การให้สีพร่องบนไฮเพอร์กราฟเคเอกรูปสองส่วนแบบบริบูรณ์และหลายส่วนแบบบริบูรณ์

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DEFECTIVE COLORINGS ON COMPLETE BIPARTITE AND MULTIPARTITE k-UNIFORM HYPERGRAPHS

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วิทยานิพนธ์ฉบับนี้เราได้ปรับเปลี่ยนบทนิยามของการให้สีพร่องและรงคเลขของการให้สีพร่องบนกราฟไปเป็นการให้สีพร่องและรงคเลขของการให้สีพร่องบนไฮเพอร์กราฟ ในส่วนแรกเราหาค่ารงคเลขของการให้สีพร่องบนไฮเพอร์กราฟเคเอกรูปสองส่วนแบบบริบูรณ์ และค่ารงคเลขของการให้สีพร่องบนไฮเพอร์กราฟเคเอกรูปสองส่วนแบบบริบูรณ์เมื่อแต่ละคลาสสีไม่บรรจุวง ในส่วนที่สองเราหาค่ารงคเลขของการให้สีพร่อง และค่ารงคเลขของการให้สีพร่องเมื่อแต่ละคลาสสีไม่บรรจุวงบนไฮเพอร์กราฟเคเอกรูปเคส่วนแบบบริบูรณ์ที่แต่ละเส้นเชื่อมประกอบด้วยเคจุดยอดจากเคส่วนแบ่งกั้นที่แตกต่างกัน ในส่วนสุดท้ายเราหาขอบเขตบนของค่ารงคเลขของการให้สีพร่องและค่ารงคเลขของการให้สีพร่องเมื่อแต่ละคลาสสีไม่บรรจุวง บนไฮเพอร์กราฟสามเอกรูปสามส่วนแบบบริบูรณ์ที่แต่ละเส้นเชื่อมประกอบด้วยสามจุดยอดจากอย่างน้อยสองส่วนแบ่งกั้นที่แตกต่างกัน

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In this thesis, we modify the definition of a defective coloring and a defective chromatic number on graphs to a defective coloring and a defective chromatic number on hypergraphs. First, we find the defective chromatic number on a complete bipartite k-uniform hypergraph and the defective chromatic number on a complete bipartite k-uniform hypergraph of which each color class is acyclic.

Second, we determine the defective chromatic number and the defective chromatic number of which each color class is acyclic on a complete k-partite k-uniform hypergraph whose each edge has k vertices from k different partite sets.

Finally, we determine the upper bound of the defective chromatic numbers and determine the defective chromatic number of which each color class is acyclic on a complete tripartite 3-uniform hypergraph whose each edge has three vertices from at least two different partite sets.

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CONTENTS

CHAPTER I
INTRODUCTION

The basic idea of the hypergraph is to consider a generalization of a graph. There are many interesting topics on graph theory. One of its is a proper vertex coloring on a graph that was generalized to a proper vertex coloring on a hypergraph as follows. A λ-coloring of a hypergraph H is a labeling $f : V(H) \to \{1, 2, 3, \ldots, \lambda\}$ and a proper λ-coloring of a hypergraph H is a λ-coloring of a hypergraph H such that no edge of H (besides singletons) has all vertices of the same color. The chromatic number of hypergraph H, denoted by $\chi(H)$, is the minimum number λ for which a proper λ-coloring exists [9].

The vertex coloring on a graph does not have only proper vertex colorings. Many researchers are also interested in defective colorings, see [3] and [4]. A defective coloring on a graph G is a coloring of a graph G in which some adjacent vertices maybe assigned the same color. A (k, d)-defective coloring of a graph G is a vertex coloring of G with k colors such that each vertex is allowed at most d neighbors having the same color. In this work, we modify the definition of a defective coloring and a defective chromatic number on graphs to a defective coloring and a defective chromatic number on hypergraphs as follows. Let $d \geq 0$, a (λ, d)-defective coloring is a λ-coloring of a hypergraph H in which there are at most d edges of H having all vertices of the same color. If H admits a (λ, d)-defective coloring, then $\chi_{\leq d}(H)$ denotes the least integer λ.

In 2016, Muaengwaeng and and Nakprasit [8] considered (λ, d)-defective colorings and chromatic numbers of graphs of which each color class induces a forest, i.e., each color class is acyclic. We also modify their idea to a (λ, d)-defective coloring and chromatic number on a hypergraph of which each color class is acyclic.

This thesis is organized as follows. In Chapter II, we recall some basic definitions and theorems involving the research.

In Chapter III, we determine the defective chromatic numbers on a complete bipartite k-uniform hypergraph and also determine the defective chromatic numbers on a complete bipartite k-uniform hypergraph of which each color class is acyclic.

In Chapter IV, we determine the defective chromatic number and the defective chromatic number of which each color class is acyclic on a complete k-partite k-uniform hypergraph whose each edge has k vertices from k different partite sets.

In Chapter V, we determine the upper bound of the defective chromatic number and determine the defective chromatic number of which each color class is acyclic on a complete tripartite 3-uniform hypergraph whose each edge has three vertices from at least two different partite sets.

CHAPTER II

PRELIMINARIES

In this chapter, we recall some definitions and notations used throughout this research. We also review some literature related to our research.

Definition 2.1. [9] A **hypergraph** H is an ordered pair $H = (V(H),\ E(H))$, where the set $V(H)$ of vertices is a nonempty finite set and the set $E(H)$ of (hyper) edges is a collection of distinct nonempty subsets of $V(H)$. Let $k > 1$. If every edge of H has size k, then H is called a ***k*-uniform hypergraph**. Note that a 2-uniform hypergraph is simply an ordinary graph.

Figure 2.1 shows a hypergraph H of 6 vertices $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and 4 edges $E(H) = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_5, v_6\}$ and $e_4 = \{v_4\}$.

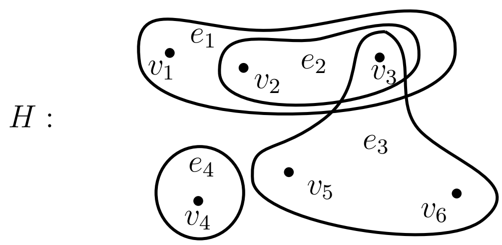


Figure 2.1: hypergraph H.

Any hypergraph $H' = (V(H'), E(H'))$ such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$ is a **subhypergraph** of H. In such a case, we write $H' \subseteq H$.

A hypergraph $H' = (V(H'), E(H'))$ is called an **induced subhypergraph** of H if $V(H') \subseteq V(H)$ and all edges of H completely contained in $V(H')$ form the family $E(H')$. Sometimes we say that H' is a subhypergraph **induced by** $V(H')$.

Figure 2.2 shows a hypergraph H of five vertices and three edges of which both H_1 and H_2 are subhypergraphs; H_1 is an induced subhypergraph of H but H_2 is not an induced subhypergraph of H.

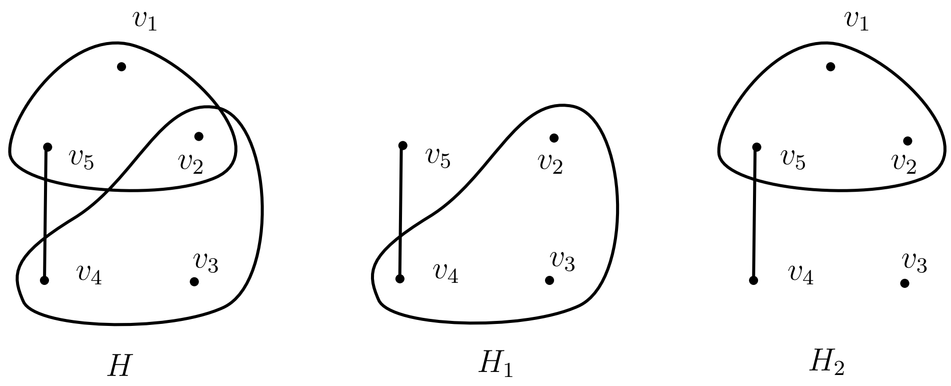


Figure 2.2: Hypergraph H, induced subhypergraph H_1 and subhypergraph H_2.

A set $U \subseteq V(H)$ is called an **independent (stable) set** if U induces no edge of H. $\{v_1, v_3, v_5\}$ is an independent set of the hypergraph H shown in Figure 2.2.

Definition 2.2. [9] For $k \geq 1$, a **complete k-uniform hypergraph** on n vertices, denoted by $K_n^{(k)}$, has all k-subsets of an n-set vertices as edges. The hypergraph $K_n^{(2)}$ is the complete graph K_n.

Example 2.3. A complete 3-uniform hypergraph $K_5^{(3)}$ with vertex set $\{v_1, v_2, v_3, v_4,$ $v_5\}$ has 10 edges as follows.
$e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_1, v_2, v_4\}, e_3 = \{v_1, v_2, v_5\}, e_4 = \{v_1, v_3, v_4\}, e_5 = \{v_1, v_3, v_5\},$
$e_6 = \{v_1, v_4, v_5\}, e_7 = \{v_2, v_3, v_4\}, e_8 = \{v_2, v_3, v_5\}, e_9 = \{v_2, v_4, v_5\}, e_{10} = \{v_3, v_4, v_5\}.$

In 2001, Jirimutu and Wang [6] introduced the concept of a complete bipartite k-uniform hypergraph $K_{m,n}^{(k)}$ as follows.

Definition 2.4. For $k \geq 1$, a **complete bipartite k-uniform hypergraph** $K_{m,n}^{(k)}$ has the vertex set $V(K_{m,n}^{(k)})$ consisting of two partite sets V_1 and V_2 such that $|V_1| = m$, $|V_2| = n$, $k \leq m+n$ and the edge set $E(K_{m,n}^{(k)}) = \{e \subseteq V_1 \cup V_2 : |e| = k, e \cap V_1 \neq \varnothing \textit{ and } e \cap V_2 \neq \varnothing\}$.

From Definition 2.4, we notice that

$$|E(K_{m,n}^{(k)})| = \sum_{p=1}^{k-1} \binom{m}{p}\binom{n}{k-p} = \binom{m+n}{k} - \binom{m}{k} - \binom{n}{k}.$$

Next, we give an example of complete bipartite 3-uniform hypergraphs, namely $K^{(3)}_{3,4}$.

Example 2.5. Let $V_1 = \{0, 1, 2\}$ and $V_2 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. The complete bipartite 3-uniform hypergraph $K^{(3)}_{3,4}$ whose vertex set is $V_1 \cup V_2$ has 30 edges as follows:
$\{0, \bar{0}, \bar{1}\}, \{0, \bar{0}, \bar{2}\}, \{0, \bar{0}, \bar{3}\}, \{0, \bar{1}, \bar{2}\}, \{0, \bar{1}, \bar{3}\}, \{0, \bar{2}, \bar{3}\}, \{1, \bar{0}, \bar{1}\}, \{1, \bar{0}, \bar{2}\}, \{1, \bar{0}, \bar{3}\},$
$\{1, \bar{1}, \bar{2}\}, \{1, \bar{1}, \bar{3}\}, \{1, \bar{2}, \bar{3}\}, \{2, \bar{0}, \bar{1}\}, \{2, \bar{0}, \bar{2}\}, \{2, \bar{0}, \bar{3}\}, \{2, \bar{1}, \bar{2}\}, \{2, \bar{1}, \bar{3}\}, \{2, \bar{2}, \bar{3}\},$
$\{0, 1, \bar{0}\}, \{0, 1, \bar{1}\}, \{0, 1, \bar{2}\}, \{0, 1, \bar{3}\}, \{0, 2, \bar{0}\}, \{0, 2, \bar{1}\}, \{0, 2, \bar{2}\}, \{0, 2, \bar{3}\}, \{1, 2, \bar{0}\},$
$\{1, 2, \bar{1}\}, \{1, 2, \bar{2}\}, \{1, 2, \bar{3}\}.$
Note that $\{0, 1, 2\}, \{\bar{0}, \bar{1}, \bar{2}\}, \{\bar{0}, \bar{1}, \bar{3}\}, \{\bar{0}, \bar{2}, \bar{3}\}$ and $\{\bar{1}, \bar{2}, \bar{3}\}$ are not edges of $K^{(3)}_{3,4}$.

