CHAPTER III

GENERALIZED SEMINEAR-FIELDS

In this chapter we shall generalize the concept of a seminear-field by giving a new definition which contains J. Hattakosol's definition as a special case.

<u>Definition 3.1</u>. A seminear-ring $(K,+,\cdot)$ is said to be a <u>generalized seminear-field</u> iff there exists an element a in K such that $(K\setminus\{a\},\cdot)$ is a group. Such an element a is called a <u>special element of K</u>.

Clearly a seminear-field is a generalized seminear-field therefore every example of a seminear-field given in chapter I and II is an example of generalized seminear-fields.

<u>Example 3.2</u>. Let $(G,\cdot)$ be a group and let $d \in G$. Let a be a symbol not representing any element of G. Let $K = G \cup \{a\}$. Define + on K and extend $\cdot$ to K by $a\cdot x = d\cdot x$ and $x\cdot a = x\cdot d$ for all $x \in G$, $a^2 = d$ and either

(1) $x + y = y$ for all $x,y \in K$ or

(2) $x + y = x$ for all $x,y \in K$.

It is easy to show that $(K,+,\cdot)$ is a seminear-field.

<u>Example 3.3</u>. Let $K = \{a,e\}$. Define + and $\cdot$ on K by

$\cdot$	e	a
e	e	a
a	a	e

and

+	e	a
e	e	a
a	e	a

Then $(K,+,\cdot)$ is a generalized seminear-field.

Example 3.4. Let D be a ratio seminear-ring. Let a be a symbol not representing any element of D and $d \in D$. Extend + and $\cdot$ from D to $D \cup \{a\}$ by

(1) $ax = dx$ and $xa = xd$ for all $x \in D$, $a^2 = d^2$,

(2) $a + x = d + x$ and $x + a = x + d$ for all $x \in D$ and

(3) $a + a = d + d$.

It is easy to check that $D \cup \{a\}$ is a generalized seminear-field.

From now on the word "seminear-field" will mean a generalized seminear-field.

Theorem 3.5. Let K be a seminear-field with a as a special element. Then exactly one of the following statements hold :

(1) $ax = xa = a$ for all $x \in K$.

(2) $ax = xa = x$ for all $x \in K$.

(3) $ax = a$ and $xa = x$ for all $x \in K$.

(4) $ax = x$ and $xa = a$ for all $x \in K$.

(5) $a^2 \neq a$ and $ae = ea = a$.

(6) $a^2 \neq a$ and $ae = ea \neq a$ where e is the identity of $(K \setminus \{a\}, \cdot)$.

Proof. Consider a^2.

Case 1. $a^2 = a$. By Theorem 1.29, we obtain (1) - (4).

Case 2. $a^2 \neq a$. Consider ae and ea.

Subcase 2.1. $ae = ea = a$. Then we obtain (5).

Subcase 2.2. $ae = a$ and $ea \neq a$. Claim that $ax = a$ for all $x \in K \setminus \{a\}$ and $ea = e$. Let $x \in K \setminus \{a\}$. Then $a = ae = a(xx^{-1}) = (ax)x^{-1}$. If $ax \neq a$ then $(ax)x^{-1} \in K \setminus \{a\}$ which is a group. This is a contradiction. Hence $ax = a$ for all $x \in K \setminus \{a\}$.

Since $ea \neq a$, there is a $y \in K \setminus \{a\}$ such that $(ea)y = e$. So $e = e(ay) = ea$. Now $a^2 = (ae)a = a(ea) = ae = a$. Contradicting the fact that $a^2 \neq a$. Hence this case cannot occur.

Subcase 2.3. $ae \neq a$ and $ea = a$. Using a proof similar to the proof of Subcase 2.2, we can show that this case cannot occur.

Subcase 2.4. $ae \neq a$ and $ea \neq a$. Then $ae = e(ae) = (ea)e = ea$. Hence we obtain (6).

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From Theorem 3.5 we see that there are 6 types of special elements in a seminear-field and we call a special element satisfying (1),(2),(3),(4),(5) or (6) a category I,II,III,IV,V or VI special element respectively.

Note that Example 3.3 is a seminear-field with a category V special element and Example 3.2 and 3.4 are seminear-fields with a as a category VI special element.

In this chapter we shall only study seminear-fields with category V and VI special elements because seminear-fields with category I,II,III and IV special elements were studied already in [1] and Chapter 2.

Theorem 3.6. If K is a seminear-field with a category V special element then $|K| = 2$.

Proof. Let K be a seminear-field with a as a category V special element and let e be the identity of $(K \setminus \{a\}, \cdot)$. Claim that $ax = a$ for all $x \in K \setminus \{a\}$. Let $x \in K \setminus \{a\}$. Then $a = a = ae = a(xx^{-1}) = (ax)x^{-1}$. If $ax \neq a$ then $(ax)x^{-1} \in K \setminus \{a\}$ which is a group. This is a contradiction. Hence $ax = a$ for all $x \in K \setminus \{a\}$. Since $a^2 \neq a$, there is a $y \in K \setminus \{a\}$ such that $a^2y = e$. So $e = a^2y =$

$a(ay) = a^2$. Hence $a^2 = e$. Suppose that $|K| > 2$. Let $z \in K \setminus \{a,e\}$. Then $z = ez = a^2z = a(az) = a^2 = e$, a contradiction. Hence $|K| = 2$. #

Remark : A seminear-field with a category V special element is a ratio seminear-ring. Now we shall find, up to isomorphism, all seminear-fields K with a category V special elementa. By Theorem 3.6, $K = \{a,e\}$. Claim that $e + e = a$ or $a + a = a$. If $e + e \neq a$ then $e + e = e$. Thus $a + a = ea + ea = (e+e)a = ea = a$.

Case 1. $e + e = a$. Then $a + a = ea + ea = (e+e)a = a^2 = e$ since $a^2 \neq a$. $e + a = e + (e+e) = (e+e) + e = a + e$. Hence $e + a = a$ or $e + a = e$. So we have 2 tables.

(1)

+	e	a
e	a	a
a	a	e

or

(2)

+	e	a
e	a	e
a	e	e

Case 2. $a + a = a$. Then $a = a + a = ea + ea = (e+e)a$. If $e + e = a$ then $a = (e+e)a = a^2 = e$, a contradiction. Hence $e + e = e$. If $e + a = a$ then $a + e = ea + a^2 = (e+a)a = a^2 = e$. If $e + a = e$ then $a + e = ea + a^2 = (e+a)a = ea = a$. So we have 2 tables.

(3)

+	e	a
e	e	a
a	e	a

or

(4)

+	e	a
e	e	e
a	a	a

It is easy to verify that (1),(2),(3) and (4) are tables of semigroups under addition. By defining $f(e) = a$ and $f(a) = e$, we have that semigroups defined by (1) and (2) are isomorphic. Therefore, up to isomorphism, there are 3 seminear-fields containing a category special element.

Remark 3.7. Let $K = \{a,e\}$ with stucture

$\cdot$	e	a
e	e	a
a	a	a

and

+	e	a
e	e	a
a	e	a

Then $(K,+,\cdot)$ is a seminear-field with a as a category I special element. And note that $(K,+,\cdot)$ is a seminear-field with e as a category II special element. In this case we see that there does not exist a unique special element of K. However, if $|K| > 2$, we do get uniqueness as the following theorem shows.

