Chapter 3

BASIC KNOWLEDGE

3.1 Mathematical Models

The first step in analyzing a control system is to derive a mathematical model of the system. A mathematical model of a dynamic system is defined as a set of equation that represents the dynamics of the system accurately or, at least, fairly well. Once such a model is obtained, various methods are available for the analysis of the system performance.

3.1.1 Flexible Link

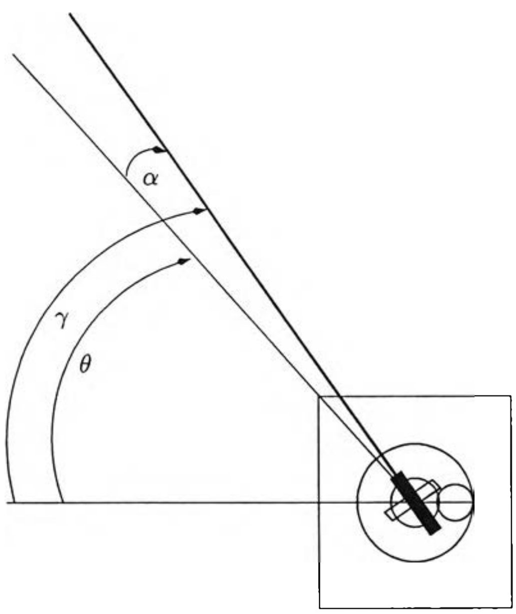

Figure 3.1: Schematic picture of flexible link.

Flexible Link plant is presented in linear systems. It is modeled in state space representation which also known to be widely used in modern control theory. In state space analysis we are concerned with three types of variables that are involved in the modeling of dynamics system, they are inputs, outputs and states variables. The schematic representation of flexible link plant from top view is depicted in Fig. 3.1. By using state-space model provided by Quanser [13], we are left with the following equations.

$$\dot{x} = Ax + Bu \tag{3.1}$$

For the combined servomotor and the flexible link module, the state variables of x is

$$x = \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_{stiff}}{J_{eq}} & -\frac{\eta_m \eta_g K_t K_m K_g^2 + B_{eq} R_m}{J_{eq} R_m} & 0 \\ 0 & \frac{K_{stiff}(J_{eq}+J_{arm})}{J_{eq} J_{arm}} & \frac{\eta_m \eta_g K_t K_m K_g^2 + B_{eq} R_m}{J_{eq} R_m} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\eta_m \eta_g K_t K_g}{J_{eq} R_m} \\ \frac{\eta_m \eta_g K_t K_g}{J_{eq} R_m} \end{bmatrix} \tag{3.3}$$

with
θ = Servo load gear angle (radians)
α = Arm deflection (radians)
ω_c = Link's damped natural frequency

The value of ω_c is taken from [13], which is claimed to be experimentally calculated. Table 3.1 is a list of the nomenclatures used in the state space equation. By putting the value of parameters from Table 3.1 to the equation (3.3) we have the following results.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 566.46 & -37.022 & 0 \\ 0 & -921.77 & 37.022 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 65.113 \\ -65.113 \end{bmatrix}.$$

This equation will be used in the development of real and simulation experiments of flexible link plant in our system.

3.1.2 Rotary Inverted Pendulum

Inverted Pendulum is such a plant that is hard to control. Of course, it is always fascinating to control a hard plant. They include some instability which prevents them to work properly on their own. To conduct inverted pendulum experiment on a remote laboratory is hard as we have to cope with the swing up problem. As complement, in this system we will develop an online simulation for this plant platform. A state space representation developed in [15] is used. The modeling is described by nonlinear equations. This nonlinear dynamical

Table 3.1: Flexible link parameters.

