\alpha}{s}\ln(v_i) -$$
$$\frac{\alpha}{s}\ln(\alpha) + \frac{\beta}{s}\ln(v_j) - \frac{\beta}{s}\ln(\beta)\cdots\frac{\Omega}{s}\ln(v_k) - \frac{\Omega}{s}\ln(\Omega) \qquad (2.43)$$

Deriving the price elasticity:

$$\tau_{x_i} = \frac{\alpha - s}{s},\ \tau_{x_j} = \frac{\beta - s}{s},\cdots \tau_{x_k} = \frac{\Omega - s}{s} \qquad (2.44)$$

If the production function exhibit constant returns to scale:

$$\varepsilon_{x_i x_i} = \alpha - 1 \quad \text{, for } s=1$$

Hence, in a Cobb-Douglas world the constant own-price elasticity is independent of output and always defined by *equation 2.44*.