Theorem 3.8. If K is a seminear-field of order greater than 2 then there exists a unique special element of K.

Proof. Assume that K is a seminear-field with $|K| > 2$ and let a and a' be special elements of K. We must show that $a = a'$. Suppose that $a \neq a'$. Let e be the identity of $(K\setminus\{a\},\cdot)$ and e' the identity of $(K\setminus\{a'\},\cdot)$

Case 1. $a^2 = a$. Since $a^2 = a \in K\setminus\{a'\}$, $a = e'$. Let $x \in K\setminus\{a,a'\}$. Then there is a $y \in K\setminus\{a'\}$ such that $xy = e' = a$. If $y = a$ then $x = xe' = xa = xy = e' = a$, a contradiction. Hence $y \neq a$, so we have $x \neq a$, $y \neq a$ and $xy = a$. This contradicts the fact that $(K\setminus\{a\},\cdot)$ is a group.

Case 2. $a^2 \neq a$. Since $a^2 \neq a$ and $(K\setminus\{a\},\cdot)$ is a group, e is the only multiplicative idempotent of K. Since $e'^2 = e'$, $e' = e$. Thus $a = ae' = ae$ and $a = e'a = ea$. Hence K is a seminear-field with a as a category V seminear-field. By Theorem 3.6, $|K| = 2$, a contradiction. Therefore $a = a'$.

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Remark 3.9. Let K be a seminear-field. Then the following statements hold :

(1) If there are elements a and b in K such that a and b are category V special elements of K then a = b.

(2) If there are elements a and b in K such that a and b are category VI special elements of K then a = b.

Proof. (1) By Theorem 3.6, $|K| = 2$. Let $K = \{a,b\}$. Since $a^2 \neq a$ and $b^2 \neq b$, $(K \setminus \{a\}, \cdot)$ is not a group, a contradiction.

(2) If $|K| > 2$ then, by Theorem 3.8, we obtain (4). If $|K| = 2$ then use the same proof as in (1).

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Remark 3.10. Let $K = \{a,e\}$ with structure :

(1)

•	e	a
e	e	a
a	e	a

and

+	e	a
e	e	a
a	e	a

or

(2)

•	e	a
e	e	e
a	a	a

and

+	e	a
e	e	e
a	a	a

Then K with structure (1) is a seminear-field with a and e as category III special elements. K with structure(2)is a seminear-field with a and e as category IV special elements. Hence category III and IV special elements are never unique.

We shall now study seminear-fields with a category VI special element.

Theorem 3.11. Let K be a seminear-field with a as a special element. Then a is a category VI special element of K if and only if there exists a unique element d in $K \setminus \{a\}$ such that $ax = dx$ and $xa = xd$ for all $x \in K$.

Proof. Let e be the identity of $(K \setminus \{a\}, \cdot)$. Assume that a is a category VI special element of K. Let $d = ae = ea$. Let x be any element of K. If $x = a$ then $ax = a^2 = ea^2 = (ea)a = da$. If $x \neq a$ then $ax = a(ex) = (ae)x = dx$. Hence $ax = dx$. Similarly, we can show that $xa = xd$. Therefore $ax = dx$ and $xa = xd$ for all $x \in K$. To show uniqueness, let $d^* \in K \setminus \{a\}$ be such that $ax = d^*x$ and $xa = xd^*$ for all $x \in K$. Then $d = de = ae = d^*e = d^*$.

Conversely, assume that there exists a unique element d in $K \setminus \{a\}$ such that $ax = dx$ and $xa = xd$ for all $x \in K$. Then $a^2 = ad = d^2 \neq a$, $ae = de = d$ and $ea = ed = d$, so a is a category VI special element of K.

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Theorem 3.12. Let $(K,+,\cdot)$ be a seminear-field with a as a category VI special element. Then $(K \setminus \{a\},+,\cdot)$ is a ratio seminear-ring.

Proof. Let e denote the identity of $(K \setminus \{a\},\cdot)$. Then $ae = ea \neq a$. To show that $(K \setminus \{a\},+,\cdot)$ is a ratio seminear-ring, it is sufficient to show that $x + y \in K \setminus \{a\}$ for all $x,y \in K \setminus \{a\}$. Let $x,y \in K \setminus \{a\}$. Suppose $x + y = a$. Then $a = x + y = xe + ye = (x+y)e = ae$, a contradiction. Hence $x + y \in K \setminus \{a\}$.

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Theorem 3.12 indicates that every seminear-field with a category VI special element comes from a ratio seminear-ring by adding an element.

Remark 3.13. Let K be a seminear-field with a as a category VI special element. Then $xy \neq a$ for all $x,y \in K$.

Proof. Let e denote the identity of $(K \setminus \{a\},\cdot)$. By assumption, $a^2 \neq a$, $ae = ea \neq a$. Let $x,y \in K$. If $x \neq a$ and $y = a$ then $xy = xa = (xe)a = x(ea) \neq a$ since $x, ea \in K \setminus \{a\}$ which is a group. Similarly, if $x = a$ and $y \neq a$ then $xy \neq a$.

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Theorem 3.14. Let K be a seminear-field with a as a category VI special element and let $d \in K \setminus \{a\}$ be such that $ax = dx$ and $xa = xd$ for all $x \in K$. Then the following statements hold :

(1) If $a + a = a$ then $(K,+)$ is a band.

(2) If $a + a \neq a$ then $a + a = d + d$.

(3) For all $x,y \in K \setminus \{a\}$, $x + x = y + y$ if and only if $x = y$.

(4) For all $x \in K \setminus \{a\}$, $x + a = a$ or $x + a = x + d$.

(5) For all $x \in K \setminus \{a\}$, $a + x = a$ or $a + x = d + x$.

Proof. (1) Assume that $a + a = a$. Let $x \in K \setminus \{a\}$. Then $x + x = ex + ex = d(d^{-1}x) + d(d^{-1}x) = a(d^{-1}x) + a(d^{-1}x) = (a+a)d^{-1}x = ad^{-1}x = d(d^{-1}x) = x$. Hence $(K,+)$ is a band.

(2) Assume that $a + a \neq a$. Then $a + a = (a+a)e = ae + ae = de + de = d + d$.

(3) Let $x,y \in K \setminus \{a\}$ be such that $x + x = y + y$. By Theorem 3.12, $(K \setminus \{a\},+)$ is a semigroup. So $x + x \in K \setminus \{a\}$. Thus $(e+e)x = x + x = y + y = (e+e)y$. Since $e + e \neq a$ and $(K \setminus \{a\},\cdot)$ is a group, it follows that $x = y$.

(4) Let $x \in K \setminus \{a\}$. Suppose that $x + a \neq a$. Then $x + a = (x+a)e = xe + ae = x + de = x + d$.

(5) The proof of (5) is similar to the proof of (4). #

Proposition 3.15. If K is a seminear-field of order greater than 2 with a category VI special element then K contains no additive zero.