Symbol	Description	Value
L	Length of flexible link (meter)	0.3810 (12 inches)
m	Mass of flexible link (kg)	0.065
K_{Gage}	Strain gage calibration factor (Volt/Inch)	1
J_{Link} or J_{Arm}	Moment of inertia of a link (assumed rigid)	0.031
K_{Stiff}	Modeled stiffness constant (experimentally calculated)	1.175
J_{eq}	Combined moment inertia of armatur and tachometer (experimentally calculated)	9.7585×10^{-5}
η_m	Motor electro mechanical efficiency	0.69
η_g	Gearbox efficiency	0.9
K_t	Motor torque constant (N.m/A)	0.00767
K_m	Motor back-emf constant (V.s/rd)	0.00767
K_g	Internal gear ratio (of the planetary gearbox)	14
B_{eq}	Equivalent viscous friction referred to the second gear	1.5×10^{-3}
R_m	Armature resistance (Ohm)	2.6

equations are linearized at three different operating points and are arranged into the state-space equation.

As mentioned in the objective of the experiment in section §2.1.2, we will observe the behavior of the plant when the first pendulum, that is the long one is balanced to the upright position and the second pendulum is lying downwards. Fig. 3.2 depicts the condition of the experiments.

According to [15], at condition as shown in Fig. 3.2, state space representation of the rotary inverted pendulum is obtained in such a particular procedure. Firstly, the nonlinear dynamical equations at the position which is pointed by objective is arranged into the following state-space equation.

$$E\dot{x} = Fx + Gu \tag{3.4}$$

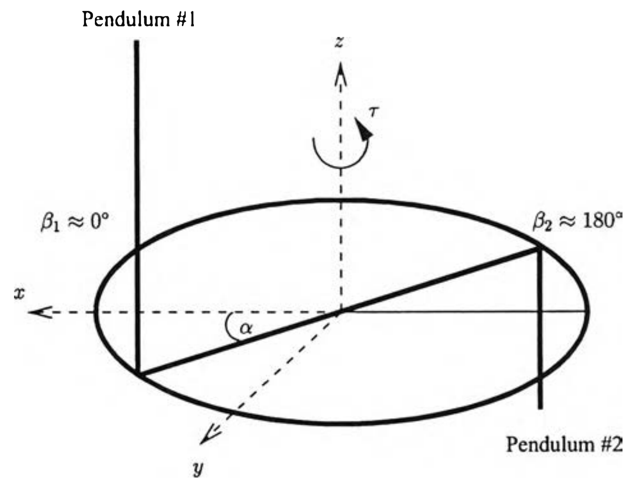

Figure 3.2: Diagram of rotary inverted pendulum used in e-laboratory system.

with

$$\begin{aligned} x &= \begin{bmatrix} \alpha & \beta_1 & \beta_2 & \dot{\alpha} & \dot{\beta}_1 & \dot{\beta}_2 \end{bmatrix}^T, \\ u &= V \end{aligned} \tag{3.5}$$

and the matrices E, F, G at this operating point are as follows.

$$E = \left[\begin{array}{c|ccc} I_{3\times3} & & 0_{3\times3} & \\ \hline & J_0 + m_1L^2 + m_2L^2 & -m_1l_1L & m_2l_2L \\ 0_{3\times3} & -m_1l_1L & J_1 + m_1l_1^2 & 0 \\ & m_2l_2L & 0 & J_2 + m_2l_2^2 \end{array}\right] \tag{3.6}$$

$$F = \left[\begin{array}{ccc|ccc} & 0_{3\times3} & & & I_{3\times3} & \\ \hline 0 & 0 & 0 & -\left(c_0 + \frac{K_mK_b}{R}\right) & 0 & 0 \\ 0 & m_1gl_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & -m_2gl_2 & 0 & 0 & -c_2 \end{array}\right] \tag{3.7}$$

$$G = \left[\begin{array}{c} 0_{3\times1} \\ \hline K_mA_m/R \\ 0 \\ 0 \end{array}\right] \tag{3.8}$$

After that we will rearange equation (3.4) into the following equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.9}$$

Putting the parameters from Table (3.1.2) we will obtain results as follows

Table 3.2: Rotary inverted pendulum parameters.

