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BOUNDS IN A COMBINATORIAL CENTRAL LIMIT THEOREM FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS

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ให้ X เป็นเวกเตอร์สุ่มที่มีการแจกแจงแบบสม่ำเสมอบน $[0,1]^3$ และกำหนดให้ f เป็นฟังก์ชันจาก $\Re^3$ ไปยัง $\Re$ ซึ่งสามารถหาปริพันธ์ได้และนิยามให้

$$\mu = Ef(X) = \int_{[0,1]^3} f(x)dx$$

ตัวประมาณค่าอย่างง่ายตัวหนึ่งของ μ คือ

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

โดยที่ $X_1, X_2, ..., X_n$ เป็นเวกเตอร์สุ่มที่เป็นอิสระต่อกันและมีการแจกแจงแบบสม่ำเสมอบน $[0,1]^3$ อย่างไรก็ตามมีวิธีในการสุ่มเลือก $X_1, X_2, ..., X_n$ อยู่หลายวิธี หนึ่งในนั้นคือ การสุ่มตัวอย่างแบบแถวเชิงตั้งฉากโดย ในปี ค.ศ. 1996 ลอฮ์ได้พิสูจน์ว่า $\hat{\mu}$ สอดคล้องทฤษฎีบทลิมิตกลางและให้ขอบเขตแบบสม่ำเสมอในการประมาณด้วยการแจกแจงปกติสำหรับ $\hat{\mu}$

ในวิทยานิพนธ์ฉบับนี้เราปรับปรุงขอบเขตแบบสม่ำเสมอของลอฮ์และให้ขอบเขตแบบไม่สม่ำเสมอโดยวิธีการของสไตน์ ยิ่งไปกว่านั้นเรายังหาอสมการเข้มข้นแบบสม่ำเสมอและแบบไม่สม่ำเสมออีกด้วย

ภาควิชา ...**คณิตศาสตร์**... ลายมือชื่อนิสิต..
สาขาวิชา ...**คณิตศาสตร์**... ลายมือชื่ออาจารย์ที่ปรึกษา..................................
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Let X be a random vector uniformly distributed on $[0,1]^3$ and let f be an integrable function from $\mathbb{R}^3$ into $\mathbb{R}$ and define

$$\mu = Ef(X) = \int_{[0,1]^3} f(x)dx.$$

A simple estimator of μ is

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

where $X_1, X_2, ..., X_n$ are independent random vectors and uniformly distributed on $[0,1]^3$. However, there are many methods to choose the points X_i's. One of those is the orthogonal array. In 1996, Loh proved that $\hat{\mu}$ obeys a central limit theorem and a uniform bound for the distribution of $\hat{\mu}$ and normal distribution was given.

In this thesis, we improve a uniform bound given by Loh and give a non-uniform bound using Stein's method. Furthermore, we also establish a uniform and a non-uniform concentration inequality.

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Field of study	...**Mathematics**...	Advisor's signature	....
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CONTENTS

CHAPTER I
INTRODUCTION

In many scientific and technological fields, especially the computer experiment, we always face the problem of computing a value of integral over a high dimensional domain. Among numerical integration techniques, Monte Carlo methods are especially useful and often competitive for high dimensional integration(see, Davis and Rabinowitz ([1]), chap. 5.10, Niederreiter([2]), Evans and Swartz ([3])). The simple Monte Carlo method may be formulated as follows:

Consider a deterministic function $Y = f(x) \in \mathbb{R}$ where $x \in [0,1]^d$ and f is known but hard to calculate. Then our aim is to estimate

$$\int_{[0,1]^d} f(x)dx,$$

that is the mean, $\mu = E(f \circ X)$, of $f \circ X$ where X is a random vector uniformly distributed on a unit hypercube $[0,1]^d$. The simplest way is to draw the samplings $X_1, X_2, ..., X_n$ independently and uniformly distributed from $[0,1]^d$ and use

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f \circ X_i$$

as an estimator of μ.

Beside the uniform sampling, there are various alternative ways to select the points X_i's for $\hat{\mu}$. For examples, lattice sampling(see Patterson([4]), Latin hypercube sampling(see, McKay, Conover and Beckman([5]), Owen([6]), the orthogonal arrays(see, Loh([7]), Owen([8]), Tang([9])), scrambled net(see, Owen([10]) and ([11])). In this work, we investigate orthogonal arrays sampling.

An orthogonal array of strength t with index λ ($\lambda \geq 1$), is an $n \times d$ matrix with elements taken from the set $\{0, 1, ..., q-1\}$ such that for any $n \times t$ submatrix, each of the q^t possible rows appears the same number λ of times where d, n, q and t are positive integers with $t \leq d$ and $q \geq 2$. Of course $n = \lambda q^t$. A class of these arrays are denoted by $OA(n, d, q, t)$(see Raghavarao([12]) for more details).

Loh(1996) considered the class $OA(n,3,q,2)$ when $n=q^2$ and constructed the sampling $X_1, X_2, ..., X_{q^2}$ on the unit cube $[0,1]^3$ as follows: Let

(a) π_1, π_2, π_3 be random permutations of $\{0,1,...,q-1\}$,

(b) $U_{i_1,i_2,i_3,j}$ be $[0,1]$ uniform random variables where $i_1, i_2, i_3 \in \{0,1,...,q-1\}$, $j \in \{1,2,3\}$; and

(c) $U_{i_1,i_2,i_3,j}$'s and π_k's be all stochastically independent.

An orthogonal array-based sample of size q^2, $\{X_1, X_2, ..., X_{q^2}\}$, is defined to be

$$\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \le i \le q^2\},$$

where, for each $i_1, i_2, i_3 \in \{0,1,...,q-1\}$ and $j \in \{1,2,3\}$,

$$X\big(i_1,i_2,i_3\big) = \big(T_1(i_1,i_2,i_3), T_2(i_1,i_2,i_3), T_3(i_1,i_2,i_3)\big),$$

$$T_j(i_1,i_2,i_3) = \frac{i_j + U_{i_1,i_2,i_3,j}}{q},$$

and $a_{i,j}$ is the $(i,j)^{th}$ element of some arbitary but fixed $A \in OA(q^2,3,q,2)$.

So the estimator $\hat{\mu}$ of μ in (1.1) can be express in the form of

$$\hat{\mu} = \frac{1}{q^2}\sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})).$$

Owen([8]) gave an expression for the asymptotic variance of $\hat{\mu}$ that if $E(f \circ X)^2 < \infty$ then, as $q \to \infty$, we have

$$q^2 Var(\hat{\mu}) = \int_{[0,1]^3} f_{rem}^2(x)dx + o(1),$$

where, for each $i_1, i_2, i_3 \in \{0,1,...,q-1\}$ and $j \in \{1,2,3\}$,

$$f_{rem}(x) = f(x) - \mu - \sum_{j=1}^{3} f_j(x_j) - \sum_{1\le k<l\le 3} f_{k,l}(x_k,x_j)$$

$$f_{k,l}(x_k,x_j) = \int_0^1 \big[f(x) - \mu - f_k(x_k) - f_l(x_l)\big] \prod_{j\ne k,l} dx_j$$

$$f_j(x_j) = \int_{[0,1]^2} \big[f(x) - \mu\big] \prod_{k\ne j} dx_k.$$

Assume that $Var(\hat{\mu}) > 0$. We define

$$W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}}.$$

From Owen[8], we note that

$$EW = 0 \text{ and } VarW = EW^2 = 1. \tag{1.1}$$

From now on, we let X be a uniform random vector on $[0,1]^3$ and Φ the standard normal distribution. Loh([7]) gave a uniform bound on the normal approximation of W in Theorem 1.1.

Theorem 1.1. *Suppose that $E(f\circ X)^r < \infty$ for some even integer $r \geq 4$. Then*

$$\sup\left\{\left|P(W\leq w)-\Phi(w)\right| : -\infty < w < \infty\right\} = O(q^{\frac{2-r}{2r-2}}) \quad as \;\; q\to\infty.$$

In Theorem 1.1 we observe that the order of a bound is $O(q^{-\frac{2}{5}})$ when we assume $E(f\circ X)^6 < \infty$.

In this work, we improve it to $O(q^{-\frac{1}{2}})$ and give a non-uniform bound. Furthermore, we find concentration inequalities of W in case of uniform and non-uniform. These are our main results.

Theorem 1.2. *(A Uniform concentration inequality)*
Assume that $E(f\circ X)^4 < \infty$. Then as $q\to\infty$,

$$P(a\leq W\leq a+\lambda) \leq 2\lambda\big(1+\frac{1}{q-1}\big) + O(\frac{1}{\sqrt{q}}),$$

for any real number a and $\lambda \geq 0$.

Theorem 1.3. *(A Non-uniform concentration inequality)*
Assume that $E(f\circ X)^4 < \infty$. Then there exists a constant C such that

$$P(z\leq W\leq z+\lambda) \leq \frac{C\lambda}{1+z} + \frac{1}{1+z}O(\frac{1}{\sqrt{q}}), \;\; as\ q\to\infty,$$

for any real number z, $\lambda \geq 0$.

Theorem 1.4. *(A Uniform bound for randomized orthogonal array sampling designs)*
Suppose that $E(f\circ X)^6 < \infty$. Then as $q\to\infty$,

$$\sup\left\{\left|P(W\leq w)-\Phi(w)\right| : -\infty < w < \infty\right\} = O(q^{-\frac{1}{2}}).$$

Theorem 1.5. *(A Non-uniform bound for randomized orthogonal array sampling designs)*
Suppose that $E(f\circ X)^r<\infty$ *for even number* $r\geq 8$. *Then for* $z\in\mathbb{R}$,

$$\big|P(W\leq z)-\Phi(z)\big|\leq \max\big(\frac{1}{(1+|z|)^{1-\frac{2}{r}}}O(\frac{1}{q^{\frac{r-8}{2r}}}),\frac{1}{(1+|z|)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}})\big)\quad \textit{as } q\to\infty.$$

In this thesis, we organize as follows. In chapter 2, we give some definitions in elementary probability theory, a background of Stein's method and some useful properties of Stein's solution. In chapter 3, we give a uniform concentration inequality. A non-uniform concentration inequality is stated in chapter 4. The proofs of a uniform bound and non-uniform bound for randomized orthogonal array are shown in chapter 5 and chapter 6 respectively.

CHAPTER II
PRELIMINARIES

In this chapter, we give some basic concepts in probability which will be used in our work. The proof is omited but can be found in Stein([13]), Petrov([14]) and Barbour and Chen([15]).

2.1 Probability Space and Random Variables

A **probability space** is a measure space $(\Omega, \mathcal{F}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure.** The set Ω will be referred as a **sample space** and its elements are called **points** or **elementary events.** The elements of $\mathcal{F}$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A**.**

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$. A **random vector** $X = (X_1, X_2, ..., X_k)$ is a finite family of random variables $X_1,\ X_2,\ X_3, ...,\ X_k$ and for each $i = 1, 2, ..., k,\ X_i$ is called **component** of the random vector. We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \le a)$ or $P(a \le X \le b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \le x)$$

is called the **distribution function** of X.

Let X be a random variable with the distribution function F. X is said to be a **discrete random variable** if the image of X is countable and X is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability function** of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{1}{2\sigma^2}(x-\mu)^2 \Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that a discrete random variable X is **uniform** with parameter n if there exist $x_1, x_2, ..., x_n$ such that $P(X = x_i) = \frac{1}{n}$ for any $i = 1, 2, ..., n$.

If X is a continuous random variable with probability function

$$f(t) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \le x \le b, \\ 0 & \text{if} \quad otherwise, \end{cases}$$

we call that X is **uniform** on $[a, b]$.

We say that a random vector X is **uniform** on $[0,1]^3$ if its distribution function is defined by

$$F(x_1, x_2, ..., x_k) = \begin{cases} 0 & \text{if at least one} \quad x_i \le a_i, \\ \prod_{i=1}^{n} \frac{c_i - a_i}{b_i - a_i} & \text{if} \quad otherwise, \end{cases}$$

where $c_i = x_i$ if $a_i < x_i \le b_i$ and $c_i = b_i$ if $x_i > b_i$.

2.2 Independence

Let $(\Omega, \Im, P)$ be a probability space and $\mathcal{F}_\alpha$ are sub σ-algebra of $\mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{1, 2, ..., k\}$ of Λ,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_m$ for $m = 1, ..., k$.

Let $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) | \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra with $\mathcal{E}_\alpha \subseteq \sigma(\mathcal{E}_\alpha)$.

We say that the set of random variables $\{X_\alpha | \ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha) | \ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B \text{ is a Borel subset of } \mathbb{R}\}$.

Theorem 2.1. *Random variables $X_1, X_2, ..., X_n$ are* **independent** *if for any Borel sets $B_1, B_2, ..., B_n$ we have*

$$P(\bigcap_{i=1}^{n} \{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2. *If* X_{ij} ; $i = 1, 2, ..., n,\ j = 1, 2, ..., m_i$ *are independent and* $f_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}$ *are measurable, then* $\{f_i(X_{i1}, X_{i2}, ..., X_{im_i}),\ i = 1, 2, ..., n\}$ *is independent.*

2.3 Expectation, Variance and Conditional Expectation

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$.
If $\int_\Omega |X|dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3.

1. *If* X *is a discrete random variable, then* $E(X) = \sum_{x \in Im X} xP(X = x)$.

2. *If* X *is a continuous random variable with probability function* f*, then*

$$E(X) = \int_\mathbb{R} xf(x)dx.$$

Proposition 2.4. *Let* X *and* Y *be random variables such that* $E(|X|) < \infty$ *and* $E(|Y|) < \infty$ *and* $a, b \in R$. *Then we have the followings:*

1. $E(aX + bY) = aE(X) + bE(Y)$.

2. *If* $X \leq Y$ *, then* $E(X) \leq E(Y)$.

3. $|E(X)| \leq E(|X|)$.

4. *If* X *and* Y *are independent , then* $E(XY) = E(X)E(Y)$.

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the k**-th moment** of X about the origin and call $E[(X - E(X))^k]$ the k**-th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X, denoted by $Var(X)$. Then

$$Var(X) = E[X - E(X)]^2.$$

We note that

1. $Var(X) = E(X^2) - E^2(X)$.

2. If $X \sim N(\mu, \sigma^2)$ then $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proposition 2.5. *If $X_1, ..., X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, ..., n$, then*

1. $E(X_1 X_2 ... X_n) = E(X_1)E(X_2)...E(X_n)$,

2. $Var(a_1 X_1 + \cdots + a_n X_n) = a_1^2 Var(X_1) + \cdots + a_n^2 Var(X_n)$ *for any real number* $a_1, ..., a_n$.

The following inequalities are useful in our work.

1. **Hölder's inequality :**

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p) E^{\frac{1}{q}}(|X|^q)$$

where $0 < p, q < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $E(|X|^p) < \infty$,$E(|Y|^q) < \infty$.

2. **Chebyshev's inequality :**

$$P(\{|X - E(X)| \geq \varepsilon\}) \leq \frac{Var(X)}{\varepsilon^2} \quad \text{for all} \quad \varepsilon > 0$$

where $E(X^2) < \infty$.

Let X be a random variable with finite expected value on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and signed-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP.$$

Then by Radon-Nikodym theorem we have $\mathcal{Q}_X \ll P_{\mathcal{D}}$ and there exists a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{F}, P)$ such that

$$\int_E E^{\mathcal{D}}(X) dP_{\mathcal{D}} = \mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } E \in \mathcal{D}.$$

We will say that $E^{\mathcal{D}}(X)$ is the **conditional expectation** of X with respect to $\mathcal{D}$.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.6. *Let X be a random variable on probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

1. *If X is random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $E^{\mathcal{D}}(X) = X$ a.s.$[P_{\mathcal{D}}]$.*
2. *$E^{\mathcal{F}}(X) = X$ a.s.$[P]$.*
3. *If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^{\mathcal{D}}(X) = E(X)$ a.s.$[P_{\mathcal{D}}]$.*

Theorem 2.7. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|)$ and $E(|Y|)$ are finite . Then for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$ the followings hold.*

1. *If $X \le Y$, then $E^{\mathcal{D}}(X) \le E^{\mathcal{D}}(Y)$ a.s. $[P_{\mathcal{D}}]$.*
2. *$E^{\mathcal{D}}(aX + bY) = aE^{\mathcal{D}}(X) + bE^{\mathcal{D}}(X)$ a.s. $[P_{\mathcal{D}}]$ for any $a,\ b \in \mathbb{R}$.*

Theorem 2.8. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|XY|)$ and $E(|Y|)$ are finite and $\mathcal{D}_1, \mathcal{D}_2$ be any sub σ-algebra of $\mathcal{F}$. If X is a random variable with respect to $\mathcal{D}_1$, then*

1. *$E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ a.s. $[P_{\mathcal{D}_1}]$.*
2. *$E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ a.s. $[P_{\mathcal{D}_2}]$.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ be a sub σ-algebra of $\mathcal{F}$. For any event A on $\mathcal{F}$, we defined the **conditional probability of A given $\mathcal{D}$** by

$$P(A|\mathcal{D}) = E^{\mathcal{D}}(1_A)$$

where 1_A is defined by

$$1_A(w) = \begin{cases} 1 & \text{if } \ w \in A \\ 0 & \text{if } \ w \notin A. \end{cases}$$

2.4 Stein's Method for Normal Approximation

In 1972, Stein introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation for dependent random variables. The technique used was novel. Stein's technique is free of Fourier methods and relied instead on the elementary differential equation. This method was adapted and applied to the Poisson approximation by Chen in 1975. Since then, Stein's method has stimulated an area of intensive research in combinatorics, probability and statistics.

In this section we give basic results on the Stein's equation and its solution.

Let Z be a standard normal distributed random variable and let $\mathcal{C}_{bd}$ be the set of continuous and piecewise continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z)| < \infty$.

For $g \in \mathcal{C}_{bd}$ and any real valued function I with $E|I(Z)| < \infty$, the equation

$$g'(w) - wg(w) = I(w) - EI(Z) \tag{2.1}$$

is called **Stein's equation**.

If $I_z(w) = 1(w \leq z)$, then the Stein's equation becomes

$$g'(w) - wg(w) = I_z(w) - \Phi(z). \tag{2.2}$$

Hence

$$E(g'(W) - Wg(W)) = P(W \leq z) - \Phi(z)$$

for any random variable W and the solution g_z of Stein's equation (2.2) is given by

$$g_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(w)[1-\Phi(z)] & \text{if } w \leq z \\ \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(z)[1-\Phi(w)] & \text{if } w \geq z. \end{cases} \tag{2.3}$$

The following properties of g_z are used in this work.

Proposition 2.9. *For all real w, u, v, we have*

1. *$wg_z(w)$ is an increasing function of w,*
2. *$|g_z'(w)| \leq 1$,*
3. *$|g_z'(w) - g_z'(v)| \leq 1$,*
4. *$0 < g_z(w) \leq \min(\frac{\sqrt{2\pi}}{4}, \frac{1}{|z|})$,*
5. *$|(w+u)g_z(w+u) - (w+v)g_z(w+v)| \leq (|w| + \frac{\sqrt{2\pi}}{4})(|u|+|v|)$,*
6. *$|g_z'(w+u) - g_z'(w+v) - \int_v^u h(w+u)du|$ $\leq 1(z - \max(u,v) < w \leq z - \min(u,v))$ where $h(w) = (wg_z(w))'$.*

Proposition 2.10. *Let* $h(w) = (wg_z(w))'$. *Then*

$$0 \leq h(w) \leq \begin{cases} 4(1+z^2)e^{\frac{z^2}{8}}(1-\Phi(z)) & \text{if } w \leq \frac{z}{2} \\ 4(1+z^2)e^{\frac{z^2}{2}}(1-\Phi(z)) & \text{if } \frac{z}{2} < w \leq z. \end{cases}$$

Proposition 2.11. *Let* (X, Y) *be an exchangeable pair in the sense that*

$$P(X \in A, Y \in B) = P(X \in B, Y \in A)$$

for every Borel measurable subsets A *and* B *of* $\mathbb{R}$. *Then, for all measurable functions* $F : \mathbb{R}^2 \to \mathbb{R}$ *that are antisymmetric in the sense that, for all* $x, x' \in \mathbb{R}$

$$F(x, x') = -F(x', x)$$

we have

$$EE^X F(X, Y) = 0,$$

provided that

$$E|F(X, Y)| < \infty.$$

CHAPTER III
A UNIFORM CONCENTRATION INEQUALITY FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS

Let X be a random variable. The function $Q_X : [0, \infty) \to \mathbb{R}$ defined by

$$Q_X(\lambda) = \sup_x P(x \leq X \leq x + \lambda)$$

is called a **uniform (L$\acute{e}$vy) concentration function** of X and the function $Q_X : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ defined by

$$Q_X(x; \lambda) = P(x \leq X \leq x + \lambda)$$

is called a **non-uniform (L$\acute{e}$vy) concentration function** of X.

Upper bounds of uniform and non-uniform concentration functions are called **uniform** and **non-uniform** concentration inequalities respectively. In this chapter, we will give a uniform concentration inequality for randomized orthogonal array sampling designs.

With the same notation as in Chapter 1, let $q \in \{2, 3, 4, ...\}$ and $A = [a_{i,j}]_{q^2 \times 3} \in OA(q^2, 3, q, 2)$ where $OA(q^2, 3, q, 2)$ is a class of the orthogonal arrays with strength 2 and index 1 and we also let

$$\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \leq i \leq q^2\}$$

be a class of the sample points with based on orthogonal array $A = [a_{i,j}]_{q^2 \times 3}$ where π_1, π_2, π_3 are random permutations of $\{0, 1, ..., q-1\}$ and $X(i_1, i_2, i_3)$ is defined as in chapter 1.

Let

$$W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}} \tag{3.1}$$

Our main result in this chapter is to give an uniform concentration inequality of W which is stated as follows.

Theorem 3.1. *Assume that* $E(f \circ X)^4 < \infty$. *Then as* $q \to \infty$,

$$P(a \le W \le a + \lambda) \le 2\lambda\big(1 + \frac{1}{q-1}\big) + O(\frac{1}{\sqrt{q}}),$$

for any real number a *and* $\lambda \ge 0$.

To prove Theorem 3.1, it suffices to prove the following theorem.

Theorem 3.2. *For any real number* $a \le b$. *Assume that* $E(f \circ X)^4 < \infty$. *Then as* $q \to \infty$,

$$P(a \le W \le b) \le 2(b-a)\Big(1 + \frac{1}{q-1}\Big) + O(\frac{1}{\sqrt{q}}).$$

Note that, if we choose $b = a + \lambda$ where $\lambda \ge 0$, then Theorem 3.1 follows directly from Theorem 3.2.

3.1 Auxiliary Results

In this section, we give auxilially results for proving Theorem 3.2.