Proof. Let a be a category VI special element of K and let e be the identity of $(K \setminus \{a\},\cdot)$. By Theorem 3.12 and Proposition 1.19, $K \setminus \{a\}$ contains no additive zero. It follows that for any $x \in K \setminus \{a\}$, x is not an additive zero of K. Suppose that a is an additive zero of K. Then $a + x = x + a = a$ for all $x \in K$. Since a is a category VI special element, there exists a unique element d in $K \setminus \{a\}$ such that $ax = dx$ and $xa = xd$ for $x \in K$. Claim that $x + e = e + x = e$ for all $x \in K\setminus\{a\}$. Let $x \in K\setminus\{a\}$. Then $a + xd = xd + a = a$. Thus $e = dd^{-1} = ad^{-1} = (a+xd)d^{-1} = ad^{-1} + x = dd^{-1} + x = e + x$. Similarly, $e = x + e$. Hence $x + e = e + x = e$ for all $x \in K\setminus\{a\}$. By Corollary 1.20, $|K\setminus\{a\}| = 1$. Thus $|K| = 2$, a contradiction. Hence K contains no additive zero.

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Proposition 3.16. If K is a seminear-field of order greater than 2 with a category VI special element then K contains no additive identity.

Proof. Let a be category VI special element of K and let e be the identity of $(K\setminus\{a\},\cdot)$. By Theorem 3.12 and Proposition 1.21, $K\setminus\{a\}$ contains no additive identity. It follows that for any $x \in K\setminus\{a\}$, x is not an additive identity of K. Suppose that a is an additive identity of K. Then $a + x = x + a = x$ for all $x \in K$. Since a is a category VI special element, by Theorem 3.11 there exists a unique element d in $K\setminus\{a\}$ such that $ax = dx$ and $xa = xd$ for all $x \in K$. Claim that $x + e = e + x = e$ for all $x \in K\setminus\{a\}$. Let $x \in K\setminus\{a\}$. $a + x^{-1}d = x^{-1}d + a = x^{-1}d$. Thus $e = (x^{-1}d)(d^{-1}x) = (a+x^{-1}d)d^{-1}x$ $= a(d^{-1}x) + e = d(d^{-1}x) + e = x + e$. Similarly, $e = e + x$. Hence $x + e = e + x = e$ for all $x \in K\setminus\{a\}$. By Corollary 1.20, $|K\setminus\{a\}| = 1$. Thus $|K| = 2$, a contradiction. Therefore K contains no additive identity.

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Theorem 3.17. Let K be a seminear-field with a as a category VI special element. Then the following statements hold :

(1) If K is L.A.C. then $x + y = y$ for all $x,y \in K\setminus\{a\}$.

(2) If K is R.A.C. then $x + y = y$ for all $x,y \in K\setminus\{a\}$.

(3) If $a + a = a$ then the following statements hold :

(3.1) K is L.A.C. if and only if $x + y = y$ for all $x,y \in K$.

(3.2) K is R.A.C. if and only if $x + y = x$ for all $x,y \in K$.

(3.3) K cannot be A.C.

<u>Proof</u>. Let e be the identity of $(K\setminus\{a\},\cdot)$ and let d be the unique element in $K\setminus\{a\}$ such that $ax = dx$ and $xa = xd$ for all $x \in K$.

(1) Assume that K is L.A.C. Claim that $z + a = a$ for all $z \in K\setminus\{a\}$. Let $z \in K\setminus\{a\}$. If $z + a \neq a$ then by Theorem 3.14 (4), $z + a = z + d$ and so $a = d$, a contradiction. Hence $z + a = a$ for all $z \in K\setminus\{a\}$. Let $x,y \in K\setminus\{a\}$. Then $xy^{-1}d + a = a$, so $y = d(d^{-1}y) = a(d^{-1}y) = (xy^{-1}d + a)d^{-1}y = x + ad^{-1}y = x + d(d^{-1}y)$ $= x + y$. Hence $x + y = y$ for all $x,y \in K\setminus\{a\}$.

The proof of (2) is similar to the proof of (1).

(3) Assume that $a + a = a$.

(3.1) Assume that K is L.A.C. Let $x,y \in K$.

<u>Case 1</u>. $x = y = a$. Then $x + y = a + a = a = y$.

<u>Case 2</u>. $x \neq a$, $y = a$. In the proof of (1), we showed that $z + a = a$ for all $z \in K\setminus\{a\}$. Thus $x + y = x + a = a = y$.

<u>Case 3</u>. $x = a$, $y \neq a$. By (1), $d + y = y$. If $a + y = a$ then $a + y = a + a$. Thus $y = a$, a contradiction. Hence $a + y = d + y$ $= y$. Therefore $x + y = a + y = d + y = y$.

<u>Case 4</u>. $x \neq a$, $y \neq a$. By (1), $x + y = y$.

Hence $x + y = y$ for all $x,y \in K$.

The converse is obvious.

The proof of (3.2) is similar to the proof of (3.1).

(3.3) Suppose that K is A.C. Thus K is L.A.C. In the proof of (1), we showed that $z + a = a$ for all $z \in K \setminus \{a\}$. Now $d + a = a$. $a + a = a = d + a$. Since K is R.A.C., $a = d$, a contradiction. Hence K is not A.C.

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Note that Example 3.2 (1) is L.A.C. and Example 3.2 (2) is R.A.C.

Theorem 3.18. Let K be a seminear-field with a as a category VI special element and let e be the identity of $(K \setminus \{a\}, \cdot)$. Then K is A.C. if and only if $K = \{a, e\}$ with the structure

·	e	a
e	e	e
a	e	e

and

+	e	a
e	e	a
a	a	e

Proof. Let d be the unique element in $K \setminus \{a\}$ such that $ax = dx$ and $xa = xd$ for all $x \in K$.

Assume that K is A.C. Claim that $a + x = x + a = a$ for all $x \in K \setminus \{a\}$. Let $x \in K \setminus \{a\}$. If $a + x = d + x$ then $a = d$. a contradiction. Thus $a + x = a$. Similarly we can show that $x + a = a$. Claim that $y + e = e + y = e$ for all $y \in K \setminus \{a\}$. Let $y \in K \setminus \{a\}$. Then $a + yd = yd + a = a$. Multiply this equation on the right by d^{-1} we get that $e + y = y + e = e$. By Theorem 3.12 and Corollary 1.20, we obtain $|K \setminus \{a\}| = 1$. Hence $|K| = 2$. Consequently $d = e$. By Theorem 3.17 (3), $a + a = d + d = e + e = e$. Therefore we have the above structure.

Conversely, it is straightforward to check that the above seminear-field is A.C.

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<u>Definition 3.19</u>. Let K be a (generalized) seminear-field with a as a special element. Let $D = K \setminus \{a\}$. Then $\{x \in D \mid x + a = a\}$ ($\{x \in D \mid a + x = a\}$) is called the <u>left (right) fundamental set of a in K</u>. The set $\{x \in D \mid x + a = a + x = a\}$ is called the <u>fundamental set of a in K</u>. If a is a category VI special element of K then we shall always denote the left (right) fundamental set of a in K by S_L (S_R). The fundamental set of a in K is denoted by S.