Later on, some researchers added more partite sets and try to set up a definition of a complete r-partite k-uniform hypergraph. However, the knowledge of hypergraph is still dynamic. Thus, some literature may use different definitions of a complete r-partite k-uniform hypergraph. In this work, we are interested in the definition that was given by Kuhl and Schroeder [7] and by Boonklurb et al. [2].

Definition 2.6. [7] For $k \geq 2$, a **complete k-partite k-uniform hypergraph** $K^{(k)}_{k\times m}$ has a vertex set as k partite sets, $V_1, V_2, V_3, \ldots, V_k$, of equal size m and $E(K^{(k)}_{k\times m})$ is the set of all $\{v_1, v_2, v_3, \ldots, v_k\}$ such that $\{v_1, v_2, v_3, \ldots, v_k\}$ intersects every partite set.

Example 2.7. Let $V_1 = \{0, 1, 2\}, V_2 = \{\bar{0}, \bar{1}, \bar{2}\}$ and $V_3 = \{\bar{\bar{0}}, \bar{\bar{1}}, \bar{\bar{2}}\}$. The complete tripartite 3-uniform hypergraph $K^{(3)}_{3\times 3}$ whose vertex set is $V_1 \cup V_2 \cup V_3$ has 27 edges as follows:

$$\begin{gathered}
\{0, \bar{0}, \bar{\bar{0}}\}, \{0, \bar{0}, \bar{\bar{1}}\}, \{0, \bar{0}, \bar{\bar{2}}\}, \{0, \bar{1}, \bar{\bar{0}}\}, \{0, \bar{1}, \bar{\bar{1}}\}, \{0, \bar{1}, \bar{\bar{2}}\}, \{0, \bar{2}, \bar{\bar{0}}\}, \{0, \bar{2}, \bar{\bar{1}}\}, \{0, \bar{2}, \bar{\bar{2}}\},\\
\{1, \bar{0}, \bar{\bar{0}}\}, \{1, \bar{0}, \bar{\bar{1}}\}, \{1, \bar{0}, \bar{\bar{2}}\}, \{1, \bar{1}, \bar{\bar{0}}\}, \{1, \bar{1}, \bar{\bar{1}}\}, \{1, \bar{1}, \bar{\bar{2}}\}, \{1, \bar{2}, \bar{\bar{0}}\}, \{1, \bar{2}, \bar{\bar{1}}\}, \{1, \bar{2}, \bar{\bar{2}}\},\\
\{2, \bar{0}, \bar{\bar{0}}\}, \{2, \bar{0}, \bar{\bar{1}}\}, \{2, \bar{0}, \bar{\bar{2}}\}, \{2, \bar{1}, \bar{\bar{0}}\}, \{2, \bar{1}, \bar{\bar{1}}\}, \{2, \bar{1}, \bar{\bar{2}}\}, \{2, \bar{2}, \bar{\bar{0}}\}, \{2, \bar{2}, \bar{\bar{1}}\}, \{2, \bar{2}, \bar{\bar{2}}\}.
\end{gathered}$$

Note that Definition 2.6 can be modified to a complete k-partite k-uniform hypergraph whose partite sets have different sizes, $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$, and there are $n_1 n_2 n_3 \cdots n_k$ edges which each edge is still of the form $e = \{v_1, v_2, v_3, \ldots, v_k\}$, where $v_i \in V_i$ for all $i \in \{1, 2, 3, \ldots, k\}$.

Notice that if $a, b \in V_1$ and $c \in V_2$, then $\{a, b, c\}$ is not an edge of $K^{(3)}_{n_1,n_2,n_3}$ defined by Kuhl and Schroeder [7]. We give a remark here that the next definition considers a more general complete r-partite k-uniform hypergraph in which $\{a, b, c\}$ is its edge even if $a, b \in V_1$ and $c \in V_2$.

Definition 2.8. [2] For $k, r \geq 2$, a **complete r-partite k-uniform hypergraph** $K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}$ has a vertex set consisting of r partite sets $V_1, V_2, V_3, \ldots, V_r$ such that $|V_1| = n_1$, $|V_2| = n_2$, $|V_3| = n_3, \ldots, |V_r| = n_r$, $k \leq n_1 + n_2 + n_3 + \cdots + n_r$ and the edge set $E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}) = \{e : e \subseteq V(H), |e| = k \text{ and } |e \cap V_i| < k \text{ for all } i \in \{1, 2, 3, \ldots, r\}\}$.

From Definition 2.8, we notice that

$$|E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r})| = \binom{\sum_{i=1}^{r} n_i}{k} - \sum_{i=1}^{r} \binom{n_i}{k}.$$

Next, we give an example of a complete tripartite 3-uniform hypergraph, namely $K^{(3)}_{2,3,4}$.

Example 2.9. Let $V_1 = \{0, 1\}, V_2 = \{\bar{0}, \bar{1}, \bar{2}\}$ and $V_3 = \{\bar{\bar{0}}, \bar{\bar{1}}, \bar{\bar{2}}, \bar{\bar{3}}\}$. The complete tripartite 3-uniform hypergraph $K^{(3)}_{2,3,4}$ whose vertex set is $V_1 \cup V_2 \cup V_3$ has 79 edges as follows:

$$\begin{gathered}
\{0,\bar{0},\bar{\bar{0}}\}, \{0,\bar{0},\bar{\bar{1}}\}, \{0,\bar{0},\bar{\bar{2}}\}, \{0,\bar{0},\bar{\bar{3}}\}, \{0,\bar{1},\bar{\bar{0}}\}, \{0,\bar{1},\bar{\bar{1}}\}, \{0,\bar{1},\bar{\bar{2}}\}, \{0,\bar{1},\bar{\bar{3}}\}, \{0,\bar{2},\bar{\bar{0}}\}, \\
\{0,\bar{2},\bar{\bar{1}}\}, \{0,\bar{2},\bar{\bar{2}}\}, \{0,\bar{2},\bar{\bar{3}}\}, \{1,\bar{0},\bar{\bar{0}}\}, \{1,\bar{0},\bar{\bar{1}}\}, \{1,\bar{0},\bar{\bar{2}}\}, \{1,\bar{0},\bar{\bar{3}}\}, \{1,\bar{1},\bar{\bar{0}}\}, \{1,\bar{1},\bar{\bar{1}}\}, \\
\{1,\bar{1},\bar{\bar{2}}\}, \{1,\bar{1},\bar{\bar{3}}\}, \{1,\bar{2},\bar{\bar{0}}\}, \{1,\bar{2},\bar{\bar{1}}\}, \{1,\bar{2},\bar{\bar{2}}\}, \{1,\bar{2},\bar{\bar{3}}\}, \{0,1,\bar{0}\}, \{0,1,\bar{1}\}, \{0,1,\bar{2}\}, \\
\{0,1,\bar{\bar{0}}\}, \{0,1,\bar{\bar{1}}\}, \{0,1,\bar{\bar{2}}\}, \{0,1,\bar{\bar{3}}\}, \{0,\bar{0},\bar{1}\}, \{1,\bar{0},\bar{1}\}, \{\bar{0},\bar{1},\bar{\bar{0}}\}, \{\bar{0},\bar{1},\bar{\bar{1}}\}, \{\bar{0},\bar{1},\bar{\bar{2}}\}, \\
\{\bar{0},\bar{1},\bar{\bar{3}}\}, \{0,\bar{0},\bar{2}\}, \{1,\bar{0},\bar{2}\}, \{\bar{0},\bar{2},\bar{\bar{0}}\}, \{\bar{0},\bar{2},\bar{\bar{1}}\}, \{\bar{0},\bar{2},\bar{\bar{2}}\}, \{\bar{0},\bar{2},\bar{\bar{3}}\}, \{0,\bar{1},\bar{2}\}, \{1,\bar{1},\bar{2}\}, \\
\{\bar{1},\bar{2},\bar{\bar{0}}\}, \{\bar{1},\bar{2},\bar{\bar{1}}\}, \{\bar{1},\bar{2},\bar{\bar{2}}\}, \{\bar{1},\bar{2},\bar{\bar{3}}\}, \{0,\bar{\bar{0}},\bar{\bar{1}}\}, \{1,\bar{\bar{0}},\bar{\bar{1}}\}, \{\bar{0},\bar{\bar{0}},\bar{\bar{1}}\}, \{\bar{1},\bar{\bar{0}},\bar{\bar{1}}\}, \{\bar{2},\bar{\bar{0}},\bar{\bar{1}}\}, \\
\{0,\bar{\bar{0}},\bar{\bar{2}}\}, \{1,\bar{\bar{0}},\bar{\bar{2}}\}, \{\bar{0},\bar{\bar{0}},\bar{\bar{2}}\}, \{\bar{1},\bar{\bar{0}},\bar{\bar{2}}\}, \{\bar{2},\bar{\bar{0}},\bar{\bar{2}}\}, \{0,\bar{\bar{0}},\bar{\bar{3}}\}, \{1,\bar{\bar{0}},\bar{\bar{3}}\}, \{\bar{0},\bar{\bar{0}},\bar{\bar{3}}\}, \{\bar{1},\bar{\bar{0}},\bar{\bar{3}}\}, \\
\{\bar{2},\bar{\bar{0}},\bar{\bar{3}}\}, \{0,\bar{\bar{1}},\bar{\bar{2}}\}, \{1,\bar{\bar{1}},\bar{\bar{2}}\}, \{\bar{0},\bar{\bar{1}},\bar{\bar{2}}\}, \{\bar{1},\bar{\bar{1}},\bar{\bar{2}}\}, \{\bar{2},\bar{\bar{1}},\bar{\bar{2}}\}, \{0,\bar{\bar{1}},\bar{\bar{3}}\}, \{1,\bar{\bar{1}},\bar{\bar{3}}\}, \{\bar{0},\bar{\bar{1}},\bar{\bar{3}}\}, \\
\{\bar{1},\bar{\bar{1}},\bar{\bar{3}}\}, \{\bar{2},\bar{\bar{1}},\bar{\bar{3}}\}, \{0,\bar{\bar{2}},\bar{\bar{3}}\}, \{1,\bar{\bar{2}},\bar{\bar{3}}\}, \{\bar{0},\bar{\bar{2}},\bar{\bar{3}}\}, \{\bar{1},\bar{\bar{2}},\bar{\bar{3}}\}, \{\bar{2},\bar{\bar{2}},\bar{\bar{3}}\}.
\end{gathered}$$

Note that $\{\bar{0},\bar{1},\bar{2}\}, \{\bar{\bar{0}},\bar{\bar{1}},\bar{\bar{2}}\}, \{\bar{\bar{0}},\bar{\bar{1}},\bar{\bar{3}}\}, \{\bar{\bar{1}},\bar{\bar{2}},\bar{\bar{3}}\}$ are not edges of $K^{(3)}_{2,3,4}$. Next, let us introduce a cycle of length q of a hypergraph H.

Definition 2.10. [1] For $q > 1$, a **cycle of length q of a hypergraph H** is defined to be a sequence $(v_1, e_1, v_2, e_2, v_3, e_3, \ldots, e_q, v_{q+1})$ such that

1. $v_1, v_2, v_3, \ldots, v_q$ are all distinct vertices of H,
2. $e_1, e_2, e_3, \ldots, e_q$ are all distinct edges of H and
3. $v_k, v_{k+1} \in e_k$ for $k \in \{1, 2, 3, \ldots, q\}$ and $v_{q+1} = v_1$.

H :

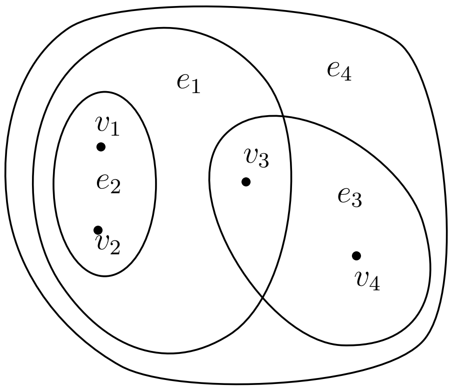


Figure 2.3: The hypergraph H with cycle $(v_1, e_2, v_2, e_1, v_3, e_3, v_4, e_4, v_1)$.

For example, $(v_1, e_2, v_2, e_1, v_3, e_3, v_4, e_4, v_1)$ is a cycle of length 4 of the hypergraph H shown in Figure 2.3.