For each $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$ and $j, k, l \in \{1, 2, 3\}$ such that $k < l$ and for any integrable function f, define

$$\begin{aligned}
\mu(i_1, i_2, i_3) &= Ef \circ X(i_1, i_2, i_3), \\
\mu_j(i_j) &= \frac{1}{q^2} \sum_{\substack{i_k=0 \\ k \ne j}}^{q-1} [\mu(i_1, i_2, i_3) - \mu], \\
\mu_{k,l}(i_k, i_l) &= \frac{1}{q} \sum_{\substack{i_j=0 \\ j \ne k,l}}^{q-1} [\mu(i_1, i_2, i_3) - \mu - \mu_k(i_k) - \mu_l(i_l)], \\
Y(i_1, i_2, i_3) &= \frac{1}{q^2\sqrt{\mathrm{Var}(\hat{\mu})}} \Big[f \circ X(i_1, i_2, i_3) - \mu - \sum_{j=1}^{3} \mu_j(i_j) - \sum_{1 \le k < l \le 3} \mu_{k,l}(i_k, i_l) \Big], \\
\tilde{\mu}(i_1, i_2, i_3) &= EY(i_1, i_2, i_3).
\end{aligned}$$

Then W in (3.1) can be rewritten as

$$W = \sum_{i=1}^{q^2} Y\big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big) \tag{3.2}$$

and

$$\sum_{i_j=0}^{q-1} \tilde{\mu}(i_1, i_2, i_3) = 0. \tag{3.3}$$

for each $j \in \{1, 2, 3\}$(Loh([7]): 1212). In 1996, Loh([7]) defined a random function ρ_π be such that

$$\big(i_1, i_2, \rho_\pi(i_1, i_2)\big) = \big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big) \tag{3.4}$$

for some $i \in \{1, ..., q^2\}$ and showed that W in (3.2) can be rewritten again as the form

$$W = \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1, i_2)\big). \tag{3.5}$$

Let I and K be uniformly distributed random variables on $\{0, 1, ..., q-1\}$, (I, K) uniformly distributed on $\big\{(i, k) | i, k = 0, 1, ..., q-1, i \neq k\big\}$ and assume that they are indenpendent of all π_1, π_2, π_3 and $U_{i_1,i_2,i_3,j}$'s defined as in Chapter 1.
Define

$$\widetilde{W} = W - S_1 - S_2 + S_3 + S_4$$

where

$$S_1 = \sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(I, i_2)\big), \qquad S_2 = \sum_{i_2=0}^{q-1} Y\big(K, i_2, \rho_\pi(K, i_2)\big),$$
$$S_3 = \sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(K, i_2)\big), \qquad S_4 = \sum_{i_2=0}^{q-1} Y\big(K, i_2, \rho_\pi(I, i_2)\big).$$

Note that $\big(W, \widetilde{W}\big)$ is an exchangeable pair(see Loh([7]): 1213).

Lemma 3.3.

1. *S_1, S_2, S_3, S_4 are identically distributed.*
2. *If $E(f \circ X)^r < \infty$ for any positive even integer r, then, for every $i = 1, 2, 3, 4$, $ES_i^r = O(q^{-\frac{r}{2}})$ as $q \to \infty$.*

Proof. 1. Clearly S_1 and S_2 have the same distribution and so do S_3 and S_4. Thus, it suffices to show that S_1 and S_3 have the same distribution. Let $S^{(q)}$ be the set of all permutations on $\{0, 1, ..., q-1\}$ and for each $i = 0, 1, ..., q-1$. Define a random variable $\rho_\pi(i, \cdot) : \{0, 1, ..., q-1\} \to \{0, 1, ..., q-1\}$ by

$$\rho_\pi(i, \cdot)(j) = \rho_\pi(i, j).$$

From (3.4) and the fact that π_1, π_2, π_3 are independent, we have

$$\rho_\pi(i,\cdot) \text{ is a random permutation on } \{0,1,...,q-1\}.$$

Then, for every $a \in \mathbb{R}$, we have

$$\begin{aligned}
P(S_3 \le a) &= P(\sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(K, i_2)\big) \le a) \\
&= \sum_{i \ne k} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \rho_\pi(k, i_2)\big) \le a, (I,K) = (i,k)) \\
&= \frac{1}{q(q-1)} \sum_{i \ne k} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \rho_\pi(k, i_2)\big) \le a) \\
&= \frac{1}{q(q-1)} \sum_{i \ne k} \sum_{\beta \in S^{(q)}} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \beta(i_2)\big) \le a, \rho_\pi(k,\cdot) = \beta) \\
&= \frac{1}{q(q-1)q!} \sum_{i \ne k} \sum_{\beta \in S^{(q)}} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \beta(i_2)\big) \le a) \\
&= \frac{1}{qq!} \sum_{i=0}^{q-1} \sum_{\beta \in S^{(q)}} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \beta(i_2)\big) \le a) \\
&= \frac{1}{q} \sum_{i=0}^{q-1} \sum_{\beta \in S^{(q)}} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \beta(i_2)\big) \le a, \rho_\pi(i,\cdot) = \beta) \\
&= \frac{1}{q} \sum_{i=0}^{q-1} P(\sum_{i_2=0}^{q-1} Y\big(i, i_2, \rho_\pi(i, i_2)\big) \le a) \\
&= P(\sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(I, i_2)\big) \le a) \\
&= P(S_1 \le a).
\end{aligned}$$

Hence S_3 has the same distribution as S_1.

2. follows from 1 and the fact that $ES_1^r = O(q^{-\frac{r}{2}})$(Loh([7]): 1215).

□

Lemma 3.4.

1. If $E(f \circ X)^2 < \infty$*, then* $\frac{1}{q} \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} EY^2(i,j,k) = 1 + O(\frac{1}{q})$ *as* $q \to \infty$.

2. If $E(f \circ X)^r < \infty$ *for some positive even integer* r*,* *then* $\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^r(i,j,k) = O(q^{3-r})$ *as* $q \to \infty$.

3. *If* $E(f \circ X)^r < \infty$ *for some positive even integer* r,

   then $E\Big(\widetilde{W} - W\Big)^r \le O(q^{-\frac{r}{2}})$ *as* $q \to \infty$.

Proof.

1 and 2 follow from Loh([7]): 1215-1217.

3 follows from Lemma 3.3(1, 2) and the fact that

$$\begin{aligned} E\big(\widetilde{W} - W\big)^r &= E\Big|\sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(I,i_2)\big) + \sum_{i_2=0}^{q-1} Y\big(K, i_2, \rho_\pi(K,i_2)\big) \\ &\qquad - \sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(K,i_2)\big) - \sum_{i_2=0}^{q-1} Y\big(K, i_2, \rho_\pi(I,i_2)\big)\Big|^r \\ &\le 4^r E\Big|\sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(I,i_2)\big)\Big|^r \\ &= 4^r E S_1^r. \end{aligned}$$

□

Lemma 3.5. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous and piecewise continuously differentiable function. Then*

$$EWg(W) = E\int_{-\infty}^{\infty} g'(W+t)M(t)dt - \Delta g(W) \tag{3.6}$$

and

$$\big|\Delta g(W)\big| \le \frac{1}{q-1}\Big[Eg^2\big(W\big)\Big]^{\frac{1}{2}}, \tag{3.7}$$

where

$$M(t) = \frac{q}{4}\big(\widetilde{W} - W\big)\Big\{\mathbb{I}\big(0 \le t \le \widetilde{W} - W\big) - \mathbb{I}\big(\widetilde{W} - W \le t \le 0\big)\Big\},$$

$$\Delta g\big(W\big) = \frac{1}{q-1} Eg\big(W\big)\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}\tilde{\mu}(i,j,\rho_\pi(i,j))$$

and $\mathbb{I}$ *is the indicator function.*

Proof. Let $\mathcal{A}$ be the $\sigma-$algebra generated by

$$\{\big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big), U_{\pi_1(a_{i,1}),\pi_2(a_{i,2}),\pi_3(a_{i,3}),j} : 1 \le i \le q^2, 1 \le j \le 3\}$$

and $F:\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(w,\tilde{w}) = (\tilde{w} - w)(g(\tilde{w}) + g(w))$$

for each $w, \tilde{w} \in \mathbb{R}$. Then F is an antisymmetric function and by Proposition 2.11, we have

$$EF(W,\widetilde{W}) = 0. \tag{3.8}$$

From (3.8) and the fact that

$$E^{\mathcal{A}}(\widetilde{W} - W) = -\frac{2}{q}W - \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\tilde{\mu}(i,j,\rho_\pi(i,j)),$$

(Loh([7]): 1217), we have

$$\begin{aligned}
0 &= EF(W,\widetilde{W}) \\
&= E(\widetilde{W} - W)(g(\widetilde{W}) + g(W)) \\
&= E(\widetilde{W} - W)(2g(W)) + E(\widetilde{W} - W)(g(\widetilde{W}) - g(W)) \\
&= 2E\Big\{g(W)E^{\mathcal{A}}(\widetilde{W} - W)\Big\} + E(\widetilde{W} - W)(g(\widetilde{W}) - g(W)) \\
&= 2Eg(W)\Big\{-\frac{2}{q}W - \frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\tilde{\mu}(i,j,\rho_\pi(i,j))\Big\} \\
&\quad + E(\widetilde{W} - W)(g(\widetilde{W}) - g(W))
\end{aligned} \tag{3.9}$$

which implies that

$$\begin{aligned}
EWg(W) &= \frac{q}{4}E(\widetilde{W} - W)(g(\widetilde{W}) - g(W)) - \frac{1}{q-1}Eg(W)\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\tilde{\mu}(i,j,\rho_\pi(i,j)) \\
&= \frac{q}{4}E(\widetilde{W} - W)(g(\widetilde{W}) - g(W)) - \Delta g(W) \qquad (3.10) \\
&= \frac{q}{4}E(\widetilde{W} - W)\Big(\int_0^{\widetilde{W}-W} g'(W+t)dt\Big) - \Delta g(W) \\
&= E\int_{-\infty}^{\infty} g'(W+t)M(t)dt - \Delta g(W).
\end{aligned}$$

From Loh([7]): 1216, we know that $\big|\Delta g(W)\big| \le \frac{1}{q-1}\Big[Eg^2(W)\Big]^{\frac{1}{2}}\Big[EW^2\Big]^{\frac{1}{2}}$.

Hence, by (1.1), $\big|\Delta g(W)\big| \le \frac{1}{q-1}\Big[Eg^2(W)\Big]^{\frac{1}{2}}$. □

To prove Lemma 3.7-3.8, we introduce the following construction which is used in K.Neammanee and J.Suntornchost([16]) and N. Chaidee([17]).

Let $\bar{I}, \bar{K}, \bar{L}$ and $\bar{M}$ be uniformly distributed random vectors on $\{0, 1, ..., q-1\}$ which satisfy the followings:

1. $(\bar{I}, \bar{K})$ and $(\bar{L}, \bar{M})$ are uniformly distributed random vectors on

$$\Big\{(i,k)|i,k = 0,1,...,q-1 \ \text{ and } \ i \neq k\Big\}.$$

2. $\Big[(\bar{I}, \bar{K}), (\bar{L}, \bar{M})\Big]$ is uniformly on

$$\Big\{\big[(i,k),(l,m)\big]|i,k,l,m = 0,1,...,q-1 \ \text{ and } \ i \neq k, l \neq m \ \text{ and } \ (i,k) \neq (l,m)\Big\}.$$

3. $(\bar{I}, \bar{K}), (\bar{L}, \bar{M})$ and $U_{i_1,i_2,i_3,j}$'s, π_1, π_2, π_3 are mutually independent.

Hence

$$P\Big([(\bar{I}, \bar{K}), (\bar{L}, \bar{M})] = \big[(i,k),(l,m)\big]\Big) = \frac{1}{q(q-1)\big[q(q-1)-1\big]}$$

for any $i,k,l,m = 0,1,...,q-1$ and $i \neq k, l \neq m$ and $(i,k) \neq (l,m)$.

For each $i,k \in \{0,1,...,q-1\}$, $\beta, \alpha \in S^{(q)}$ and $\delta > 0$, we also let

$$z_\delta[(i,k),(\beta,\alpha)] = \Big|\sum_{j=0}^{q-1}\big\{Y(i,j,\beta(j)) + Y(k,j,\alpha(j)) - Y(i,j,\alpha(j)) - Y(k,j,\beta(j))\big\}\Big|$$

$$\times \min\Big(\delta, \Big|\sum_{j=0}^{q-1}\big\{Y(i,j,\beta(j)) + Y(k,j,\alpha(j)) - Y(i,j,\alpha(j)) - Y(k,j,\beta(j))\big\}\Big|\Big),$$

$$\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] = z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))],$$

$$Z_\delta = \sum_{i\neq k}\hat{z}_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))],$$

and

$$\begin{aligned}\widetilde{Z}_\delta = Z_\delta &- \hat{z}_\delta[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\\ &- \hat{z}_\delta[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\\ &+ \hat{z}_\delta[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\\ &+ \hat{z}_\delta[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))].\end{aligned}$$

We recall that an array

$$\begin{array}{cccc} c_{11} & c_{12} & \cdots & c_{1q} \\ c_{21} & c_{22} & \cdots & c_{2q} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pq} \end{array}$$

is an *Latin rectangle* if each row contains the numbers $0, 1, ..., q-1$ in some order and each column does not contain any digit repeated. To prove Lemma 3.7, we need a property of Latin rectangle in Theorem 3.6.

Theorem 3.6. *(Hall([18]): 50-51) The number of ways of adding a row to an $r \times q$ Latin rectangle to give an $(r+1) \times q$ Latin rectangle is at least $(q-r)!$.*

First, we let

$$T_1 = S^{(q)} \text{ where } S^{(q)} \text{ is the set of all permutations on } \{0, 1, ..., q-1\}$$

and, for $i = 2, 3, ...,$

$$\begin{aligned} T_i &= \big\{(\beta_1, \beta_2, ..., \beta_i) \big| \ \beta_j \in S^{(q)} \text{ for } j = 1, 2, ..., i \text{ and } \beta_j\text{'s are pairwise disjoint}\big\} \\ &= \bigcup_{(\beta_1, \beta_2, ..., \beta_{i-1}) \in T_{i-1}} \big\{(\beta_1, \beta_2, ..., \beta_{i-1}, \alpha) \big| \ \alpha \in S(\beta_1, \beta_2, ..., \beta_{i-1})\big\} \end{aligned}$$

where $\quad S(\beta_1, \beta_2, ..., \beta_{i-1}) = \big\{\alpha \in S^{(q)} \big| \ \alpha \cap \beta_j = \varnothing \ \text{ for } j = 1, 2, ..., i-1\big\}.$

Lemma 3.7. *Let $\delta > 0$. Then*

1. $\big(Z_\delta, \widetilde{Z}_\delta\big)$ is an exchangeable pair.

2. Assume that $E(f \circ X)^2 < \infty$. Then, as $q \to \infty$,

$$E\big[Z_\delta \sum_{i \neq k} \sum_{l \neq m} \hat{z}_\delta[(i,k), (\rho_\pi(l, \cdot), \rho_\pi(m, \cdot))]\big] \leq \delta^2 O(q^4).$$

Proof. First, we will show that if $M \in OA(q^2, 3, q, 2)$ then $\pi(M) \in OA(q^2, 3, q, 2)$, where $\pi(M) = [\pi_j(a_{ij})]$. To do this, we suppose that $\pi(M)$ is not an orthogonal array. Then there exist $i, k \in \{0, 1, ..., q-1\}$ and $j_1, j_2 \in \{1, 2, 3\}$ such that $i \neq k$ and $j_1 \neq j_2$ and $\big(\pi_1(a_{i,j_1}), \pi_2(a_{i,j_2})\big) = \big(\pi_1(a_{k,j_1}), \pi_2(a_{k,j_2})\big)$. So $(a_{i,j_1}, a_{i,j_2}) = (a_{k,j_1}, a_{k,j_2})$ which is a contradiction to the property of A. Hence, $\pi(M)$ is an orthogonal array. So, it implies that $\rho_\pi(1,\cdot), \rho_\pi(2,\cdot), ..., \rho_\pi(q,\cdot)$ are pairwise disjoint. Then the exchangeability of Z_δ and $\widetilde{Z}_\delta$ follows from the following fact. For $a, b \in \mathbb{R}$,

$$P(Z_\delta \leq a, \widetilde{Z}_\delta \leq b)$$

$$= \sum_{i \neq k} \sum_{\substack{l \neq m \\ (l,m) \neq (i,k)}} \sum_{(\beta,\alpha,\gamma,\eta) \in T_4} P\Big(Z_\delta \leq a, \widetilde{Z}_\delta \leq b,$$
$$\Big[(\bar{I}, \bar{K}), (\bar{L}, \bar{M})\Big] = \big[(i,k),(l,m)\big],\ (\rho_\pi(i,\cdot), \rho_\pi(k,\cdot), \rho_\pi(l,\cdot), \rho_\pi(m,\cdot)) = (\beta, \alpha, \gamma, \eta)\Big)$$

$$= \sum_{i \neq k} \sum_{\substack{l \neq m \\ (l,m) \neq (i,k)}} \sum_{(\beta,\alpha,\gamma,\eta) \in T_4} P\Big(Z_\delta \leq a, \widetilde{Z}_\delta \leq b,$$
$$\Big[(\bar{I}, \bar{K}), (\bar{L}, \bar{M})\Big] = \big[(i,k),(l,m)\big],\ (\rho_\pi(i,\cdot), \rho_\pi(k,\cdot), \rho_\pi(l,\cdot), \rho_\pi(m,\cdot)) = (\gamma, \eta, \beta, \alpha)\Big)$$

$$= \sum_{i \neq k} \sum_{\substack{l \neq m \\ (l,m) \neq (i,k)}} \sum_{(\beta,\alpha,\gamma,\eta) \in T_4} P\Big(\widetilde{Z}_\delta \leq a, Z_\delta \leq b,$$
$$\Big[(\bar{I}, \bar{K}), (\bar{L}, \bar{M})\Big] = \big[(i,k),(l,m)\big],\ (\rho_\pi(i,\cdot), \rho_\pi(k,\cdot), \rho_\pi(l,\cdot), \rho_\pi(m,\cdot)) = (\gamma, \eta, \beta, \alpha)\Big)$$

$$= P(\widetilde{Z}_\delta \leq a, Z_\delta \leq b).$$

2. Note that

$$\begin{aligned} &E\Big[Z_\delta \sum_{i \neq k} \sum_{l \neq m} \hat{z}[(i,k), (\rho_\pi(l,\cdot), \rho_\pi(m,\cdot))]\Big] \\ &= \sum_{i \neq k} \sum_{l \neq m} \sum_{u \neq v} E\hat{z}_\delta[(i,k), (\rho_\pi(l,\cdot), \rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v), (\rho_\pi(u,\cdot), \rho_\pi(v,\cdot))] \\ &= \sum_{A} E\big\{\hat{z}_\delta[(i,k), (\rho_\pi(l,\cdot), \rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v), (\rho_\pi(u,\cdot), \rho_\pi(v,\cdot))]\big\}, \end{aligned} \tag{3.11}$$

where $A = \Big\{(i,k,l,m,u,v) \big| \;\; i,k,l,m,u,v \in \{0,1,...,q-1\}$

$\text{and } i \neq k, \;\; l \neq m, \;\; u \neq v\Big\}$

$= \bigcup_{i=1}^{7} A_i$

and $A_1 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u = l, \;\; u \neq m, \;\; v \neq l, \;\; v = m\big\}$

$A_2 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u \neq l, \;\; u = m, \;\; v = l, \;\; v \neq m\big\}$

$A_3 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u \neq l, \;\; u \neq m, \;\; v \neq l, \;\; v = m\big\}$

$A_4 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u \neq l, \;\; u = m, \;\; v \neq l, \;\; v \neq m\big\}$

$A_5 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u \neq l, \;\; u \neq m, \;\; v = l, \;\; v \neq m\big\}$

$A_6 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u = l, \;\; u \neq m, \;\; v \neq l, \;\; v \neq m\big\}$

$A_7 = \big\{(i,k,l,m,u,v) \in A \;\big|\;\; u \neq l, \;\; u \neq m, \;\; v \neq l, \;\; v \neq m\big\}.$

We first consider the sum on A_1. Note that

$$
\begin{aligned}
&\sum_{A_1} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\Big\} \\
&\qquad\qquad \times \Big\{z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Ez_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\} \\
&\leq \sum_{i\neq k}\sum_{l\neq m} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\} \\
&\quad + \sum_{i\neq k}\sum_{l\neq m} Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Ez_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] \\
&= R_{11} + R_{12}.
\end{aligned}
\tag{3.12}
$$

where

$$R_{11} = \sum_{i\neq k}\sum_{l\neq m} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\}$$

and

$$R_{12} = \sum_{i \neq k} \sum_{l \neq m} E z_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))] E z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))].$$

By Lemma 3.4(2, 3) and Cauchy's inequality, we note that

$$R_{11} = \sum_{i \neq k} \sum_{l \neq m} E\Big\{ z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\}$$

$$\leq \frac{1}{2} \sum_{i \neq k} \sum_{l \neq m} E\big\{ z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2 + \frac{1}{2} \sum_{i \neq k} \sum_{l \neq m} E\big\{ z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2$$

$$= \frac{1}{2} \sum_{i \neq k} \sum_{l \neq m} E\big\{ z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2 + \frac{q(q-1)}{2} \sum_{l \neq m} E\big\{ z_\delta[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\big\}^2$$

$$\leq \frac{\delta^2}{2} \sum_{i \neq k} \sum_{l \neq m} E\Big\{ \Big| \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2)) + \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(m,i_2)) - \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(m,i_2)) - \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(l,i_2)) \Big| \Big\}^2$$
$$+ \frac{\delta^2 q(q-1)}{2} \sum_{i \neq k} E\Big\{ \Big| \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(i,i_2)) + \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(k,i_2)) - \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(k,i_2)) - \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(i,i_2)) \Big| \Big\}^2$$

$$\leq 8\delta^2 (q-1)^2 \sum_{i=0}^{q-1} \sum_{l=0}^{q-1} E\Big\{ \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2)) \Big\}^2 + \frac{\delta^2 q^2 (q-1)^2}{2} E(\widetilde{W} - W)^2 \tag{3.13}$$

$$= 8\delta^2 q^2 (q-1)^2 E\Big\{ \sum_{i_2=0}^{q-1} Y(\bar{I},i_2,\rho_\pi(\bar{L},i_2)) \Big\}^2 + \frac{\delta^2 q^2 (q-1)^2}{2} O(q^{-1})$$

$$= \delta^2 O(q^3) \tag{3.14}$$

and

$$R_{12} = \Big\{ \sum_{i \neq k} E z_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))] \Big\}^2$$

$$
\begin{aligned}
&\leq \delta^2\Big\{\sum_{i\neq k} E\Big|\sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(i,i_2)) + \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(k,i_2)) \\
&\quad - \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(k,i_2)) - \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(i,i_2))\Big|\Big\}^2 \\
&= q^2(q-1)^2\delta^2\Big\{E\Big|\sum_{i_2=0}^{q-1} Y(I,i_2,\rho_\pi(I,i_2)) + \sum_{i_2=0}^{q-1} Y(K,i_2,\rho_\pi(K,i_2)) \\
&\quad - \sum_{i_2=0}^{q-1} Y(I,i_2,\rho_\pi(K,i_2)) - \sum_{i_2=0}^{q-1} Y(K,i_2,\rho_\pi(I,i_2))\Big|\Big\}^2 \\
&= q^2(q-1)^2\delta^2\Big\{E\big(\widetilde{W}-W\big)\Big\}^2 \\
&= \delta^2 O(q^3). \qquad (3.15)
\end{aligned}
$$

