CHAPTER II

INVERTIBLE MATRICES OVER A BOOLEAN SEMIRING

Characterizations of invertible matrices over the Boolean algebra of 2 elements, over any Boolean algebra of sets and over any idempotent semiring were given in [2], [3] and [4], respectively. Boolean algebras of sets are a generalization of the Boolean algebra of 2 elements and idempotent semirings are a generalization of Boolean algebras of sets. In this chapter, invertible matrices are studied extensively in this line. We characterize invertible matrices over any Boolean semiring.

Some examples of Boolean semirings are as follows : Boolean algebras, Boolean rings and $([0,1],\max,\min)$ (which is the semiring $([0,1],\oplus,\odot)$ where $x\oplus y = \max\{x,y\}$ and $x\odot y = \min\{x,y\}$ for all x,y in $[0,1]$).

Boolean rings containing more than one element are not idempotent semirings. Also, the semiring $(\{0\}\cup[\frac{1}{2},1],\oplus,\min)$ where $x\oplus y = \frac{1}{2}$ for all $x,y \in [\frac{1}{2},1]$ and $0\oplus x = x\oplus 0 = x$ for all $x \in \{0\}\cup[\frac{1}{2},1]$ is a Boolean semiring which is not an idempotent semiring. More generally, if S is a set of positive real numbers with maximum element M, minimum element m and $M \neq m$, then $(S\cup\{0\},\oplus,\min)$ where $x\oplus y = m$ for all $x,y \in S$ and $0\oplus x = x\oplus 0 = x$ for all $x \in S\cup\{0\}$ is a Boolean semiring which is not an idempotent semiring.

We first give a necessary condition for a matrix over any semiring to be invertible.

Proposition 2.1. Let S be a semiring and A an $n\times n$ matrix over S. If the matrix A is invertible over S, then $A_{ij}A_{ik}$ is additively invertible in S for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$.

Proof : First, we note that if a and b are additively invertible elements of S, then $xa + yb$ is additively invertible in S for all $x,y \in S$.

Let B be an $n\times n$ matrix over S such that $AB = BA = I_n$. For $i,j \in \{1,2,\ldots,n\}$, if $i \neq j$, then $0 = (BA)_{ij} = \sum_{t=1}^{n} B_{it}A_{tj}$ which implies that $B_{it}A_{tj}$ is additively invertible in S for all t in $\{1,2,\ldots,n\}$. This proves that $B_{it}A_{tj}$ is additively invertible in S for all $i,j,t \in \{1,2,\ldots,n\}$, $i \neq j$.

Next, let $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$. Then $B_{ki}A_{ij}$ is additively invertible in S and $B_{ti}A_{ik}$ is additively invertible in S for all $t \in \{1,2,\ldots,n\}$, $t \neq k$. Since

$$\begin{aligned}
A_{ij}A_{ik} &= (A_{ij}A_{ik})(AB)_{ii} \\
&= (A_{ij}A_{ik})\Big(\sum_{t=1}^{n} A_{it}B_{ti}\Big) \\
&= \sum_{t=1}^{n} A_{ij}A_{ik}A_{it}B_{ti} \\
&= A_{ij}A_{ik}A_{ik}B_{ki} + \sum_{\substack{t=1 \\ t\neq k}}^{n} A_{ij}A_{ik}A_{it}B_{ti} \\
&= A_{ik}^{2}(B_{ki}A_{ij}) + \sum_{\substack{t=1 \\ t\neq k}}^{n} A_{ij}A_{it}(B_{ti}A_{ik}),
\end{aligned}$$

we have that $A_{ij}A_{ik}$ is additively invertible in S. #

The following corollary is obtained easily from the following fact : A square matrix A over a semiring S is invertible over S if and only if A^T is invertible over S.

<u>Corollary 2.2.</u> Let S be a semiring and A an nxn matrix over S. If the matrix A is invertible over S, then $A_{ij}A_{kj}$ is additively invertible in S for all $i,j,k \in \{1,2,...,n\}$, $i \neq k$.