<u>Proposition 3.20</u>. Let K be a seminear-field with a as a category VI special element and $D = K \setminus \{a\}$.

(1) If $y \in D \setminus S_L$ then y is not L.A.C.

(2) If $y \in D \setminus S_R$ then y is not R.A.C.

(Therefore if $y \in D \setminus S$ then y is not A.C.)

<u>Proof</u>. Let $d \in K \setminus \{a\}$ be such that $ax = dx$ and $xa = xd$ for all $x \in K$.

(1) If $y \in D \setminus S_L$ then $y + a = y + d$. Since $a \neq d$, y is not L.A.C.

(2) If $y \in D \setminus S_R$ then $a + y = d + y$. Since $a \neq d$, y is not R.A.C. #

<u>Proposition 3.21</u>. Let K be a seminear-field with a as a category VI special element, $D = K \setminus \{a\}$ and let $d \in D$ be such that $ax = dx$ and $xa = xd$ for all $x \in K$. Then the following statements hold :

(1) $S_L \subseteq LI_D(d)$ and $S_R \subseteq RI_D(d)$. (Therefore $S \subseteq I_D(d)$.)

(2) $S_L = \emptyset$ or S_L is a filter in $(D,+)$. (Hence $D \setminus S_L = \emptyset$ or $D \setminus S_L$ is a completely prime ideal of $(D,+)$.)

(3) $S_R = \emptyset$ or S_R is a filter in $(D,+)$. (Hence $D \setminus S_R = \emptyset$ or $D \setminus S_R$ is a completely prime ideal of $(D,+)$.)

(4) $S = \emptyset$ or S is a filter in $(D,+)$. (Hence $D \setminus S = \emptyset$ or $D \setminus S$ is a completely prime ideal of $(D,+)$.)

(5) If $d \in S_L$ then $S_L = LI_D(d)$.

(6) If $d \in S_R$ then $S_R = RI_D(d)$.

(Therefore if $d \in S$ then $S = I_D(d)$.)

Proof. Let e be the identity of $(D,\cdot)$.

(1) To show that $S_L \subseteq LI_D(d)$, let $x \in S_L$. Then $x + a = a$. Multiply on the right by e, we obtain that $x + d = d$. Thus $x \in LI_D(d)$. Hence $S_L \subseteq LI_D(d)$. Similarly, we can show that $S_R \subseteq RI_D(d)$.

(2) Suppose that $S_L \neq \emptyset$. To show that S_L is a filter in $(D,+)$, let $x,y \in D$. Assume that $x,y \in S_L$. Then $x + a = y + a = a$, so $(x+y) + a = x + (y+a) = x + a = a$. Thus $x + y \in S_L$. Conversely, assume that $x + y \in S_L$. Then $(x+y) + a = a$. If $y + a \neq a$ then $x + (y+a) \in D$ which is an additive semigroup. This is a contradiction. Thus $y + a = a$. Consequently $x + a = a$. Hence S_L is a filter in $(D,+)$.

The proofs of (3) and (4) are similar to the proof of (2).

(5) Assume that $d \in S_L$. Then $d + a = a$. By (1), it suffices to show that $LI_D(d) \subseteq S_L$. Let $y \in LI_D(d)$. Then $y + d = d$, so $y + a = y + (d+a) = (y+d) + a = d + a = a$. Hence $y \in S_L$. Therefore $LI_D(d) \subseteq S_L$.

The proof of (6) is similar to the proof of (5).

Proposition 3.22. Let K be a seminear-field with a as a category VI special element, $D = K \setminus \{a\}$ and $d \in D$ such that $ax = dx$ and $xa = xd$ for all $x \in K$.

(1) If $S_L = \emptyset$ and S_R is a filter in $(D,+)$ then $d \in S_R$ iff $a + a = a$.

(2) If $S_R = \emptyset$ and S_L is a filter in $(D,+)$ then $d \in S_L$ iff $a + a = a$.

(3) If $\emptyset \neq S_R \subset D$ and $S_L \subset S_R$ then $d \in S_R$ iff $a + a = a$.

(4) If $\emptyset \neq S_L \subset D$ and $S_R \subset S_L$ then $d \in S_L$ iff $a + a = a$.

(5) If $S_L \not\subseteq S_R$ and $S_R \not\subseteq S_L$ then $a + a = d + d$.

Proof. (1) Assume that $S_L = \emptyset$ and S_R is a filter in $(D,+)$. Suppose that $d \in S_R$. Then $d = d + d$ and $a + d = a$. Since $S_L = \emptyset$, $d + a = d + d = d$. Thus $a + a = (a+d) + a = a + (d+a) = a + d = a$. Hence $a + a = a$.

Conversely, assume that $a + a = a$. If $S_R = D$ then $d \in S_R$. Suppose that $S_R \subset D$. To show that $d \in S_R$, suppose that $d \in D \setminus S_R$. Then $a + d \neq a$, so $a + d = d + d$. Let $x \in S_R$. Then $a + x = a$ and $x + a = x + d$. Thus $a = a + a = (a+x) + a = a + (x+a) = a + (x+d) = (a+x) + d = a + d = d + d$, a contradiction. Hence $d \in S_R$.

The proof of (2) is similar to the proof of (1).

(3) Assume that $\emptyset \neq S_R \subset D$ and $S_L \subset S_R$. Suppose that $d \in S_R$. Then $a + d = a$. Let $x \in S_R \setminus S_L$. Then $a + x = a$ and $x + a = x + d$. Thus $a + a = (a+x) + a = a + (x+a) = a + (x+d) = (a+x) + d = a + d = a$.

Conversly, assume that $d \in D \setminus S_R$. Then $a + d = d + d$. To show that $a + a \neq a$, let $y \in S_R \setminus S_L$. Then $a + y = a$ and $y + a = y + d$. Thus $a + a = (a+y) + a = a + (y+a) = a + (y+d) = (a+y) + d = a + d = d + d \neq a$. Hence if $a + a = a$ then $d \in S_R$.

The proof of (4) is similar to the proof of (3).

(5) Assume that $S_L \not\subseteq S_R$ and $S_R \not\subseteq S_L$. To show that $a + a = d + d$. Claim that $a + d = d + d$. Since $S_L \not\subseteq S_R$, there is an element x in $S_L \setminus S_R$. Thus $x + a = a$ and $a + x = d + x$. Since $S_L \subseteq LI_D(d)$, $x + d = d$. Thus $a + d = a + (x+d) = (a+x) + d = (d+x) + d = d + (x+d) = d + d$. Since $S_R \not\subseteq S_L$, there is an element y in $S_R \setminus S_L$. Then $a + y = a$ and $y + a = y + d$. Since

$S_R \subseteq RI_D(d)$, $d + y = d$. So $a + a = (a+y) + a = a + (y+a) =$ $a + (y+d) = (a+y) + d = a + d = d + d$.

#

<u>Theorem 3.23.</u> Let D be a ratio seminear-ring. Let a be a symbol not representing any element of D and let $d \in D$. Let $F_L \subseteq LI_D(d)$ be either $\emptyset$ or a filter in $(D,+)$ and let $F_R \subseteq RI_D(d)$ be either $\emptyset$ or a filter in $(D,+)$. Then the binary operations on D can be extended to $K = D \cup \{a\}$ in such a way that the following properties hold :

(1) K is a seminear-field containing a as a category VI special element.