It follows from (3.12),(3.14) and (3.15) that

$$
\sum_{A_1} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \leq \delta^2 O(q^3). \qquad (3.16)
$$

Similar to A_1, we can conclude that

$$
\sum_{A_2} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \leq \delta^2 O(q^3). \qquad (3.17)
$$

Next, we consider the sum on A_3.

$$
\begin{aligned}
&\sum_{A_3} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u \\ u\neq l,m}} \hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u \\ u\neq l,m}} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\Big\} \\
&\qquad \times\Big\{z_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))] - Ez_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]\Big\} \\
&\leq \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u \\ u\neq l,m}} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]z_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]\Big\} \\
&\quad + (q-2)\sum_{i\neq k}\sum_{u\neq m} Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Ez_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))] \\
&= R_{31} + R_{32}. \qquad (3.18)
\end{aligned}
$$

where

$$R_{31} = \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\\ u\neq l,m}} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]z_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]\Big\}$$

and

$$R_{32} = (q-2)\sum_{i\neq k}\sum_{u\neq m} Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Ez_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]$$

Obviously,

$$R_{32} = (q-2)R_{12} \leq \delta^2 O(q^4). \tag{3.19}$$

Now, it remains to bound R_{31}. By Lemma 3.4(2, 3).

$$\begin{aligned}
R_{31} &= \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\\ u\neq l,m}} E\Big\{z_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]z_\delta[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]\Big\}\\
&\leq \delta^2 \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\\ u\neq l,m}} E\Big\{\Big|\sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2)) + \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(m,i_2))\\
&\quad - \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(m,i_2)) - \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(l,i_2))\Big|\Big\}\\
&\quad \times \Big\{\Big|\sum_{t_2=0}^{q-1} Y(u,t_2,\rho_\pi(u,t_2)) + \sum_{t_2=0}^{q-1} Y(m,t_2,\rho_\pi(m,t_2))\\
&\quad - \sum_{t_2=0}^{q-1} Y(u,t_2,\rho_\pi(m,t_2)) - \sum_{t_2=0}^{q-1} Y(m,t_2,\rho_\pi(u,t_2))\Big|\Big\}\\
&\leq \frac{1}{2}\delta^2 \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\\ u\neq l,m}} E\Big\{\Big|\sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2)) + \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(m,i_2))\\
&\quad - \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(m,i_2)) - \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(l,i_2))\Big|\Big\}^2\\
&\quad + \frac{1}{2}\delta^2 \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\\ u\neq l,m}} E\Big\{\Big|\sum_{t_2=0}^{q-1} Y(u,t_2,\rho_\pi(u,t_2)) + \sum_{t_2=0}^{q-1} Y(m,t_2,\rho_\pi(m,t_2))\\
&\quad - \sum_{t_2=0}^{q-1} Y(u,t_2,\rho_\pi(m,t_2)) - \sum_{t_2=0}^{q-1} Y(m,t_2,\rho_\pi(u,t_2))\Big|\Big\}^2\\
&\leq \frac{\delta^2}{2}(q-2)\sum_{i\neq k}\sum_{l\neq m} E\Big\{\Big|\sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(l,i_2)) + \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(m,i_2))\\
&\quad - \sum_{i_2=0}^{q-1} Y(i,i_2,\rho_\pi(m,i_2)) - \sum_{i_2=0}^{q-1} Y(k,i_2,\rho_\pi(l,i_2))\Big|\Big\}^2\\
&\quad + \frac{\delta^2}{2}q(q-1)(q-2)\sum_{u\neq m} E\Big\{\Big|\sum_{t_2=0}^{q-1} Y(u,t_2,\rho_\pi(u,t_2)) + \sum_{t_2=0}^{q-1} Y(m,t_2,\rho_\pi(m,t_2))
\end{aligned}$$

$$- \sum_{t_2=0}^{q-1} Y(u, t_2, \rho_\pi(m, t_2)) - \sum_{t_2=0}^{q-1} Y(m, t_2, \rho_\pi(u, t_2))\Big|\Big\}^2$$

$$\leq 8\delta^2(q-1)^2(q-2) \sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\Big|\sum_{i_2=0}^{q-1} Y(i, i_2, \rho_\pi(l, i_2))\Big|^2$$

$$+ \frac{\delta^2}{2} q^2(q-1)^2(q-2)E\Big\{\Big|\sum_{t_2=0}^{q-1} Y(\bar{I}, t_2, \rho_\pi(\bar{I}, t_2)) + \sum_{t_2=0}^{q-1} Y(\bar{K}, t_2, \rho_\pi(\bar{K}, t_2))$$

$$-Y(\bar{I}, t_2, \rho_\pi(\bar{K}, t_2)) - Y(\bar{K}, t_2, \rho_\pi(\bar{I}, t_2))\Big\}\Big|\Big\}^2 \quad (3.20)$$

$$\leq 8\delta^2 q^2(q-1)^2(q-2)E\Big|\sum_{i_2=0}^{q-1}\big\{Y(\bar{I}, i_2, \rho_\pi(\bar{L}, i_2))\Big|^2$$

$$+ 4\delta^2 q^2(q-1)^2(q-2)E(\widetilde{W} - W)^2$$

$$\leq \delta^2 O(q^4). \quad (3.21)$$

By (3.18),(3.19) and (3.21),

$$\sum_{A_3} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \leq \delta^2 O(q^3). \quad (3.22)$$

By the same argement of (3.22), we obtain

$$\sum_{A_4} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \leq \delta^2 O(q^3), \quad (3.23)$$

$$\sum_{A_5} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \leq \delta^2 O(q^3), \quad (3.24)$$

$$\sum_{A_6} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \leq \delta^2 O(q^3). \quad (3.25)$$

Next, we consider the sums over A_7.
Observe that, for each $u, v \in \{0, 1, ..., q-1\}$ such that $u \neq v$,

$$\sum_{u\neq v}\sum_{(\gamma,\eta)\in T_2} E\hat{z}_\delta[(u,v),(\gamma,\eta)]$$

$$= \sum_{u\neq v}\sum_{(\gamma,\eta)\in T_2} Ez_\delta[(u,v),(\gamma,\eta)] - |T_2|\sum_{u\neq v} Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]$$

$$= \sum_{u\neq v}\sum_{(\gamma,\eta)\in T_2} Ez_\delta[(u,v),(\gamma,\eta)] - \sum_{u\neq v}\sum_{(\gamma,\eta)\in T_2} Ez_\delta[(u,v),(\gamma,\eta)]$$

$$= 0. \quad (3.26)$$

Then, from (3.26), we have

$$
\begin{aligned}
&\sum_{A_7} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\neq v\\ u\neq l,m\\ v\neq l,m}} E\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \\
&= \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u\neq v\\ u\neq l,m\\ v\neq l,m}} \frac{1}{|\,T_4|}\sum_{(\beta,\alpha,\gamma,\eta)\in T_4} E\hat{z}_\delta[(i,k),(\beta,\alpha)]\hat{z}_\delta[(u,v),(\gamma,\eta)] \\
&= \frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha,\gamma,\eta)\in T_4} E\hat{z}_\delta[(i,k),(\beta,\alpha)]E\hat{z}_\delta[(u,v),(\gamma,\eta)] \\
&= \frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]\Big\{\sum_{(\gamma,\eta)\in T_2} E\hat{z}_\delta[(u,v),(\gamma,\eta)] \\
&\quad - E\hat{z}_\delta[(u,v),(\beta,\alpha)] - E\hat{z}_\delta[(u,v),(\alpha,\beta)] \\
&\quad - \sum_{\gamma\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\gamma,\alpha)] - \sum_{\eta\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\alpha,\eta)] \\
&\quad - \sum_{\gamma\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\gamma,\beta)] - \sum_{\eta\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\beta,\eta)]\Big\} \\
&= B_1 + B_2 + B_3 + B_4 + B_5 + B_6
\end{aligned}
\tag{3.27}
$$

where

$$B_1 = -\frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]E\hat{z}_\delta[(u,v),(\beta,\alpha)]$$

$$B_2 = -\frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]E\hat{z}_\delta[(u,v),(\alpha,\beta)]$$

$$B_3 = -\frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]\sum_{\gamma\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\gamma,\alpha)]$$

$$B_4 = -\frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]\sum_{\eta\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\alpha,\eta)]$$

$$B_5 = -\frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]\sum_{\gamma\in S(\beta,\alpha)} E\hat{z}[(u,v),(\gamma,\beta)]$$

$$B_6 = -\frac{(q-2)(q-3)}{|\,T_4|}\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]\sum_{\eta\in S(\beta,\alpha)} E\hat{z}_\delta[(u,v),(\beta,\eta)].$$

To complete the lemma, we have to bound the terms B_i's in (3.27).

We now consider the term B_1.

$$
\begin{aligned}
-&\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} E\hat{z}_\delta[(i,k),(\beta,\alpha)]E\hat{z}_\delta[(u,v),(\beta,\alpha)]\\
&= -\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} \big\{Ez_\delta[(i,k),(\beta,\alpha)] - Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\big\}\\
&\quad \times \big\{Ez_\delta[(u,v),(\beta,\alpha)] - Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\}\\
&\le \sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} \Big\{Ez_\delta[(i,k),(\beta,\alpha)]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\\
&\quad + Ez_\delta[(u,v),(\beta,\alpha)]Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\Big\}\\
&= 2\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} Ez_\delta[(i,k),(\beta,\alpha)]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\\
&= 2\big|T_2\big|\sum_{i\neq k}\sum_{u\neq v} Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\\
&= 2\big|T_2\big|R_{12}\\
&\le \big|T_2\big|\delta^2 O(q^3)
\end{aligned}
$$

where we have used (3.15) in the last inequality. From this fact and the fact that (see theorem 3.6), for any positive integer i,

$$\big|T_{i+1}\big| \ge \big|T_i\big|(q-i)!. \tag{3.28}$$

Thus

$$B_1 \le \frac{(q-2)(q-3)}{\big|\,T_4\big|}\big|T_2\big|\delta^2 O(q^3) \le \frac{(q-2)(q-3)}{\big|\,T_2\big|(q-2)!(q-3)!}\big|T_2\big|\delta^2 O(q^3) \le \delta^2 O(q^4). \tag{3.29}$$

By using the same argument of B_1, we can show that

$$B_2 \le \delta^2 O(q^4). \tag{3.30}$$

Next, we consider the term B_3. By (3.15),we note that

$$
\begin{aligned}
-&\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha,\gamma)\in T_3} E\hat{z}_\delta[(i,k),(\beta,\alpha)]E\hat{z}_\delta[(u,v),(\gamma,\alpha)]\\
&= -\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha,\gamma)\in T_3} \big\{Ez_\delta[(i,k),(\beta,\alpha)] - Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\big\}
\end{aligned}
$$

$$\times \left\{Ez_\delta[(u,v),(\gamma,\alpha)] - Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\right\}$$

$$\leq \sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha,\gamma)\in T_3} \Big\{Ez_\delta[(i,k),(\beta,\alpha)]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] + Ez_\delta[(u,v),(\gamma,\alpha)]Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\Big\}$$

$$= \sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} \big|S(\beta,\alpha)\big| Ez_\delta[(i,k),(\beta,\alpha)]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]$$

$$\leq q!\sum_{i\neq k}\sum_{u\neq v}\sum_{(\beta,\alpha)\in T_2} Ez_\delta[(i,k),(\beta,\alpha)]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]$$

$$= q!\big|T_2\big|\sum_{i\neq k}\sum_{u\neq v} Ez_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Ez_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]$$

$$= q!\big|T_2\big|R_{12}$$

$$\leq q!\big|T_2\big|\delta^2 O(q^3).$$

From this fact and (3.28),

$$B_3 \leq \frac{(q-2)(q-3)q!\big|T_2\big|\delta^2 O(q^3)}{\big|T_4\big|} \leq \frac{(q-2)(q-3)q!\big|T_2\big|\delta^2 O(q^3)}{\big|T_2\big|(q-2)!(q-3)!} \leq \delta^2 O(q^4). \tag{3.31}$$

Using the same technique, we obtain

$$B_4 \leq \delta^2 O(q^4), \quad B_5 \leq \delta^2 O(q^4) \text{ and } B_6 \leq \delta^2 O(q^4). \tag{3.32}$$

Thus, by (3.27)-(3.32)

$$\sum_{A_7} E\big[\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{z}_\delta[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big] \leq \delta^2 O(q^4). \tag{3.33}$$

Hence, by (3.11),(3.16),(3.17),(3.22)-(3.25) and (3.33)

$$E\big[\sum_{i\neq k}\sum_{l\neq m}\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]Z_\delta\big] \leq \delta^2 O(q^4).$$

□

Lemma 3.8. *Let* $\delta > 0$ *and*

$$G_\delta = \sum_{i\neq k} z_\delta\big[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))\big].$$

Assume that $E(f\circ X)^4 < \infty$. *Then as* $q\to\infty$,

$$Var(G_\delta) \leq \delta^2 O(q^2).$$

Proof. Since $Var(G_\delta) = EZ_\delta^2$, it suffices to show that $EZ_\delta^2 < \delta^2 O(q^2)$.
By Lemma 3.3(2), we observe that

$$
\begin{aligned}
E\Big|\sum_{j=0}^{q-1} Y(\bar{I}, j, \rho_\pi(\bar{L}, j))\Big|^2 \\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\Big|\sum_{j=0}^{q-1} Y(i, j, \rho_\pi(l, j))\Big|^2 \\
&= \frac{1}{q^2}\sum_{i=0}^{q-1} E\Big|\sum_{j=0}^{q-1} Y(i, j, \rho_\pi(i, j))\Big|^2 + \frac{1}{q^2}\sum_{i\neq l} E\Big|\sum_{j=0}^{q-1} Y(i, j, \rho_\pi(l, j))\Big|^2 \\
&= \frac{1}{q}E\Big|\sum_{j=0}^{q-1} Y(\bar{I}, j, \rho_\pi(\bar{I}, j))\Big|^2 + \frac{q-1}{q}E\Big|\sum_{j=0}^{q-1} Y(\bar{I}, j, \rho_\pi(\bar{K}, j))\Big|^2 \\
&= \frac{1}{q}ES_1^2 + \frac{q-1}{q}ES_3^2 \\
&= O(\frac{1}{q}).
\end{aligned}
\tag{3.34}
$$

From (3.34) and Lemma 3.4(3), we have

$$
\begin{aligned}
E\big[\widetilde{Z}_\delta - Z_\delta\big]^2 \\
&= E\Big[\hat{z}_\delta[(\bar{I}, \bar{K}), (\rho_\pi(\bar{I}, \cdot), \rho_\pi(\bar{K}, \cdot))] + \hat{z}_\delta[(\bar{L}, \bar{M}), (\rho_\pi(\bar{L}, \cdot), \rho_\pi(\bar{M}, \cdot))] \\
&\quad - \hat{z}_\delta[(\bar{I}, \bar{K}), (\rho_\pi(\bar{L}, \cdot), \rho_\pi(\bar{M}, \cdot))] - \hat{z}_\delta[(\bar{L}, \bar{M}), (\rho_\pi(\bar{I}, \cdot), \rho_\pi(\bar{K}, \cdot))]\Big]^2 \\
&\leq 8E\Big[\big(z_\delta[(\bar{I}, \bar{K}), (\rho_\pi(\bar{L}, \cdot), \rho_\pi(\bar{M}, \cdot))]\big)^2 + E\big(z_\delta[(\bar{I}, \bar{K}), (\rho_\pi(\bar{I}, \cdot), \rho_\pi(\bar{K}, \cdot))]\big)^2\Big] \\
&\leq 8\delta^2\Big\{E\Big|\sum_{j=0}^{q-1} Y(\bar{I}, j, \rho_\pi(\bar{L}, j)) + Y(\bar{K}, j, \rho_\pi(\bar{M}, j)) \\
&\qquad - Y(\bar{I}, j, \rho_\pi(\bar{M}, j)) - Y(\bar{K}, j, \rho_\pi(\bar{L}, j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1} Y(\bar{I}, j, \rho_\pi(\bar{I}, j)) + Y(\bar{K}, j, \rho_\pi(\bar{K}, j)) \\
&\qquad - Y(\bar{I}, j, \rho_\pi(\bar{K}, j)) - Y(\bar{K}, j, \rho_\pi(\bar{I}, j))\Big|^2\Big\} \\
&\leq 8\delta^2\Big\{16E\Big|\sum_{j=0}^{q-1} Y(\bar{I}, j, \rho_\pi(\bar{L}, j))\Big|^2 + E|\widetilde{W} - W|^2\Big\} \\
&= \delta^2 O(\frac{1}{q}).
\end{aligned}
\tag{3.35}
$$

Let $\mathcal{B}$ be the $\sigma-$algebra generated by π_i's and $U_{i_1,i_2,i_3,j}$'s. Hence

$$\begin{aligned}
&E^{\mathcal{B}}\big[\widetilde{Z}_\delta - Z_\delta\big]\\
&= E^{\mathcal{B}}\Big[- \hat{z}_\delta[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))] - \hat{z}_\delta[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\\
&\qquad + \hat{z}_\delta[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] + \hat{z}_\delta[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{q(q-1)[q(q-1)-1]}\sum_{i\neq k}\sum_{\substack{l\neq m\\(l,m)\neq(i,k)}} E^{\mathcal{B}}\hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\\
&\quad - \frac{2}{q(q-1)}\sum_{i\neq k} E^{\mathcal{B}}\hat{z}_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\\
&= \frac{2}{q(q-1)[q(q-1)-1]}\sum_{i\neq k}\sum_{\substack{l\neq m\\(l,m)\neq(i,k)}} \hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\\
&\quad - \frac{2}{q(q-1)}\sum_{i\neq k} \hat{z}_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\\
&= \frac{2}{q(q-1)[q(q-1)-1]}\sum_{i\neq k}\sum_{l\neq m} \hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\\
&\quad - \frac{2}{q(q-1)-1}\sum_{i\neq k} \hat{z}_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]\\
&= \frac{2}{q(q-1)[q(q-1)-1]}\sum_{i\neq k}\sum_{l\neq m} \hat{z}_\delta[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - \frac{2Z_\delta}{q(q-1)-1}. \qquad (3.36)
\end{aligned}$$

From this fact and the exchangeability of $\big(Z_\delta,\widetilde{Z}_\delta\big)$, we can use the same argument of (3.10) with $g(x)=x$ to show that

$$\begin{aligned}
EZ_\delta^2 =& \frac{q(q-1)}{4}E\big(\widetilde{Z}_\delta - Z_\delta\big)^2\\
&+ \frac{1}{q(q-1)}\sum_{i\neq k}\sum_{l\neq m} E\hat{z}_\delta[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Z_\delta. \qquad (3.37)
\end{aligned}$$

Hence, by (3.35), (3.37) and Lemma 3.7(2),

$$\begin{aligned}
EZ_\delta^2 &\leq \frac{q(q-1)}{4}(\delta^2 O(\frac{1}{q})) + \frac{1}{q(q-1)}(\delta^2 O(q^4))\\
&\leq \delta^2 O(q^2).
\end{aligned}$$

□

Now, we are ready to prove the uniform concentration inequality stated in Theorem 3.2.

3.2 Proof of Theorem 3.2

Proof.
Let $\delta = \max(\frac{q}{4}E|\widetilde{W} - W|^3,\ \frac{1}{\sqrt{q}})$ and G_δ defined as in Lemma 3.8.
First, we note from Lemma 3.4(3) that $\delta \le O(\frac{1}{\sqrt{q}})$
and

$$E\big|\widetilde{W} - W\big| \min(\delta, \big|\widetilde{W} - W\big|) = \frac{1}{q(q-1)} EG_\delta.$$

Using the same argument of (3.10) with $g(x) = x$, we have

$$\frac{q}{4}E\big(\widetilde{W} - W\big)^2 = EW^2 + \Delta W.$$

Hence, from this fact, (3.7) and $EW^2 = 1$,

$$\frac{q}{4}E\big(\widetilde{W} - W\big)^2 \ge 1 - \frac{1}{q-1}. \tag{3.38}$$

By (3.38) and the fact that $\min(a, b) \ge b - \frac{b^2}{4a}$ for any $a, b > 0$, we have

$$\begin{aligned}
EG_\delta &= q(q-1)E\big|\widetilde{W} - W\big| \min(\delta, \big|\widetilde{W} - W\big|) \\
&\ge q(q-1)E\big(\widetilde{W} - W\big)^2 - \frac{q(q-1)}{4\delta}E\big|\widetilde{W} - W\big|^3 \\
&\ge 4(q-1)\{1 - \frac{1}{q-1}\} - (q-1) \\
&= 3q - 7.
\end{aligned} \tag{3.39}$$

Define function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} -\frac{1}{2}(b-a) - \delta & \text{if} \quad t < a - \delta, \\ -\frac{1}{2}(b+a) + t & \text{if} \quad a - \delta \le t \le b + \delta, \\ \frac{1}{2}(b-a) + \delta & \text{if} \quad b + \delta < t. \end{cases} \tag{3.40}$$

We observe that

$$\begin{aligned}
&E\int_{-\infty}^{\infty} f'(W+t)M(t)dt \\
&\geq E\mathbb{I}(a \leq W \leq b)\int_{|t|\leq\delta} M(t)dt \\
&= \frac{q}{4}E\mathbb{I}(a \leq W \leq b)|\widetilde{W}-W|\min(\delta,|\widetilde{W}-W|) \\
&= \frac{1}{4(q-1)}E\mathbb{I}(a \leq W \leq b)G_\delta \\
&\geq \frac{1}{4(q-1)}E\mathbb{I}(a \leq W \leq b)G_\delta\mathbb{I}(G_\delta > q-1) \\
&\geq \frac{1}{4}E\mathbb{I}(a \leq W \leq b)\mathbb{I}(G_\delta > q-1) \\
&= \frac{1}{4}E\{\mathbb{I}(a \leq W \leq b) - \mathbb{I}(a \leq W \leq b, G_\delta \leq q-1)\} \\
&\geq \frac{1}{4}P(a \leq W \leq b) - \frac{1}{4}P(G_\delta \leq q-1).
\end{aligned}$$

Hence

$$P(a \leq W \leq b) \leq 4E\int_{-\infty}^{\infty} f'(W+t)M(t)dt + P(G_\delta \leq q-1). \tag{3.41}$$

Using the same argument of (3.6), we have

$$EWf(W) = E\int_{-\infty}^{\infty} f'(W+t)M(t)dt - \Delta f(W) \tag{3.42}$$

and

$$\left|\Delta f(W)\right| \leq \frac{1}{q-1}\Big[Ef^2(W)\Big]^{\frac{1}{2}}.$$

Thus by (3.41), (3.42), Lemma 3.8 and the fact that $\delta \leq O(\frac{1}{\sqrt{q}})$, we have

$$\begin{aligned}
&P(a \leq W \leq b) \\
&\leq 4EWf(W) + P(G_\delta \leq q-1) + \frac{4}{q-1}\{Ef^2(W)\}^{\frac{1}{2}} \\
&\leq 4\Big\{\frac{1}{2}(b-a)+\delta\Big\} + P(EG_\delta - G_\delta \geq 3q-7-q+1) + \frac{4}{q-1}\{Ef^2(W)\}^{\frac{1}{2}} \\
&= 2(b-a) + 4\delta + \frac{VarG_\delta}{(2q-6)^2} + \frac{4}{q-1}\Big\{\Big(\frac{1}{2}(b-a)+\delta\Big)^2\Big\}^{\frac{1}{2}} \\
&\leq 2(b-a) + 4\delta + \frac{O(q)}{(2q-6)^2} + \frac{4}{q-1}\Big(\frac{1}{2}(b-a)+\delta\Big)
\end{aligned}$$

$$
\begin{aligned}
&= 2(b-a)\Big(1+\frac{1}{q-1}\Big) + 4\delta + \frac{4\delta}{q-1} + O(\frac{1}{q}) \\
&\leq 2(b-a)\Big(1+\frac{1}{q-1}\Big) + O(\frac{1}{\sqrt{q}}).
\end{aligned}
$$

□

CHAPTER IV
A NON-UNIFORM CONCENTRATION INEQUALITY FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS

In this chapter, we give a non-uniform concentration inequality for orthogonal array of W defined in (3.5). Now, we state the main result of this chapter as follows.