Symbol	Description	Value
α	Angular displacement of the rotating disc (rad)	
β_1	1st pendulum angle (rad)	
β_2	2nd pendulum angle (rad)	
V	PWM signal sent to the motor drive circuit	
J_0	Moment of inertia of rotating disc (kg $\cdot$m^2)	0.06
J_1	Moment of inertia of 1st pendulum (kg $\cdot$m^2)	0.008
J_2	Moment of inertia of 2nd pendulum (kg $\cdot$m^2)	0.002
m_1	Mass of 1st pendulum mass (kg)	0.25
m_2	Mass of 2nd pendulum mass (kg)	0.13
L	Radius of rotating disc (m)	0.172
A_m	PWM constant	0.043
K_m	Torque constant (Nm/A)	0.374
K_b	Back-emf constant (Vs/rad)	0.374
R	Resistance (Ohm)	8.26
c_0	Friction coefficient of rotating disc	0.04
c_1	Friction coefficient of 1st pendulum	0.0031
c_2	Friction coefficient of 2nd pendulum	0.00088

$$A = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 4.2017 & 1.7791 & -0.3247 & -0.0222 & 0.0095 \\ 0 & 28.1858 & 0.8197 & -0.1496 & -0.1486 & 0.0044 \\ 0 & -2.9100 & -40.6937 & 0.2249 & 0.0153 & -0.2162 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0302 \\ 0.0139 \\ -0.0209 \end{bmatrix} \tag{3.10}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This result will be used in the development of rotary inverted pendulum plant in our system. The controller that applied is designed for this model.

3.1.3 Belt Conveyor

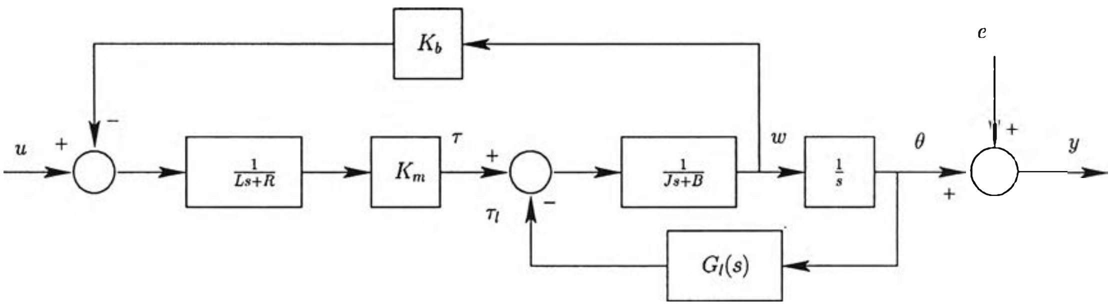

Figure 3.3: Block diagram of belt conveyor.

According to [16], the belt conveyor system is modeled as Fig. 3.3. We can see that the model is mainly constructed from two blocks of transfer function. Those blocks of transfer function are representing the motor model and mechanical conveyor model. The parameters that included in that model is listed in Table 3.3

Table 3.3: Belt conveyor parameters.

Symbol	Description	Value
R	Resistance (Ω)	0.6
L	Inductance (H)	1.24×10^{-3}
J	Hub inertia (Kg $\cdot$m^2)	1.2×10^{-5}
K_m	Torque constant (N.m/A)	3.42×10^{-2}
K_b	Back emf. constant (Volt/rad)	3.42×10^{-2}

By applying the value of parameter from Table 3.3 to the Fig. 3.3 we obtain the nominal model of Belt Conveyor as shown by Fig. 3.4. After that we will assume that the load dynamic transfer function $G_l(s)$ is as follows

$$G_l(s) = \frac{2s+6}{s^2+20s+125} \tag{3.11}$$

The model in Fig. 3.4 will be subsequently used as the experiment platform in the developed system, and as representation of the plant in the simulation.

Beside this developed transfer function, we also present the state space representation of the system. State-space representation is an important aspect of modern control system. Many controller can be created base on state space model, for example proper state feedback which will make unstable systems can be stabilized and damping of oscillatory systems can