The next proposition gives some general properties of any Boolean semiring.

<u>Proposition 2.3.</u> Let S be a Boolean semiring. Then the following statements hold :

(1) For every $x \in S$, $2x = 4x$.

(2) For $x,y \in S$, if $x+y = 0$, then $2x = 2y = 0$.

(3) For $x,y \in S$, if $xy = 1$, then $x = y = 1$.

<u>Proof</u> : (1) If x is an element of S, then $2x = x+x = (x+x)^2 = x^2 + x^2 + x^2 + x^2 = x+x+x+x = 4x$.

(2) Let $x,y \in S$ be such that $x+y = 0$. Then $2x+2y = 0$ and by (1), $2x = 4x$ and $2y = 4y$. Hence $2x = 2x+0 = 2x+2x+2y = 2x+2y = 0$, and $2y = 0$ can be shown similarly.

(3) Let $x,y \in S$ be such that $xy = 1$. Then $x = x1 = xxy = xy = 1$, and $y = 1$ can be shown similarly. #

The next theorem gives a characterization of invertible matrices over a Boolean semiring. To prove this theorem, a lemma related to finite permutation groups is required as follows :

Lemma 2.4. Let n be a positive integer, $\mathcal{S}_n$ the permutation group of degree n, $\mathcal{A}_n$ the alternating group of degree n and $\mathcal{B}_n = \mathcal{S}_n \setminus \mathcal{A}_n$. If $i,j \in \{1,2,\ldots,n\}$ are such that $i \neq j$, then the map $\sigma \mapsto (\sigma(i),\sigma(j))\sigma$ is a 1-1 map from $\mathcal{A}_n$ onto $\mathcal{B}_n$.

Proof : Assume that $i,j \in \{1,2,\ldots,n\}$ are such that $i \neq j$. If $\sigma \in \mathcal{A}_n$, then $(\sigma(i),\sigma(j)) \in \mathcal{B}_n$ (since $\sigma(i) \neq \sigma(j)$) which implies that $(\sigma(i),\sigma(j))\sigma \in \mathcal{B}_n$. Let $\rho,\delta \in \mathcal{A}_n$ be such that $(\rho(i),\rho(j))\rho = (\delta(i),\delta(j))\delta$. Then $(\rho(i),\rho(j))\rho(k) = ((\rho(i),\rho(j))\rho)(k) = ((\delta(i),\delta(j))\delta)(k) = (\delta(i),\delta(j))\delta(k)$ for all $k \in \{1,2,\ldots,n\}$. Therefore $\rho(i) = (\rho(i),\rho(j))\rho(j) = (\delta(i),\delta(j))\delta(j) = \delta(i)$, $\rho(j) = (\rho(i),\rho(j))\rho(i) = (\delta(i),\delta(j))\delta(i) = \delta(j)$ and if $k \in \{1,2,\ldots,n\} \setminus \{i,j\}$, then $\rho(k) \neq \rho(i)$, $\rho(k) \neq \rho(j)$, $\delta(k) \neq \delta(i)$ and $\delta(k) \neq \delta(j)$ which imply that $\rho(k) = (\rho(i),\rho(j))\rho(k) = (\delta(i),\delta(j))\delta(k) = \delta(k)$. Hence $\rho = \delta$. This proves that $\sigma \mapsto (\sigma(i),\sigma(j))\sigma$ is a 1-1 map from $\mathcal{A}_n$ into $\mathcal{B}_n$. Since $|\mathcal{A}_n| = |\mathcal{B}_n| = \frac{n!}{2}$, it follows that the map $\sigma \mapsto (\sigma(i),\sigma(j))\sigma$ is a 1-1 map from $\mathcal{A}_n$ onto $\mathcal{B}_n$. #

Theorem 2.5. Let S be a Boolean semiring and A an n×n matrix over S. Then the matrix A is invertible over S if and only if

(i) $\text{per}(A) = 1$ and

(ii) $2A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$.