(2) F_L is the left fundamental set of a in K and F_R is the right fundamental set of a in K.

(3) If $(D,+)$ is not a band then $a + a = d + d$.

(4) If $(D,+)$ is a band then

$$a + a = \begin{cases} a \text{ or } d & \text{if } F_L = F_R = \emptyset, \\ a & \text{if } F_L = \emptyset,\ F_R = D \text{ (in this case } (D,+) \text{ is a right zero semigroup.)}, \\ a & \text{if } F_L = \emptyset,\ \emptyset \neq F_R \subset D,\ d \in F_R, \\ d & \text{if } F_L = \emptyset,\ \emptyset \neq F_R \subset D,\ d \in D \setminus F_R, \\ a \text{ or } d & \text{if } F_L = F_R = D \text{ (in this case } D = \{e\}), \\ a & \text{if } F_L = D,\ F_R = \emptyset \text{ (in this case } (D,+) \text{ is a left zero semigroup.)}, \\ a & \text{if } \emptyset \neq F_L \subset D,\ d \in F_L,\ F_R = \emptyset, \\ d & \text{if } \emptyset \neq F_L \subset D,\ d \in D \setminus F_L,\ F_R = \emptyset, \\ a \text{ or } d & \text{if } \emptyset \neq F_L \subset D,\ F_L = F_R, \\ a & \text{if } \emptyset \neq F_L \subset D,\ \emptyset \neq F_R \subset D, \text{ (either } F_L \subset F_R,\ d \in F_R \text{ or } F_R \subset F_L,\ d \in F_L), \\ d & \text{if } \emptyset \neq F_L \subset D,\ \emptyset \neq F_R \subset D, \text{ (either } F_L \subset F_R,\ d \in D \setminus F_R \text{ or } F_R \subset F_L,\ d \in D \setminus F_L), \\ d & \text{if } F_L \nsubseteq F_R,\ F_R \nsubseteq F_L. \end{cases}$$

Furthermore , any extension of addition on D to K such that (1) and (2) hold must be as given above.

<u>Proof</u>. Suppose that $F_L = F_R = \emptyset$. Extend + and $\cdot$ from D to K by

(1) $xa = xd$ and $ax = dx$ for all $x \in D$, $a^2 = d^2$,

(2) $x + a = x + d$ and $a + x = d + x$ for all $x \in D$ and

(3) $a + a = \begin{cases} a \text{ or } d & \text{if } (D,+) \text{ is a band ,} \\ d + d & \text{if } (D,+) \text{ is not a band.} \end{cases}$

To show that K is a seminear-field, we must show that (a_1) $x(yz) = (xy)z$ for all $x,y,z \in K$, (b_1) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_3) $(x+y)z = xz + yz$ for all $x,y,z \in K$. To prove (a_1), let $x,y,z \in K$.

<u>Case 1</u>. $x = y = z = a$.

$x(yz) = a(a^2) = ad^2 = dd^2 = d^2d = d^2a = a^2a = (xy)z$.

<u>Case 2</u>. $x = y = a$, $z \neq a$.

$x(yz) = a(az) = a(dz) = d(dz) = d^2z = a^2z = (xy)z$.

<u>Case 3</u>. $x = z = a$, $y \neq a$.

$x(yz) = a(ya) = a(yd) = d(yd) = (dy)d = (dy)a = (ay)a = (xy)z$

<u>Case 4</u>. $x \neq a$, $y = z = a$.

$x(yz) = x(a^2) = xd^2 = (xd)d = (xd)a = (xa)a = (xy)z$.

<u>Case 5</u>. $x \neq a$, $y \neq a$, $z = a$.

$x(yz) = x(ya) = x(yd) = (xy)d = (xy)a = (xy)z$.

<u>Case 6</u>. $x \neq a$, $y = a$, $z \neq a$.

$x(yz) = x(az) = x(dz) = (xd)z = (xa)z = (xy)z$.

<u>Case 7</u>. $x = a$, $y \neq a$, $z \neq a$.

$x(yz) = a(yz) = d(yz) = (dy)z = (ay)z = (xy)z$.

<u>Case 8</u>. $x \neq a$, $y \neq a$, $z \neq a$.

$x(yz) = (xy)z$.

To prove (b_1), let $x,y,z \in K$. Consider the following cases.

<u>Case 1</u>. $x = y = z = a$

<u>Subcase 1.1</u>. $a + a = a$.

$x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z$.

<u>Subcase 1.2</u>. $a + a = d$.

$x + (y+z) = a + (a+a) = a + d = d + d = d + a = (a+a) + a = (x+y) + z$.

<u>Subcase 1.3</u>. $a + a = d + d$.

$x + (y+z) = a + (a+a) = a + (d+d) = d + (d+d) = (d+d) + a = (d+d) + a = (a+a) + a = (x+y) + z$.

<u>Case 2</u>. $x = y = a$, $z \neq a$.

<u>Subcase 2.1</u>. $a + a = a$. Then $(D,+)$ is a band.

$x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z$,

$(x+y) + z = (a+a) + z = a + z = d + z$.

<u>Subcase 2.2</u>. $a + a = d$. Then $(D,+)$ is a band.

$x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z$,

$(x+y) + z = (a+a) + z = d + z$.

<u>Subcase 2.3</u>. $a + a = d + d$.

$x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = (a+a) + z = (x+y) + z$.

<u>Case 3</u>. $x = z = a$, $y \neq a$.

$x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z$.

<u>Case 4</u>. $x \neq a$, $y = z = a$.

This proof is similar to the proof of Case 2.

<u>Case 5</u>. $x \neq a$, $y \neq a$, $z = a$.

$x + (y+z) = x + (y+a) = x + (y+d) = (x+y) + d = (x+y) + a = (x+y) + z$.

<u>Case 6</u>. $x \neq a$, $y = a$, $z \neq a$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = (x+a) + z = (x+y) + z.$

<u>Case 7.</u> $x = a,\ y \neq a,\ z \neq a.$

$x + (y+z) = a + (y+z) = d + (y+z) = (d+y) + z = (a+y) + z = (x+y) + z.$

<u>Case 8.</u> $x \neq a,\ y \neq a,\ z \neq a.$

$x + (y+z) = (x+y) + z.$

To prove (c_1), let $x,y,z \in K$.

<u>Case 1.</u> $x = y = z = a.$

<u>Subcase 1.1.</u> $a + a = a$. Then $(D,+)$ is a band.

$(x+y)z = (a+a)a = a^2 = d^2 = d^2 + d^2 = a^2 + a^2 = xz + yz.$

<u>Subcase 1.2.</u> $a + a = d$. Then $(D,+)$ is a band.

$(x+y)z = (a+a)a = da = d^2 = d^2 + d^2 = a^2 + a^2 = xz + yz.$

<u>Subcase 1.3.</u> $a + a = d + d.$

$(x+y)z = (a+a)a = (d+d)a = (d+d)d = d^2 + d^2 = a^2 + a^2 = xz + yz.$

<u>Case 2.</u> $x = y = a,\ z \neq a.$

<u>Subcase 2.1.</u> $a + a = a$. Then $(D,+)$ is a band.