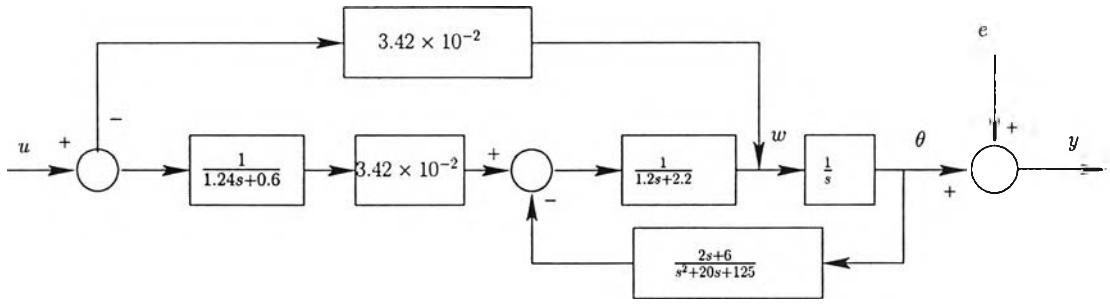

Figure 3.4: Nominal model of belt conveyor.

be improved. One basic approach is known as the pole-placement design. Pole-placement design allows the placement of poles at specified locations, provided the system is controllable.

Using a result from [17], a state space representation of belt conveyor is snatched as showed in equation (3.12). The developed state space is developed in several condition, for example, in case $n = 1$ and the parameters shown in the Table 3.3, we get the system matrices A, B, C, D. For information, n is the assumption of the number of the section of belt conveyor consist of.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -80 & 40 & 0 & 0 & 4 & 0 & 0 \\ 2000 & -4000 & 0 & 0 & 200 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 200 & 200 & 0 & 0 & -40 & 0 & 200 \\ 0 & 0 & 0 & 0 & 0 & -0.2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.12}$$

3.2 Identification

Identification process is an important step in control system. Before one can control such a plant they should identify and model it in a form that easy to be analyzed. Two method of identification are as follows.

1. Nonparametric methods
   Model is determined by direct techniques without selecting set of models.

2. Parametric methods
   Set of models are studied. Each of the models employs a finite dimensional parameter vector. These parameters must be searched for describing the system.

3.2.1 Non Parametric Methods

These methods described how to estimate the frequency functions directly, without neccessary need to select sets of models selection to determine the model. We can say that nonparametric methods are easy to apply but give only moderately accurate models. When high accuracy is needed a parametric method has to be used. So for nonparametric methods can be used to get a first crude model, which give useful information on how to apply the parametric method.

Transient Analysis

We can understand this method relatively easy and get the model of the data plot which depicted by Fig. 3.5

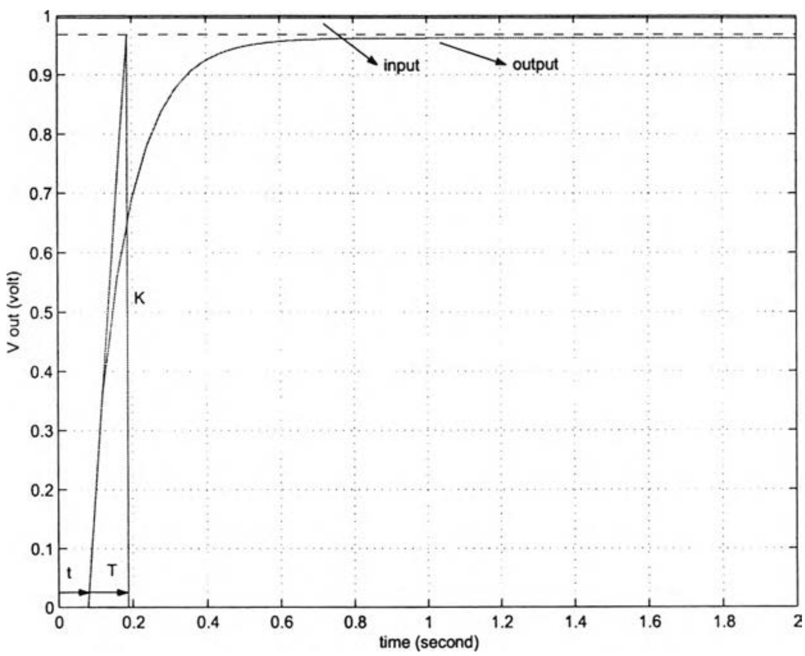

Figure 3.5: Step response with transient analysis.