In general, the idea of the hypergraph is to generalize a graph. Also, in a vertex coloring, there is a definition of hypergraph colorings which generalize the respective graph concepts as follows.

Definition 2.11. [9] A **$\boldsymbol{\lambda}$-coloring** of a hypergraph H is a labeling $f : V(H) \to \{1, 2, 3, \ldots, \lambda\}$ and a **proper $\boldsymbol{\lambda}$-coloring** of a hypergraph H is a λ-coloring of a hypergraph H such that no edge of H (besides singletons) has all vertices of the same color. The **chromatic number** of a hypergraph H, denoted by $\chi(H)$, is the minimum number λ for which a proper λ-coloring exists.

An edge of a hypergraph is said to be **monochromatic** if all its vertices have the same color.

Example 2.12. $K_{10}^{(3)}$ has 10 vertices, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$.

Let $f : V(K_{10}^{(3)}) \to \{1, 2, 3, 4, 5\}$ be a labeling defined by $f(v_i) = 1$ for $i \in \{1, 10\}$, $f(v_i) = 2$ for $i \in \{2, 9\}$, $f(v_i) = 3$ for $i \in \{3, 8\}$, $f(v_i) = 4$ for $i \in \{4, 7\}$ and $f(v_i) = 5$ for $i \in \{5, 6\}$. Because of all edges having size 3, there are no edges being monochromatic. Then, $K_{10}^{(3)}$ has a proper 5-coloring.

If we assign only four colors, then we can use the Pigeonhole principle to obtain that there is a color class containing at least 3 vertices. Then, there is monochromatic edges. Thus, $K_{10}^{(3)}$ has no proper 4-coloring. Therefore, $\chi(K_{10}^{(3)}) = 5$.

Example 2.13. Consider a complete 4-partite 3-uniform hypergraph $K_{2,2,3,3}^{(3)}$ of which each edge has 3 vertices from at least 2 different partite sets.

Assign 4 colors to all vertices of $K_{2,2,3,3}^{(3)}$ with only vertices of the same partite set got the same color. Thus, $K_{2,2,3,3}^{(3)}$ has a proper 4-coloring. Now, assign 3 colors to $K_{2,2,3,3}^{(3)}$. Since $K_{2,2,3,3}^{(3)}$ has 10 vertices, there are one color class contains at least 4 vertices and such 4 vertices have at least two vertices from different partite sets. Thus, there is a monochromatic edge and hence, $K_{2,2,3,3}^{(3)}$ has no proper 3-coloring. Therefore, $\chi(K_{2,2,3,3}^{(3)}) = 4$.

Example 2.14. Consider $K_{2,2,2}^{(4)}$. Let $V_1 = \{v_1, v_2\}$, $V_2 = \{\bar{v_1}, \bar{v_2}\}$ and $V_3 = \{\bar{\bar{v_1}}, \bar{\bar{v_2}}\}$ be the partite sets of $K_{2,2,2}^{(4)}$. Assign two colors to all vertices of $K_{2,2,2}^{(4)}$ as follows.

Let $f : V(K_{2,2,2}^{(4)}) \to \{1, 2\}$ be a labeling defined by $f(i) = 1$ for $i \in \{v_1, v_2, \bar{v_1}\}$ and $f(i) = 2$ for $i \in \{\bar{v_2}, \bar{\bar{v_1}}, \bar{\bar{v_2}}\}$. Because of all edges having size 4, there are no edges being monochromatic. Then, $K_{2,2,2}^{(4)}$ has a proper 2-coloring. It is easy to see that $K_{2,2,2}^{(4)}$ has no proper 1-coloring. Therefore, $\chi(K_{2,2,2}^{(4)}) = 2$.

These three examples leads to the following two lemmas.

Lemma 2.15. *For $n \geq k \geq 2$. $\chi(K_n^{(k)}) = \left\lceil \dfrac{n}{k-1} \right\rceil$.*

Proof. Assign $\left\lceil \dfrac{n}{k-1} \right\rceil$ colors to all n vertices of $K_n^{(k)}$ by coloring each color to at most $k-1$ vertices. Since every edge of $K_n^{(k)}$ contains k vertices, $K_n^{(k)}$ has a proper $\left\lceil \dfrac{n}{k-1} \right\rceil$-coloring.

Now, suppose that $K_n^{(k)}$ has a proper $\left(\left\lceil \frac{n}{k-1} \right\rceil - 1\right)$-coloring. Since any k-subset of $V(K_n^{(k)})$ is an edge, the number of vertices assigned the same colors must be at most $k-1$ vertices. Thus, $\left(\left\lceil \frac{n}{k-1} \right\rceil - 1\right)(k-1) \geq n$.

Let $n = (k-1)q + r$, where $0 \leq r < k-1$.

If $r = 0$, then $\left\lceil \frac{n}{k-1} \right\rceil - 1 = q - 1$ and $\left(\left\lceil \frac{n}{k-1} \right\rceil - 1\right)(k-1) = (q-1)(k-1) < q(k-1) = n$, a contradiction.

If $0 < r < k-1$, then $\left\lceil \frac{n}{k-1} \right\rceil - 1 = q$ and $\left(\left\lceil \frac{n}{k-1} \right\rceil - 1\right)(k-1) = q(k-1) = n - r < n$, a contradiction.

Thus, $K_n^{(k)}$ has no proper $\left(\left\lceil \frac{n}{k-1} \right\rceil - 1\right)$-coloring. Therefore, $\chi(K_n^{(k)}) = \left\lceil \frac{n}{k-1} \right\rceil$. □

Lemma 2.16. *Let $k, r \geq 2$. $\chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}) \leq r$, where $K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}$ is the hypergraph defined in Definition 2.8.*

Proof. Assign r colors to all vertices of $K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}$ with only vertices of the same partite set got the same colors. Thus, $K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}$ has a proper r-coloring. Therefore, $\chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}) \leq r$. □

Note that if we consider the complete multipartite k-uniform hypergraph defined in Definition 2.6, then we can also use the same idea to proof Lemma 2.16 to conclude that $\chi(K^{(k)}_{k\times m}) \leq k$.

There are many research articles that studied coloring on hypergraphs. For example, in 2013, Frieze and Mubayi [5] considered a simple k-uniform hypergraph for an integer $k \geq 3$ (a k-uniform hypergraph is **simple** if every two edges share at most one vertex). They showed that there is a constant c depending only on k such that every simple k-uniform hypergraph H with maximum degree $\triangle$ has chromatic number satisfying

$$\chi(H) < c\left(\frac{\triangle}{\log\triangle}\right)^{\frac{1}{k-1}}.$$

A defective coloring on a graph is one of the generalized idea of proper coloring on a graph and it is one of the interesting concepts that has been studied by many researchers, see [3] and [4]. It can be used to solve the scheduling problem. Some researchers added the condition that each color class of a defective coloring is acyclic and found the defective chromatic number of this type. In the following definition, we modify the definition of a defective coloring and a defective chromatic number on graphs to a defective coloring and a defective chromatic number on hypergraphs as follows.

Definition 2.17. Let $d \geq 0$. A $\boldsymbol{(\lambda, d)}$**-defective coloring** of a hypergraph H is a λ-coloring of a hypergraph H in which there are at most d monochromatic edges. If H admits a (λ, d)-defective coloring, then $\chi_{\leq d}(H)$ denotes the least integer λ.

Note that $(\lambda, 0)$-defective coloring of a hypergraph H is a proper λ-coloring of such hypergraph.

Example 2.18. Consider a hypergraph $K_{10}^{(3)}$ from Example 2.12. We obtain that a proper 5-coloring of a hypergraph $K_{10}^{(3)}$ has a $(5, 0)$-defective coloring of $K_{10}^{(3)}$.

Let $d = 1$. Since $d = 1$, there is **at most** 1 edge of $K_{10}^{(3)}$ being monochromatic. Then, we get immediately that $K_{10}^{(3)}$ has a $(5, 1)$-defective coloring. Now, we suppose that $K_{10}^{(3)}$ has a $(4, 1)$-defective coloring. By Example 2.12, $K_{10}^{(3)}$ has no $(4, 0)$-defective coloring. Then, $K_{10}^{(3)}$ has exactly one monochromatic edge containing 3 vertices. Consider the other 7 vertices outside this monochromatic edge. Since there are 3 colors left to be assigned, we use the Pigeonhole principle to obtain that there is a color class containing at least $\left\lceil \frac{7}{3} \right\rceil = 3$ vertices, impossible. Therefore, $\chi_{\leq 1}(K_{10}^{(3)}) = 5$.

Let $d = 2$. We show that $K_{10}^{(3)}$ has a $(4, 2)$-defective coloring. Let $f : V(K_{10}^{(3)}) \rightarrow \{1, 2, 3, 4\}$ be a labeling defined by $f(v_i) = 1$ for $i \in \{1, 2, 3\}$, $f(v_i) = 2$ for $i \in \{4, 5, 6\}$, $f(v_i) = 3$ for $i \in \{7, 8\}$ and $f(v_i) = 4$ for $i \in \{9, 10\}$. Since all edges having size 3, there are exactly two edges being monochromatic. Thus, $K_{10}^{(3)}$ has a $(4, 2)$-defective coloring. Now, we suppose that $K_{10}^{(3)}$ has a $(3, 2)$-defective coloring. Since $K_{10}^{(3)}$ has no $(4, 1)$-defective coloring, we get immediately that $K_{10}^{(3)}$ has no $(3, 1)$-defective coloring. Thus, $K_{10}^{(3)}$ has exactly two disjoint monochromatic edges of two different colors if two such edges are not disjoint, then it will occur at least

4 monochromatic edges. Consider the other 4 vertices outside these two disjoint edges. Since there is only one color left, we obtain that there is a color class containing 4 vertices, impossible. Therefore, $\chi_{\leq 2}(K_{10}^{(3)}) = 4$. For $d = 3, 4$ and 5, we obtain $\chi_{\leq d}(K_{10}^{(3)}) = 4$ in a similar way.

Let $d = 6$. We show that $K_{10}^{(3)}$ has a $(3, 6)$-defective coloring. Let $f : V(K_{10}^{(3)}) \to \{1, 2, 3\}$ be a labeling defined by $f(v_i) = 1$ for $i \in \{1, 2, 3, 4\}$, $f(v_i) = 2$ for $i \in \{5, 6, 7\}$ and $f(v_i) = 3$ for $i \in \{8, 9, 10\}$. Since all edges having size 3, there are four edges containing all vertices of color 1 and the other two edges containing all vertices of colors 2 and 3. Thus, there are six edges being monochromatic. Hence, $K_{10}^{(3)}$ has a $(3, 6)$-defective coloring. Now, If we assign only two colors, then it occurs at least 20 monochromatic edges, impossible. Therefore, $\chi_{\leq 6}(K_{10}^{(3)}) = 3$.

From this example, we see that the chromatic number of a hypergraph H decreases when the value of d is increasing.

Next, let us consider relation between a complete hypergraph and a complete multipartite hypergraph as follows.