$(x+y)z = (a+a)z = az = dz = dz + dz = az + az = xz + yz.$

<u>Subcase 2.2.</u> $a + a = d$. Then $(D,+)$ is a band.

$(x+y)z = (a+a)z = dz = dz + dz = az + az = xz + yz.$

<u>Subcase 2.3.</u> $a + a = d + d.$

$(x+y)z = (a+a)z = (d+d)z = dz + dz = az + az = xz + yz.$

<u>Case 3.</u> $x = z = a,\ y \neq a.$

$(x+y)z = (a+y)a = (d+y)a = (d+y)d = d^2 + yd = a^2 + ya = xz + yz.$

<u>Case 4.</u> $x \neq a,\ y = z = a.$

This proof is similar to Case 3.

<u>Case 5.</u> $x \neq a,\ y \neq a,\ z = a.$

$(x+y)z = (x+y)a = (x+y)d = xd + yd = xa + ya = xz + yz.$

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$(x+y)z = (x+a)z = (x+d)z = xz + dz = xz + az = xz + yz.$

Case 7. $x = a$, $y \neq a$, $z \neq a$.

This proof is similar to Case 6.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$(x+y)z = xz + yz.$

Hence K is a seminear-field and we obtain (1) - (4).

Suppose that $F_L = \emptyset$ and $F_R = D$. Since $d \in F_R = RI_D(d)$, $d + d = d$. Therefore $(D,+)$ is a band. Extend $\cdot$ and $+$ from D to K by

(1) $xa = xd$ and $ax = dx$ for all $x \in D$, $a^2 = d^2$,

(2) $x + a = x + d$ and $a + x = a$ for all $x \in D$ and

(3) $a + a = a$.

To show that K is a seminear-field. We shall show that (a_2) $x(yz) = (xy)z$ for all $x,y,z \in K$, (b_2) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and $(x+y)z = xz + yz$ for all $x,y,z \in K$. The proof of (a_2) is the same as the proof of (a_1). To prove (b_2), let $x,y,z \in K$. Note that $a + t = a$ for all $t \in K$.

Case 1. $x = a$.

$x + (y+z) = a + (y+z) = a$, $(x+y) + z = (a+y) + z = a + z = a.$

Case 2. $x \neq a$.

Subcase 2.1. $y = z = a$.

$x + (y+z) = x + (a+a) = x + a = x + d$, $(x+y) + z = (x+a) + a = (x+d) + d = x + (d+d) = x + d.$

Subcase 2.2. $y = a$, $z \neq a$. Since $z \in RI_D(d)$, $d + z = d$.

$x + (y+z) = x + (a+z) = x + a = x + d$, $(x+y) + z = (x+a) + z = (x+d) + z = x + (d+z) = x + d.$

Subcase 2.3. $y \neq a$, $z = a$.

$x + (y+z) = x + (y+a) = x + (y+d) = (x+y) + d$, $(x+y) + z = (x+y) + a = (x+y) + d$.

Subcase 2.4. $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_3), let $x,y,z \in K$.

Case 1. $x = y = z = a$.

$(x+y)z = (a+a)a = a^2 = d^2 = d^2 + d^2 = a^2 + a^2 = xz + yz$.

Case 2. $x = y = a$, $z \neq a$.

$(x+y)z = (a+a)z = az = dz = dz + dz = az + az = xz + yz$.

Case 3. $x = z = a$, $y \neq a$. Since $y \in RI_D(d)$, $d + y = d$.

$(x+y)z = (a+y)a = a^2 = d^2 = (d+y)d = d^2 + yd = a^2 + ya = xz + yz$.

Case 4. $x \neq a$, $y = z = a$.

$(x+y)z = (x+a)a = (x+d)a = (x+d)d = xd + d^2 = xa + a^2 = xz + yz$.

Case 5. $x \neq a$, $y \neq a$, $z = a$.

$(x+y)z = (x+y)a = (x+y)d = xd + yd = xa + ya = xz + yz$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$(x+y)z = (x+a)z = (x+d)z = xz + dz = xz + az = xz + yz$.

Case 7. $x = a$, $y \neq a$, $z \neq a$. Since $y \in RI_D(d)$, $d + y = d$.

$(x+y)z = (a+y)z = az = dz = (d+y)z = dz + yz = az + yz = xz + yz$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1),(2) and (4)

Suppose that $F_L = \emptyset$ and F_R is a proper filter in $(D,+)$. Then $D \setminus F_R$ is an ideal of $(D,+)$. Extend $+$ and $\cdot$ from D to K by

(1) $xa = xd$ and $ax = dx$ for all $x \in D$, $a^2 = d^2$,

(2) $x + a = x + d$ for all $x \in D$,

$a + x = a$ for all $x \in F_R$, $a + x = d + x$ for all $x \in D \setminus F_R$ and

$$(3)\quad a + a = \begin{cases} a & \text{if } (D,+) \text{ is a band and } d \in F_R\text{,} \\ d & \text{if } (D,+) \text{ is a band and } d \in D\setminus F_R\text{,} \\ d + d & \text{if } (D,+) \text{ is not a band .} \end{cases}$$

To show that K is a seminear-field, we shall show that (a_3) $x(yz) = (xy)z$ for all $x,y,z \in K$, (b_3) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_3) $(x+y)z = xz + yz$ for all $x,y,z \in K$. The proof of (a_3) is the same as the proof of (a_1). Note that $D\setminus F_R$ is an ideal of $(D,+)$.

To prove (b_3), let $x,y,z \in K$. Consider the following cases :

Case 1. $x = y = z = a$.

Subcase 1.1. $(D,+)$ is a band, $d \in F_R$. Then $a + a = a$.
$x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z$.

Subcase 1.2. $(D,+)$ is a band, $d \in D\setminus F_R$. Then $a + a = d$.
$x + (y+z) = a + (a+a) = a + d = d + d = d + a = (a+a) + a = (x+y) + z$.

Subcase 1.3. $(D,+)$ is not a band, Then $a + a = d + d$ and $d \in D\setminus F_R$.
$x + (y+z) = a + (a+a) = a + (d+d) = d + (d+d) = (d+d) + d = (d+d) + a = (a+a) + a = (x+y) + z$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $a + a = a$. Then $(D,+)$ is a band.
If $z \in F_R$ then $a + z = a$. Thus $x + (y+z) = a + (a+z) = a + a = a = a + z = (a+a) + z = (x+y) + z$. If $z \in D\setminus F_R$ then $d + z \in D\setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + a = d + z = a + z = (a+a) + a = (x+y) + z$.

Subcase 2.2. $a + a = d$. Then $(D,+)$ is a band.
If $z \in F_R$ then $d + z = d$ since $F_R \subseteq RI_D(d)$.
$x + (y+z) = a + (a+z) = a + a = d = d + z = (a+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $d + z \in D \setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z = (a+a) + z = (x+y) + z$.

<u>Subcase 2.3.</u> $a + a = d + d$.