From the Fig. 3.5, we can see that the step response of the plant can be described by first order differential equation, with transfer function G(s) given by this equation:

$$G(s) = \frac{K}{1+sT}e^{-st} \tag{3.13}$$

The value of the parameter can be found from the figure above, where t = 0.08s, $T = 0.1s$ and $K = 0.97$. Thus, the transfer function can be written as

$$G(s) = \frac{0.97}{1+0.1s}e^{-0.08s} \tag{3.14}$$

Frequency Analysis

The input to the plant is sinusoidal of different frequency. If the input signal is a sinusoid:

$$u(t) = a \sin(\omega t) \tag{3.15}$$

and the system is asymptotically stable, then the output is also sinusoidal but with different amplitude and phase.

$$y(t) = b \sin(\omega t + \varphi) \tag{3.16}$$

where

$$\begin{aligned} b &= a|G(i\omega)| \\ \varphi &= \arg|G(i\omega)| \end{aligned} \tag{3.17}$$

we can find these parameters by plotting the data as shown by Fig. 3.6

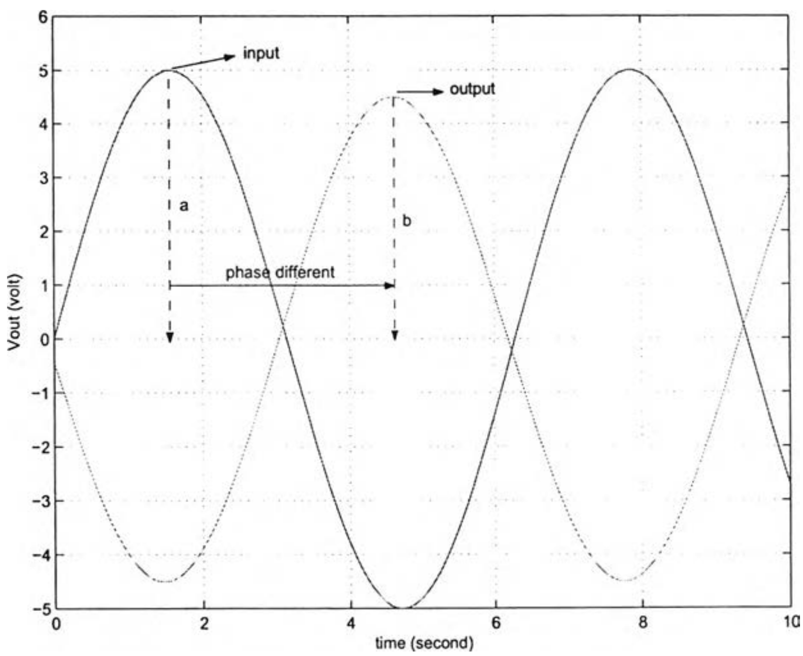

Figure 3.6: Output and input of the plant.

Spectral Analysis

Consider the system

$$y(t) = G(q)u(t) + v(t) \tag{3.18}$$

where G is the transfer function of the system and $v(t) = H(q)e(t)$ ($e(t)$ is white noise). This implies the following relationship.

$$\begin{aligned} \Phi_y(\omega) &= |G(e^{j\omega})|^2\Phi_u(\omega) + \Phi_v(\omega) \\ \Phi_{yu}(\omega) &= G(e^{j\omega})\Phi_u(\omega) \end{aligned} \tag{3.19}$$

The estimation of the transfer function can be obtained by firstly, estimating the covariance functions, $\hat{R}_y(\tau)$, $\hat{R}_u(\tau)$, and $\hat{R}_{yu}(\tau)$ where,

$$\hat{R}_{yu}(\tau) = \frac{1}{N}\sum_{t=1}^{N} y(t+\tau)u(t) \tag{3.20}$$

and similarly with the others. Then, their spectra can be obtained by

$$\hat{\Phi}_{yu}(\omega) = \sum_{\tau=-M}^{M} \hat{R}_{yu}(\tau)W_M(\tau)e^{-j\omega\tau} \tag{3.21}$$

and similarly for $\hat{\Phi}_y$ and $\hat{\Phi}_u$. $W_M(\tau)$ is called *lag window* and M describes the width of lag window. Large M gives small bias but high variance for the estimation. In spectral analysis, this lag window is called *frequency window*. For a wide window (small M), many different frequencies will be weighted to estimate at the selected frequency, ω_0. This leads to a small variance of the transfer function at ω_0, while the estimation has a large error at ω_0.