Theorem 4.1. *Assume that $E(f \circ X)^4 < \infty$. Then there exists a constant C such that*

$$P(z \leq W \leq w) \leq \frac{C}{1+z}(w-z) + \frac{1}{1+z} O(\frac{1}{\sqrt{q}}), \ textas \ q \to \infty,$$

for any real number $0 < z \leq w$.

From Theorem 4.1 we have the following theorem which is a non-uniform concentration inequality for W.

Theorem 4.2. *Assume that $E(f \circ X)^4 < \infty$. Then there exists a some constant C, we have*

$$P(z \leq W \leq z + \lambda) \leq \frac{C\lambda}{1+z} + \frac{1}{1+z} O(\frac{1}{\sqrt{q}}), \ as \ q \to \infty,$$

for any real number $z, \ \lambda \geq 0$.

4.1 Auxiliary Results

With the same notation as in chapter 3, for each i, j and $k \in \{1, 2, ..., q\}$, and $z \geq 0$, we let

$$Y_z(i,j,k) = Y(i,j,k)\mathbb{I}\big(\big|Y(i,j,k)\big| > 1+z\big),$$

$$\widehat{Y}_z(i,j,k) = Y(i,j,k)\mathbb{I}\big(\big|Y(i,j,k)\big| \leq 1+z\big),$$

$$\widehat{Y} = \sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j))$$

and

$$\widetilde{Y} = \widehat{Y} - \widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}$$

where $$\widehat{S}_{1,z}=\sum_{j=0}^{q-1}\widehat{Y}_z(I,j,\rho_\pi(I,j)),\quad \widehat{S}_{2,z}=\sum_{j=0}^{q-1}\widehat{Y}_z(K,j,\rho_\pi(K,j)),$$
$$\widehat{S}_{3,z}=\sum_{j=0}^{q-1}\widehat{Y}_z(I,j,\rho_\pi(K,j)),\quad \widehat{S}_{4,z}=\sum_{j=0}^{q-1}\widehat{Y}_z(K,j,\rho_\pi(I,j)).$$

From Lemma 3.4(2), we note that

$$\begin{aligned}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}E|Y(i,j,k)^mY_z(i,j,k)^n| &\le \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\frac{E|Y(i,j,k)|^{m+n+t}}{(1+z)^t}\\ &\le \frac{O(q^{3-m-n-t})}{(1+z)^t}\end{aligned}\tag{4.1}$$

for any integers m,n and t which $m\ge 0, n,t>0$ and $m+n+t$ is an even number. The following lammas are useful tools for proving Theorem 4.1 .

Lemma 4.3. *Let $g:\mathbb{R}\to\mathbb{R}$ be a measurable function. Then*

1. $$E\widehat{Y}g(\widehat{Y})=\frac{q-1}{4}E(\widetilde{Y}-\widehat{Y})(g(\widetilde{Y})-g(\widehat{Y}))+\widetilde{\Delta}g(\widehat{Y})$$
where
$$\widetilde{\Delta}g(\widehat{Y})=\frac{1}{q}Eg(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k).\tag{4.2}$$

2. If g is a continuous and piecewise continuously differrentiable function, then

$$E\widehat{Y}g(\widehat{Y})=E\int_{-\infty}^{\infty}g'(\widehat{Y}+t)K(t)dt+\widetilde{\Delta}g(\widehat{Y})$$

where

$$K(t)=\frac{q-1}{4}\Big(\widetilde{Y}-Y\Big)\Big(\mathbb{I}(0\le t\le\widetilde{Y}-Y)-\mathbb{I}(\widetilde{Y}-Y\le t<0)\Big).$$

3. $|\widetilde{\Delta}\widehat{Y}|\le O(\frac{1}{q})$ *where* $\widetilde{\Delta}\widehat{Y}$ *is defined in (4.2) for* $g(x)=x$.

Proof. 1. Let $\mathcal{B}$ be σ-algebra generated by π_i's and $U_{i_1,i_2,i_3,j}$'s. Note that

$$\begin{aligned}
&E^{\mathcal{B}}\big[\widetilde{Y}-\widehat{Y}\big]\\
&=E^{\mathcal{B}}\Big[-\widehat{S}_{1,z}-\widehat{S}_{2,z}+\widehat{S}_{3,z}+\widehat{S}_{4,z}\Big]\\
&=-\frac{2}{q}\sum_{i=0}^{q-1}E^{\mathcal{B}}\Big[\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))\Big]+\frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{\substack{k=0\\k\neq i}}^{q-1}E^{\mathcal{B}}\Big[\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j))\Big]\\
&=-\frac{2}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))+\frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{\substack{k=0\\k\neq i}}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j))\\
&=(-\frac{2}{q}-\frac{2}{q(q-1)})\widehat{Y}+\frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j))\\
&=-\frac{2}{q-1}\widehat{Y}+\frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)
\end{aligned}$$

and we can show that $(\widetilde{Y},\widehat{Y})$ is an exchangeable pair by using the same technique for proving the exchangeability of $(Z_\delta,\widetilde{Z}_\delta)$ in Lemma 3.7. From these facts and the fact that

$$F(w,\tilde{w})=(\tilde{w}-w)(g(\tilde{w})+g(w)),$$

is anti-symmetric, we have

$$\begin{aligned}
0&=E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})+g(\widehat{Y})\big)\\
&=E\big(\widetilde{Y}-\widehat{Y}\big)(2g(\widehat{Y}))+E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&=2Eg(\widehat{Y})E^{\mathcal{B}}\big[\widetilde{Y}-\widehat{Y}\big]+E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&=2Eg(\widehat{Y})\{-\frac{2}{q-1}\widehat{Y}+\frac{2}{q(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\}\\
&\quad+E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&=-\frac{4}{q-1}E\widehat{Y}g(\widehat{Y})+\frac{4}{q(q-1)}Eg(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\\
&\quad+E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big).
\end{aligned}$$

Then

$$\begin{aligned}
E\widehat{Y}g(\widehat{Y})&=\frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)+\frac{1}{q}Eg(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\\
&=\frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)+\widetilde{\Delta}g(\widehat{Y}).
\end{aligned}$$

2. Follows directly from 1 and the fact that

$$\begin{aligned}
&\frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\big(g(\widetilde{Y})-g(\widehat{Y})\big)\\
&\qquad=\frac{q-1}{4}E\big(\widetilde{Y}-\widehat{Y}\big)\int_0^{\widetilde{Y}-\widehat{Y}} g'(\widehat{Y}+t)dt\\
&\qquad=E\int_{-\infty}^{\infty} g'(\widehat{Y}+t)K(t)dt+\widetilde{\Delta}g\big(\widehat{Y}\big).
\end{aligned}$$

3. Note that

$$\begin{aligned}
\widetilde{\Delta}\widehat{Y} &= \frac{1}{q}E\widehat{Y}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\\
&= \frac{1}{q^2}E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\Big)\Big(\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}\widehat{Y}_z(l,m,n)\Big)\\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}E\big\{Y(i,j,k)Y(l,m,n)-Y(i,j,k)Y_z(l,m,n)\\
&\quad -Y(l,m,n)Y_z(i,j,k)+Y_z(i,j,k)Y_z(l,m,n)\big\}\\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}E\big\{Y^2(i,j,k)-2Y(i,j,k)Y_z(i,j,k)+Y_z^2(i,j,k)\big\}\\
&\quad +\frac{1}{q^2}E\sum_{i,j,k}\sum_{\substack{l,m,n\\(l,m,n)\neq(i,j,k)}}\big\{Y(i,j,k)Y(l,m,n)-2Y(i,j,k)Y_z(l,m,n)\\
&\quad +Y_z(i,j,k)Y_z(l,m,n)\big\}\\
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^2(i,j,k)-\frac{2}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY(i,j,k)Y_z(i,j,k)\\
&\quad +\frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY_z^2(i,j,k)\\
&\quad +\frac{1}{q^2}\sum_{i,j,k}\sum_{\substack{l,m,n\\(l,m,n)\neq(i,j,k)}}\Big\{\tilde{\mu}(i,j,k)\tilde{\mu}(l,m,n)-2\tilde{\mu}(i,j,k)EY_z(l,m,n)\\
&\quad +EY_z(i,j,k)Y_z(l,m,n)\Big\}
\end{aligned}$$

$$= A_1 + A_2 + A_3 + A_4 + A_5 \tag{4.3}$$

where

$$A_1 = \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k)$$

$$A_2 = -\frac{2}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY(i,j,k)Y_z(i,j,k)$$

$$A_3 = \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1} EY_z(i,j,k)Y_z(l,m,n)$$

$$A_4 = \frac{1}{q^2}\sum_{\substack{i,j,k \\ (i,j,k)\neq}}\sum_{\substack{l,m,n \\ (l,m,n)}} \tilde{\mu}(i,j,k)\tilde{\mu}(l,m,n)$$

and

$$A_5 = -\frac{2}{q^2}\sum_{i,j,k}\sum_{\substack{l,m,n \\ (l,m,n)\neq(i,j,k)}} \tilde{\mu}(i,j,k)EY_z(l,m,n).$$

From Lemma 3.4(1), we have

$$\begin{aligned} A_1 &= \frac{1}{q}\Big(\frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k)\Big) \\ &= \frac{1}{q}(1 + O(\frac{1}{q})) \\ &= O(\frac{1}{q}). \end{aligned} \tag{4.4}$$

By (4.1),

$$\begin{aligned} |A_2| &= \frac{2}{q^2}\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY(i,j,k)Y_z(i,j,k)\Big| \\ &\leq \frac{2}{q^2(1+z)^2}O(q^{3-1-1-2}) \\ &= \frac{1}{(1+z)^2}O(\frac{1}{q^3}) \end{aligned} \tag{4.5}$$

and

$$
\begin{aligned}
|A_3| &= \frac{1}{q^2}\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY_z(i,j,k)\Big)^2 \\
&\leq q\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY_z^2(i,j,k) \\
&\leq q\cdot\frac{1}{(1+z)^4}O(q^{3-6}) \\
&= \frac{1}{(1+z)^4}O(\frac{1}{q^2}).
\end{aligned} \tag{4.6}
$$

Now, we consider $|A_4|$ and $|A_5|$, by (3.3) and Lemma 3.4(2),

$$
\begin{aligned}
|A_4| &= \frac{1}{q^2}\Big|\sum_{i,j,k}\sum_{\substack{l,m,n\\(l,m,n)\neq(i,j,k)}} \tilde{\mu}(i,j,k)\tilde{\mu}(l,m,n)\Big| \\
&= \frac{1}{q^2}\Big|-\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\tilde{\mu}^2(i,j,k)\Big| \\
&\leq \frac{1}{q^2}\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k)\Big| \\
&= O(\frac{1}{q}),
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
|A_5| &= \frac{2}{q^2}\Big|\sum_{i,j,k}\sum_{\substack{l,m,n\\(l,m,n)\neq(i,j,k)}} \tilde{\mu}(i,j,k)EY_z(l,m,n)\Big| \\
&= \frac{2}{q^2}\Big|-\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}\tilde{\mu}(l,m,n)EY_z(l,m,n)\Big| \\
&\leq \frac{1}{q^2}\Big|\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1}\tilde{\mu}^2(l,m,n) + \sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1} EY_z^2(l,m,n)\Big| \\
&\leq \frac{2}{q^2}\Big|\sum_{l=0}^{q-1}\sum_{m=0}^{q-1}\sum_{n=0}^{q-1} EY^2(l,m,n)\Big| \\
&= O(\frac{1}{q}).
\end{aligned} \tag{4.8}
$$

Then 3 follows from (4.3)-(4.8).

□

Lemma 4.4.

1. If $E(f\circ X)^2<\infty$*, then* $\big|E\widehat{Y}^2-1\big|\le O(\frac{1}{q})$ *as* $q\to\infty$*.*

2. If $E(f\circ X)^r<\infty$*, then, for any even number,* $r\ge 2$ $E|\widetilde{Y}-\widehat{Y}|^r\le O(\frac{1}{q^{\frac{r}{2}}})$ *as* $q\to\infty$*.*

3. If $E(f\circ X)^{r+1}<\infty$*, then, for any odd number,* $r\ge 3$ $E|\widetilde{Y}-\widehat{Y}|^r\le O(\frac{1}{q^{\frac{r}{2}}})$ *as* $q\to\infty$*.*

Proof. Let

$$S_{1,z}=\sum_{j=0}^{q-1}Y_z(I,j,\rho_\pi(I,j)) \quad , \quad S_{2,z}=\sum_{j=0}^{q-1}Y_z(K,j,\rho_\pi(K,j))$$
$$S_{3,z}=\sum_{j=0}^{q-1}Y_z(I,j,\rho_\pi(K,j)) \quad , \quad S_{4,z}=\sum_{j=0}^{q-1}Y_z(K,j,\rho_\pi(I,j)).$$

From Lemma 3.3(1), we know that

$$S_1,S_2,S_3 \text{ and } S_4$$

are identically distributed and we can use the same argument to show that

$$S_{1,z},S_{2,z},S_{3,z} \text{ and } S_{4,z}$$

are identically distributed. Note that

$$\begin{aligned}
ES_1^2 &= E\Big(\sum_{j=0}^{q-1}Y(I,j,\rho_\pi(I,j))\Big)^2\\
&= \frac{1}{q}\sum_{i=0}^{q-1}E\Big(\sum_{j=0}^{q-1}Y(i,j,\rho_\pi(i,j))\Big)^2\\
&= \frac{1}{q}\sum_{i=0}^{q-1}E\Big(\sum_{j=0}^{q-1}Y^2(i,j,\rho_\pi(i,j))+\sum_{j_1=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}EY(i,j_1,\rho_\pi(i,j_1))Y(i,j_2,\rho_\pi(i,j_2))\Big)\\
&= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}EY^2(i,j,\rho_\pi(i,j))\\
&\quad +\frac{1}{q}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}EY(i,j_1,\rho_\pi(i,j_1))Y(i,j_2,\rho_\pi(i,j_2))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q^2}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k) \\
&\quad + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}\sum_{k_1=0}^{q-1}\sum_{\substack{k_2=0\\ k_2\neq k_1}}^{q-1} EY(i,j_1,k_1)Y(i,j_2,k_2) \\
&= \frac{1}{q}(1+O(\frac{1}{q})) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{k_1=0}^{q-1}\tilde{\mu}(i,j_1,k_1)\Big(\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}\sum_{\substack{k_2=0\\ k_2\neq k_1}}^{q-1}\tilde{\mu}(i,j_2,k_2)\Big) \\
&= \frac{1}{q}(1+O(\frac{1}{q})) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j_1=0}^{q-1}\sum_{k_1=0}^{q-1}\tilde{\mu}(i,j_1,k_1)\Big(-\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{q-1}\tilde{\mu}(i,j_2,k_1)\Big) \\
&= \frac{1}{q}(1+O(\frac{1}{q})) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\tilde{\mu}^2(i,j,k) \\
&= \frac{1}{q}(1+O(\frac{1}{q})) + \frac{1}{q^2(q-1)}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^2(i,j,k) \\
&= \frac{1}{q}(1+O(\frac{1}{q})) + O(\frac{1}{q^2}) \\
&= \frac{1}{q} + O(\frac{1}{q^2}).
\end{aligned}
$$

Note that, for each $r, s \in \mathbb{N}$ such that $r+s$ is an even number

$$
\begin{aligned}
E|S_{1,z}|^r &= E\Big|\sum_{j=0}^{q-1} Y_z(I,j,\rho_\pi(I,j))\Big|^r \\
&\le q^{r-1}\sum_{j=0}^{q-1} E|Y_z^r(I,j,\rho_\pi(I,j))| \\
&= q^{r-3}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E|Y_z^r(i,j,k)| \\
&= q^{r-3}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{EY^{r+s}(i,j,k)}{(1+z)^s} \\
&\le \frac{1}{(1+z)^s} O(\frac{1}{q^s}).
\end{aligned} \tag{4.9}
$$

Hence

$$
ES_1^2 = ES_2^2 = ES_3^2 = ES_4^2 = \frac{1}{q} + O(\frac{1}{q^2}) \tag{4.10}
$$

and from (4.9), we choose $r = 2$ and $s = 4$

$$ES_{1,z}^2 = ES_{2,z}^2 = ES_{3,z}^2 = ES_{4,z}^2 \leq \frac{1}{(1+z)^4} O(\frac{1}{q^4}). \tag{4.11}$$

From (4.10) and (4.11) ,

$$\begin{aligned} E|S_1 S_{1,z}| &\leq \Big\{ES_1^2\Big\}^{\frac{1}{2}} \Big\{ES_{1,z}^2\Big\}^{\frac{1}{2}} \\ &= O(\frac{1}{\sqrt{q}}) \Big\{\frac{1}{(1+z)^4} O(\frac{1}{q^4})\Big\}^{\frac{1}{2}} \\ &= \frac{1}{(1+z)^2} O(\frac{1}{q^2\sqrt{q}}). \end{aligned} \tag{4.12}$$

By using the same argument as in (4.12) and the fact that $S_{1,z}, S_{2,z}, S_{3,z}$ and $S_{4,z}$ have the same distribution, we can conclude that

$$E|S_i S_{j,z}| \leq \frac{1}{(1+z)^2} O(\frac{1}{q^2\sqrt{q}}) \tag{4.13}$$

for $i, j = 1, 2, 3, 4$.

Next, we will bound ES_iS_j for $1 \leq i < j \leq 4$.

Let $\mathcal{A}$ be the $\sigma-$algebra generated by

$$\{(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})), U_{\pi_1(a_{i,1}),\pi_2(a_{i,2}),\pi_3(a_{i,3}),j} : 1 \leq i \leq q^2, 1 \leq j \leq 3\}.$$

From Loh([7]): 1220-1221, we have $E\big|E^{\mathcal{A}} S_i S_j\big| = O(\frac{1}{q^2})$ for $1 \leq i < j \leq 4$. Thus

$$\begin{aligned} |ES_iS_j| &= \big|EE^{\mathcal{A}} S_i S_j\big| \\ &\leq E\big|E^{\mathcal{A}} S_i S_j\big| \\ &= O(\frac{1}{q^2}). \end{aligned} \tag{4.14}$$

Note from Lemma 4.3(1) that

$$E\widehat{Y}^2 = \frac{q-1}{4} E\big(\widetilde{Y} - \widehat{Y}\big)^2 + \widetilde{\Delta}\widehat{Y} \tag{4.15}$$

and by (4.10) we have

$$\begin{aligned} E\big(\widetilde{Y} - \widehat{Y}\big)^2 &= E\big(\widehat{S}_{1,z} + \widehat{S}_{2,z} - \widehat{S}_{3,z} - \widehat{S}_{4,z}\big)^2 \\ &= E\big(S_1 + S_2 - S_3 - S_4 - (S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z})\big)^2 \end{aligned}$$

$$\begin{aligned}
&= E\big(S_1 + S_2 - S_3 - S_4\big)^2 \\
&\quad - 2E\big(S_1 + S_2 - S_3 - S_4\big)\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big) \\
&\quad + E\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big)^2 \\
&= \sum_{k=1}^{4} ES_k^2 + 2E\{S_1S_2 - S_1S_3 - S_1S_4 - S_2S_3 - S_2S_4 + S_3S_4\} \\
&\quad - 2E\big(S_1 + S_2 - S_3 - S_4\big)\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big) \\
&\quad + E\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big)^2 \\
&= \frac{4}{q} + O(\frac{1}{q^2}) + 2E\{S_1S_2 - S_1S_3 - S_1S_4 - S_2S_3 - S_2S_4 + S_3S_4\} \\
&\quad - 2E\big(S_1 + S_2 - S_3 - S_4\big)\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big) \\
&\quad + E\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big)^2. \qquad (4.16)
\end{aligned}$$

Hence, (4.15) and (4.16),

$$\begin{aligned}
E\widehat{Y}^2 &= (1 - \frac{1}{q}) + O(\frac{1}{q}) \\
&\quad + \frac{q-1}{2}E\{S_1S_2 - S_1S_3 - S_1S_4 - S_2S_3 - S_2S_4 + S_3S_4\} \\
&\quad - \frac{q-1}{2}E\big(S_1 + S_2 - S_3 - S_4\big)\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big) \\
&\quad + \frac{q-1}{4}E\big(S_{1,z} + S_{2,z} - S_{3,z} - S_{4,z}\big)^2 + \widetilde{\Delta}\widehat{Y}, \qquad (4.17)
\end{aligned}$$

which implies that

$$\begin{aligned}
|E\widehat{Y}^2 - 1| &\le O(\frac{1}{q}) + q\Big(\sum_{1\le i<j\le 4} E|S_iS_j| + \sum_{i=1}^{4}\sum_{j=1}^{4} E|S_iS_{j,z}| + \sum_{i=1}^{4} ES_{i,z}^2\Big) + \widetilde{\Delta}\widehat{Y} \\
&= O(\frac{1}{q}) + q\Big(\sum_{1\le i<j\le 4} E|S_iS_j| + \sum_{i=1}^{4}\sum_{j=1}^{4} E|S_iS_{j,z}| + \sum_{i=1}^{4} ES_{i,z}^2\Big) \qquad (4.18)
\end{aligned}$$

where we have used Lemma 4.3(3) in the last equality.
We obtain 1 from (4.11), (4.13)-(4.15).