Proof : First, assume that A is an invertible matrix over S. Then $AB = BA = I_n$ for some n×n matrix B over S. By Theorem 1.1, there exists an element r of S such that

$$\det{}^+(AB) = (\det{}^+A)(\det{}^+B)+(\det{}^-A)(\det{}^-B)+r$$

and

$$\det{}^-(AB) = (\det{}^+A)(\det{}^-B)+(\det{}^-A)(\det{}^+B)+r.$$

But $\det^+(AB) = \det^+ I_n = 1$ and $\det^-(AB) = \det^- I_n = 0$, so we have that

$$1 = (\det^+A)(\det^+B)+(\det^-A)(\det^-B)+r \quad \ldots\ldots (1)$$

and

$$0 = (\det^+A)(\det^-B)+(\det^-A)(\det^+B)+r. \quad \ldots\ldots (2)$$

Then (1)+(2) gives

$$1 = (\det^+A)(\det^+B)+(\det^+A)(\det^-B)+(\det^-A)(\det^+B)+ (\det^-A)(\det^-B)+2r$$

which implies that

$$1 = (\det^+A+\det^-A)(\det^+B+\det^-B)+2r$$
$$= \operatorname{per}(A)\operatorname{per}(B)+2r.$$

It follows from (2) and Proposition 2.3(2) that $2r = 0$. Hence $\operatorname{per}(A)\operatorname{per}(B) = 1$. By Proposition 2.3(3), $\operatorname{per}(A) = 1$, so (i) holds.

Since A is invertible over S, by Proposition 2.1, $A_{ij}A_{ik}$ is additively invertible in S for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$. Since S is a Boolean semiring, by Proposition 2.3(2), $2A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$. Hence (ii) holds.

To prove the converse, assume that (i) and (ii) hold. If $n = 1$, then A is the 1×1 matrix whose element is 1, so A is invertible over S. Assume that $n > 1$ and let B be the $n\times n$ matrix over S defined by

$$B_{ij} = \sum_{\substack{\sigma\in \mathcal{S}_n \\ \sigma(j)=i}} \Big(\prod_{\substack{k=1 \\ k\neq j}}^{n} A_{k\sigma(k)} \Big)$$

for all $i,j \in \{1,2,\ldots,n\}$. Claim that $AB = I_n$. Let $i,j \in \{1,2,\ldots,n\}$.

Then

$$(AB)_{ij} = \sum_{t=1}^{n} A_{it} B_{tj}$$

$$= \sum_{t=1}^{n} A_{it} \Big(\sum_{\substack{\sigma \varepsilon \mathcal{S}_n \\ \sigma(j)=t}} \big(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \big) \Big)$$

$$= \sum_{t=1}^{n} \Big(\sum_{\substack{\sigma \varepsilon \mathcal{S}_n \\ \sigma(j)=t}} A_{i\sigma(j)} \big(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \big) \Big).$$

Since $\mathcal{S}_n = \bigcup_{t=1}^{n} \{\sigma \,\varepsilon\, \mathcal{S}_n \mid \sigma(j) = t\}$ is a disjoint union, it follows that

$$\sum_{t=1}^{n} \Big(\sum_{\substack{\sigma \varepsilon \mathcal{S}_n \\ \sigma(j)=t}} A_{i\sigma(j)} \big(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \big) \Big) = \sum_{\sigma \varepsilon \mathcal{S}_n} A_{i\sigma(j)} \big(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \big).$$

Thus

$$(AB)_{ij} = \sum_{\sigma \varepsilon \mathcal{S}_n} A_{i\sigma(j)} \big(\prod_{\substack{k=1 \\ k \neq j}}^{n} A_{k\sigma(k)} \big).$$