The estimation of the transfer function are

$$\hat{G}_N(e^{j\omega}) = \frac{\hat{\Phi}_{yu}(\omega)}{\hat{\Phi}_u(\omega)} \tag{3.22}$$

Empirical Transfer-function Estimate

The empirical transfer-function estimate (ETFE) is simply introduced by

$$\hat{\hat{G}}_N(e^{j\omega}) = \frac{Y_N(\omega)}{U_N(\omega)} \tag{3.23}$$

where

$$Y_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} y(t)e^{-jt\omega} \quad , \quad U_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} u(t)e^{-jt\omega}$$

The smooth version can be performed by applying a lag window during the FFT process and its effect is similar to the lag window's effect in the spectral analysis.

3.2.2 Parametric Methods

The general structure for a SISO system with input u and output y can be written by

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t-nk) + \frac{C(q)}{D(q)}e(t) \tag{3.24}$$

Then we have some estimation methods to enable us to find the model from that SISO structure.

Prediction error

Let $y(t)$ be the measured data and $\hat{y}(t|\theta)$ be the predictor with its parameters θ. The prediction error is given by

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t|\theta) \tag{3.25}$$

Let $V(\theta, Z)$ be a well-defined scalar-valued function of the model parameter θ and the collection of measured data, Z, the estimate $\hat{\theta}$ is defined by minimization of

$$\hat{\theta} = \arg\min_{\theta} V(\theta, Z) \tag{3.26}$$

The general term *prediction-error identification methods (PEM)* is used for any approach that use (3.26) as a criterion in the identification.

Least squares

In case the model structure can be written in the linear regression form, it employs a predictor

$$\hat{y}(t|\theta) = \varphi^T\theta \tag{3.27}$$

Therefore, the prediction error becomes

$$\varepsilon(t,\theta) = y(t) - \varphi^T\theta \tag{3.28}$$

The *least-square criterion* is introduced by

$$V(\theta, Z) = \frac{1}{N}\sum_{t=1}^{N}\frac{1}{2}\varepsilon^2 \tag{3.29}$$

Using quadratic form in θ gives the advantage in the sense that the solution can be minimized analytically.

$$\hat{\theta}^{LS} = \left[\frac{1}{N}\varphi(t)\varphi^T(t)\right]^{-1}\frac{1}{N}\sum_{t=1}^{N}\varphi(t)y(t) \tag{3.30}$$

Instrumental Variables Methods

From the linear regression, suppose that the data actually comes with some sequence $v(t)$

$$y(t) = \varphi^T(t)\theta + v(t) \tag{3.31}$$

Being correlation between $\varphi(t)$ and $v(t)$, $\hat{\theta}$ will not converge to the true value of θ. Therefore, a general correlation vector ζ is introduced, and then called *instrumental variable* in the following solution.

$$\hat{\theta}^{IV} = \text{sol}\left\{\frac{1}{N}\sum_{t=1}^{N}\zeta(t)[y(t) - \varphi^T(t)\theta] = 0\right\} \tag{3.32}$$

or

$$\hat{\theta}^{IV} = \left[\frac{1}{N}\zeta(t)\varphi^T(t)\right]^{-1} \frac{1}{N}\sum_{t=1}^{N}\zeta(t)y(t) \tag{3.33}$$

provided the requirement

$$\begin{aligned} \overline{E}\zeta(t)\varphi^T(t) \quad &\text{be nonsingular} \\ \overline{E}\zeta(t)v(t) &= 0 \end{aligned}$$

Subspace Methods

A linear system can be written in state space form as

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(1) &= Cx(t) + Du(t) + v(t) \end{aligned} \tag{3.34}$$

If we know u, y and x, (3.34) becomes a linear regression with the unknown parameters, all matrices, A, B, C, and D.