2. From (4.9), we choose $r = s$, we have $E|S_{1,z}|^r \le \frac{1}{(1+z)^r}O(\frac{1}{q^r})$.
Since $S_{1,z}, S_{2,z}, S_{3,z}$ and $S_{4,z}$ are the same distribution, we have

$$E|S_{1,z}|^r = E|S_{2,z}|^r = E|S_{3,z}|^r = E|S_{4,z}|^r \le \frac{1}{(1+z)^r}O(\frac{1}{q^r}). \qquad (4.19)$$

Hence 2 follows from Lemma 3.4(3), (4.19) and the fact that

$$
\begin{aligned}
&E|\widehat{Y}-\widetilde{Y}|^r \\
&= E|\widehat{S}_{1,z}-\widehat{S}_{2,z}+\widehat{S}_{3,z}+\widehat{S}_{4,z}|^r \\
&= E|S_1+S_2-S_3-S_4-(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z})|^r \\
&= E|(\widetilde{W}-W)-(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z})|^r \\
&\le C\{E|\widetilde{W}-W|^r+E|S_{1,z}|^r+E|S_{2,z}|^r+E|S_{3,z}|^r+E|S_{4,z}|^3\}
\end{aligned}
$$

for some constant C.

3 can be shown by using the same argument as in 2. $\square$

Lemma 4.5. *Assume that $E(f\circ X)^4<\infty$. Let $\gamma = max(\frac{q}{4}E|\widehat{Y}-\widetilde{Y}|^3,\ \frac{1}{\sqrt{q}})$ and*

$$
U_\gamma = \sum_{i\neq k}\Big|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i,j,\rho_\pi(i,j))+\widehat{Y}_z(k,j,\rho_\pi(k,j))-\widehat{Y}_z(i,j,\rho_\pi(k,j))-\widehat{Y}_z(k,j,\rho_\pi(i,j))\}\Big|
$$

$$
\times \min\Big(\gamma,\sum_{i\neq k}\Big|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i,j,\rho_\pi(i,j))+\widehat{Y}_z(k,j,\rho_\pi(k,j))-\widehat{Y}_z(i,j,\rho_\pi(k,j))-\widehat{Y}_z(k,j,\rho_\pi(i,j))\}\Big|\Big).
$$

If $(1+z)\gamma<1$ then as $q\to\infty$,

1. $EU_\gamma \ge 3q+O(1)$,
2. $Var(U_\gamma)\le \frac{1}{1+z}O(q\sqrt{q})$.

Proof. First, from Lemma 4.4(3), we note that $\gamma \le O(\frac{1}{\sqrt{q}})$.

1. From (4.16), we have

$$
E\big(\widetilde{Y}-\widehat{Y}\big)^2 = \frac{4}{q}+O(\frac{1}{q^2})+EM_1 \tag{4.20}
$$

where

$$
\begin{aligned}
M_1 =& 2\{S_1S_2-S_1S_3-S_1S_4-S_2S_3-S_2S_4+S_3S_4\} \\
&-2\big(S_1+S_2-S_3-S_4\big)\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big) \\
&+E\big(S_{1,z}+S_{2,z}-S_{3,z}-S_{4,z}\big)^2.
\end{aligned}
$$

By (4.11),(4.13) and (4.14), we have

$$
q(q-1)|EM_1| \le O(1). \tag{4.21}
$$

Hence, by (4.20), (4.21) and the fact that $\min(a,b) \geq b - \dfrac{b^2}{4a}$ for any $a, b > 0$,

$$
\begin{aligned}
EU_\gamma &= q(q-1)E(\widehat{Y} - \widetilde{Y})\min(\gamma, \big|\widehat{Y} - \widetilde{Y}\big|) \\
&\geq q(q-1)E(\widehat{Y} - \widetilde{Y})^2 - \frac{q(q-1)}{4\gamma}E\big|\widehat{Y} - \widetilde{Y}\big|^3 \\
&= 4q - 4 + O(1) + q(q-1)|EM_1| - q + 1 \\
&= 3q + O(1).
\end{aligned}
$$

2. For each $i, k \in \{0, 1, ..., q-1\}, \beta, \alpha \in S^{(q)}$ we let

$$
\begin{aligned}
s_\gamma[(i,k),(\beta,\alpha)] = \Big|\sum_{j=0}^{q-1}\big\{\widehat{Y}_z(i,j,\beta(j)) + \widehat{Y}_z(k,j,\alpha(j)) - \widehat{Y}_z(i,j,\alpha(j)) - \widehat{Y}_z(k,j,\beta(j))\big\}\Big| \\
\times \min\Big(\gamma, \Big|\sum_{j=0}^{q-1}\big\{\widehat{Y}_z(i,j,\beta(j)) + \widehat{Y}_z(k,j,\alpha(j)) - \widehat{Y}_z(i,j,\alpha(j)) - \widehat{Y}_z(k,j,\beta(j))\big\}\Big|\Big)
\end{aligned}
$$

$$
\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] = s[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Es[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]
$$

$$
T_\gamma = \sum_{i \neq k} \hat{s}[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]
$$

and

$$
\begin{aligned}
\widetilde{T}_\gamma = T_\gamma &- \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))] \\
&- \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] \\
&+ \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] \\
&+ \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))].
\end{aligned}
$$

First, we note that $\big(T_\gamma, \widetilde{T}_\gamma\big)$ is an exchangeable pair and $Var(U_\gamma) = ET_\gamma^2$.

By the same argument of (3.37), we have

$$
ET_\gamma^2 = \frac{q(q-1)}{4}E(\widetilde{T}_\gamma - T_\gamma)^2 + \frac{1}{q(q-1)}\sum_{i\neq k}\sum_{l \neq m} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]T_\gamma. \quad (4.22)
$$

$$
\begin{aligned}
&E(\widetilde{T}_\gamma - T_\gamma)^2 \\
&= E\Big\{\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))] + \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] \\
&\quad - \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))] - \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\Big\}^2 \\
&\le C\Big\{E\Big|\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\Big|^2 + E\Big|\hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\Big|^2\Big\} \\
&\le C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{L},j)) + \widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{M},j)) \\
&\quad - \widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{M},j)) - \widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{L},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{I},j)) + \widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{K},j)) \\
&\quad - \widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{K},j)) - \widehat{Y}_z(\bar{K},j,\rho_\pi(\bar{I},j))\Big|^2\Big\} \\
&\le C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}\widehat{Y}_z(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2\Big\} \\
&= C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1}Y(\bar{I},j,\rho_\pi(\bar{L},j)) - \sum_{j=0}^{q-1}Y_z(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}Y(\bar{I},j,\rho_\pi(\bar{I},j)) - \sum_{j=0}^{q-1}Y_z(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}Y(\bar{I},j,\rho_\pi(\bar{K},j)) - \sum_{j=0}^{q-1}Y_z(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2\Big\} \\
&= C\gamma^2\Big\{E\Big|\sum_{j=0}^{q-1}Y(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2 + E|\sum_{j=0}^{q-1}Y_z(\bar{I},j,\rho_\pi(\bar{L},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}Y(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 + E\Big|\sum_{j=0}^{q-1}Y_z(\bar{I},j,\rho_\pi(\bar{I},j))\Big|^2 \\
&\quad + E\Big|\sum_{j=0}^{q-1}Y(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2 + E\Big|\sum_{j=0}^{q-1}Y_z(\bar{I},j,\rho_\pi(\bar{K},j))\Big|^2\Big\}
\end{aligned}
$$

$$\leq C\gamma^2\big(2ES_3^2 + ES_1^2 + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(L,j))|^4}{(1+z)^2}$$

$$+ q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(I,j))|^4}{(1+z)^2} + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(K,j))|^4}{(1+z)^2}\big) \tag{4.23}$$

Hence, by (4.23), we have

$$\begin{aligned}
E(\widetilde{T}_\gamma - T_\gamma)^2 &\leq C\gamma^2\Big\{O(\frac{1}{q}) + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(L,j))|^4}{(1+z)^2} \\
&\quad + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(I,j))|^4}{(1+z)^2} \\
&\quad + q\sum_{j=0}^{q-1}\frac{E|Y(I,j,\rho_\pi(K,j))|^4}{(1+z)^2}\Big\} \\
&\leq C\gamma^2\Big\{O(\frac{1}{q}) + \frac{q}{q^2}\sum_{i,j,k}\frac{E|Y(i,j,k)|^4}{(1+z)^2}\Big\} \\
&= \gamma^2\Big\{O(\frac{1}{q}) + \frac{q^{-1}}{(1+z)^2}O(q^{3-4})\Big\} \\
&= \gamma^2\Big\{O(\frac{1}{q}) + \frac{1}{(1+z)^2}O(\frac{1}{q^2})\Big\}.
\end{aligned}$$

From this fact, $(1+z)\gamma < 1$, (4.22) and if we can show that

$$\sum_{i\neq k}\sum_{l\neq m} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot)),\rho_\pi(m,\cdot)]T_\gamma \leq \gamma^2 O(q^4), \tag{4.24}$$

then

$$\begin{aligned}
Var(U_\gamma) &= ET_\gamma^2 \\
&\leq \frac{q(q-1)}{4}\Big\{C\gamma^2 O(\frac{1}{q}) + \frac{C\gamma^2}{(1+z)^2}O(\frac{1}{q^2})\Big\} \\
&\quad + \frac{1}{q(q-1)}\Big\{\gamma^2 O(q^4)\Big\} \\
&\leq \frac{\gamma}{1+z}O(q) + \frac{\gamma}{(1+z)^3}O(1) + \frac{\gamma}{1+z}O(q^2) \\
&\leq \frac{\gamma}{1+z}O(q^2).
\end{aligned}$$

To prove (4.24), we use the same idea of Lemma 3.7(2).

We note that

$$
\begin{aligned}
& E\Big[\sum_{i\neq k}\sum_{l\neq m}\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]T_\gamma\Big] \\
&= E\big\{\sum_{i\neq k}\sum_{l\neq m}\sum_{u\neq v}\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\} \\
&= \sum_{A} E\big\{\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\}, \qquad (4.25)
\end{aligned}
$$

where $A = \Big\{(i,k,l,m,u,v)\big|\ \ i,k,l,m,u,v,r,s \in \{0,1,...,q-1\}$ and $i\neq k,\ l\neq m,\ u\neq v\Big\}$

$$
= \bigcup_{i=1}^{7} A_i
$$

and

$$
\begin{aligned}
A_1 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u=l,\ u\neq m,\ v\neq l,\ v=m\big\} \\
A_2 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u\neq l,\ u=m,\ v=l,\ v\neq m\big\} \\
A_3 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u\neq l,\ u\neq m,\ v\neq l,\ v=m\big\} \\
A_4 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u\neq l,\ u=m,\ v\neq l,\ v\neq m\big\} \\
A_5 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u\neq l,\ u\neq m,\ v=l,\ v\neq m\big\} \\
A_6 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u=l,\ u\neq m,\ v\neq l,\ v\neq m\big\} \\
A_7 &= \big\{(i,k,l,m,u,v)\in A\big|\ \ u\neq l,\ u\neq m,\ v\neq l,\ v\neq m\big\}.
\end{aligned}
$$

Using the same argument (3.12), we have

$$
\sum_{A_1} E\big\{\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\} \leq \mathcal{R}_{11}+\mathcal{R}_{12} \qquad (4.26)
$$

where

$$
\begin{aligned}
\mathcal{R}_{11} &= \sum_{i\neq k}\sum_{l\neq m} E\Big\{s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\Big\} \\
\mathcal{R}_{12} &= \sum_{i\neq k}\sum_{l\neq m} Es_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Es_\gamma[(l,m),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))].
\end{aligned}
$$

By the same argument of (3.13) and using Lemma 3.4(2) and Lemma 4.4(2), we have

$$\begin{aligned}
\mathcal{R}_{11} &\le 8\gamma^2(q-1)^2\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}E\Big\{\Big|\sum_{i_2=0}^{q-1}\widehat{Y}_z(i,i_2,\rho_\pi(l,i_2))\Big\}^2 \\
&\quad + \frac{\gamma^2q^2(q-1)^2}{2}E(\widetilde{Y}-\hat{Y})^2 \\
&\le 8\gamma^2(q-1)^2\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}E\big\{\sum_{i_2=0}^{q-1}Y(i,i_2,\rho_\pi(l,i_2)) - \sum_{i_2=0}^{q-1}Y_z(i,i_2,\rho_\pi(l,i_2))\big\}^2 \\
&\quad + \frac{\gamma^2q^2(q-1)^2}{2}O(\frac{1}{q}) \\
&\le 8\gamma^2(q-1)^2\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}E\big\{\sum_{i_2=0}^{q-1}Y(i,i_2,\rho_\pi(l,i_2))\big\}^2 \\
&\quad + \gamma^2(q-1)^2\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}E\big\{\sum_{i_2=0}^{q-1}Y_z(i,i_2,\rho_\pi(l,i_2))\big\}^2 + \gamma^2O(q^3) \\
&\le 8\gamma^2(q-1)^2q^2E\big\{\sum_{i_2=0}^{q-1}Y(\bar{I},i_2,\rho_\pi(\bar{L},i_2))\big\}^2 \\
&\quad + \gamma^2(q-1)^2q\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}\sum_{i_2=0}^{q-1}EY_z^2(i,i_2,\rho_\pi(l,i_2)) + \gamma^2O(q^3) \\
&\le \gamma^2(q-1)^2q^2)O(\frac{1}{q}) + \gamma^2(q-1)^2q\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}\sum_{i_2=0}^{q-1}EY_z^2(i,i_2,l) + \gamma^2O(q^3) \\
&\le \gamma^2O(q^3) + \gamma^2(q-1)^2q\sum_{i=0}^{q-1}\sum_{l=0}^{q-1}\sum_{i_2=0}^{q-1}\frac{Y^4(i,i_2,l)}{(1+z)^2} \\
&\le \gamma^2O(q^3). \qquad (4.27)
\end{aligned}$$

By the same argument of (3.15), we have

$$\mathcal{R}_{12} = \gamma^2O(q^3). \qquad (4.28)$$

Hence, by (4.26)-(4.28), it implies that

$$\sum_{A_1}E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2O(q^3). \qquad (4.29)$$

Similar to A_1, we can conclude that

$$\sum_{A_2}E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2O(q^3). \qquad (4.30)$$

Next, we consider sum on A_3.

Using the same argument (3.18), we have

$$\sum_{A_3} E\big\{\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]\big\} \le \mathcal{R}_{31}+\mathcal{R}_{32} \qquad (4.31)$$

where

$$\mathcal{R}_{31} = \sum_{i\neq k}\sum_{l\neq m}\sum_{\substack{u=0\\ u\neq l,m}}^{q-1} E\Big\{s_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]\Big\}$$

$$\begin{aligned}\mathcal{R}_{32} &= (q-2)\sum_{i\neq k}\sum_{u\neq m} Es_\gamma[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]Es_\gamma[(u,m),(\rho_\pi(u,\cdot),\rho_\pi(m,\cdot))]\\ &\le (q-2)\mathcal{R}_{12}\\ &\le \gamma^2 O(q^4) \qquad (4.32)\end{aligned}$$

where we have used (4.28) in the last inequality.

Using the same argument of (3.20) and (4.27), Lemma 3.4(2) and Lemma 4.4(2), we have

$$\begin{aligned}\mathcal{R}_{31} &\le 8\gamma^2(q-1)^2(q-2)\sum_{i=0}^{q-1}\sum_{l=0}^{q-1} E\Big\{\big|\sum_{i_2=0}^{q-1}\widehat{Y}_z(i,i_2,\rho_\pi(l,i_2))\Big\}^2\\ &\quad + 4\gamma^2q^2(q-1)^2(q-2)E(\widetilde{Y}-\hat{Y})^2\\ &\le \gamma^2 O(q^4). \qquad (4.33)\end{aligned}$$

Hence, by (4.31)-(4.33), it implies that

$$\sum_{A_3} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4). \qquad (4.34)$$

Similar to A_3, we can conclude that

$$\sum_{A_4} Es_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4), \qquad (4.35)$$

$$\sum_{A_5} Es_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4), \qquad (4.36)$$

$$\sum_{A_6} Es_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4). \qquad (4.37)$$

For the last sumation

$$\sum_{A_7} Es_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]s_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))],$$

we note that, by the same argument of (3.27),

$$\sum_{A_7} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))]$$
$$= \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5 + \mathcal{B}_6.$$

where

$$\mathcal{B}_1 \le 2|T_2|\mathcal{R}_{12}, \quad \mathcal{B}_2 \le 2|T_2|\mathcal{R}_{12}, \quad \mathcal{B}_3 \le 2q!|T_2|\mathcal{R}_{12}$$
$$\mathcal{B}_4 \le 2q!|T_2|\mathcal{R}_{12}, \quad \mathcal{B}_5 \le 2q!|T_2|\mathcal{R}_{12}, \quad \mathcal{B}_6 \le 2q!|T_2|\mathcal{R}_{12}$$

Hence, by (3.28),

$$\sum_{A_7} E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]\hat{s}_\gamma[(u,v),(\rho_\pi(u,\cdot),\rho_\pi(v,\cdot))] \le \gamma^2 O(q^4). \tag{4.38}$$

From (4.25), (4.29), (4.30) and (4.34)-(4.38), the lamma is proved. □

4.2 Proof of Theorem 4.1

Proof. First, we note that

$$P(z \le W \le w) \le P(W \ne \widehat{Y}) + P(z \le \widehat{Y} \le w) \tag{4.39}$$

and by Lemma 3.4(2),

$$\begin{aligned}
P(W \ne \widehat{Y}) &= P\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z) \ge 1\Big) \\
&\le E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z)\Big) \\
&= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E\mathbb{I}(|Y(i,j,k)| > 1+z) \\
&\le \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{E|Y(i,j,k)|^4}{(1+z)^4} \\
&\le \frac{1}{q}\frac{1}{(1+z)^4} O(q^{3-4}) \\
&= \frac{1}{(1+z)^4} O(\frac{1}{q^2}).
\end{aligned} \tag{4.40}$$

Thus it remains to bound the term $P(z \leq \widehat{Y} \leq w)$.

Let γ be defined as in Lemma 4.5.

Case 1 $(1+z)\gamma \geq 1$.

By Lemma 4.2(1) and the fact that $\gamma \geq \dfrac{1}{1+z}$.

$$\begin{aligned}P(z \leq \widehat{Y}) &= P(1+z \leq 1+\widehat{Y})\\ &\leq \frac{E|\widehat{Y}+1|^2}{(1+z)^2}\\ &\leq C\frac{E|\widehat{Y}|^2}{(1+z)^2} + \frac{C}{(1+z)^2}\\ &\leq \frac{C}{(1+z)^2}\\ &\leq \frac{C\gamma}{1+z}.\end{aligned}$$

Thus $P(z \leq \widehat{Y} \leq w) \leq P(z \leq \widehat{Y}) \leq \dfrac{C\gamma}{1+z}$.

Hence, by $\gamma \leq O(\dfrac{1}{\sqrt{q}})$, we have

$$P(z \leq W \leq w) \leq \frac{1}{(1+z)} O(\frac{1}{\sqrt{q}}).$$

Case 2 $(1+z)\gamma < 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{if} \quad t < a-\gamma,\\ (1+t+\gamma)(t-z+\gamma) & \text{if} \quad z-\gamma \leq t \leq w+\gamma,\\ (1+t+\gamma)(w-z+2\gamma) & \text{if} \quad t > w+\gamma.\end{cases} \tag{4.41}$$

Then f is a non decreasing function satisfying

$$f'(t) \geq \begin{cases} (1+z) & \text{for} \quad z-\gamma < t < w+\gamma,\\ 0 & \text{otherwise .}\end{cases}$$

By the same argument of Lemma 4.3(2),

$$E\widehat{Y}f(\widehat{Y}) = E\int_{-\infty}^{\infty} f'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}f(\widehat{Y}). \tag{4.42}$$

We observe that

$$
\begin{aligned}
& E\int_{-\infty}^{\infty} f'(\widehat{Y}+t)K(t)dt \\
& \geq (1+z)E\mathbb{I}(z \leq \widehat{Y} \leq w)\int_{|t|\leq\gamma} K(t)dt \qquad (4.43) \\
& = \frac{(q-1)(1+z)}{4}E\mathbb{I}(z \leq \widehat{Y} \leq w)|\widehat{Y}-\widetilde{Y}|\min(\gamma, |\widehat{Y}-\widetilde{Y}|) \\
& = \frac{(q-1)(1+z)}{4q(q-1)}E\mathbb{I}(z \leq \widehat{Y} \leq w)U_\gamma \\
& \qquad\qquad \text{where } U_\gamma \text{ was defined in Lemma 4.5.} \\
& \geq \frac{(1+z)}{4q}E\mathbb{I}(z \leq \widehat{Y} \leq w)U_\gamma\mathbb{I}(U_\gamma \geq q) \\
& \geq \frac{(1+z)}{4}E\mathbb{I}(z \leq \widehat{Y} \leq w)\mathbb{I}(U_\gamma \geq q) \\
& = \frac{(1+z)}{4}E\big\{E\mathbb{I}(z \leq \widehat{Y} \leq w) - \mathbb{I}(z \leq \widehat{Y} \leq w, U_\gamma \leq q)\big\} \\
& \geq \frac{(1+z)}{4}\big\{P(z \leq \widehat{Y} \leq w) - P(U_\gamma \leq q)\big\}.
\end{aligned}
$$

By this fact, (4.42), and Lemma 4.5(1), we have

$$
\begin{aligned}
P(z \leq \widehat{Y} \leq w) &\leq \frac{4}{(1+z)}E\widehat{Y}f(\widehat{Y}) - \frac{4}{(1+z)}\widetilde{\Delta}f(\widehat{Y}) + P(U_\gamma \leq q) \\
&\leq \frac{4}{(1+z)}(w-z+2\gamma)E|\widehat{Y}||1+\gamma+\widehat{Y}| \\
&\quad + \frac{4}{(1+z)}|\widetilde{\Delta}f(\widehat{Y})| + P(EU_\gamma - U_\gamma \geq 3q + O(1) - q) \\
&\leq \frac{4}{(1+z)}(w-z+2\gamma)\Big\{E|\widehat{Y}| + E|\widehat{Y}|^2\Big\} \\
&\quad + \frac{4}{(1+z)}|\widetilde{\Delta}f(\widehat{Y})| + \frac{C}{q^2}E(U_\gamma - EU_\gamma)^2. \qquad (4.44)
\end{aligned}
$$

By Lemma 4.4(1), Lemma 4.5(2), (4.44) and the fact that $\gamma \leq O(\frac{1}{\sqrt{q}})$,

$$
P(z \leq \widehat{Y} \leq w) \leq \frac{C}{(1+z)}(w-z) + \frac{4}{(1+z)}|\widetilde{\Delta}f(\widehat{Y})|. \qquad (4.45)
$$