Theorem 2.19. *Let $r \geq 2$, $n = n_1 + n_2 + n_3 + \cdots + n_r \geq k \geq 2$ and $d \geq 0$. If $\binom{n_1}{k} + \binom{n_2}{k} + \binom{n_3}{k} + \cdots + \binom{n_r}{k} \leq d$, then $\chi_{\leq d}(K_n^{(k)}) \leq \chi(K_{n_1,n_2,n_3,\ldots,n_r}^{(k)})$.*

Proof. Let $d \geq 0$ and $\chi(K_{n_1,n_2,n_3,\ldots,n_r}^{(k)}) = t$. Consider $K_n^{(k)}$. Since $n = n_1 + n_2 + n_3 + \cdots + n_r$, we divide n vertices of $K_n^{(k)}$ into r parts, V_i where $|V_i| = n_i$ for all $i \in \{1, 2, 3, \ldots, r\}$. Now, we show that $K_n^{(k)}$ has a (t, d)-defective coloring. Since $K_{n_1,n_2,n_3,\ldots,n_r}^{(k)}$ has a proper t-coloring, we can color r parts of $K_n^{(k)}$ similar to such coloring of $K_{n_1,n_2,n_3,\ldots,n_r}^{(k)}$. Next, consider a coloring in each part. Since $K_n^{(k)}$ is a complete k-uniform hypergraph, there are edges containing vertices in each part. Since each part has at most n_i vertices assigned the same color, the number of monochromatic edges is $\binom{n_1}{k} + \binom{n_2}{k} + \binom{n_3}{k} + \cdots + \binom{n_r}{k} \leq d$. Thus, $K_n^{(k)}$ has a (t, d)-defective coloring. Therefore, $\chi_{\leq d}(K_n^{(k)}) \leq \chi(K_{n_1,n_2,n_3,\ldots,n_r}^{(k)})$. □

Corollary 2.20. *Let $r \geq 2$, $n = n_1 + n_2 + n_3 + \cdots + n_r \geq k \geq 2$ and $d \geq 0$. If $\binom{n_1}{k} + \binom{n_2}{k} + \binom{n_3}{k} + \cdots + \binom{n_r}{k} \leq d < \binom{\lceil \frac{n}{r-1} \rceil}{k}$, then $\chi_{\leq d}(K_n^{(k)}) = \chi(K_{n_1,n_2,n_3,\ldots,n_r}^{(k)})$.*

Proof. By Theorem 2.19 and Lemma 2.16, $\chi_{\le d}(K_n^{(k)}) \le \chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}) \le r$. Let us assign $r-1$ colors to all vertices of $K_n^{(k)}$. Since $K_n^{(k)}$ has n vertices, by the Pigeonhole principle, there are $\left\lceil \dfrac{n}{r-1} \right\rceil$ vertices assigned the same colors. Since such $\left\lceil \dfrac{n}{r-1} \right\rceil$ vertices induce $\binom{\lceil \frac{n}{r-1} \rceil}{k}$ monochromatic edges which is more than d. Thus, in this case $\chi_{\le d}(K_n^{(k)}) > r-1$ and hence, $r-1 < \chi_{\le d}(K_n^{(k)}) \le \chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}) \le r$. Therefore, $\chi_{\le d}(K_n^{(k)}) = \chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_r}) = r$. □

Example 2.21. Consider $K_{13}^{(3)}$ and $K_{4,4,5}^{(3)}$.

Let $d = 20$. Since $n = 13$, $n_1 = 4, n_2 = 4, n_3 = 5$, $\binom{n_1}{k} + \binom{n_2}{k} + \binom{n_3}{k} = \binom{4}{3} + \binom{4}{3} + \binom{5}{3} = 4 + 4 + 10 = 18$ and $\binom{\lceil \frac{n}{r-1} \rceil}{k} = \binom{\lceil \frac{13}{3-1} \rceil}{3} = \binom{7}{3} = 35$. Then, $\binom{n_1}{k} + \binom{n_2}{k} + \binom{n_3}{k} \le d < \binom{\lceil \frac{n}{r-1} \rceil}{k}$. By Corollary 5.4, $\chi_{\le 20}(K_{13}^{(3)}) = \chi(K_{4,4,5}^{(3)}) \le 3$.

Now, we assign two colors to all vertices of $K_{13}^{(3)}$. Since there are 13 vertices with two colors, we use the Pigeonhole principle to obtain that there is a color class containing at least $\left\lceil \dfrac{13}{2} \right\rceil = 7$ vertices of the same colors and then those 7 vertices induces $\binom{7}{3} = 35$ monochromatic edges which is more than d. Thus, $K_{13}^{(3)}$ has no $(2, 20)$-defective coloring and hence, $2 < \chi_{\le 20}(K_{13}^{(3)}) \le \chi(K_{4,4,5}^{(3)}) \le 3$. Therefore, $\chi_{\le 20}(K_{13}^{(3)}) = \chi(K_{4,4,5}^{(3)}) = 3$.

Example 2.22. Consider $K_{13}^{(3)}$ and $K_{4,4,5}^{(3)}$.

Let $d = 56$, that is, $d \ge max\{\binom{n_1}{k} + \binom{n_2}{k} + \binom{n_3}{k}, \binom{\lceil \frac{n}{r-1} \rceil}{k}\}$. From the previous example, $\chi(K_{4,4,5}^{(3)}) = 3$. By Theorem 2.19, $\chi_{\le 56}(K_{13}^{(3)}) \le \chi(K_{4,4,5}^{(3)}) = 3$. However, we can assign two colors to the vertices of $K_{13}^{(3)}$ by coloring 1 to 6 vertices and 2 to other 7 vertices. Then, it occurs $\binom{6}{3} + \binom{7}{3} = 20 + 35 = 55$ monochromatic edges which is less than d. Thus, $K_{13}^{(3)}$ has a $(2, 55)$-defective coloring. Therefore, $\chi_{\le 56}(K_{13}^{(3)}) < \chi(K_{4,4,5}^{(3)})$.

From Theorem 2.19 and its corollary, we can see that the number of vertices in $K_n^{(k)}$ determine the number of colors to be assigned.

Theorem 2.23. *Let $k \geq 3$. If $0 \leq d < k+1$, then there is an integer n such that $\chi_{\leq d}(K_n^{(k)}) = l$ for all $l \in \mathbb{N}$.*

Proof. Let $0 \leq d < k+1$ and $l \in \mathbb{N}$. Choose $n = (l-1)(k-1) + kd$. Consider $K_n^{(k)}$. Color all vertices of $K_n^{(k)}$ in such a way that there are at most d edges of $K_n^{(k)}$ being monochromatic edges. Since $d < k+1$, all of d monochromatic edges of $K_n^{(k)}$ are pairwise disjoint. If all of d monochromatic edges are not pairwise disjoint, then there are at least $k+1$ vertices having the same color and such $k+1$ vertices form $\binom{k+1}{k} = k+1$ monochromatic edges which is more than d. Now, we assign d colors to kd vertices by each color is assigned to exactly k vetices. Then, there are d monochromatic edges of $K_n^{(k)}$. Next, color other $n-kd$ vertices by using $\left\lceil \dfrac{n-kd}{k-1} \right\rceil$ colors. By Lemma 2.15, each edge form by these $n-kd$ vertices is not a monochromatic edge. Thus, $\chi_{\leq d}(K_n^{(k)}) = \left\lceil \dfrac{n-kd}{k-1} \right\rceil + 1 = \left\lceil \dfrac{(l-1)(k-1)+kd-kd}{k-1} \right\rceil = (l-1)+1 = l$. □

Theorem 2.24. *Let $k \geq 3$. For any hypergraph H, if $0 \leq d < k+1$, then*

$$\chi_{\leq d}(K_n^{(k)}) = \begin{cases} \left\lceil \dfrac{n}{k} \right\rceil, & \textit{if } \ kd \geq n, \\[2em] \left\lceil \dfrac{n-kd}{k-1} \right\rceil + d, & \textit{if } \ kd < n. \end{cases}$$

Proof. We know from the proof of Theorem 2.23 that if $d < k+1$, then d monochromatic edges are pairwise disjoint.

Case 1: $kd \geq n$. Assign $\left\lceil \frac{n}{k} \right\rceil$ colors to n vertices in such a way that at most k vertices having the same color. Since $kd \geq n$, $d \geq \left\lceil \frac{n}{k} \right\rceil$. Thus, $K_n^{(k)}$ has a $\left(\left\lceil \frac{n}{k} \right\rceil, d\right)$-defective coloring. Suppose that $K_n^{(k)}$ has a $\left(\left\lceil \frac{n}{k} \right\rceil - 1, d\right)$-defective coloring. Since each monochromatic edge is pairwise disjoint, the number of vertices assigned by the same color must be at most k. Thus, $\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) k \geq n$.

Let $n = kq + r$ where $0 \leq r < k$.

If $r = 0$, then $\left\lceil \frac{n}{k} \right\rceil = q$ and $\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) k = (q-1)k < qk = n$, a contradiction.

If $0 < r < k$, then $\left\lceil \frac{n}{k} \right\rceil = q{+}1$ and $\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) k = (q{+}1{-}1)k = kq = n{-}r < n$, a contradiction.

Thus, $\chi_{\leq d}(K_n^{(k)}) = \left\lceil \frac{n}{k} \right\rceil$.

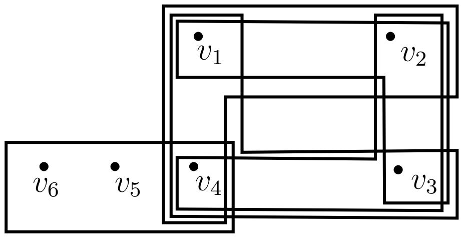
Case 2: $kd < n$. Assign d colors to kd vertices in such a way that at most k vertices having the same color. Since $kd < n$, there are $n{-}kd$ vertices left to be colored. Assign $\left\lceil \frac{n-kd}{k-1} \right\rceil$ colors to those left over vertices in such a way that at most $k{-}1$ vertices having the same color. Thus, $K_n^{(k)}$ has a $\left(\left\lceil \frac{n-kd}{k-1} \right\rceil + d, d\right)$-defective coloring.

Next, let us try to assign $\left\lceil \frac{n-kd}{k-1} \right\rceil + d{-}1$ colors to each vertex of $K_n^{(k)}$. If there are $d{-}i$ monochromatic edges where $0 \leq i \leq d$, then those $d{-}i$ monochromatic edges contain $k(d{-}i)$ vertices and then there are $n{-}kd + ki$ vertices left to be colored. Thus, there are other $\left\lceil \frac{n-kd}{k-1} \right\rceil + (d-1) - (d-i) = \left\lceil \frac{n-kd}{k-1} \right\rceil + (i-1)$ colors that can be assigned to those $n{-}kd + ki$ vertices and all edges constructed from those vertices are not monochromatic. However, for a proper vertex coloring, we need at least $\left\lceil \frac{n-kd+ki}{k-1} \right\rceil$ colors. Since $(k+i){-}1 > 2$, $\left\lceil \frac{n-kd+ki}{k-1} \right\rceil > \left\lceil \frac{(n-kd+ki)-(k+i)+1}{k-1} \right\rceil = \left\lceil \frac{n-kd}{k-1} \right\rceil + (i{-}1)$. Hence, $K_n^{(k)}$ has no $\left(\left\lceil \frac{n-kd}{k-1} \right\rceil + d-1, d\right)$-defective coloring. Thus, $\chi_{\leq d}(K_n^{(k)}) = \left\lceil \frac{n-kd}{k-1} \right\rceil + d$. □

Theorem 2.24 indicate that if a k-uniform hypergraph contains a large complete k-uniform hypergraph as its subhypergraph, then the number of colors to be assigned need to be at least greater or equal to the number of vertices contained in this induced subhyergraph. Now, let us introduce the clique number of a k-uniform hypergraph as follows.

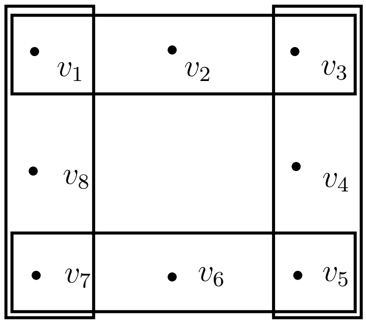
Definition 2.25. [1] Let $k \geq 2$. For a k-uniform hypergraph H, a nonempty set $A \subset V(H)$ is defined to be a **clique** if either $|A| < k$ or A is a complete k-uniform subhypergraph of H. The **clique number**, denoted by $\omega(H)$, of a k-uniform hypergraph H is the largest cardinality of a subset of $V(H)$ inducing a complete k-uniform hypergraph.

Example 2.26. Let H be a 3-uniform hypergraph with 6 vertices, as follows.



We see that $\{v_1, v_2, v_3, v_4\}$ induces $K_4^{(3)}$ and it is easy to see that 4 is the largest number of a subset of $V(H)$ inducing a complete 3-uniform hypergraph. Thus, $\omega(H) = 4$.

Example 2.27. Let H be a 3-uniform hypergraph with 8 vertices, as follows.



We see that $\{v_1, v_2, v_3\}$ induces $K_3^{(3)}$ and it is easy to see that 3 is the largest number of a subset of $V(H)$ inducing a complete 3-uniform hypergraph. Thus, $\omega(H) = 3$.

Next, we study relation between $\chi_{\leq d}(H)$ and $\omega(H)$ for any k-uniform hypergraph and $0 \leq d < k+1$.