If $z \in F_R$ then $d + z = d$ since $F_R \subseteq RI_D(d)$. Thus $x + (y+z) = a + (a+z) = a + a = d + d = d + (d+z) = (d+d) + z = (a+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $d + z \in D \setminus F_R$ which is an ideal of $(D,+)$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = (a+a) + z = (x+y) + z$.

<u>Case 3.</u> $x = z = a$, $y \neq a$.

<u>Subcase 3.1.</u> $(D,+)$ is a band, $d \in F_R$. Then $a + a = a$.

<u>Subcase 3.1.1.</u> $y + d \in F_R$. Since F_R is a filter in $(D,+)$, $y \in F_R$.

$x + (y+z) = a + (y+a) = a + (y+d) = a = a + a = (a+y) + a = (x+y) + z$.

<u>Subcase 3.1.2.</u> $y + d \in D \setminus F_R$. Since $d \in F_R$, $y \in D \setminus F_R$.

$x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z$.

<u>Subcase 3.2.</u> $(D,+)$ is a band, $d \in D \setminus F_R$. Then $a + a = d$ and $y + d \in D \setminus F_R$.

If $y \in F_R$ then $d + y = d$. Thus $x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = d + d = d$, $(x+y) + z = (a+y) + a = a + a = d$.

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z$.

Subcase 3.3. (D,+) is not a band. Then $a + a = d + d$ and $d \in D \setminus F_R$. Thus $y + d \in D \setminus F_R$.

If $y \in F_R$ then $d + y = d$. Thus $x + (y+z) = a + (y+a) = a + (y+d)$ $= d + (y+d) = (d+y) + d = d + d = a + a = (a+y) + a = (x+y) + z$.

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) =$ $(d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z$.

Case 4. $x \neq a$, $y = z = a$.

Subcase 4.1. $a + a = a$. Then (D,+) is a band.

$x + (y+z) = x + (a+a) = x + a = x + d = x + (d+d) = (x+d) + d =$ $(x+d) + a = (x+a) + a = (x+y) + z$.

Subcase 4.2. $a + a = d$. Then (D,+) is a band.

$x + (y+z) = x + (a+a) = x + d$, $(x+y) + z = (x+a) + a = (x+d) + a$ $= (x+d) + d = x + (d+d) = x + d$.

Subcase 4.3. $a + a = d + d$.

$x + (y+z) = x + (a+a) = x + (d+d) = (x+d) + d = (x+d) + a =$ $(x+a) + a = (x+y) + z$.

Case 5. $x \neq a$, $y \neq a$, $z = a$.

$x + (y+z) = x + (y+a) = x + (y+d) = (x+y) + d = (x+y) + a =$ $(x+y) + z$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

If $z \in F_R$ then $d = d + z$ since $F_R \subseteq RI_D(d)$.

$x + (y+z) = x + (a+z) = x + a = x + d = x + (d+z) = (x+d) + z =$ $(x+a) + z = (x+y) + z$.

If $z \in D \setminus F_R$ then $x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z =$ $(x+a) + z = (x+y) + z$.

Case 7. $x = a$, $y \neq a$, $z \neq a$.

Subcase 7.1. $y + z \in F_R$. Since F_R is a filter in (D,+), $y,z \in F_R$.

$x + (y+z) = a + (y+z) = a = a + z = (a+y) + z = (x+y) + z.$

Subcase 7.2. $y + z \in D \setminus F_R$.

If $y \in F_R$ then $z \in D \setminus F_R$. Since $F_R \subseteq RI_D(d)$, $d + y = d$.

$x + (y+z) = a + (y+z) = d + (y+z) = (d+y) + z = d + z = a + z = (a+y) + z = (x+y) + z.$

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+z) = d + (y+z) = (d+y) + z = (a+y) + z = (x+y) + z.$

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z.$

To prove (c_3), let $x,y,z \in K$. Consider the following cases :

Case 1. $x = y = z = a.$

This proof is the same as the proof of case 1 in (c_1).

Case 2. $x = y = a$, $z \neq a$.

This proof is the same as the proof of case 2 in (c_1).

Case 3. $x = z = a$, $y \neq a$.

If $y \in F_R$ then $d + y = d$ since $F_R \subseteq RI_D(d)$.

$(x+y)z = (a+y)a = a^2 = d^2 = (d+y)d = d^2 + yd = a^2 + ya = xz + yz.$

If $y \in D \setminus F_R$ then $(x+y)z = (a+y)a = (d+y)a = (d+y)d = d^2 + yd = a^2 + ya = xz + yz.$

Case 4. $x \neq a$, $y = z = a$.

$(x+y)z = (x+a)a = (x+d)a = (x+d)d = xd + d^2 = xa + a^2 = xz + yz.$

Case 5. $x \neq a$, $y \neq a$, $z = a$.

$(x+y)z = (x+y)a = (x+y)d = xd + yd = xa + ya = xz + yz.$

Case 6. $x \neq a$, $y = a$, $z \neq a$.

$(x+y)z = (x+a)z = (x+d)z = xz + dz = xz + az = xz + yz.$

Case 7. $x = a$, $y \neq a$, $z \neq a$.

If $y \in F_R$ then $d = d + y$ since $F_R \subseteq RI_D(d)$.

$(x+y)z = (a+y)z = az = dz = (d+y)z = dz + yz = az + yz = az + yz = xz + yz.$

If $y \in D \setminus F_R$ then $(x+y)z = (a+y)z = (d+y)z = dz + yz = az + yz = xz + yz$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1) - (4).

Suppose that $F_L = F_R = D$. Then $D = I_D(d) = \{x \in D \mid x + d = d + x = d\}$. Claim that $D = \{e\}$. Let $x \in D$. Then $xd + d = d + xd = d$. Multiply this equation on the right by d^{-1}, we obtain that $x + e = e + x = e$. Hence $x + e = e + x = e$ for all $x \in D$. By Corollary 1.20, $D = \{e\}$. Consequently, $d = e$. Note that $(D,+)$ is a band. Extend + and · from D to K by (1) $ea = ae = a^2 = e$, (2) $e + a = a + e = a$ and (3) $a + a = a$ or e. So $K = \{a,e\}$ has one of the following two structures.

·	e	a
e	e	e
a	e	e

and

+	e	a
e	e	a
a	a	a

or

·	e	a
e	e	e
a	e	e

and

+	e	a
e	e	a
a	a	e

.

It is easy to check that K is a seminear-field. And we obtain (1) - (4).

For the cases ($F_L = D$ and $F_R = \emptyset$) and (F_L is a proper filter in $(D,+)$ and $F_R = \emptyset$), the proofs are similar to proofs of cases ($F_L = \emptyset$ and $F_R = D$) and ($F_L = \emptyset$ and F_R is a proper filter in $(D,+)$), respectively.

Suppose that $F_L = D$ and F_R is a proper filter in $(D,+)$. Now we have $LI_D(d) = D$ and F_R is a filter in $(D,+)$. By Proposition 1.24 (4.5), $F_R = D = \{e\}$, a contradiction. Hence this case cannot occur. Similarly, we can show that the case (F_L is a proper filter in $(D,+)$ and $F_R = D$) cannot occur.

Suppose that F_L and F_R are proper filters in $(D,+)$.