$$Y(t) = \Theta\Phi(t) + E(t) \tag{3.35}$$

where,

$$\begin{aligned} Y(t) &= \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix}, \Theta = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ \Phi(t) &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad E(t) = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \end{aligned} \tag{3.36}$$

Therefore, the next problem is how to observe the state x. Consider the k-step predictor of the impulse response,

$$\hat{y}(t|t-k) = \sum_{j=k}^{\infty}[h_u(j)u(t-j) + h_e(j)e(t-j)] \tag{3.37}$$

$$\begin{aligned} \hat{Y}_r(t) &= \begin{bmatrix} \hat{y}(t|t-1) \\ \cdots\hat{y}(t+r-1|t-1) \end{bmatrix} \\ \hat{\mathbf{Y}} &= \begin{bmatrix} \hat{Y}_r(1) & \ldots & \hat{Y}_r(N) \end{bmatrix} \end{aligned} \tag{3.38}$$

The order of system can be determined from rank $\hat{\mathbf{Y}}$ such that the system in (3.34) has the n^{th} order if and only if

$$\text{rank}\hat{\mathbf{Y}} = n \quad \forall r \geq n$$

The state vector x can be obtained by choosing a row basis of $\hat{\mathbf{Y}}$,

$$x(t) = L\hat{Y}_r(t) \tag{3.39}$$

where $L\hat{\mathbf{Y}}$ spans $\hat{\mathbf{Y}}$. Thus, the next step is to construct the k-step predictor in (3.37). But in reality, the predictor should be calculated from the finite number of past data.

$$\hat{y}(t+k-1|t-1) = \alpha_1 y(t-1) + \ldots + \alpha_{s_1} y(t-s_1) + \beta_1 u(t-1) + \ldots + \beta_{s_2} u(t-s_2) \tag{3.40}$$

which the parameters of the predictors can be determined by linear least squares.

In conclusion, the algorithm of subspace method can be synthesized in backward direction of the above explanation.

1. Construct the k-step predictor
2. Form $\hat{Y}_r(t)$ and $\hat{\mathbf{Y}}$
3. Estimate the rank n of $\hat{\mathbf{Y}}$ and determine L to get $x(t)$
4. Estimate A, B, C, D using Least square method

3.3 Controller Design

Two famous control strategies is discussed here. PID control scheme is investigated in section §3.3.1 and LQR control technique is introduced right away at section §3.3.2.

3.3.1 PID Controller

PID stands for Proportional, Integral, Derivative. Its controllers are designed to eliminate the need for continuous operator attention. Although being included as classic control design, in fact it is by far the widest type of automatic control used in industry. According to [18] more than half of the industrial controllers in use today utilize PID or modified PID control schemes. The usefulness of PID controls especially lies in their general applicability to most control systems. This reality emphasizes the importance of studying PID controller.

Consider the following feedback system with the plant is the system to be controlled, as depicted by Fig. 3.7.

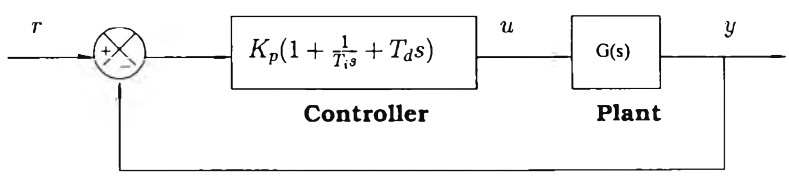

Figure 3.7: PID control of plant.

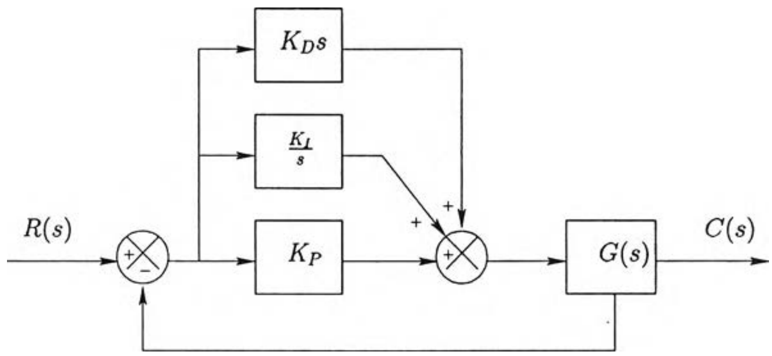

Figure 3.8: PID controller.