If $i = j$, then

$$(AB)_{ij} = (AB)_{ii}$$

$$= \sum_{\sigma \varepsilon \mathcal{S}_n} A_{i\sigma(i)} \big(\prod_{\substack{k=1 \\ k \neq i}}^{n} A_{k\sigma(k)} \big)$$

$$= \sum_{\sigma \varepsilon \mathcal{S}_n} \big(\prod_{k=1}^{n} A_{k\sigma(k)} \big)$$

$$= \mathrm{per}(A)$$

$$= 1.$$

Next, assume that $i \neq j$. If $n = 2$, then

$$(AB)_{ij} = \sum_{\sigma\in\mathcal{S}_2} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{2} A_{k\sigma(k)}\Big)$$

$$= \sum_{\sigma\in\mathcal{S}_2} A_{i\sigma(j)}A_{i\sigma(i)}$$

$$= A_{ij}A_{ii}+A_{ii}A_{ij}$$

$$= 2A_{ii}A_{ij}$$

$$= 0 \quad \text{(by the assumption (ii))}.$$

Assume further that $n > 2$. Then

$$(AB)_{ij} = \sum_{\sigma\in\mathcal{S}_n} A_{i\sigma(j)}\Big(\prod_{\substack{k=1\\k\neq j}}^{n} A_{k\sigma(k)}\Big)$$

$$= \sum_{\sigma\in\mathcal{S}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big)$$

$$= \sum_{\sigma\in\mathcal{A}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big) +$$

$$\sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big).$$

For each $\sigma \in \mathcal{A}_n$, let $\bar{\sigma} = (\sigma(i),\sigma(j))\sigma$. Then $\sigma \mapsto \bar{\sigma}$ is a 1-1 map from $\mathcal{A}_n$ onto $\mathcal{B}_n$ (Lemma 2.4). Therefore

$$\sum_{\sigma\in\mathcal{B}_n} A_{i\sigma(j)}A_{i\sigma(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}\Big) = \sum_{\sigma\in\mathcal{A}_n} A_{i\bar{\sigma}(j)}A_{i\bar{\sigma}(i)}\Big(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\bar{\sigma}(k)}\Big),$$

so we have that

$$(AB)_{ij} = \sum_{\sigma\varepsilon\mathcal{A}_n} A_{i\sigma(j)}A_{i\sigma(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}) + \sum_{\sigma\varepsilon\mathcal{A}_n} A_{i\bar{\sigma}(j)}A_{i\bar{\sigma}(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\bar{\sigma}(k)})$$

$$= \sum_{\sigma\varepsilon\mathcal{A}_n} [A_{i\sigma(j)}A_{i\sigma(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}) + A_{i\bar{\sigma}(j)}A_{i\bar{\sigma}(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\bar{\sigma}(k)})].$$

For each $\sigma \varepsilon \mathcal{A}_n$, $\bar{\sigma}(i) = ((\sigma(i),\sigma(j))\sigma)(i) = \sigma(j)$, $\bar{\sigma}(j) = ((\sigma(i),\sigma(j))\sigma)(j) = \sigma(i)$ and if $k \varepsilon \{1,2,\ldots,n\}\setminus\{i,j\}$, then $\sigma(k) \neq \sigma(i)$ and $\sigma(k) \neq \sigma(j)$ which imply that $\bar{\sigma}(k) = ((\sigma(i),\sigma(j))\sigma)(k) = \sigma(k)$. Hence for each $\sigma \varepsilon \mathcal{A}_n$,

$$A_{i\bar{\sigma}(j)}A_{i\bar{\sigma}(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\bar{\sigma}(k)}) = A_{i\sigma(j)}A_{i\sigma(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)}).$$

Thus

$$(AB)_{ij} = \sum_{\sigma\varepsilon\mathcal{A}_n} [2A_{i\sigma(j)}A_{i\sigma(i)}(\prod_{\substack{k=1\\k\neq i,j}}^{n} A_{k\sigma(k)})].$$

Since $i \neq j$, $\sigma(i) \neq \sigma(j)$ for all $\sigma \varepsilon \mathcal{A}_n$, so it follows by (ii) that $2A_{i\sigma(j)}A_{i\sigma(i)} = 0$ for all $\sigma \varepsilon \mathcal{A}_n$. Hence $(AB)_{ij} = 0$.