Theorem 2.28. *Let $k \geq 3$. For any k-uniform hypergraph H, if $0 \leq d < k+1$, then*

$$\chi_{\leq d}(H) \geq \begin{cases} \left\lceil \dfrac{\omega(H)}{k} \right\rceil, & \text{if } kd \geq \omega(H), \\[2ex] \left\lceil \dfrac{\omega(H) - kd}{k-1} \right\rceil + d, & \text{if } kd < \omega(H). \end{cases}$$

Proof. Let $\omega(H) = m$. Then, H contains $K_m^{(k)}$.

Case 1: Let $kd \geq \omega(H)$. Since H contains $K_m^{(k)}, kd \geq m$ and $0 \leq d < k+1$, by

Theorem 2.24, $\left\lceil \frac{m}{k} \right\rceil$ colors are required to color just the clique. Thus, $\chi_{\leq d}(H) \geq \left\lceil \frac{\omega(H)}{k} \right\rceil$.

Case 2: Let $kd < \omega(H)$. Since H contains $K_m^{(k)}, kd < m$ and $0 \leq d < k+1$, by Theorem 2.24, $\left\lceil \frac{m-kd}{k-1} \right\rceil + d$ colors are required to color just the clique. Thus, $\chi_{\leq d}(H) \geq \left\lceil \frac{\omega(H)-kd}{k-1} \right\rceil + d$. □

Example 2.29. Consider the 3-uniform hypergraph H from Example 2.26. The hypergraph H has $k = 3$.

If $d = 0$, then $kd = 0 < 4 = \omega(H)$.
By Theorem 2.28, $\chi(H) \geq \left\lceil \frac{\omega(H)-kd}{k-1} \right\rceil + d = \left\lceil \frac{4-0}{2} \right\rceil + 0 = 2$.

If $d = 1$, then $kd = 3 < 4 = \omega(H)$.
By Theorem 2.28, $\chi_{\leq 1}(H) \geq \left\lceil \frac{\omega(H)-kd}{k-1} \right\rceil + d = \left\lceil \frac{4-3}{2} \right\rceil + 1 = 1 + 1 = 2$.

If $2 \leq d \leq 4$. Then, $kd \geq 4 = \omega(H)$.
By Theorem 2.28, $\chi_{\leq d}(H) \geq \left\lceil \frac{\omega(H)}{k} \right\rceil = \left\lceil \frac{4}{3} \right\rceil = 2$.

Example 2.30. Consider a hypergraph $K_{10}^{(3)}$. It is obvious that $\omega(K_{10}^{(3)}) = 10$.

If $d = 0$, then $kd = 0 < 10 = \omega(K_{10}^{(3)})$. By Theorem 2.28, $\chi(K_{10}^{(3)}) \geq \left\lceil \frac{\omega(K_{10}^{(3)})-kd}{k-1} \right\rceil + d = \left\lceil \frac{10-0}{2} \right\rceil + 0 = 5$.

If $d = 1$, then $kd = 3 < 10 = \omega(K_{10}^{(3)})$. By Theorem 2.28, $\chi_{\leq 1}(K_{10}^{(3)}) \geq \left\lceil \frac{\omega(K_{10}^{(3)})-kd}{k-1} \right\rceil + d = \left\lceil \frac{10-3}{2} \right\rceil + 1 = 4 + 1 = 5$.

If $d = 2$, then $kd = 6 < 10 = \omega(K_{10}^{(3)})$. By Theorem 2.28, $\chi_{\leq 2}(K_{10}^{(3)}) \geq \left\lceil \frac{\omega(K_{10}^{(3)})-kd}{k-1} \right\rceil + d = \left\lceil \frac{10-6}{2} \right\rceil + 2 = 2 + 2 = 4$.

If $d = 3$, then $kd = 9 < 10 = \omega(K_{10}^{(3)})$. By Theorem 2.28, $\chi_{\leq 3}(K_{10}^{(3)}) \geq \left\lceil \frac{\omega(K_{10}^{(3)})-kd}{k-1} \right\rceil + d = \left\lceil \frac{10-9}{2} \right\rceil + 3 = 1 + 3 = 4$.

If $d = 4$, then $kd = 12 \geq 10 = \omega(K_{10}^{(3)})$. By Theorem 2.28, $\chi_{\leq 4}(K_{10}^{(3)}) \geq \left\lceil \frac{\omega(K_{10}^{(3)})}{k} \right\rceil = \left\lceil \frac{10}{3} \right\rceil = 4$.

In 2016, for $d \geq 0$, Muaengwaeng and Nakprasit [8] considered (λ, d)-defective colorings on a graph and each color class induces a forest, i.e., each color class is acyclic. We also modify their idea as follows.

Definition 2.31. Let $d \geq 0$. A **(λ, d)-defective coloring without monochromatic cycle** of a hypergraph H is a (λ, d)-defective coloring of a hypergraph of which each color class which is a set of the same color vertices induces an acyclic subhypergraph of H. If H admits a (λ, d)-defective coloring without monochromatic cycle, then $\chi^T_{\leq d}(H)$ denotes the least integer λ. If $d = 0$, we use $\chi^T(H)$ instead of $\chi^T_{\leq 0}(H)$.

Notices that if hypergraph H contains cycle, then $\chi^T_{\leq d}(H) \neq 1$.

Example 2.32. $K_5^{(3)}$ has 5 vertices, $\{v_1, v_2, v_3, v_4, v_5\}$, and 10 edges,
$e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_1, v_2, v_4\}, e_3 = \{v_1, v_2, v_5\}, e_4 = \{v_1, v_3, v_4\}, e_5 = \{v_1, v_3, v_5\}$,
$e_6 = \{v_1, v_4, v_5\}, e_7 = \{v_2, v_3, v_4\}, e_8 = \{v_2, v_3, v_5\}, e_9 = \{v_2, v_4, v_5\}$ and
$e_{10} = \{v_3, v_4, v_5\}$.

Let $d = 4$ and $f : V(K_5^{(3)}) \to \{1, 2\}$ be a labeling defined by $f(v_i) = 1$ for $i \in \{1, 2, 3, 4\}$, $f(v_5) = 2$. Then, e_1, e_2, e_4 and e_7 are monochromatic. Thus, $K_5^{(3)}$ has a $(2, 4)$-defective coloring. However, there is the cycle $(v_1, e_1, v_2, e_2, v_4, e_4, v_1)$, that v_1, v_2 and v_4 are assigned the same color, namely 1. Thus, this $(2, 4)$-defective coloring does not satisfy Definition 2.31.

Example 2.33. The hypergraph H shown in Figure 2.4 has 8 vertices and 4 edges.

Let $f : V(H) \to \{1, 2\}$ be a labeling defined by $f(v_i) = 1$ for $i \in \{3, 7, 8\}$ and $f(v_i) = 2$ for $i \in \{1, 2, 4, 5, 6\}$.

Consider vertices in the color class 1, there is only one edge of H containing v_3, v_7, v_8. Thus, this color class is acyclic.

Now, consider color class 2, $\{v_1, v_2, v_4, v_5, v_6\}$. This color class is an independent set. Hence, $\{v_1, v_2, v_4, v_5, v_6\}$ induces an acyclic subhypergraph.

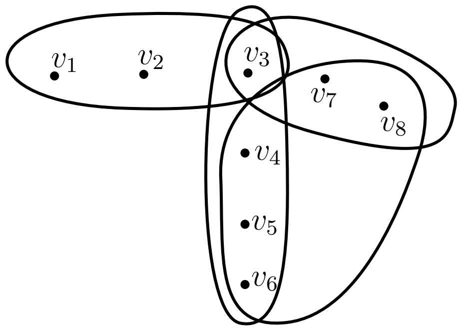


Figure 2.4: Hypergraph H.

Therefore, H has a $(2, 1)$-defective coloring in which each color class is acyclic. It is obvious that we cannot assing $(1, 1)$-defective coloring for H. Thus, $\chi^T_{\leq 1}(H) = 2$.

Example 2.34. Consider a hypergraph $K^{(3)}_{10}$ from Example 2.12.

Let $d = 6$. We show that $K^{(3)}_{10}$ has a $(4, 6)$-defective coloring without monochromatic cycle. Let $f : V(K^{(3)}_{10}) \to \{1, 2, 3, 4\}$ be a labeling defined by $f(v_i) = 1$ for $i \in \{1, 2, 3\}$, $f(v_i) = 2$ for $i \in \{4, 5, 6\}$, $f(v_i) = 3$ for $i \in \{7, 8, 9\}$ and $f(v_{10}) = 4$. Because of all edges having size 3, there are exactly three edges being monochromatic and then each color class does not induce a cycle. Thus, $K^{(3)}_{10}$ has a $(4, 3)$-defective coloring without monochromatic cycle. Also, $K^{(3)}_{10}$ has a $(4, 6)$-defective coloring without monochromatic cycle. Now, we suppose that $K^{(3)}_{10}$ has a $(3, 6)$-defective coloring without monochromatic cycle. Note that if there is a color class of size at least 4, then such color class induces a cycle. Thus, we allow at most 3 vertices labeled the same color. Hence, $K^{(3)}_{10}$ has no $(3, 6)$-defective coloring without monochromatic cycle because $K^{(3)}_{10}$ has 10 vertices. Therefore, $\chi^T_{\leq 6}(K^{(3)}_{10}) = 4$.

Example 2.35. Consider a hypergraph $K^{(3)}_{2,2,3,3}$ of which each edge has 3 vertices from at least 2 different partite sets.

Let $d = 4$. we show that $K^{(3)}_{2,2,3,3}$ has a $(4, 4)$-defective coloring without monochromatic cycle. Since $K^{(3)}_{2,2,3,3}$ has 4 partite sets, we can assign different colors to vertices of different partite sets. Since each edge has vertices from at least two partite sets, no edges of $K^{(3)}_{2,2,3,3}$ is monochromatic. $K^{(3)}_{2,2,3,3}$ has a $(4, 4)$-defective coloring without monochromatic cycle. Now, we suppose that $K^{(3)}_{2,2,3,3}$ has $(3,4)$-defective coloring without monochromatic cycle. If we assign three colors, then there are at least one color class containing four vertices, v_1, v_2, v_3 and v_4. Since $K^{(3)}_{2,2,3,3}$ has no partite set with four colors, these four vertices are contained in at least two partite sets. Suppose that v_1 and v_2 are in different partite sets. There are different edges $e_1 = \{v_1, v_2, v_3\}$ and $e_2 = \{v_1, v_2, v_4\}$. Hence, these four vertices induce cycle $(v_1, e_1, v_2, e_2, v_1)$, impossible. Therefore, $\chi^T_{\leq 4}(K^{(3)}_{2,2,3,3}) = 4$.

From Examples 2.12, 2.18, 2.34 and 2.35, we can conclude the theorem as follows.

Theorem 2.36. *Let $d > 0$. For any hypergraph H, $\chi(H) \geq \chi^T_{\leq d}(H) \geq \chi_{\leq d}(H)$.*

Proof. Since a proper vertex coloring of the hypergraph H forces each color class being an independent set, the chromatic number of H is greater than or equal to the defective chromatic number of H of which each color class is acyclic, i.e., $\chi(H) \geq \chi^T_{\leq d}(H)$.

Since $\chi_{\leq d}(H)$ is the minimum number λ for which a (λ, d)-defective coloring exists and $\chi^T_{\leq d}(H)$ is the minimum number λ for which a (λ, d)-defective coloring without monochromatic cycle exists, $\chi^T_{\leq d}(H)$ cannot be less than $\chi_{\leq d}(H)$. Thus, $\chi^T_{\leq d}(H) \geq \chi_{\leq d}(H)$. □

Note that for $d = 0$, $\chi(H) = \chi_{\leq 0}(H)$. Thus, $\chi(H) = \chi^T_{\leq 0}(H) = \chi_{\leq 0}(H)$.

Next, if we consider the proof of Theorem 2.24 and the proof of theorem 2.28, then each pairwise disjoint monochromatic edges cannot form any monochromatic cycle. Thus, we can conclude the following.