Note that, by (4.1) and Lemma 4.4(1),

$$
\begin{aligned}
|\widetilde{\Delta} f(\widehat{Y})| &= \Big|\frac{1}{q} E f(\widehat{Y}) \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\le \frac{1}{q}(w-z+2\gamma) E\Big|(1+\gamma+\widehat{Y}) \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\le \frac{1}{q}(w-z+2\gamma) E\Big|(1+\gamma) \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\quad + \frac{1}{q}(w-z+2\gamma) E\Big|\widehat{Y} \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\le (w-z+2\gamma)(1+\gamma)\frac{1}{q}\sum_{k=0}^{q-1} E\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\quad + \frac{1}{q}(w-z+2\gamma) E\Big|\widehat{Y}\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k) - \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\Big)\Big| \\
&\le (w-z+2\gamma)(1+\gamma) E\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j))\Big| \\
&\quad + \frac{1}{q}(w-z+2\gamma) E\Big|\widehat{Y} \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k)\Big| \\
&\quad + \frac{1}{q}(w-z+2\gamma) E\Big|\widehat{Y} \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\Big| \\
&\le (w-z+2\gamma)(1+\gamma) E|\widehat{Y}| \\
&\quad + \frac{1}{q}(w-z+2\gamma)\{E\widehat{Y}^2\}^{\frac{1}{2}}\{E(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k))^2\}^{\frac{1}{2}} \\
&\quad + \frac{1}{q}(w-z+2\gamma)\{E\widehat{Y}^2\}^{\frac{1}{2}}\{E(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k))^2\}^{\frac{1}{2}} \\
&\le (w-z+2\gamma)(1+\gamma)\{E\widehat{Y}^2\}^{\frac{1}{2}} \\
&\quad + \frac{1}{q}(w-z+2\gamma)\{1\}^{\frac{1}{2}}\Big\{\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^2(i,j,k) \\
&\quad + E\Big(\sum_{i_1,j_1,k_1}\ \sum_{\substack{i_2,j_2,k_2 \\ (i_1,j_1,k_1)\neq(i_2,j_2,k_2)}} Y(i_1,j_1,k_1)Y(i_2,j_2,k_2)\Big)\Big\}^{\frac{1}{2}} \\
&\quad + \frac{1}{q}(w-z+2\gamma)\{1\}^{\frac{1}{2}}\{q^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY_z^2(i,j,k)\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (w-z+2\gamma)(1+\gamma) \\
&\quad + \frac{1}{q}(w-z+2\gamma)\big\{O(q) \\
&\quad + \sum_{i_1,j_1,k_1} \sum_{\substack{i_2,j_2,k_2 \\ (i_1,j_1,k_1)\neq(i_2,j_2,k_2)}} \tilde{\mu}(i_1,j_1,k_1)\tilde{\mu}(i_2,j_2,k_2)\big\}^{\frac{1}{2}} \\
&\quad + \frac{1}{q}(w-z+2\gamma)\{q^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{EY^4(i,j,k)}{(1+z)^2}\}^{\frac{1}{2}} \\
&\leq (w-z+2\gamma)(1+\gamma) \\
&\quad + \frac{1}{q}(w-z+2\gamma)\big\{O(q) + \sum_{i_1,j_1,k_1} \tilde{\mu}^2(i_1,j_1,k_1)\big\}^{\frac{1}{2}} \\
&\quad + \frac{(w-z+2\gamma)}{q(1+z)^2}\{q^3(q^{3-4})\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq (w-z+2\gamma)(1+\gamma) \\
&\quad + \frac{1}{q}(w-z+2\gamma)\big\{O(q) + \sum_{i_1,j_1,k_1} EY^2(i_1,j_1,k_1)\big\}^{\frac{1}{2}} \\
&\quad + \frac{(w-z+2\gamma)}{q(1+z)^2}\{q^2\}^{\frac{1}{2}} \\
&\leq (w-z+2\gamma)(1+\gamma) + \frac{1}{q}(w-z+2\gamma)\{O(q)\}^{\frac{1}{2}} \\
&\quad + \frac{(w-z+2\gamma)}{(1+z)^2}O(1) \\
&\leq (w-z+2\gamma)O(1).
\end{aligned}
$$

From this fact and (4.45), we can conclude that

$$P(z \leq \widehat{Y} \leq w) \leq \frac{C}{(1+z)}(w-z) + \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}).$$

□

CHAPTER V
A UNIFORM BOUND FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS

In this chapter, we use the same notation as in the previous chapter. In 1996, Loh([7]) gave a uniform bound on the normal approximation of

$$W = \sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1,i_2)\big)$$

and the following is his result.

Theorem 5.1. *Suppose that $E(f\circ X)^r < \infty$ for some even integer $r \geq 4$. Then*

$$\sup\left\{\left|P(W \leq z) - \Phi(z)\right| : -\infty < z < \infty\right\} = O(q^{\frac{2-r}{2r-2}}) \quad \textit{as} \;\; q \to \infty.$$

In this chapter, we improve a bound in Theorem 5.1 in case of $r \geq 6$. Here is our main result.

Theorem 5.2. *Suppose that $E(f\circ X)^6 < \infty$. Then*

$$\sup\left\{\left|P(W \leq z) - \Phi(z)\right| : -\infty < z < \infty\right\} = O(q^{-\frac{1}{2}}) \quad \textit{as} \;\; q \to \infty.$$

We will prove our main result by using Stein's method. His method starts on the differential equation

$$g'(w) - wg(w) = h(w) - Eh(\mathcal{N}) \tag{5.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and piecewise continuously differentiable function, $h : \mathbb{R} \to \mathbb{R}$ is a test function and $\mathcal{N}$ is the standard normal random variable.

For any real number z, let h be an indicator function defined by

$$h_z(x) = \mathbb{I}(x \leq z) \tag{5.2}$$

then Stein's equation (5.1) for h_z has a unique solution $g_z : \mathbb{R} \to \mathbb{R}$ defined in (2.3). Thus, we have

$$g_z'(W) - Wg_z(W) = h_z(W) - \Phi(z)$$

which implies that

$$P(W \le z) - \Phi(z) = Eg_z'(W) - EWg_z(W)$$

where Φ denote the distribution function of the standard normal distribution. Thus, it suffices to bound

$$Eg_z'(W) - EWg_z(W)$$

instead of

$$P(W \le z) - \Phi(z).$$

By the same argument of (3.6), it is easy to show that

$$EWg_z(W) = E\int_{-\infty}^{\infty} g_z'(W+t)M(t)dt - \Delta g_z(W) \tag{5.3}$$

and

$$\begin{aligned} \left|\Delta g_z(W)\right| &\le \frac{1}{q-1}\{Eg_z^2(W)\}^{\frac{1}{2}} \\ &= O(\frac{1}{q-1}) \end{aligned} \tag{5.4}$$

where we have used Proposition 2.9(4) in the last equality.
Thus, from (5.3), (5.4) and Propositon 2.9(2),

$$\begin{aligned} \left|P(W \le z) - \Phi(z)\right| &= \left|Eg_z'(W) - EWg_z(W)\right| \\ &\le \left|E\int_{-\infty}^{\infty}\{g_z'(W) - g_z'(W+t)\}M(t)dt\right| \\ &\quad + \left|Eg_z'(W)E\int_{-\infty}^{\infty}M(t)dt - Eg_z'(W)\int_{-\infty}^{\infty}M(t)dt\right| \\ &\quad + \left|Eg_z'(W)\right|\left|1 - E\int_{-\infty}^{\infty}M(t)dt\right| + \left|\Delta g_z(W)\right| \\ &\le \left|E\int_{-\infty}^{\infty}\{g_z'(W) - g_z'(W+t)\}M(t)dt\right| \\ &\quad + \left|Eg_z'(W)E\int_{-\infty}^{\infty}M(t)dt - Eg_z'(W)\int_{-\infty}^{\infty}M(t)dt\right| \\ &\quad + \left|1 - E\int_{-\infty}^{\infty}M(t)dt\right| + O(\frac{1}{q-1}). \end{aligned} \tag{5.5}$$

From Loh([7]) we know that

$$\big|Eg_z'(W)\big|\Big|1 - E\int_{-\infty}^{\infty} M(t)dt\Big| = O(\frac{1}{q}), \tag{5.6}$$

and

$$\big|Eg_z'(W)E\int_{-\infty}^{\infty} M(t)dt - Eg_z'(W)\int_{-\infty}^{\infty} M(t)dt\big| = O(\frac{1}{\sqrt{q}}). \tag{5.7}$$

Hence, by (5.5)-(5.7)

$$\big|P(W \le z) - \Phi(z)\big| \le O(\frac{1}{\sqrt{q}}) + \big|E\int_{-\infty}^{\infty}\{g_z'(W) - g_z'(W+t)\}M(t)dt\big|. \tag{5.8}$$

From Lemma 3.4(3), we note that

$$E\int_{-\infty}^{\infty}|t|M(t)dt = \frac{q}{8}E|\widetilde{W} - W|^3 \le \frac{q}{8}\{E|\widetilde{W} - W|^4\}^{\frac{3}{4}} = O(\frac{1}{\sqrt{q}}). \tag{5.9}$$

If we can show that

$$E\int_{-\infty}^{\infty}\mathbb{I}(z - t \le W \le z)M(t)dt \le O(\frac{1}{\sqrt{q}})$$

then, by Proposition 2.9(5), Lemma 3.4(3) and (5.9),

$$\begin{aligned}
&\Big|E\int_{\mathbb{R}}\{g_z'(W) - g_z'(W+t)\}M(t)dt\Big| \\
&\le E\int_{\substack{W\le z\\ W+t>z}} M(t)dt + E\int_{t\le 0}(|W| + \frac{\sqrt{2\pi}}{4})(0 + |t|)M(t)dt \\
&\le E\int_{t>0}\mathbb{I}(z - t \le W \le z)M(t)dt + E\int_{t\le 0}|W||t|M(t)dt + \frac{\sqrt{2\pi}}{4}E\int_{t\le 0}|t|M(t)dt \\
&\le O(\frac{1}{\sqrt{q}}) + \frac{q}{4}E|W|\frac{|\widetilde{W} - W|^3}{2} + O(\frac{1}{\sqrt{q}}) \\
&= O(\frac{1}{\sqrt{q}}) + \frac{q}{8}(EW^2)^{\frac{1}{2}}(E|\widetilde{W} - W|^6)^{\frac{1}{2}} + O(\frac{1}{\sqrt{q}}) \qquad (5.10) \\
&\le O(\frac{1}{\sqrt{q}}) + \frac{q}{8}(O(q^{-3}))^{\frac{1}{2}} \\
&= O(\frac{1}{\sqrt{q}}). \qquad (5.11)
\end{aligned}$$

Hence, by (5.8) and (5.11), we have

$$\big|P(W \le z) - \Phi(z)\big| \le O(\frac{1}{\sqrt{q}}).$$

Now, it remains to show that

$$E\int_{-\infty}^{\infty}\mathbb{I}(z-t\le W\le z)M(t)dt \le O(\frac{1}{\sqrt{q}}).$$

For each $\delta \ge 0$ and $a,\ b\in\mathbb{R}$ which is $a<b$, define a function f_δ by

$$f_\delta(t)=\begin{cases} -\frac{1}{2}(b-a)-\delta & \text{if} \quad t<a-\delta, \\ -\frac{1}{2}(b+a)+t & \text{if} \quad a-\delta\le t\le b+\delta, \\ \frac{1}{2}(b-a)+\delta & \text{if} \quad b+\delta<t. \end{cases}$$

It is easy to see that, for every $t\in\mathbb{R}$,

$$|f_\delta(t)| \le \frac{1}{2}(b-a)+\delta \tag{5.12}$$

and

$$\begin{aligned} E&\int_{|t|\le\delta}\mathbb{I}(a\le W\le b)M(t)dt \\ &\le E\int_{-\infty}^{\infty} f'_\delta(W+t)M(t)dt \\ &= EWf_\delta(W)+\Delta f_\delta(W). \end{aligned} \tag{5.13}$$

where we have used the same argument of (3.6) in the last equality. So, by the same argument of (3.7),(5.12), (5.13) and the fact that $M(t)=0$ for $|t|>|\widetilde{W}-W|$, we have

$$\begin{aligned} E&\int_{-\infty}^{\infty}\mathbb{I}(z-t\le W\le z)M(t)dt \\ &= E\int_{|t|\le|\widetilde{W}-W|}\mathbb{I}(z-t\le W\le z)M(t)dt \\ &\le E\int_{|t|\le|\widetilde{W}-W|}\mathbb{I}(z-|\widetilde{W}-W|\le W\le z)M(t)dt \\ &\le EWf_{|\widetilde{W}-W|}(W)+\Delta f_{|\widetilde{W}-W|}(W) \\ &\le E|W|\big(\frac{1}{2}(z-z+|\widetilde{W}-W|)+|\widetilde{W}-W|\big)+\frac{1}{q-1}\{Ef^2_{|\widetilde{W}-W|}(W)\}^{\frac{1}{2}} \\ &\le CE|W||\widetilde{W}-W|+\frac{1}{q-1}\{E\big(\frac{1}{2}(z-z+|\widetilde{W}-W|)+|\widetilde{W}-W|\big)^2\}^{\frac{1}{2}} \\ &\le C(EW^2)^{\frac{1}{2}}(E|\widetilde{W}-W|^2)^{\frac{1}{2}}+\frac{C}{q}\big(E|\widetilde{W}-W|^2\big)^{\frac{1}{2}} \\ &= O(\frac{1}{\sqrt{q}}). \end{aligned}$$

where we have used Lemma 3.4(3) in the last equality. □

Remark : From (5.10), if we can show that

$$EW^4 < C$$

then

$$E|W||\widetilde{W} - W|^3 \leq \left(EW^4\right)^{\frac{1}{4}} \left(E|\widetilde{W} - W|^4\right)^{\frac{3}{4}} \leq O(\frac{1}{q^{\frac{3}{2}}}).$$

This implies that we can relax the assumption

$$E(f \circ X)^6 < \infty$$

in the main theorem to the condition

$$E(f \circ X)^4 < \infty.$$

□

CHAPTER VI
A NON-UNIFORM BOUND FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING DESIGNS

In this chapter, we use the same notation as in the previous chapter. Loh([7]) gave a uniform bound on the normal approximation of W in 1996 and the following is his result.

Theorem 6.1. *Suppose that $E(f \circ X)^r < \infty$ for some even integer $r \geq 4$. Then*

$$\sup\left\{\left|P(W \leq w) - \Phi(w)\right| : -\infty < w < \infty\right\} = O(q^{\frac{2-r}{2r-2}}) \quad as \;\; q \to \infty.$$

In chapter 5, we improve a uniform bound of Theorem 6.1 and our result is stated as follows:

Theorem 6.2. *Suppose that $E(f \circ X)^6 < \infty$. Then*

$$\sup\left\{\left|P(W \leq w) - \Phi(w)\right| : -\infty < w < \infty\right\} = O(q^{-\frac{1}{2}}) \quad as \;\; q \to \infty$$

In this chapter, we give a non-uniform bound for the approximation of W by Φ:

Theorem 6.3. *Suppose that $E(f \circ X)^r < \infty$ for even number $r \geq 8$. Then, as $q \to \infty$, for $z \in \mathbb{R}$,*

$$\left|P(W \leq z) - \Phi(z)\right| \leq \max\Big(\frac{1}{(1+|z|)^{1-\frac{2}{r}}} O(\frac{1}{q^{\frac{r-8}{2r}}}), \frac{1}{(1+|z|)^{\frac{11}{12}}} O(\frac{1}{q^{\frac{1}{6}}})\Big).$$

Note that : To bound $\left|P(W \leq z) - \Phi(z)\right|$, it suffices to consider $z > 0$ as we have used the fact that $\Phi(z) = 1 - \Phi(-z)$ and apply the result to $-W$ when $z \leq 0$. So, from now on, we assume $z > 0$.

To prove the main theorem of this chapter(Theorem 6.3), we need the following lemmas:

Lemma 6.4.

1. If $E(f\circ X)^2<\infty$, *then* $E\big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k)\big)^2\le O(q)$ *as* $q\to\infty$.

2. If $E(f\circ X)^4<\infty$, *then* $E\big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k)\big)^4\le O(q^2)$ *as* $q\to\infty$.

Proof.

1. From (3.3) and Lemma 3.4(2), we have

$$
\begin{aligned}
&E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3)\Big)^2\\
&\quad=\sum_{i_1,i_2,i_3}EY^2(i_1,i_2,i_3)+\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}EY(i_1,i_2,i_3)Y(j_1,j_2,j_3)\\
&\quad=\sum_{i_1,i_2,i_3}EY^2(i_1,i_2,i_3)+\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\\
&\quad=\sum_{i_1,i_2,i_3}EY^2(i_1,i_2,i_3)-\sum_{i_1,i_2,i_3}\tilde{\mu}^2(i_1,i_2,i_3)\\
&\quad\le O(q).
\end{aligned}
\tag{6.1}
$$

2. Note that

$$
E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3)\Big)^4=C_1M_1+C_2M_2+C_3M_3+C_4M_4+C_5M_5 \tag{6.2}
$$

where

$$
\begin{aligned}
M_1&=\sum_{i_1,i_2,i_3}EY^4(i_1,i_2,i_3),\\
M_2&=\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}EY(i_1,i_2,i_3)Y^3(j_1,j_2,j_3),
\end{aligned}
$$

$$M_3 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY^2(i_1,i_2,i_3)Y^2(j_1,j_2,j_3),$$

$$M_4 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3) \\ (k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} EY^2(i_1,i_2,i_3)Y(j_1,j_2,j_3)Y(k_1,k_2,k_3),$$

$$M_5 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{l_1,l_2,l_3 \\ (l_1,l_2,l_3)\neq(i_1,i_2,i_3) \\ (l_1,l_2,l_3)\neq(k_1,k_2,k_3)}}$$

$$EY(i_1,i_2,i_3)Y(j_1,j_2,j_3)Y(k_1,k_2,k_3)Y(l_1,l_2,l_3)$$

and C_1, C_2, C_3, C_4, C_5 are some constants.

From (3.3) and Lemma 3.4(2), we know that

$$M_1 = O(\frac{1}{q}), \tag{6.3}$$

$$\begin{aligned}
\big|M_2\big| &= \big| \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}(i_1,i_2,i_3)EY^3(j_1,j_2,j_3)\big| \\
&= \big| - \sum_{j_1,j_2,j_3} \tilde{\mu}(j_1,j_2,j_3)EY^3(j_1,j_2,j_3)\big| \\
&\leq \sum_{j_1,j_2,j_3} \big\{\tilde{\mu}^2(j_1,j_2,j_3) + EY^6(j_1,j_2,j_3)\big\} \\
&\leq \sum_{j_1,j_2,j_3} \big\{EY^2(j_1,j_2,j_3) + EY^6(j_1,j_2,j_3)\big\} \\
&\leq \big\{O(q) + O(\frac{1}{q^3})\big\} \\
&= O(q),
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
M_3 &= \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY^2(i_1,i_2,i_3)EY^2(j_1,j_2,j_3) \\
&\leq \big(\sum_{i_1,i_2,i_3} EY^2(i_1,i_2,i_3)\big)^2 \\
&\leq O(q^2),
\end{aligned} \tag{6.5}$$

$$M_4 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{l_1,l_2,l_3 \\ (l_1,l_2,l_3)\neq(i_1,i_2,i_3) \\ (l_1,l_2,l_3)\neq(j_1,j_2,j_3)}} EY^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(l_1,l_2,l_3)$$

$$= -\sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(i_1,i_2,i_3) - \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(j_1,j_2,j_3)$$

$$= \sum_{i_1,i_2,i_3} EY^2(i_1,i_2,i_3)\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(i_1,i_2,i_3) + \Big(\sum_{i_1,i_2,i_3} EY^2(i_1,i_2,i_3)\Big)\Big(\sum_{j_1,j_2,j_3} \tilde{\mu}^2(j_1,j_2,j_3)\Big)$$

$$\leq \sum_{i_1,i_2,i_3} \big\{EY^4(i_1,i_2,i_3) + \tilde{\mu}^4(i_1,i_2,i_3)\big\} + \Big(\sum_{i_1,i_2,i_3} EY^2(i_1,i_2,i_3)\Big)^2$$
$$\leq \sum_{i_1,i_2,i_3} \big\{EY^4(i_1,i_2,i_3) + EY^4(i_1,i_2,i_3)\big\} + \big(O(q)\big)^2$$
$$\leq O(\frac{1}{q}) + O(q^2)$$
$$\leq O(q^2), \tag{6.6}$$

and

$$\big|M_5\big| = \Big|\sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3) \\ (k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} \sum_{\substack{l_1,l_2,l_3 \\ (l_1,l_2,l_3)\neq(i_1,i_2,i_3) \\ (l_1,l_2,l_3)\neq(j_1,j_2,j_3) \\ (l_1,l_2,l_3)\neq(k_1,k_2,k_3)}} \tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(k_1,k_2,k_3)\tilde{\mu}(l_1,l_2,l_3)\Big|$$

$$= \Big| -3 \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3) \\ (k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} \tilde{\mu}^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(k_1,k_2,k_3)\Big|$$

$$= \Big|3 \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(i_1,i_2,i_3)$$

$$+ 3 \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(j_1,j_2,j_3)\Big|$$

$$= \Big| -3 \sum_{i_1,i_2,i_3} \tilde{\mu}^3(i_1,i_2,i_3)\tilde{\mu}(i_1,i_2,i_3)$$

$$+ 3\big(\sum_{i_1,i_2,i_3} \tilde{\mu}^2(i_1,i_2,i_3)\big)\big(\sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}^2(j_1,j_2,j_3)\big)\Big|$$

$$\leq 3 \sum_{i_1,i_2,i_3} \tilde{\mu}^4(i_1,i_2,i_3) + 3\big(\sum_{i_1,i_2,i_3} \tilde{\mu}^2(i_1,i_2,i_3)\big)^2$$

$$\leq 3 \sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) + 3\big(\sum_{i_1,i_2,i_3} EY^2(i_1,i_2,i_3)\big)^2$$

$$\leq O(q^{3-4}) + (O(q))^2$$

$$\leq O(q^2). \tag{6.7}$$

Hence from (6.2)-(6.7), we have 2.

□

Lemma 6.5. *Suppose that $E(f \circ X)^4 < \infty$. Then*

1. $E\big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\big)^2 \leq \frac{1}{(1+z)^2} O(\frac{1}{q})$ *as* $q \to \infty$.
2. $E\big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\big)^4 \leq \frac{1}{(1+z)^2} O(\frac{1}{q^2})$ *as* $q \to \infty$.