Therefore $AB = I_n$. By Theorem 1.2, $AB = BA = I_n$. Hence A is invertible over S. #

The following corollary is clearly obtained from the facts that for any square matrix A over a semiring S, $per(A) = per(A^T)$ and A is invertible over S if and only if A^T is invertible over S.

Corollary 2.6. Let S be a Boolean semiring and A an $n\times n$ matrix over S. Then the matrix A is invertible over S if and only if

(i) per(A) = 1 and

(ii) $2A_{ij}A_{kj} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $i \neq k$.

If S is an idempotent semiring, then $2x = x$ for all $x \in S$. Thus for any n×n matrix A over an idempotent semiring S and for $i,j,k \in \{1,2,\ldots,n\}$, $2A_{ij}A_{ik} = 0$ if and only if $A_{ij}A_{ik} = 0$. Hence the following result which was proved in [4] is obtained as a consequence of Theorem 2.5.

<u>Corollary 2.7.</u> Let S be an idempotent semiring and A an n×n matrix over S. Then the matrix A is invertible over S if and only if

(i) per(A) = 1 and

(ii) $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$.

<u>Corollary 2.8.</u> Let S be an idempotent semiring and A an n×n matrix over S. Then the matrix A is invertible over S if and only if

(i) per(A) = 1 and

(ii) $A_{ij}A_{kj} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $i \neq k$.

<u>Proof</u> : It follows from Corollary 2.7 and the facts that $\text{per}(A) = \text{per}(A^T)$ and A is invertible over S if and only if A^T is invertible over S. #

It is known that a square matrix A over the Boolean algebra of 2 elements B is invertible over B if and only if A is a permutation matrix (see [2]). We shall prove this known result by Corollary 2.7 and Corollary 2.8.

Corollary 2.9. Let B be the Boolean algebra of 2 elements and A a square matrix over B. Then the matrix A is invertible over B if and only if A is a permutation matrix.

Proof : Let A be an $n \times n$ matrix over B.

First, assume that A is invertible over B. By Corollary 2.7 and Corollary 2.8, per(A) = 1, $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$ and $A_{ij}A_{kj} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $i \neq k$. Since

$$\text{per}(A) = \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{k=1}^{n} A_{k\sigma(k)} \right) = \text{per}(A^T) = \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{k=1}^{n} A_{\sigma(k)k} \right) = 1,$$ it follows

that every row and every column of A has at least one element 1. Since $A_{ij}A_{ik} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $j \neq k$, we have that every row has exactly one element 1. Also, every column of A having exactly one element 1 follows from the fact that $A_{ij}A_{kj} = 0$ for all $i,j,k \in \{1,2,\ldots,n\}$, $i \neq k$. Hence A is a permutation matrix.

For the converse, see Chapter I, page 5. #

A characterization of invertible matrices over a commutative ring with identity is given in [1] . Theorem 4 of Chapter 5 in [1] states that a square matrix A over a commutative ring R with identity is invertible over R if and only if det(A) is multiplicatively invertible in R. If R is a Boolean ring with identity 1, then 1 is the only multiplicatively invertible element in R (Proposition 2.3(3)). It follows that a square matrix A over a Boolean ring R with identity 1 is invertible over R if and only if det(A) = 1. However, this known result follows independently by Theorem 2.5 since $2x = 0$ for all x in any Boolean ring and det(A) = per(A) if A is a square matrix over a Boolean ring with identity.

<u>Corollary 2.10</u>. Let R be a Boolean ring with identity 1 and A a square matrix over R. Then the matrix A is invertible over R if and only if $\det(A) = 1$.