Theorem 2.37. *Let $k \geq 3$. For any hypergraph H, if $0 \leq d < k+1$, then*

$$\chi^T_{\leq d}(K^{(k)}_n) = \begin{cases} \left\lceil \dfrac{n}{k} \right\rceil, & \text{if } kd \geq n, \\ \left\lceil \dfrac{n-kd}{k-1} \right\rceil + d, & \text{if } kd < n. \end{cases}$$

Theorem 2.38. *Let $k \geq 3$. For any k-uniform hypergraph, if $0 \leq d < k+1$, then*

$$\chi_{\leq d}(H) \geq \begin{cases} \left\lceil \dfrac{\omega(H)}{k} \right\rceil, & \text{if } \; kd \geq \omega(H), \\ \\ \left\lceil \dfrac{\omega(H) - kd}{k-1} \right\rceil + d, & \text{if } \; kd < \omega(H). \end{cases}$$

The following theorems are some observations on $\chi(H)$ and $\chi_{\leq d}(H)$.

Theorem 2.39. *For any hypergraph H in which each edge has only one vertex, $\chi(H) = 1$.*

Proof. For a hypergraph H. Since each edge has only one vertex, we can use one color to assign to all vertices. Thus, $\chi(H) = 1$. □

Theorem 2.40. *Let $d \geq 0$ and $k > 1$. For any hypergraph H in which each edge has at least k vertices, $\chi_{\leq d}(H) = 1$ if and only if $d \geq |E(H)|$.*

Proof. We get immediately that if $d \geq |E(H)|$, then $\chi_{\leq d}(H) = 1$.
Conversely, let $\chi_{\leq d}(H) = 1$, that is, H has a $(1, d)$-defective coloring. Then, all edges of H are monochromatic edges or singletons. Since $k > 1$, there is no singleton. Thus, $d \geq |E(H)|$. □

CHAPTER III
DEFECTIVE COLORING ON COMPLETE BIPARTITE k-UNIFORM HYPERGRAPH

In this chapter, we focus on a complete bipartite k-uniform hypergraph and provide values of $\chi_{\leq d}(K_{m,n}^{(k)})$ and $\chi_{\leq d}^{T}(K_{m,n}^{(k)})$, for $d \geq 0$.

Throughout this chapter, we let V_1 and V_2 be partite sets of $V(K_{m,n}^{(k)})$ with $|V_1| = m$, $|V_2| = n$ and we always assume that $m \leq n$.

Theorem 3.1. *For a hypergraph $K_{m,n}^{(k)}$, let $k > 1$ and $d \geq 0$. Then*

$$\chi_{\leq d}(K_{m,n}^{(k)}) = \begin{cases} 1, & \text{if } d \geq |E(K_{m,n}^{(k)})|, \\ 2, & \text{if } d < |E(K_{m,n}^{(k)})|. \end{cases}$$

Proof. **Case 1:** $d \geq |E(K_{m,n}^{(k)})|$. By Theorem 2.40, $\chi_{\leq d}(K_{m,n}^{(k)}) = 1$.
Case 2: $d < |E(K_{m,n}^{(k)})|$. Assign all vertices of partite set V_1 color a and assign the other vertices color b. Since each edge has vertices from both partite sets, no edge of $K_{m,n}^{(k)}$ is monochromatic. Thus, $\chi(K_{m,n}^{(k)}) \leq 2$. By Theorem 2.36, $\chi_{\leq d}(K_{m,n}^{(k)}) \leq \chi(K_{m,n}^{(k)}) \leq 2$. By Theorem 2.40, since $k > 1$ and $|E(K_{m,n}^{(k)})| > d$, $\chi_{\leq d}(K_{m,n}^{(k)}) \neq 1$. Therefore, $\chi_{\leq d}(K_{m,n}^{(k)}) = 2$. $\square$

Example 3.2. Let $V_1 = \{0, 1, 2\}$ and $V_2 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be two partite sets of vertex set of $K_{3,4}^{(4)}$. Note that $|E(K_{3,4}^{(4)})| = \binom{7}{4} - \binom{3}{4} - \binom{4}{4} = 35 - 0 - 1 = 34$.

(i) Let $d = 35$. Then, $d > |E(K_{3,4}^{(4)})|$. We can use one color to assign to all vertices of $K_{3,4}^{(4)}$ and $\chi_{\leq 35}(K_{3,4}^{(4)}) = 1$.

(ii) Let $d = 30$. Then, $d < |E(K_{3,4}^{(4)})|$. By Theorem 3.1, we can assign the color a to all vertices in V_1 and the color b to all vertices in V_2. Since each edge has vertices from both partite sets, no edges of $K_{3,4}^{(4)}$ has all vertices of the same color and $\chi_{\leq 30}(K^{(4)}_{3,4}) = 2$.

Next, we find chromatic numbers of defective coloring on $K^{(k)}_{m,n}$ in which each color class is acyclic.

Theorem 3.3. *Let $k \geq 3$.*

(i) $\chi^T(K^{(k)}_{m,n}) = 2$.

(ii) *For $d \geq 1$, then*

$$\chi^T_{\leq d}(K^{(k)}_{m,n}) = \begin{cases} 1, & \text{if } \ m+n = k, \\ 2, & \text{if } \ m+n > k. \end{cases}$$

Proof. (i) We know that $\chi(K^{(k)}_{m,n}) = 2$. Then, $\chi^T(K^{(k)}_{m,n}) = \chi(K^{(k)}_{m,n}) = 2$.

(ii) **Case 1:** $m + n = k$. Since $K^{(k)}_{m,n}$ is a k-uniform hypergraph and $k \geq 3$, there is one edge. Since $|E(K^{(k)}_{m,n})| = 1$ and $d \geq 1$, we can only assign the vertices by one color. Thus, $\chi^T_{\leq d}(K^{(k)}_{m,n}) = 1$.

Case 2: $m+n > k$. We know that $\chi(K^{(k)}_{m,n}) \leq 2$. By Theorem 2.36, $\chi^T_{\leq d}(K^{(k)}_{m,n}) \leq \chi(K^{(k)}_{m,n}) \leq 2$. Now, we show that $\chi^T_{\leq d}(K^{(k)}_{m,n}) \neq 1$. If $d < |E(K^{(k)}_{m,n})|$, then we cannot assign one color to this hypergraph. Suppose that $d \geq |E(K^{(k)}_{m,n}|$. Consider the hypergraph $K^{(k)}_{m,n}$ with two partite sets $V_1 = \{v_1, \ldots, v_m\}$ and $V_2 = \{u_1, \ldots, u_n\}$. Since $K^{(k)}_{m,n}$ is a complete bipartite k-uniform hypergraph and $k \geq 3$, there are two different edges containing both v_1 and u_1 , namely e_1 and e_2. Thus, $K^{(k)}_{m,n}$ contains cycle $(v_1, e_1, u_1, e_2, v_1)$. Since $K^{(k)}_{m,n}$ contains a cycle, we cannot assign only one color to this hypergraph. Thus, $\chi^T_{\leq d}(K^{(k)}_{m,n}) \neq 1$. Therefore, $\chi^T_{\leq d}(K^{(k)}_{m,n}) = 2$. □

Example 3.4. Let $V_1 = \{0, 1, 2\}$ and $V_2 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be two partite sets of vertex set of $K^{(4)}_{3,4}$. Note that $|E(K^{(4)}_{3,4})| = 34$.

Let $d = 35$. Then, $d > |E(K^{(4)}_{3,4})|$. We can assign the color a to all vertices in V_1 and the color b to all vertices in V_2. Since each edge has vertices from both partite sets, no edges of $K^{(4)}_{3,4}$ are monochromatic and each color class is an independent set. Thus, $\chi^T_{\leq 35}(K^{(4)}_{3,4}) = 2$.

Example 3.5. Let $d > 0$ and let $V_1 = \{0, 1, 2\}$ and $V_2 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be two partite sets of vertex set of $K^{(7)}_{3,4}$. Since $m + n = 3 + 4 = 7$, $\chi^T_{\leq d}(K^{(7)}_{3,4}) = 1$.

CHAPTER IV
DEFECTIVE COLORING ON COMPLETE k-PARTITE k-UNIFORM HYPERGRAPH

In this chapter, we focus on a complete k-partite k-uniform hypergraph of which each edge contains vertices from k different partite sets according to Kuhl and Schroeder [7] and provide values of $\chi_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})$ and $\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})$. Note that $|E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})| = n_1n_2n_3\cdots n_k$.

Throughout this chapter, we assume that $n_1 \leq n_2 \leq n_3 \leq \cdots \leq n_k$.

Theorem 4.1. *For a hypergraph $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$, let $k \geq 2$, $d \geq 0$. Then*

$$\chi_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = \begin{cases} 1, & \text{if } d \geq |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|, \\ 2, & \text{if } d < |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|. \end{cases}$$

Proof. **Case 1:** $d \geq |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|$. By Theorem 2.40, $\chi_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 1$.
Case 2: $d < |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|$. Assign all vertices of one partite set color a and then assign the other vertices color b. Since each edge has vertices from all partites, no edge of this hypergraph has all vertices of the same color. Thus, $\chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \leq 2$. By Theorem 2.36, $\chi_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \leq \chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \leq 2$. Since $d < |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|$, by Theorem 2.40 $\chi_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \neq 1$. Thus, $\chi_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$. □

Example 4.2. Consider $K^{(7)}_{1,2,3,3,4,5,5}$.

(i) Let $d = 1800$. Then, $d = |E(K^{(7)}_{1,2,3,3,4,5,5})|$. We can use one color to assign to all vertices of $K^{(7)}_{1,2,3,3,4,5,5}$ and $\chi_{\leq 1800}(K^{(7)}_{1,2,3,3,4,5,5}) = 1$.

(ii) Let $d = 1234$. Then, $d < |E(K^{(7)}_{1,2,3,3,4,5,5})|$. By Theorem 4.1, we can assign color a to all vertices in V_1 and color b to all vertices in the other partite sets. Since each edge has vertices from all partite sets, no edges of $K^{(7)}_{1,2,3,3,4,5,5}$ has all vertices of the same color and $\chi_{\leq 1234}(K^{(7)}_{1,2,3,3,4,5,5}) = 2$.

Next, we find chromatic numbers of defective coloring on $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$ in which each color class is acyclic.

Theorem 4.3. *Let $k \geq 3$.*

(i) $\chi^T(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$.

(ii) *For $d \geq 1$,*

$$\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = \begin{cases} 1, & \text{if } |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})| = 1, \\ 2, & \text{if } |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})| \geq 2. \end{cases}$$

Proof. (i) We get from Theorem 4.1 that $\chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$.
Then, $\chi^T(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = \chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$.

(ii) **Case 1:** $|E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})| = 1$. Since $d \geq 1$, we can only assign the vertices by one color. Thus, $\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 1$.
Case 2: $|E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})| \geq 2$. We get from Theorem 4.1 that $\chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$. By Theorem 2.36, $\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \leq \chi(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$.

Next, we show that $\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \neq 1$. If $d < |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|$, then we cannot assign one color to this hypergraph. Suppose that $d \geq |E(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k})|$. Consider the hypergraph $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$ with k partite sets $V_1 = \{v_{11}, v_{12}, \ldots, v_{1n_1}\}$, $V_2 = \{v_{21}, v_{22}, \ldots, v_{2n_2}\}, \ldots, V_k = \{v_{k1}, v_{k2}, \ldots, v_{kn_k}\}$. Since $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$ is a complete k-partite k-uniform hypergraph and $k \geq 3$, there are two different edges containing both v_{11} and v_{21} , namely e_1 and e_2. Thus, $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$ contains cycle $(v_{11}, e_1, v_{21}, e_2, v_{11})$. Since $K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}$ contains a cycle, we can not assign only one color to this hypergraph. Thus, $\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) \neq 1$. Therefore, $\chi^T_{\leq d}(K^{(k)}_{n_1,n_2,n_3,\ldots,n_k}) = 2$. □

Example 4.4. Consider $K^{(7)}_{1,2,3,3,4,5,5}$. Then, by Theorem 4.3, $\chi^T(K^{(7)}_{1,2,3,3,4,5,5}) = 2$ and since $1 \times 2 \times 3^2 \times 4 \times 5^2 \neq 1$, by Theorem 4.2, $\chi^T_{\leq d}(K^{(7)}_{1,2,3,3,4,5,5}) = 2$ for $d > 0$.