Proof. 1. follows from (4.1) that

$$E\big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y_z(i_1,i_2,i_3)\big)^2$$

$$= \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3) + \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z(i_1,i_2,i_3)Y_z(j_1,j_2,j_3)$$

$$\leq \frac{1}{(1+z)^2} \sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) + \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3)$$

$$\leq \frac{1}{(1+z)^2} O(\frac{1}{q}) + \big(\frac{1}{(1+z)^3} O(\frac{1}{q})\big)^2$$

$$\leq \frac{1}{(1+z)^2} O(\frac{1}{q}).$$

2. By the same argument as in (6.2) we see that

$$E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3)\Big)^4 = C_1M_{1,z}+C_2M_{2,z}+C_3M_{3,z}+C_4M_{4,z}+C_5M_{5,z} \quad (6.8)$$

where

$$\begin{aligned}
M_{1,z} &= \sum_{i_1,i_2,i_3} EY_z^4(i_1,i_2,i_3),\\
M_{2,z} &= \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z(i_1,i_2,i_3)Y_z^3(j_1,j_2,j_3),\\
M_{3,z} &= \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z^2(i_1,i_2,i_3)Y_z^2(j_1,j_2,j_3),\\
M_{4,z} &= \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\sum_{\substack{k_1,k_2,k_3\\(k_1,k_2,k_3)\neq(i_1,i_2,i_3)\\(k_1,k_2,k_3)\neq(j_1,j_2,j_3)}}\\
&\quad EY_z^2(i_1,i_2,i_3)Y_z(j_1,j_2,j_3)Y_z(k_1,k_2,k_3),\\
M_{5,z} &= \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\sum_{\substack{k_1,k_2,k_3\\(k_1,k_2,k_3)\neq(i_1,i_2,i_3)}}\sum_{\substack{l_1,l_2,l_3\\(l_1,l_2,l_3)\neq(i_1,i_2,i_3)\\(l_1,l_2,l_3)\neq(k_1,k_2,k_3)}}\\
&\quad EY_z(i_1,i_2,i_3)Y_z(j_1,j_2,j_3)Y_z(k_1,k_2,k_3)Y_z(l_1,l_2,l_3)
\end{aligned}$$

and C_1,C_2,C_3,C_4,C_5 are some positive constants. From (4.1),

$$M_{1,z} \le \frac{1}{(1+z)^2O(q^3)}, \quad (6.9)$$

$$\begin{aligned}
\big|M_{2,z}\big| &\le \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} E|Y_z(i_1,i_2,i_3)|E|Y_z^3(j_1,j_2,j_3)|\\
&\le \Big\{\sum_{i_1,i_2,i_3} E|Y_z(i_1,i_2,i_3)|\Big\}\Big\{\sum_{j_1,j_2,j_3} E|Y_z^3(j_1,j_2,j_3)|\Big\}\\
&\le \Big\{\frac{1}{(1+z)^3}O(\frac{1}{q})\Big\}\Big\{\frac{1}{1+z}O(\frac{1}{q})\Big\}\\
&= \frac{1}{(1+z)^4}O(\frac{1}{q^2}), \qquad (6.10)
\end{aligned}$$

$$
\begin{aligned}
M_{3,z} &= \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z^2(i_1,i_2,i_3)Y_z^2(j_1,j_2,j_3) \\
&\leq \big\{ \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3)\big\}^2 \\
&\leq \big\{ \frac{1}{(1+z)^2} O(\frac{1}{q})\big\}^2 \\
&= \frac{1}{(1+z)}^4 O(\frac{1}{q^2}),
\end{aligned} \tag{6.11}
$$

$$
\begin{aligned}
|M_{4,z}| &\leq \big\{ \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3)\big\}\big\{ \sum_{j_1,j_2,j_3} |EY_z(j_1,j_2,j_3)|\big\}\big\{ \sum_{l_1,l_2,l_3} |EY_z(l_1,l_2,l_3)|\big\} \\
&\leq \big\{ \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3\big\}\big\{ \sum_{j_1,j_2,j_3} E|Y_z(j_1,j_2,j_3)|\big\}\big\{ \sum_{l_1,l_2,l_3} E|Y_z(l_1,l_2,l_3)|\big\} \\
&\leq \big\{\frac{1}{(1+z)^2} O(\frac{1}{q})\big\}\big\{\frac{1}{(1+z)^3} O(\frac{1}{q})\big\}^2 \\
&= \frac{1}{(1+z)^5} O(\frac{1}{q^3}),
\end{aligned} \tag{6.12}
$$

and

$$
\begin{aligned}
|M_{5,z}| \leq &\big(\sum_{i_1,i_2,i_3} E|Y_z(i_1,i_2,i_3|)\big)\big(\sum_{j_1,j_2,j_3} E|Y_z(j_1,j_2,j_3)|\big) \\
&\big(\sum_{k_1,k_2,k_3} E|Y_z(k_1,k_2,k_3|)\big)\big(\sum_{l_1,l_2,l_3} E|Y_z(l_1,l_2,l_3)|\big) \\
\leq &\big\{\frac{1}{(1+z)^3} O(\frac{1}{q^4})\big\}^4.
\end{aligned} \tag{6.13}
$$

Hence from (6.8)-(6.13), we have 2.

□

Lemma 6.6. *For each $z \geq 0$, let $h : \mathbb{R} \to \mathbb{R}$ be defined by*

$$h(w) = (wg_z(w))' \tag{6.14}$$

If $E(f \circ X)^r < \infty$ for some positive even number $r \geq 2$. Then

$$E \int_{-\infty}^{\infty} \int_t^0 h(\widehat{Y} + u)K(t)dudt \leq \frac{1}{(1+z)^{1-\frac{2}{r}}} O(\frac{1}{\sqrt{q}}), \ \text{as } q \to \infty.$$

Proof. From Proposition 2.10 and the fact that

$$1-\Phi(z)\le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z} \quad \text{for} \ \ z\ge 0 \tag{6.15}$$

(see Barbour and Chen([15]): 23), we have

$$h(w)\le \begin{cases} C(1+z) & \text{if} \quad \frac{z}{2}\le w\le z, \\ \frac{C}{(1+z)^2} & \text{if} \quad w<\frac{z}{2} \ \text{ or } \ w>z. \end{cases} \tag{6.16}$$

From (6.16) and the fact that

$$E\int_{-\infty}^{\infty}\int_t^0 K(t)dudt \le E\int_{-\infty}^{\infty}|t|K(t)dt\le qE|\widetilde{Y}-\widehat{Y}|^3\le O(\frac{1}{\sqrt{q}}), \tag{6.17}$$

we have

$$\begin{aligned} &E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)dudt \\ &\quad = E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)\mathbb{I}(\widehat{Y}+u<\frac{z}{2} \ \text{ or } \ \widehat{Y}+u>z)dudt \\ &\qquad + E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)\mathbb{I}(\frac{z}{2}\le \widehat{Y}+u\le z)dudt \\ &\quad \le \frac{C}{(1+z)^2}O(\frac{1}{\sqrt{q}})+C(1+z)E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+u>\frac{z}{2})dudt. \end{aligned} \tag{6.18}$$

Note that, for $|t|>|\widehat{Y}-\widetilde{Y}|$, we see that

$$K(t)=0 \tag{6.19}$$

Hence, Lemma 4.4(1,2) and (6.19)

$$\begin{aligned} &(1+z)E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+u>\frac{z}{2})dudt \\ &\qquad \le (1+z)E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+4|\widetilde{Y}-\widehat{Y}|>\frac{z}{2})dudt \\ &\qquad \le q(1+z)E|\widetilde{Y}-\widehat{Y}|^3\mathbb{I}(\widehat{Y}+|\widetilde{Y}-\widehat{Y}|>\frac{z}{2}) \\ &\qquad \le q(1+z)\{E|\widetilde{Y}-\widehat{Y}|^{3r}\}^{\frac{1}{r}}\{P(\widehat{Y}+|\widetilde{Y}-\widehat{Y}|\ge\frac{z}{2})\}^{\frac{r-1}{r}} \\ &\qquad \le q(1+z)\{O(\frac{1}{q^{\frac{3r}{2}}})\}^{\frac{1}{r}}\{\frac{E(\widehat{Y}+|\widetilde{Y}-\widehat{Y}|)^2}{z^2}\}^{\frac{r-1}{r}} \\ &\qquad \le q(1+z)O(\frac{1}{q\sqrt{q}})\{\frac{E\widehat{Y}^2+|\widetilde{Y}-\widehat{Y}|^2}{z^2}\}^{\frac{r-1}{r}} \\ &\qquad \le (1+z)O(\frac{1}{\sqrt{q}})\{\frac{1+O(\frac{1}{q})}{z^2}\}^{\frac{r-1}{r}} \end{aligned}$$

$$\leq \frac{(1+z)}{z^{\frac{2r-2}{r}}} O(\frac{1}{\sqrt{q}})$$
$$\leq \frac{1}{z^{1-\frac{2}{r}}} O(\frac{1}{\sqrt{q}}).$$

□

Lemma 6.7. *Let $\mathcal{B}$ be the $\sigma-$algebra generated by π_1, π_2, π_3 and $U_{i_1,i_2,i_3;j}$'s. Then, for any $j, k = 1, 2, 3, 4$,*

1. $E\big|E^{\mathcal{B}}\{qS_k^2 - 1\}\big| = O(\frac{1}{\sqrt{q}})$ as $q \to \infty$.

2. $qE\big|E^{\mathcal{B}} S_j S_k\big| = O(\frac{1}{q})$ as $q \to \infty$.

Proof. To prove the lamma, we use the same idea from Loh([7]): 1218-1221.

1. From Cauchy-Schwarz inequality, we note that

$$\Big\{E\big|E^{\mathcal{B}}\{qS_1^2 - 1\}\big|\Big\}^2$$
$$= \Big\{E\big|E^{\mathcal{B}}\{q\big(\sum_{i_2=0}^{q-1} Y\big(I, i_2, \rho_\pi(I, i_2)\big)\big)^2 - 1\}\big|\Big\}^2$$
$$\leq E\Big\{\sum_{i_1=0}^{q-1}\big\{\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1, i_2)\big)\big\}^2\Big\}^2 - 2E\sum_{i_1=0}^{q-1}\big\{\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1, i_2)\big)\big\}^2 + 1. \tag{6.20}$$

From Loh([7]): 1218-1219, we have

$$E\sum_{i_1=0}^{q-1}\big\{\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1, i_2)\big)\big\}^2 = 1 + O(\frac{1}{q}), \tag{6.21}$$

and

$$E\Big\{\sum_{i_1=0}^{q-1}\big\{\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1, i_2)\big)\big\}^2\Big\}^2 = 1 + O(\frac{1}{q}) \tag{6.22}$$

as $q \to \infty$. Thus we conclude from (6.20)-(6.22)

$$E\big|E^{\mathcal{B}}\{qS_1^2 - 1\}\big| = O(\frac{1}{\sqrt{q}}). \tag{6.23}$$

By the symmetry of S_1 and S_2, we have

$$E\big|E^{\mathcal{B}}\{qS_2^2 - 1\}\big| = O(\frac{1}{\sqrt{q}}). \tag{6.24}$$

Next, we consider

$$
\begin{aligned}
&E\big|E^{\mathcal{B}}[qS_3^2-1]\big| \\
&= E\Big|E^{\mathcal{B}}\Big[q\big\{\sum_{i_2=0}^{q-1} Y\big(I,i_2,\rho_\pi(K,i_2)\big)\big\}^2-1\Big]\Big| \\
&= E\Big|E^{\mathcal{B}}\Big[\frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{\substack{j_1=0\\ j_1\neq i_1}}^{q-1}\sum_{i_2=0}^{q-1} Y^2\big(i_1,i_2,\rho_\pi(j_1,i_2)\big) \\
&\quad + \frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{\substack{j_1=0\\ j_1\neq i_1}}^{q-1}\sum_{i_2=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq i_2}}^{q-1} \tilde{\mu}\big(i_1,i_2,\rho_\pi(j_1,i_2)\big)\tilde{\mu}\big(i_1,j_2,\rho_\pi(j_1,j_2)\big)-1\Big]\Big| \\
&\leq E\big|S_{3;1}-1\big| + E\big|S_{3;2}\big| \\
&\leq \big\{E\big(S_{3;1}-1\big)^2\big\}^{\frac{1}{2}} + \big\{E\big(S_{3;2}\big)^2\big\}^{\frac{1}{2}}, \qquad (6.25)
\end{aligned}
$$

where

$$
\begin{aligned}
S_{3;1} &= \frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{\substack{j_1=0\\ j_1\neq i_1}}^{q-1}\sum_{i_2=0}^{q-1} Y^2\big(i_1,i_2,\rho_\pi(j_1,i_2),\\
S_{3;2} &= \frac{1}{q-1}\sum_{i_1=0}^{q-1}\sum_{\substack{j_1=0\\ j_1\neq i_1}}^{q-1}\sum_{i_2=0}^{q-1}\sum_{\substack{j_2=0\\ j_2\neq i_2}}^{q-1} \tilde{\mu}\big(i_1,i_2,\rho_\pi(j_1,i_2)\big)\tilde{\mu}\big(i_1,j_2,\rho_\pi(j_1,j_2)\big).
\end{aligned}
$$

By Loh([7]): 1219-1220,

$$E(S_{3;1}-1)^2 = O(\frac{1}{q}), \text{as } q\to\infty, \qquad (6.26)$$

and

$$E(S_{3;2})^2 = O(\frac{1}{q}), \text{as } q\to\infty. \qquad (6.27)$$

Thus we conclude from (6.25)-(6.27) that

$$E\big|E^{\mathcal{B}}[qS_3^2-1]\big| = O(\frac{1}{\sqrt{q}}), \text{as } q\to\infty. \qquad (6.28)$$

By the symmetry of S_3 and S_4, we have

$$E\big|E^{\mathcal{B}}[qS_4^2-1]\big| = O(\frac{1}{\sqrt{q}}), \text{as } q\to\infty. \qquad (6.29)$$

2. First, we observe that

$$
\begin{aligned}
&qE\big|E^{\mathcal{B}}(S_1S_2)\big| + qE\big|E^{\mathcal{B}}(S_3S_4)\big| \\
&= \frac{1}{q-1}E\Big|\sum_{i_1=0}^{q-1}\sum_{\substack{j_1=0\\ j_1\neq i_1}}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_2=0}^{q-1} Y\big(i_1,i_2,\rho_\pi(i_1,i_2)\big)Y\big(j_1,j_2,\rho_\pi(j_1,j_2)\big)\Big| \\
&\quad + \frac{1}{q-1}E\Big|E\Big(\sum_{i_1=0}^{q-1}\sum_{\substack{j_1=0\\ j_1\neq i_1}}^{q-1}\sum_{i_2=0}^{q-1}\sum_{j_2=0}^{q-1} Y\big(i_1,i_2,\rho_\pi(j_1,i_2)\big)Y\big(j_1,j_2,\rho_\pi(i_1,j_2)\big)\Big)\Big| \\
&= O(\frac{1}{q}), \text{as } q\to\infty,
\end{aligned}
\tag{6.30}
$$

where we have used Loh([7]): 1220-1221, in the last equality.
By the same argument of Loh([7]): 1221,

$$qE\big|E^{\mathcal{B}}(S_1S_3)\big| = O(\frac{1}{q}), \text{as } q\to\infty.$$

Thus it follows by symmetry that

$$qE\big|E^{\mathcal{B}}(S_1S_3)\big| + qE\big|E^{\mathcal{B}}(S_1S_4)\big| = O(\frac{1}{q}) \tag{6.31}$$

and

$$qE\big|E^{\mathcal{B}}(S_2S_3)\big| + qE\big|E^{\mathcal{B}}(S_2S_4)\big| = O(\frac{1}{q}) \tag{6.32}$$

Hence, by (6.30)-(6.32), we have 2. □

Lemma 6.8. *Let $\mathcal{B}$ be the $\sigma-$algebra generated by π_1, π_2, π_3 and $U_{i_1,i_2,i_3;j}$'s. Then*
1. $E\big|E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big| \le O(\frac{1}{\sqrt{q}})$ *as* $q\to\infty$.
2. $E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^4 = O(1)$ *as* $q\to\infty$.
3. $E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^2 = (1+z)^{\frac{1}{6}}O(\frac{1}{\sqrt[3]{q}})$ *as* $q\to\infty$.

Proof. 1. By (4.11) and (4.13),

$$
\begin{aligned}
qE|E^{\mathcal{B}}(S_{j,z}S_k)| &\le qEE^{\mathcal{B}}|S_{j,z}S_k| \\
&= qE|S_{j,z}S_k| \\
&\le q\big\{ES_{j,z}^2\big\}^{\frac{1}{2}}\big\{ES_k^2\big\}^{\frac{1}{2}} \\
&\le \frac{1}{(1+z)^2}O(\frac{1}{q\sqrt{q}}),
\end{aligned}
\tag{6.33}
$$

and

$$\begin{aligned} qE|E^{\mathcal{B}}(S_{j,z}S_{k,z})| &\le qE|S_{j,z}S_{k,z}| \\ &\le q\{ES_{j,z}^2\}^{\frac{1}{2}}\{ES_{k,z}^2\}^{\frac{1}{2}} \\ &\le \frac{1}{(1+z)^4}O(\frac{1}{q^3}). \end{aligned} \tag{6.34}$$

for any $j,k = 1,2,3,4$.

Thus $E\big|E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big|$

$$\begin{aligned} &= E\big|E^{\mathcal{B}}\{1 - \frac{(q-1)}{4}(\widehat{S}_{1,z} + \widehat{S}_{2,z} - \widehat{S}_{3,z} - \widehat{S}_{4,z})^2\}\big| \\ &\le \frac{1}{4}\sum_{k=1}^{4} E\big|E^{\mathcal{B}}((q-1)\widehat{S}_{k,z}^2 - 1)\big| + \frac{q-1}{2}\sum_{1\le j<k\le 4} E\big|E^{\mathcal{B}}(\widehat{S}_{j,z}\widehat{S}_{k,z})\big| \\ &= \frac{1}{4}\sum_{k=1}^{4} E\big|E^{\mathcal{B}}\{(q-1)(S_k - S_{k,z})^2 - 1\}\big| \\ &\quad + \frac{q-1}{2}\sum_{1\le j<k\le 4} E\big|E^{\mathcal{B}}(S_j - S_{j,z})(S_k - S_{k,z})\big| \\ &\le \frac{1}{4}\sum_{k=1}^{4} E\big|E^{\mathcal{B}}((q-1)S_k^2 - 1)\big| + \frac{(q-1)}{2}\sum_{k=1}^{4} E\big|E^{\mathcal{B}}S_{k,z}S_k\big| \\ &\quad + \frac{(q-1)}{4}\sum_{k=1}^{4} E\big|E^{\mathcal{B}}S_{k,z}^2\big| \\ &\quad + \frac{q-1}{2}\sum_{1\le j<k\le 4} E\big|E^{\mathcal{B}}(S_jS_k - S_{j,z}S_k - S_jS_{k,z} + S_{j,z}S_{k,z})\big| \\ &\le O(\frac{1}{\sqrt{q}}), \end{aligned} \tag{6.35}$$

where we have used Lemma 6.7(1, 2), (6.33) and (6.34) in the last equality.

2. By Lemma 3.3(2), (4.9), we have

$$\begin{aligned} &E\big(E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big)^4 \\ &= E\Big\{E^{\mathcal{B}}\{1 - \frac{(q-1)}{4}(\widehat{S}_{1,z} + \widehat{S}_{2,z} - \widehat{S}_{3,z} - \widehat{S}_{4,z})^2\}\Big\}^4 \\ &= E\Big\{\frac{1}{q(q-1)}\sum_{i=0}^{q-1}\sum_{\substack{k=0\\k\ne i}}^{q-1} E^{\mathcal{B}}\big[1 - \frac{(q-1)}{4}(\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j)) \\ &\quad + \sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(i,j)))^2\big]\Big\}^4 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{q(q-1)} \sum_{i=0}^{q-1} \sum_{\substack{k=0 \\ k \neq i}}^{q-1} E\Big\{E^{\mathcal{B}}\big[1 - \frac{(q-1)}{4}\big(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j)) \\
&\quad + \sum_{j=0}^{q-1} \widehat{Y}_z(k,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1} \widehat{Y}_z(k,j,\rho_\pi(i,j))\big)^2\big]\Big\}^4 \\
&= \frac{1}{q(q-1)} \sum_{i=0}^{q-1} \sum_{\substack{k=0 \\ k \neq i}}^{q-1} E\Big\{1 - \frac{(q-1)}{4}\big(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j)) + \sum_{j=0}^{q-1} \widehat{Y}_z(k,j,\rho_\pi(k,j)) \\
&\quad - \sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1} \widehat{Y}_z(k,j,\rho_\pi(i,j))\big)^2\Big\}^4 \\
&\leq \frac{1}{q(q-1)} \sum_{i=0}^{q-1} \sum_{\substack{k=0 \\ k \neq i}}^{q-1} \Big\{1 + \frac{(q-1)^4}{4^4} E\big(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j))\big)^8 \\
&\quad + \frac{(q-1)^4}{4^4} E\big(\sum_{j=0}^{q-1} \widehat{Y}_z(k,j,\rho_\pi(k,j))\big)^8 + \frac{(q-1)^4}{4^4} E\big(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(k,j))\big)^8 \\
&\quad + \frac{(q-1)^4}{4^4} E\big(\sum_{j=0}^{q-1} \widehat{Y}_z(k,j,\rho_\pi(i,j))\big)^8 \\
&= 1 + \frac{2(q-1)^4}{4^4}\Big(\frac{1}{q} \sum_{i=0}^{q} E\big(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j))\big)^8\Big) \\
&\quad + \frac{2(q-1)^4}{4^4}\Big(\frac{1}{q(q-1)} \sum_{i=0}^{q} \sum_{\substack{k=0 \\ k \neq i}}^{q-1} E\big(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(k,j))\big)^8\Big) \\
&= 1 + \frac{2(q-1)^4}{4^4} E\widehat{S}_{1,z}^{\,8} + \frac{2(q-1)^4}{4^4} E\widehat{S}_{3,z}^{\,8} \\
&= 1 + \frac{2(q-1)^4}{4^4} E(S_1 - S_{1,z})^8 + \frac{2(q-1)^4}{4^4} E(S_3 - S_{3,z})^8 \\
&\leq 1 + \frac{2(q-1)^4}{4^4}\big\{ES_1^8 + ES_{1,z}^8\big\} \\
&= O(1). \qquad (6.36)
\end{aligned}
$$

3. From 1, 2 and Chebyshev's inequality, we have

$$
\begin{aligned}
&E\big(E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big)^2 \\
&\quad = E\big(E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big)^2 \mathbb{I}\big(|E^{\mathcal{B}}(1 - \int_{-\infty}^{\infty} K(t)dt)| < q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\big) \\
&\quad\quad + E\big(E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big)^2 \mathbb{I}\big(|E^{\mathcal{B}}(1 - \int_{-\infty}^{\infty} K(t)dt)| \geq q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\big)
\end{aligned}
$$

$$
\begin{aligned}
&\le E\big|E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big|q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\\
&\quad+\big\{E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^4\big\}^{\frac{1}{2}}\big\{P\big(|E^{\mathcal{B}}(1-\int_{-\infty}^{\infty}K(t)dt)|\ge q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\big)\big\}^{\frac{1}{2}}\\
&\le O(\frac{1}{\sqrt{q}})q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}+\frac{E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^4}{\big(q^{\frac{2}{3}}(1+z)^{\frac{2}{3}}\big)^{\frac{1}{2}}}\\
&\le (1+z)^{\frac{1}{6}}O(\frac{1}{q^{\frac{1}{2}-\frac{1}{6}}})+\frac{C}{q^{\frac{1}{3}}(1+z)^{\frac{1}{3}}}\\
&\le (1+z)^{\frac{1}{6}}O(\frac{1}{\sqrt[3]{q}}).
\end{aligned}
$$

□

Proof of Theorem 6.3

Let $\gamma = \frac{q}{4}E|\widehat{Y}-\widetilde{Y}|^3$. From Lemma 4.4(3), we have $\gamma = O(\frac{1}{\sqrt{q}})$. In the view of Theorem 6.2 and the fact that

$$1=\frac{(1+z)^{1-\frac{2}{r}}}{(1+z)^{1-\frac{2}{r}}}\le\frac{2}{(1+z)^{1-\frac{2}{r}}}\quad\text{for}\quad 0<z\le 1,$$

it suffices to prove the main theorem only the case $z>1$.
Note that

$$|P(W\le z)-\Phi(z)|\le P(W\ne\widehat{Y})+|P(\widehat{Y}\le z)-\Phi(z)|. \tag{6.37}$$

By (4.40), we need to bound only the second term on the right hand side of (6.37) $|P(\widehat{Y}\le z)-\Phi(z)|$.