A PID controller is shown in Fig. 3.8 and its transfer function looks like the following equation

$$K_P + \frac{K_I}{s} + K_Ds = \frac{K_Ds^2 + K_Ps + K_I}{s} \tag{3.41}$$

with

- K_P = Proportional gain
- K_I = Integral gain
- K_D = Differential gain

First, the PID controller works in a closed-loop system using the schematic shown in Fig. 3.7. The variable e represents the tracking error, the difference between the desired input value r and the actual output y. This error signal e will be sent to the PID controller, and the controller computes both the derivative and the integral of this error signal. The signal u which just past the controller is now equal to the proportional gain K_P times the magnitude of the error plus the integral gain K_I times the integral of the error plus the derivative gain K_D times the derivative of the error.

$$u = K_Pe + K_I \int edt + K_D\frac{de}{dt} \tag{3.42}$$

This signal u will be sent to the plant, and the new output y will be obtained. This new output y will be sent back to the sensor again to find the new error signal e. The controller takes this new error signal and computes its derivative and its integral again. This process goes on and on.

Next we will discuss what is the characteristics of P, I, and D controllers. A proportional controller K_P will have the effect of reducing the rise time and will reduce, but never eliminate, the steady-state error. An integral control K_I will have the effect of eliminating the steady-state error, but it may make the transient response worse. A derivative control

K_D will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response. Effects of each of controllers K_P, K_D and K_I on a closed-loop system are summarized as shown in Table 3.4

Table 3.4: K_P, K_I, and K_D effect on closed-loop system.

Closed-loop response	Rise Time	Overshoot	Settling Time	S-s Error
K_P	Decrease	Increase	Small change	Decrease
K_I	Decrease	Increase	Increase	Eleminate
K_D	Small change	Decrease	Decrease	Small change

Note that these correlations may not be exactly accurate, because K_P, K_I, and K_D are dependent of each other. In fact, changing one of these variables can change the effect of the other two. For this reason, the table should only be used as a reference when you are determining the values for K_P, K_I, and K_D.

General tips for designing a PID controller for a given system are described by the following steps.

1. Obtain an open-loop response and determine what needs to be improved.
2. Add a proportional control to improve the rise time.
3. Add a derivative control to improve the overshoot.
4. Add an integral control to eliminate the steady-state error.
5. Adjust each of K_P, K_I, and K_D until we obtain a desired overall response.

3.3.2 Linear Quadratic Regulator Controller

LQR control is a modern state-space technique for designing optimal dynamic regulators. It refers to a linear system and a quadratic performance index. LQR control bases its controller on state space representation of the system.

$$\dot{x} = Ax + Bu \tag{3.43}$$

It calculates the optimal gain matrix K such that the state-feedback law $u = Kx$ minimizes the quadratic cost function

$$J(u) = \int_{t0}^{T} (x^T Qx + u^T Ru)dt \tag{3.44}$$

The quadratic performance index tries to limit the maximum magnitudes of the state and control variables, but this performance index does not place a hard limit on those values. A real physical system will have hard limits, particularly on actuator and sensor ranges, and therefore iteration of the design is often necessary to satisfy all the requirements. In this case, the weighting matrices are really parameters that are adjusted by the designer in order to get acceptable performance, rather than fixed quantities representing optimality. Even in this case, however, the LQR design paradigm provides a very convenient method for obtaining the feedback gain matrices that allow the performance specifications to be satisfied.

The control law for the LQR is specified as

$$u = -R^{-1}B^T\bar{P}x \tag{3.45}$$

where $\bar{P}=\bar{P}^T \geqslant 0$ solves the algebraic Ricatti equation

$$\dot{P} = PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{3.46}$$

The gain vector $K = R^{-1}B^T\bar{P}$ determines the amount of control feedback into the system. The matrices R and Q will balance the relative importance of the control input and state in the cost function (J) being optimized with a condition that both Q and R are positive definite matrices.