CHAPTER V

DEFECTIVE COLORING ON COMPLETE TRIPARTITE 3-UNIFORM HYPERGRAPH

Throughout this chapter, we consider $K^{(3)}_{3\times m}$, a complete tripartite 3-uniform hypergraph, in which the vertex set $V(K^{(3)}_{3\times m})$ consists of 3 partite sets V_1, V_2 and V_3 of the same size m, $m \geq 3$, and each edge has 3 vertices from at least 2 different partite sets.

Remark 5.1 Before finding the defective chromatic number of $K^{(3)}_{3\times m}$, we consider vertex coloring in $K^{(3)}_{3\times m}$ which uses only two colors. Note that there are five possible ways to assign two colors, namely a and b, to the vertices of $K^{(3)}_{3\times m}$ as follows.

(i) Assign all vertices of partite sets V_1 and V_2 color a and then assign the other vertices color b, see Figure 5.1. The number of monochromatic edges is

$$\binom{m}{1}\binom{m}{2}+\binom{m}{2}\binom{m}{1}=m^3-m^2$$

.

V_1	V_2	V_3
a	a	b
a	a	b
a	a	b
a	a	b
$\vdots$	$\vdots$	$\vdots$
a	a	b

Figure 5.1: The first colored assignment.

(ii) For $1 \leq p \leq m-1$, assign all vertices of partite set V_1 and V_2 color a and assign p vertices of V_3 color a and then assign the rest $m-p$ vertices of V_3 color b, see Figure 5.2. The number of monochromatic edges is

$$pm^2+\binom{m}{1}\binom{m}{2}+\binom{m}{2}\binom{m}{1}+\binom{m}{1}\binom{p}{2}+\binom{m}{2}\binom{p}{1}+\binom{m}{1}\binom{p}{2}$$
$$+\binom{m}{2}\binom{p}{1}=m^3+(2p-1)m^2+(p^2-2p)m.$$

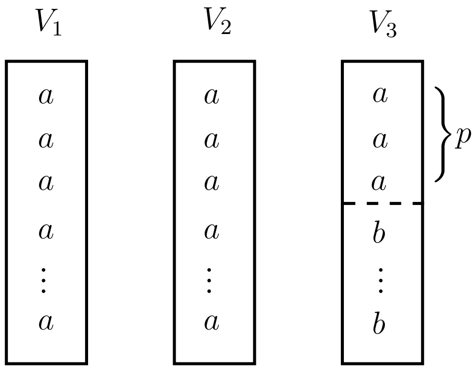


Figure 5.2: The second colored assignment.

(iii) For $1 \leq p \leq m-1$, assign all vertices of partite set V_1 color a, and all vertices of partite set V_2 color b and then assign p vertices of V_3 color a and assign the rest $m-p$ vertices of V_3 color b, see Figure 5.3. The number of monochromatic edges is

$$\binom{m}{1}\binom{p}{2}+\binom{m}{2}\binom{p}{1}+\binom{m}{1}\binom{m-p}{2}+\binom{m}{2}\binom{m-p}{1}$$
$$=m^3-(p+1)m^2+p^2m.$$

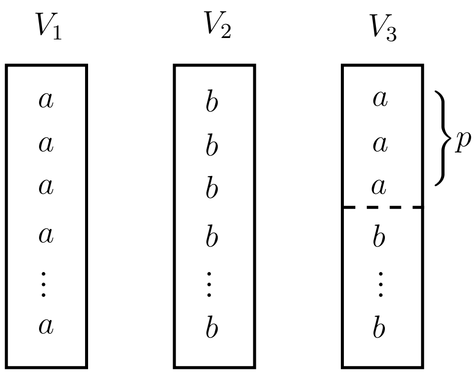


Figure 5.3: The third colored assignment.

(iv) For $1 \leq p, q \leq m-1$, assign all vertices of the partite set V_1 color a, assign q vertices of V_2 color a and assign the rest $m-q$ vertices of V_2 color b. Next, assign p vertices of V_3 color a and assign the rest $m-p$ vertices of V_3 color b, see Figure 5.4. The number of monochromatic edges is

$$\begin{aligned} & pqm + \binom{m}{1}\binom{p}{2} + \binom{m}{2}\binom{p}{1} + \binom{m}{1}\binom{q}{2} + \binom{m}{2}\binom{q}{1} + \binom{p}{1}\binom{q}{2} \\ & + \binom{p}{2}\binom{q}{1} + \binom{m-p}{1}\binom{m-q}{2} + \binom{m-p}{2}\binom{m-q}{1} \\ & = m^3 - (p+q+1)m^2 + (p^2+q^2+3pq)m - 2pq. \end{aligned}$$

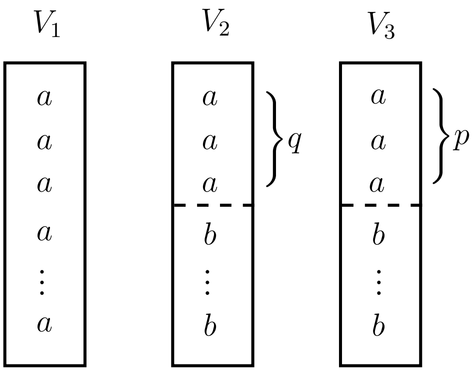


Figure 5.4: The fourth colored assignment.

(v) For $1 \leq p, q, r \leq m-1$, for each partite set, assign r, q and p vertices of V_1, V_2 and V_3 color a, respectively and assign the other vertices color b, see Figure 5.5.

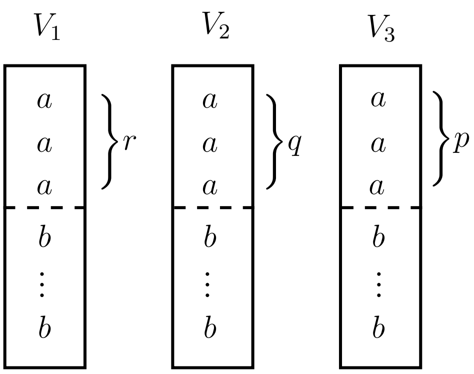


Figure 5.5: The fifth colored assignment.

The number of monochromatic edges is

$$
\begin{aligned}
&pqr + \binom{r}{1}\binom{q}{2} + \binom{r}{2}\binom{q}{1} + \binom{r}{1}\binom{p}{2} + \binom{r}{2}\binom{p}{1} + \binom{p}{1}\binom{q}{2} + \binom{p}{2}\binom{q}{1} \\
&+ (m-r)(m-q)(m-p) + \binom{m-r}{1}\binom{m-q}{2} + \binom{m-r}{2}\binom{m-q}{1} \\
&+ \binom{m-r}{1}\binom{m-p}{2} + \binom{m-r}{2}\binom{m-p}{1} + \binom{m-p}{1}\binom{m-q}{2} \\
&+ \binom{m-p}{2}\binom{m-q}{1} \\
&= 4m^3 - (4p+4q+4r+3)m^2 + (p^2+q^2+r^2+2p+2q+2r+3pq)m \\
&+ (3rp+3rq)m - (2rq+2pq+2rp).
\end{aligned}
$$

Notice from these five cases that if we assign only two colors, then it must occur some monochromatic edges.

Theorem 5.1. *Let $m \geq 3$, $d > 0$ and $t = \left\lfloor \frac{m}{2} \right\rfloor$.*

(i) $\chi(K^{(3)}_{3\times m}) = 3$.

(ii)

$$
\chi_{\leq d}(K^{(3)}_{3\times m}) = \begin{cases} 1, & \text{if } d \geq 4m^3 - 3m^2, \\ 2, & \text{if } 4m^3 - 3m^2 > d \geq m^3 - (t+1)m^2 + t^2m. \end{cases}
$$

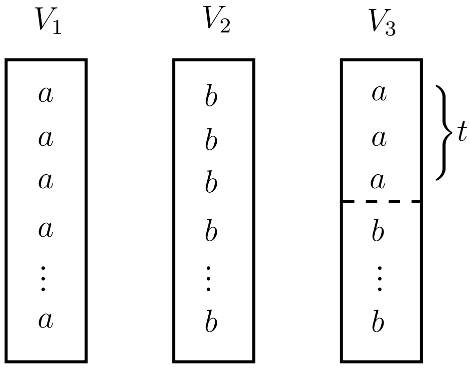
(iii) $\chi_{\leq d}(K^{(3)}_{3\times m}) \leq 3$ if $m^3 - (t+1)m^2 + t^2m > d$.

Proof. (i) Assign the color a to all vertices in the partite set V_1, the color b to all vertices in the partite set V_2 and the color c to all vertices in the partite set V_3. Since each edge contains vertices from at least two partite sets, $K^{(3)}_{3\times m}$ admits a proper 3-coloring. Thus, $\chi(K^{(3)}_{3\times m}) \leq 3$.

Now, we show that $\chi(K^{(3)}_{3\times m}) \neq 2$. Assume that $K^{(3)}_{3\times m}$ is a proper 2-coloring. From Remark 5.1, there is the edge whose contains all vertices with the same color. Thus, $\chi(K^{(3)}_{3\times m}) \neq 2$. Therefore, $\chi(K^{(3)}_{3\times m}) = 3$.

(ii) **Case 1:** $d \geq 4m^3 - 3m^2$. We know that $|E(K^{(3)}_{3\times m})| = m^3 + 3!\binom{m}{2}\binom{m}{1}\binom{m}{0}$ $= 4m^3 - 3m^2$. Then, we can assign one color to $K^{(3)}_{3\times m}$. Thus, $\chi_{\leq d}(K^{(3)}_{3\times m}) = 1$.

Case 2: $4m^3 - 3m^2 > d \geq m^3 - (t+1)m^2 + t^2m$. Assign the color a to vertices in partite set V_1, the color b to vertices in partite set V_2 and then divide the vertices of partite set V_3 into two groups of size $t = \left\lfloor \frac{m}{2} \right\rfloor$ and size $m - t$, the vertices of the group of size t is assigned the color a and the vertices of another group is assigned the color b.



The number of edges being monochromatic is

$$\binom{m}{1}\binom{t}{2} + \binom{m}{2}\binom{t}{1} + \binom{m}{1}\binom{m-t}{2} + \binom{m}{2}\binom{m-t}{1}.$$

Then,

$$\begin{aligned}
&\binom{m}{1}\binom{t}{2} + \binom{m}{2}\binom{t}{1} + \binom{m}{1}\binom{m-t}{2} + \binom{m}{2}\binom{m-t}{1} \\
&= m\binom{t}{2} + t\binom{m}{2} + m\binom{m-t}{2} + (m-t)\binom{m}{2} \\
&= m\binom{m}{2} + m\binom{t}{2} + m\binom{m-t}{2} \\
&= m\frac{m(m-1)}{2} + m\frac{t(t-1)}{2} + m\frac{(m-t)(m-t-1)}{2} \\
&= \frac{m}{2}(m^2 - m + t^2 - t + m^2 - tm - m - tm + t^2 + t) \\
&= \frac{m}{2}(2m^2 - 2(t+1)m + 2t^2) \\
&= m^3 - (t+1)m^2 + t^2m.
\end{aligned}$$

Since $d \geq m^3 - (t+1)m^2 + t^2m$, we can assign two colors to the vertices of $K^{(3)}_{3\times m}$. Thus, $\chi_{\leq d}(K^{(3)}_{3\times m}) \leq 2$.

Next, we show that $\chi_{\leq d}(K^{(3)}_{3\times m}) \neq 1$. Assume that $K^{(3)}_{3\times m}$ admits a $(1,d)$-defective coloring. Since $|E(K^{(3)}_{3\times m})| = 4m^3 - 3m^2$ and $d < 4m^3 - 3m^2$, there must be at least one edge which does not be monochromatic. Thus, we cannot assign only one color to $K^{(k)}_{m,n}$. Thus, $\chi_{\leq d}(K^{(3)}_{3\times m}) \neq 1$. Therefore, $\chi_{\leq d}(K^{(3)}_{3\times m}) = 2$.