Case 1 $(1+z)\gamma\ge 1$.
By Lemma 4.4(1) and the fact that $\gamma\ge\frac{1}{1+z}$,

$$
\begin{aligned}
P(\widehat{Y}>z)&=P(\widehat{Y}+1>1+z)\\
&\le\frac{E|\widehat{Y}+1|^2}{(1+z)^2}\\
&\le C\frac{E|\widehat{Y}|^2}{(1+z)^2}+\frac{C}{(1+z)^2}\\
&\le\frac{C}{(1+z)^2}\\
&\le\frac{C\gamma}{1+z}.
\end{aligned}\tag{6.38}
$$

Hence, from (6.38) and

$$1-\Phi(z) \leq \frac{C}{(1+z)^2} \quad \text{for some constant } C,$$

we have

$$\begin{aligned}
|P(\widehat{Y} \leq z) - \Phi(z)| &= |1 - P(\widehat{Y} > z) - \Phi(z)| \\
&= P(\widehat{Y} > z) + (1 - \Phi(z)) \\
&\leq \frac{C\gamma}{1+z} + \frac{C\gamma}{1+z} \\
&= \frac{1}{1+z} O(\frac{1}{\sqrt{q}}).
\end{aligned}$$

Case 2 $(1+z)\gamma < 1$.

Let g_z be the Stein solution of the Stein equation

$$g_z'(\widehat{Y}) - \widehat{Y} g_z(\widehat{Y}) = h(\widehat{Y}) - \Phi(z).$$

By using the idea of the proof of theorem 5.2, we have

$$\begin{aligned}
\big|P(\widehat{Y} \leq z) - \Phi(z)\big| &= \big|Eg_z'(\widehat{Y}) - E\widehat{Y} g_z(\widehat{Y})\big| \\
&\leq T_1 + T_2 + T_3 + T_4 \qquad (6.39)
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \big|E \int_{-\infty}^{\infty} \{g_z'(\widehat{Y}) - g_z'(\widehat{Y} + t)\} K(t) dt\big|, \\
T_2 &= \big|Eg_z'(\widehat{Y}) E \int_{-\infty}^{\infty} K(t) dt - Eg_z'(\widehat{Y}) \int_{-\infty}^{\infty} K(t) dt\big|, \\
T_3 &= \Big|Eg_z'(\widehat{Y}) - Eg_z'(\widehat{Y}) E \int_{-\infty}^{\infty} K(t) dt\Big|, \\
T_4 &= |\widetilde{\Delta} g_z(\widehat{Y})|,
\end{aligned}$$

and

$$K(t) = \frac{q-1}{4}\Big(\widetilde{Y} - Y\Big)\Big(\mathbb{I}\big(0 \leq t \leq \widetilde{Y} - Y\big) - \mathbb{I}\big(\widetilde{Y} - Y \leq t < 0\big)\Big).$$

By Proposition 2.9(4), (3.3) and Lemma 3.4(2), we note that

$$
\begin{aligned}
T_4 &= \big|\widetilde{\Delta} g_z(\widehat{Y})\big| \\
&= \frac{1}{q}\big|E g_z(\widehat{Y})\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}\sum_{k=1}^{q-1}\widehat{Y}_z(i,j,k)\big| \\
&\le \frac{1}{q}\big\{E g_z^2(\widehat{Y})\big\}^{\frac{1}{2}}\big\{E(\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}\sum_{k=1}^{q-1}\widehat{Y}_z(i,j,k))^2\big\}^{\frac{1}{2}} \\
&\le \frac{1}{qz}\big\{\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}\sum_{k=1}^{q-1}E\widehat{Y}_z^2(i,j,k) \\
&\quad + \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} E\widehat{Y}_z(i_1,i_2,i_3)\widehat{Y}_z(j_1,j_2,j_3)\big\}^{\frac{1}{2}} \\
&\le \frac{1}{qz}\Big\{\sum_{i=1}^{q-1}\sum_{j=1}^{q-1}\sum_{k=1}^{q-1}EY^2(i,j,k) \\
&\quad + \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \big(EY(i_1,i_2,i_3)-EY_z(i_1,i_2,i_3)\big) \\
&\quad \big(EY(j_1,j_2,j_3)-EY_z(j_1,j_2,j_3)\big)\Big\}^{\frac{1}{2}} \\
&\le \frac{1}{qz}\Big\{O(q)+\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\big\{\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3) \\
&\quad - \tilde{\mu}(i_1,i_2,i_3)EY_z(j_1,j_2,j_3) - EY_z(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3) \\
&\quad + EY_z(i_1,i_2,i_3)EY_z(j_1,j_2,j_3)\big\}\Big\}^{\frac{1}{2}} \\
&\le \frac{1}{qz}\Big\{O(q)-\sum_{i_1,i_2,i_3}\tilde{\mu}^2(i_1,i_2,i_3)+\sum_{j_1,j_2,j_3}\tilde{\mu}(j_1,j_2,j_3)EY_z(j_1,j_2,j_3) \\
&\quad + \sum_{i_1,i_2,i_3}\tilde{\mu}(i_1,i_2,i_3)EY_z(i_1,i_2,i_3)+\big(\sum_{i_1,i_2,i_3}E|Y_z(i_1,i_2,i_3)|\big)^2\Big\}^{\frac{1}{2}} \\
&\le \frac{1}{qz}\Big\{O(q)+\sum_{j_1,j_2,j_3}EY_z^2(j_1,j_2,j_3) \\
&\quad + \big(\frac{1}{(1+z)^3}\sum_{i_1,i_2,i_3}EY^4(i_1,i_2,i_3)\big)^2\Big\}^{\frac{1}{2}} \\
&\le \frac{1}{qz}\Big\{O(q)+\sum_{j_1,j_2,j_3}EY^2(j_1,j_2,j_3)+\frac{1}{(1+z)^6}O(\frac{1}{q^2})\Big\}^{\frac{1}{2}} \\
&\le \frac{2}{q(1+z)}\big\{O(q)+\frac{1}{(1+z)^6}O(\frac{1}{q^2})\big\}^{\frac{1}{2}} \\
&= \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \qquad (6.40)
\end{aligned}
$$

and from Lemma 4.3(3) and Lemma 4.4(1),

$$\begin{aligned}
T_3 &= \Big|Eg_z'(\widehat{Y}) - Eg_z'(\widehat{Y})E\int_{-\infty}^{\infty} K(t)dt\Big| \\
&= \big|Eg_z'(\widehat{Y})\big|\Big|1 - E\int_{-\infty}^{\infty} K(t)dt\Big| \\
&\le \frac{C}{(1+z)^2}\big|1 - E\widehat{Y}^2 + \Delta\widehat{Y}\big| \\
&\le \frac{C}{(1+z)^2}\big|1 - E\widehat{Y}^2\big| + \frac{C}{(1+z)^2}\big|\Delta\widehat{Y}\big| \\
&\le \frac{C}{(1+z)^2}\big(O(\frac{1}{q}) + \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}})\big) + \frac{C}{(1+z)^2}O(\frac{1}{q}) \\
&= \frac{1}{(1+z)^2}O(\frac{1}{q}) + \frac{1}{(1+z)^3}O(\frac{1}{\sqrt{q}}) \\
&\le \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \qquad (6.41)
\end{aligned}$$

Let $\mathcal{B}$ be the $\sigma-$algebra generated by π_1, π_2, π_3 and $U_{i_1,i_2,i_3;j}$'s.

$$\begin{aligned}
T_2 &= \big|Eg_z'(\widehat{Y})E\int_{-\infty}^{\infty} K(t)dt - Eg_z'(\widehat{Y})\int_{-\infty}^{\infty} K(t)dt\big| \\
&= \big|Eg_z'(\widehat{Y})E^{\mathcal{B}}\Big\{E\int_{-\infty}^{\infty} K(t)dt - \int_{-\infty}^{\infty} K(t)dt\Big\}\big| \\
&= \big|Eg_z'(\widehat{Y})E^{\mathcal{B}}\Big\{E\int_{-\infty}^{\infty} K(t)dt - 1 + 1 - \int_{-\infty}^{\infty} K(t)dt\Big\}\big| \\
&\le \frac{C}{(1+z)^2}\big|E\int_{-\infty}^{\infty} K(t)dt - 1\big| + \big|Eg_z'(\widehat{Y})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\le \frac{C}{(1+z)^2}\big|1 - E\widehat{Y}^2 + \widetilde{\Delta}\widehat{Y}\big| + \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} \le \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\quad + \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} < \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&= \frac{1}{(1+z)^2}O(\frac{1}{q}) + T_{21} + T_{22} \qquad (6.42)
\end{aligned}$$

where

$$\begin{aligned}
T_{21} &= \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} \le \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
T_{22} &= \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} > \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big|.
\end{aligned}$$

By Proposition 2.10, Lemma 6.8(1) and the fact that $e^{-\frac{3z^2}{8}} \leq \dfrac{C}{1+z}$ for some constant C,

$$
\begin{aligned}
T_{21} &= \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} \leq \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\leq \big|E\big\{\sqrt{2\pi}(\frac{z}{2})e^{\frac{z^2}{8}}\Phi(\frac{z}{2}) + 1\big\}\big\{1 - \Phi(z)\big\}\mathbb{I}(\widehat{Y} \leq \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\leq \big\{\sqrt{2\pi}(\frac{z}{2})e^{\frac{z^2}{8}} + 1\big\}\big\{\frac{e^{-\frac{z^2}{2}}}{z}\big\}E\big|E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\leq \frac{\sqrt{2\pi}}{2}e^{-\frac{3z^2}{8}}O(\frac{1}{\sqrt{q}}) \\
&\leq \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \qquad (6.43)
\end{aligned}
$$

From Proposition 2.9(2), Lemma 4.4(1) and Lemma 6.8(3),

$$
\begin{aligned}
T_{22} &= \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} > \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\leq \big|E\mathbb{I}(\widehat{Y} > \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\leq \big\{P(\widehat{Y} > \frac{z}{2})\big\}^{\frac{1}{2}}\big\{E\big|E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big|^2\big\}^{\frac{1}{2}} \\
&\leq \big\{\frac{4E|\widehat{Y}|^2}{z^2}\big\}^{\frac{1}{2}}\big\{(1+z)^{\frac{1}{6}}O(\frac{1}{q^{\frac{1}{3}}})\big\}^{\frac{1}{2}} \\
&\leq \frac{1}{(1+z)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}}). \qquad (6.44)
\end{aligned}
$$

Hence, by (6.42)-(6.44), we have

$$T_2 \leq \frac{1}{(1+z)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}}). \qquad (6.45)$$

To finish the proof of Theorem 6.3, it remains to bound T_1. By the fact that

$$\big|g_z'(w+s) - g_z'(w+s) - \int_t^s h(w+u)du\big| \leq \mathbb{I}\big(z - max(s,t) < w < z - min(s,t)\big),$$

we have

$$T_1 = T_{11} + T_{12} + T_{13} \qquad (6.46)$$

where

$$T_{11} = E\mathbb{I}(|\widetilde{Y}-\widehat{Y}| < \frac{1+z}{2})\int_{-\infty}^{\infty}\mathbb{I}(z-\max(0,t)<\widehat{Y}<z-\min(0,t))K(t)dt,$$
$$T_{12} = E\mathbb{I}(|\widetilde{Y}-\widehat{Y}| < \frac{1+z}{2})\int_{-\infty}^{\infty}\int_{t}^{0}h(\widehat{Y}+u)K(t)dudt,$$
$$T_{13} = E\mathbb{I}(|\widetilde{Y}-\widehat{Y}| \geq \frac{1+z}{2})\int_{-\infty}^{\infty}\{g_z'(\widehat{Y})-g_z'(\widehat{Y}+t)\}K(t)dt|$$

and h is definded as in Lemma 6.6 .

First, we consider T_{11}. For $\delta > 0$, let $f_\delta : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_\delta(t) = \begin{cases} 0 & \text{if} \quad t < z-2\delta, \\ (1+t+\delta)(t-z+2\delta) & \text{if} \quad z-2\delta \leq t \leq z+2\delta, \\ 4\delta(1+t+\delta) & \text{if} \quad t > z+2\delta. \end{cases} \tag{6.47}$$

Note that f_δ is a non decreasing function and

$$f_\delta'(t) \geq \begin{cases} 1+z-\delta & \text{if} \quad z-2\delta \leq t \leq z+2\delta, \\ 0 & \text{otherwise} . \end{cases} \tag{6.48}$$

By the same argument as in Lemma 4.3(2), we have

$$E\int_{-\infty}^{\infty} f_\delta'(\widehat{Y}+t)K(t)dt = E\widehat{Y}f_\delta(\widehat{Y}) - \widetilde{\Delta}f_\delta(\widehat{Y}). \tag{6.49}$$

By Lemma 4.4(1-3), (6.47), f is non decreasing, we observe that

$$\begin{aligned}
&E|\widehat{Y}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})| \\
&\leq 4E\widehat{Y}(1+\widehat{Y}+|\widetilde{Y}-\widehat{Y}|)|\widetilde{Y}-\widehat{Y}| \\
&\leq 4E\widehat{Y}|\widetilde{Y}-\widehat{Y}| + 4E\widehat{Y}^2|\widetilde{Y}-\widehat{Y}| + 4E\widehat{Y}|\widetilde{Y}-\widehat{Y}|^2 \\
&\leq 4\{E\widehat{Y}^2\}^{\frac{1}{2}}\{E|\widetilde{Y}-\widehat{Y}|^2\}^{\frac{1}{2}} + 4E\widehat{Y}^2|\widetilde{Y}-\widehat{Y}| + 4\{E\widehat{Y}^2\}^{\frac{1}{2}}\{E|\widetilde{Y}-\widehat{Y}|^4\}^{\frac{1}{2}} \\
&\leq 4\{E|\widetilde{Y}-\widehat{Y}|^2\}^{\frac{1}{2}} + 4\big\{E(\widehat{Y}^2)^{\frac{r}{r-1}}\big\}^{\frac{r-1}{r}}\big\{E|\widetilde{Y}-\widehat{Y}|^r\big\}^{\frac{1}{r}} + 4\{E|\widetilde{Y}-\widehat{Y}|^4\}^{\frac{1}{2}} \\
&\leq 4\{O(\frac{1}{q})\}^{\frac{1}{2}} + 4\{O(\frac{1}{q^2})\}^{\frac{1}{2}} + 4\big\{E\widehat{Y}^2|\widehat{Y}|^{\frac{2}{r-1}}\big\}^{\frac{r-1}{r}}\big\{E|\widetilde{Y}-\widehat{Y}|^r\big\}^{\frac{1}{r}} \\
&\leq O(\frac{1}{\sqrt{q}}) + 4\big\{(q^2(1+z))^{\frac{2}{r-1}}E\widehat{Y}^2\big\}^{\frac{r-1}{r}}\big\{O(\frac{1}{q^{\frac{r}{2}}})\big\}^{\frac{1}{r}} \\
&\leq O(\frac{1}{\sqrt{q}}) + (1+z)^{\frac{2}{r}}O(\frac{1}{q^{\frac{r-8}{2r}}}).
\end{aligned} \tag{6.50}$$

By Lemma 4.4(1,2), Lemma 6.4(1,2), Lemma 6.5(1,2), (6.47) and f is non-decreasing, we note that

$$
\begin{aligned}
\widetilde{\Delta} f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y}) &\leq \frac{1}{q}|Ef_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\sum_{i,j,k}\widehat{Y}_z(i,j,k)| \\
&\leq \frac{4}{q}\big|E(1+\widehat{Y}+|\widetilde{Y}-\widehat{Y}|)|\widetilde{Y}-\widehat{Y}|\sum_{i,j,k}\widehat{Y}_z(i,j,k)\big| \\
&\leq \frac{4}{q}\big|E(1+\widehat{Y})|\widetilde{Y}-\widehat{Y}|\big(\sum_{i,j,k}Y(i,j,k)-\sum_{i,j,k}Y_z(i,j,k)\big)\big| \\
&\quad +\frac{4}{q}\big|E|\widetilde{Y}-\widehat{Y}|^2\sum_{i,j,k}Y(i,j,k)-\sum_{i,j,k}Y_z(i,j,k)\big| \\
&\leq \frac{4}{q}\big|E(1+\widehat{Y})|\widetilde{Y}-\widehat{Y}|\sum_{i,j,k}Y(i,j,k)\big| + \frac{4}{q}\big|E(1+\widehat{Y})|\widetilde{Y}-\widehat{Y}|\sum_{i,j,k}Y_z(i,j,k)\big| \\
&\quad +\frac{4}{q}\big|E|\widetilde{Y}-\widehat{Y}|^2\sum_{i,j,k}Y(i,j,k)\big| + \frac{4}{q}\big|E|\widetilde{Y}-\widehat{Y}|^2\sum_{i,j,k}Y_z(i,j,k)\big| \\
&\leq \frac{4}{q}\big\{E(1+\widehat{Y})^2\big\}^{\frac{1}{2}}\big\{E|\widetilde{Y}-\widehat{Y}|^2(\sum_{i,j,k}Y(i,j,k))^2\big\}^{\frac{1}{2}} \\
&\quad +\frac{4}{q}\big\{E(1+\widehat{Y})^2\big\}^{\frac{1}{2}}\big\{E|\widetilde{Y}-\widehat{Y}|^2(\sum_{i,j,k}Y_z(i,j,k))^2\big\}^{\frac{1}{2}} \\
&\quad +\frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{2}}\big\{E(\sum_{i,j,k}Y(i,j,k))^2\big\}^{\frac{1}{2}} \\
&\quad +\frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{2}}\big\{E(\sum_{i,j,k}Y_z(i,j,k))^2\big\}^{\frac{1}{2}} \\
&\leq \frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{4}}\big\{E(\sum_{i,j,k}Y(i,j,k))^4\big\}^{\frac{1}{4}} \\
&\quad +\frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{4}}\big\{E(\sum_{i,j,k}Y_z(i,j,k))^4\big\}^{\frac{1}{4}} \\
&\quad +\frac{4}{q}\big\{O(\frac{1}{q^2})\big\}^{\frac{1}{2}}\big\{O(q)\big\}^{\frac{1}{2}} + \frac{4}{q}\big\{O(\frac{1}{q^2})\big\}^{\frac{1}{2}}\big\{\frac{1}{(1+z)^2}O(\frac{1}{q})\big\}^{\frac{1}{2}} \\
&\leq \frac{4}{q}\big\{O(\frac{1}{q^2})\big\}^{\frac{1}{4}}\big\{O(q^2)\big\}^{\frac{1}{4}} + \frac{4}{q}\big\{O(\frac{1}{q^2})\big\}^{\frac{1}{4}}\big\{\frac{1}{(1+z)^2}O(\frac{1}{q^2})\big\}^{\frac{1}{4}} + O(\frac{1}{q\sqrt{q}}) \\
&= O(\frac{1}{q}). \qquad (6.51)
\end{aligned}
$$

Thus, from (6.48)-(6.51), we have

$$\begin{aligned}
T_{11} &= E\int_{-\infty}^{\infty}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})\mathbb{I}(z-\max(0,t)<\widehat{Y}<z-\min(0,t))K(t)dt \\
&= E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})\mathbb{I}(z-\max(0,t)<\widehat{Y}<z-\min(0,t))K(t)dt \\
&\le E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})\mathbb{I}(z-2|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+2|\widetilde{Y}-\widehat{Y}|)K(t)dt \\
&= E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})(\frac{1+z-|\widetilde{Y}-\widehat{Y}|}{1+z-|\widetilde{Y}-\widehat{Y}|}) \\
&\quad \mathbb{I}(z-|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+|\widetilde{Y}-\widehat{Y}|)K(t)dt \\
&\le \frac{2}{1+z}E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})(1+z-|\widetilde{Y}-\widehat{Y}|) \\
&\quad \mathbb{I}(z-|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+|\widetilde{Y}-\widehat{Y}|)K(t)dt \\
&\le \frac{2}{1+z}E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2}) \\
&\quad f'_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y}+t)\mathbb{I}(z-|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+|\widetilde{Y}-\widehat{Y}|)K(t)dt \\
&\le E\Big|\widehat{Y}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\Big|+\Big|\widetilde{\Delta}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\Big| \\
&\le \frac{1}{(1+z)^{1-\frac{2}{r}}}O(\frac{1}{q^{\frac{r-8}{2r}}}). \qquad (6.52)
\end{aligned}$$

By Lemma 6.6,

$$T_{12}\le\frac{1}{(1+z)^{1-\frac{2}{r}}}O(\frac{1}{\sqrt{q}}). \qquad (6.53)$$

By Proposition 2.9(3) and Lemma 4.4(2u)

$$\begin{aligned}
T_{13} &= E\mathbb{I}(|\widetilde{Y}-\widehat{Y}|\ge\frac{1+z}{2})\int_{-\infty}^{\infty}\{g_z'(\widehat{Y})-g_z'(\widehat{Y}+t)\}K(t)dt\Big| \\
&\le E\mathbb{I}(|\widetilde{Y}-\widehat{Y}|\ge\frac{1+z}{2})\int_{-\infty}^{\infty}K(t)dt \\
&\le qE\mathbb{I}(|\widetilde{Y}-\widehat{Y}|\ge\frac{1+z}{2})|\widetilde{Y}-\widehat{Y}|^2 \\
&\le q\{P(|\widetilde{Y}-\widehat{Y}|\ge\frac{1+z}{2})\}^{\frac{1}{2}}\{E|\widetilde{Y}-\widehat{Y}|^4\}^{\frac{1}{2}} \\
&\le q\{\frac{E|\widetilde{Y}-\widehat{Y}|^2}{(1+z)^2}\}^{\frac{1}{2}}\{O(\frac{1}{q^2})\}^{\frac{1}{2}} \\
&\le q\{\frac{1}{(1+z)^2}O(\frac{1}{q})\}^{\frac{1}{2}}O(\frac{1}{q}) \\
&= \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \qquad (6.54)
\end{aligned}$$

From (6.46), (6.52)-(6.54), we have the main theorem.

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