(iii) Let $m^3 - (t+1)m^2 + t^2m > d$. We know that $\chi(K^{(3)}_{3\times m}) = 3$. By Theorem 2.36, $\chi_{\leq d}(K^{(3)}_{3\times m}) \leq \chi(K^{(3)}_{3\times m}) = 3$. Thus, $\chi_{\leq d}(K^{(3)}_{3\times m}) \leq 3$. $\square$

Example 5.2. Consider $K^{(3)}_{3\times 7}$ which $m = 7$ and $t = \left\lfloor \dfrac{7}{2} \right\rfloor = 3$. Then,

$$4m^3 - 3m^2 = 4(7^3) - 3(7^2) = 1225$$

$$m^3 - (t+1)m^2 + t^2m = 7^3 - (4)7^2 + (3^2)7 = 210.$$

Thus, by Theorem 5.1, we have

$$\begin{aligned}
\chi(K^{(3)}_{3\times 7}) &= 3,\\
\chi_{\leq d}(K^{(3)}_{3\times 7}) &= 1 \text{ if } d \geq 1225,\\
\chi_{\leq d}(K^{(3)}_{3\times 7}) &= 2 \text{ if } 1225 > d \geq 210,\\
\chi_{\leq d}(K^{(3)}_{3\times 7}) &\leq 3 \text{ if } 210 > d > 0.
\end{aligned}$$

Example 5.3. Let $d = 200$. Consider $K^{(3)}_{3\times 7}$ which $m = 7$ and $t = \left\lfloor \dfrac{7}{2} \right\rfloor = 3$. From Example 5.2, we have $\chi_{\leq 200}(K^{(3)}_{3\times 7}) \leq 3$. Assume that $K^{(3)}_{3\times 7}$ is $(2,200)$-defective coloring. From Remark 5.1, we can consider 5 cases as follow.

(i) Assign the color a to all vertices in partite sets V_1 and V_2 and then assign the color b to all vertices in partite set V_3. We can see that there are

$$\binom{7}{1}\binom{7}{2} + \binom{7}{2}\binom{7}{1} = 294$$

monochromatic edges and more than d.

(ii) For $1 \leq p \leq 6$, assign the color a to all vertices in partite set V_1 and

V_2 and then assign the color a to p vertices of V_3 and assign the color b to the rest $7-p$ vertices of V_3. We can see that the number of monochromatic edges is $49p + \binom{7}{1}\binom{7}{2} + \binom{7}{2}\binom{7}{1} + \binom{7}{1}\binom{p}{2} + \binom{7}{2}\binom{p}{1} + \binom{7}{1}\binom{p}{2} + \binom{7}{2}\binom{p}{1} = 7^3 + (2p-1)7^2 + (p^2-2p)7 = 294 + 84p + 7p^2$ edges which is more than d.

(iii) For $1 \leq p \leq 6$, assign the color a to all vertices in partite set V_1, and the color b to all vertices in partite set V_2 and then assign the color a to p vertices of V_3 and assign the color b to the rest $7-p$ vertices of V_3. We can see that the number of monochromatic edges is $\binom{7}{1}\binom{p}{2} + \binom{7}{2}\binom{p}{1} + \binom{7}{1}\binom{7-p}{2} + \binom{7}{2}\binom{7-p}{1} = 7^3 - (p+1)7^2 + (p^2)7 = 294 - 49p + 7p^2 = 7(p - \frac{7}{2})^2 + 208.25$ edges which is more than d.

(iv) For $1 \leq p, q \leq 6$, assign the color a to all vertices in the partite set V_1, assign the color a to q vertices of V_2 and assign the color b to the rest $7-q$ vertices of V_2. Next, assign the color a to p vertices of V_3 and assign the color b to the rest $7-p$ vertices of V_3. We can see that the number of monochromatic edges is $7pq + \binom{7}{1}\binom{p}{2} + \binom{7}{2}\binom{p}{1} + \binom{7}{1}\binom{q}{2} + \binom{7}{2}\binom{q}{1} + \binom{p}{1}\binom{q}{2} + \binom{p}{2}\binom{q}{1} + \binom{7-p}{1}\binom{7-q}{2} + \binom{7-p}{2}\binom{7-q}{1} = 7^3 - (p+q+1)7^2 + (p^2+q^2+3pq)7 - 2pq = 294 + 19pq - 49(p+q) + 7(p^2+q^2)$ edges. For $1 \leq p, q \leq 6$, all possible values of $294 + 19pq - 49(p+q) + 7(p^2+q^2)$ are

$$220, 225, 229, 230, 244, 254, 277,$$
$$292, 297, 324, 344, 354, 410, 425,$$
$$430, 510, 520, 624, 629, 752, 894$$

which they are more than d.

(v) For $1 \leq p, q, r \leq 6$, for each partite set, assign the color a to r, q *and* p vertices of V_1, V_2 and V_3, respectively and assign the color b to the rest vertices of each partite set. We can see that the number of monochromatic edges is $pqr + \binom{r}{1}\binom{q}{2} + \binom{r}{2}\binom{q}{1} + \binom{r}{1}\binom{p}{2} + \binom{r}{2}\binom{p}{1} + \binom{p}{1}\binom{q}{2} + \binom{p}{2}\binom{q}{1} + (7-r)(7-q)(7-p) + \binom{7-r}{1}\binom{7-q}{2} + \binom{7-r}{2}\binom{7-q}{1} + \binom{7-r}{1}\binom{7-p}{2} + \binom{7-r}{2}\binom{7-p}{1} + \binom{7-p}{1}\binom{7-q}{2} + \binom{7-p}{2}\binom{7-q}{1} = 4(7^3) - (4p+4q+4r+3)7^2 + (p^2+q^2+$

$r^2 + 2p + 2q + 2r + 3pq + 3rp + 3rq)7 - (2rq + 2pq + 2rp) = 1225 + 19(pq + pr + qr) - 182(p + q + r) + 7(p^2 + q^2 + r^2)$ edges. For $1 \leq p, q, r \leq 6$, all possible values of $1225 + 19(pq + pr + qr) - 182(p + q + r) + 7(p^2 + q^2 + r^2)$ are

$$240, 245, 250, 254, 260, 265, 269,$$
$$270, 274, 282, 284, 289, 302, 312,$$
$$317, 322, 349, 364, 369, 374, 430,$$
$$440, 445, 525, 530, 634, 757$$

which they are more than d.

Therefore, $K^{(3)}_{3\times 7}$ is not $(2, 200)$-defective coloring. Hence, $\chi_{\leq 200}(K^{(3)}_{3\times 7}) = 3$

Remark 5.2 Actually, for $m^3 - (t + 1)m^2 + t^2m > d > 0$, If we try to assign only two colors for vertices of $K^{(3)}_{3\times m}$, then, according to Remark 5.1, there are five cases as we can see in Example 5.3. We can generally prove that in cases (i) - (iv), the number of monochromatic edges is more than $m^3 - (t + 1)m^2 + t^2m$ and hence, more than d as follows.

Case 1: Since $m \geq t$, $tm \geq t^2$. Then, $-m^2 \geq -(t+1)m^2 + t^2m$. Thus, $m^3 - m^2 \geq m^3 - (t + 1)m^2 + t^2m$.

Case 2: First, we claim that $m^3 + (2p - 1)m^2 + (p^2 - 2p)m > m^3 - m^2$. Since $m^3 + (2p-1)m^2 + (p^2 - 2p)m = (m^3 - m^2) + (2pm^2 + p^2m - 2pm)$ and $m^3 - m^2 > 0$, it is sufficient to show that $2pm^2 + p^2m - 2pm > 0$. The assertion is true because $2pm^2 + p^2m > 2pm^2 > 2pm$ whenever $m \geq 3$ and we obtain the claim. Next, by Case 1, we can conclude that $m^3 + (2p - 1)m^2 + (p^2 - 2p)m > m^3 - m^2 \geq m^3 - (t + 1)m^2 + t^2m$.

Case 3: Let $t = \left\lfloor \frac{m}{2} \right\rfloor$ and $1 \leq p \leq m - 1$. We consider two cases as follows.

Case 3.1: $p \leq t$. Then, $t + p \leq m$ and $(t - p)(t + p)m \leq (t - p)m^2$. Thus, $m^3 - (t + 1)m^2 + t^2m \leq m^3 - (p + 1)m^2 + p^2m$.

Case 3.2: $t < p$. Then, $p + t \geq m$ and $(p + t)(p - t)m \geq (p - t)m^2$. Thus, $m^3 - (p + 1)m^2 + p^2m \geq m^3 - (t + 1)m^2 + t^2m$.

From both cases, we conclude that $m^3 - (p+1)m^2 + p^2m \geq m^3 - (t+1)m^2 + t^2m$.

Case 4: Let $t = \left\lfloor \frac{m}{2} \right\rfloor$ and $1 \leq p, q \leq m - 1$. We consider two cases as follows.

Case 4.1: $p+q \le t$. Then, $t+(p+q) \le m$ and $(t-(p+q))(t+(p+q))m \le (t-(p+q))m^2$. Since $m \ge 3$, we have $(t^2-(p+q)^2)m-(m-2)pq < (t^2-(p+q)^2)m \le (t-(p+q))m^2 = (t+1-(p+q+1))m^2$. Thus, $m^3-(t+1)m^2+t^2m < m^3-(p+q+1)m^2+(p^2+q^2+3pq)m-2pq$.

Case 4.2: $t < p+q$. Then, $(p+q)+t \ge m$ and $((p+q)-t)((p+q)+t)m \ge ((p+q)-t)m^2$. Since $m \ge 3$, we have $((p+q)^2-t^2)m+(m-2)pq > ((p+q)^2-t^2)m \ge ((p+q)-t)m^2 = ((p+q+1)-(t+1))m^2$. Thus, $m^3-(p+q+1)m^2+(p^2+q^2+3pq)m-2pq > m^3-(t+1)m^2+t^2m$.

From both cases, we conclude that $m^3-(p+q+1)m^2+(p^2+q^2+3pq)m-2pq > m^3-(t+1)m^2+t^2m$.

As we can see that in general these four cases can almost lead us to the conclusion that $\chi_{\le d}K^{(3)}_{3\times m} = 3$ for $m^3-(t+1)m^2+t^2m > d > 0$. Now, we consider the last case. In this case, we want to show that $4m^3-(4p+4q+4r+3)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr)m-(2pq+2pr+2qr) > m^3-(t+1)m^2+t^2m$. Since $t = \left\lfloor \frac{m}{2} \right\rfloor$, we consider two cases as follows.

If $t = \frac{m}{2}$, then $4m^3-(4p+4q+4r+3)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr)m-(2pq+2pr+2qr)-(m^3-(t+1)m^2+t^2m) = \frac{13}{4}m^3-(4p+4q+4r+2)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr)m-(2pq+2pr+2qr)$. We only know that for a sufficiently large m, $\frac{13}{4}m^3-(4p+4q+4r+2)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr)m-(2pq+2pr+2qr) > 0$.

If $t = \frac{m-1}{2}$, then $4m^3-(4p+4q+4r+3)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr)m-(2pq+2pr+2qr)-(m^3-(t+1)m^2+t^2m) = \frac{13}{4}m^3-(4p+4q+4r+2)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr-\frac{1}{4})m-(2pq+2pr+2qr)$. We only know that for a sufficiently large m, $\frac{13}{4}m^3-(4p+4q+4r+2)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr-\frac{1}{4})m-(2pq+2pr+2qr) > 0$.

Therefore, at this point, we can just conclude that $4m^3-(4p+4q+4r+3)m^2+(p^2+q^2+r^2+2p+2q+2r+3pq+3pr+3qr)m-(2pq+2pr+2qr) > m^3-(t+1)m^2+t^2m$ provided that m is large enough. We also implement a computer program and varies several values of m and it confirm that the desire inequality holds.

Next, we determine a chromatic number of defective coloring on $K^{(3)}_{3\times m}$ of which each color class is acyclic.

Theorem 5.4. *Let* $m \geq 3$. $\chi^T_{\leq d}(K^{(3)}_{3\times m}) = 3$, *for all* $d \geq 0$.

Proof. It is obvious that we can assign three colors. Thus, $\chi^T_{\leq d}(K^{(3)}_{3\times m}) \leq 3$. By Remark 5.1, there are five cases to assign two colors. The cases each occurs a color class which induces a cycle. Then, we cannot assign only two colors to this hypergraph. Thus, $\chi^T_{\leq d}(K^{(3)}_{3\times m}) \neq 2$. Therefore, $\chi^T_{\leq d}(K^{(3)}_{3\times m}) = 3$. □

From Theorem 5.4, $\chi^T_{\leq d}(K^{(3)}_{3\times m}) = \chi(K^{(3)}_{3\times m}) = 3$, we see that the value of d does not affect the result.

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