Case I $F_L = F_R$.

Extend + and $\cdot$ from D to K by

(1) $xa = xd$ and $ax = dx$ for all $x \in D$, $a^2 = d^2$,

(2) $x + a = a + x = a$ for all $x \in F_L$,

$x + a = x + d$ and $a + x = d + x$ for all $x \in D \setminus F_L$ and

(3) $a + a = \begin{cases} a \text{ or } d & \text{if } (D,+) \text{ is a band,} \\ d + d & \text{if } (D,+) \text{ is not a band.} \end{cases}$

To show that K is a seminear-field, we shall show that

(a_4) $x(yz) = (xy)z$ for all $x,y,z \in K$,

(b_4) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and

(c_4) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

The proof of (a_4) is the same as the proof of (a_1).

To prove (b_4), let $x,y,z \in K$. Consider the following cases.

Case 1. $x = y = z = a$.

Subcase 1.1. $a + a = a$.

$x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z$.

Subcase 1.2. $a + a = d$.

If $d \in F_L$ then $x + (y+z) = a + (a+a) = a + d = d + a = (a+a) + a = (x+y) + z$.

If $d \in D \setminus F_L$ then $x + (y+z) = a + (a+a) = a + d = d + d = d + a = (a+a) + a = (x+y) + z$.

Subcase 1.3. $a + a = d + d$.

If $d \in F_L$ then $d + d \in F_L$ which is an additive semigroup.

$x + (y+_z) = a + (a+a) = a + (d+d) = (d+d) + a = (a+a) + a = (x+y) + z$.

If $d \in D \setminus F_L$ then $d + d \in D \setminus F_L$ which is an ideal of $(D,+)$.

$x + (y+z) = a + (a+a) = a + (d+d) = d + (d+d) = (d+d) + d =$
$(d+d) + a = (a+a) + a = (x+y) + z.$

Case 2. $x = y = a,\ z \neq a.$

Subcase 2.1. $a + a = a$. Then $(D,+)$ is a band.

If $z \in F_L$ then $x + (y+z) = a + (a+z) = a + a = a = a + z =$
$(a+a) + z = (x+y) + z.$

If $z \in D\setminus F_L$ then $d + z \in D\setminus F_L$ which is an ideal of $(D,+)$.
$x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z,$
$(x+y) + z = (a+a) + z = a + z = d + z.$

Subcase 2.2. $a + a = d$. Then $(D,+)$ is a band.

If $z \in F_L$ then $d = d + z$ since $F_L = F_R \subseteq RI_D(d)$.
$x + (y+z) = a + (a+z) = a + a = d,\ (x+y) + z = (a+a) + z = d + z = d.$

If $z \in D\setminus F_L$ then $d + z \in D\setminus F_L$ which is an ideal of $(D,+)$.
$x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z,$
$(x+y) + z = (a+a) + z = d + z.$

Subcase 2.3. $a + a = d + d.$

If $z \in F_L$ then $d = d + z$ since $F_L = F_R \subseteq RI_D(d)$.
$x + (y+z) = a + (a+z) = a + a = d + d,\ (x+y) + z = (a+a) + z =$
$(d+d) + z = d + (d+z) = d + d.$

If $z \in D\setminus F_L$ then $d + z \in D\setminus F_L$ which is an ideal of $(D,+)$.
$x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z =$
$(a+a) + z = (x+y) + z.$

Case 3. $x = z = a,\ y \neq a.$

If $y \in F_L$ then $x + (y+z) = a + (y+a) = a + a = (a+y) + a = (x+y) + z.$

If $y \in D\setminus F_L$ then $d + y,\ y + d \in D\setminus F_L$ which is an ideal of $(D,+)$.
$x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d =$
$(d+y) + a = (a+y) + d = (x+y) + z.$

Case 4. $x \neq a,\ y = z = a.$

This proof is similar to Case 2.

<u>Case 5</u>. $x \neq a$, $y \neq a$, $z = a$.

<u>Subcase 5.1</u>. $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$.

$x + (y+z) = x + (y+a) = x + a = a$, $(x+y) + z = (x+y) + a = a$.

<u>Subcase 5.2</u>. $x + y \in D \setminus F_L$.

If $y \in F_L$ then $x \in D \setminus F_L$ and $y + d = d$. Thus $x + (y+z) = x + (y+a) = x + a = x + d$, $(x+y) + z = (x+y) + a = (x+y) + d = x + (y+d) = x + d$.

If $y \in D \setminus F_L$ then $x + (y+z) = x + (y+a) = x + (y+d) = (x+y) + d = (x+y) + a = (x+y) + z$.

<u>Case 6</u>. $x \neq a$, $y = a$, $z \neq a$.

<u>Subcase 6.1</u>. $x,z \in F_L$. Then $x + a = a = a + z$.

$x + (y+z) = x + (a+z) = x + a = a = a + z = (x+a) + z = (x+y) + z$.

<u>Subcase 6.2</u>. $x \in F_L$, $z \in D \setminus F_L$. Since $F_L \subseteq LI_D(d)$, $x + d = d$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = d + z = a + z = (x+a) + z = (x+y) + z$.

<u>Subcase 6.3</u>. $x \in D \setminus F_L$, $z \in F_L$.

This proof is similar to Subcase 6.2.

<u>Subcase 6.4</u>. $x,z \in D \setminus F_L$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = (x+a) + z = (x+y) + z$.

<u>Case 7</u>. $x = a$, $y \neq a$, $z \neq a$. This proof is similar to Case 5.

<u>Case 8</u>. $x \neq a$, $y \neq a$, $z \neq a$. $x + (y+z) = (x+y) + z$.

To prove (c_4), let $x,y,z \in K$. Consider the following cases.

<u>Case 1</u>. $x = y = z = a$.

This proof is the same as the proof of Case 1 in (c_1)

<u>Case 2</u>. $x = y = a$, $z \neq a$.

This proof is the same as the proof of Case 2 in (c_1).

<u>Case 3</u>. $x = z = a$, $y \neq a$.

If $y \in F_L$ then $d = d + y$ since $F_L = F_R \subseteq RI_D(d)$. Thus $(x+y)z =$

$(a+y)a = a^2 = d^2 = (d+y)d = d^2 + yd = a^2 + ya = xz + yz.$

If $y \in D \setminus F_L$ then $(x+y)z = (a+\ )a = (d+y)a = (d+y)d = d^2 + yd = a^2 + ya = xz + yz.$

Case 4. $x \neq a,\ y = z = a.$

This proof is similar to Case 3.

Case 5. $x \neq a,\ y \neq a,\ z = a.$

$(x+y)z = (x+y)a = (x+y)d = xd + yd = xa + ya = xz + yz.$

Case 6. $x \neq a,\ y = a,\ z \neq a.$

If $x \in F_L$ then $x + d = d$ since $F_L \subseteq LI_D(d)$. Thus $(x+y)z = (x+a)z = az = dz = (x+d)z = xz + dz = xz + az = xz + yz.$

If $x \in D \setminus F_L$ then $(x+y)z = (x+a)z = (x+d)z = xz + dz = xz + az = xz + yz.$

Case 7. $x = a,\ y \neq a,\ z \neq a.$

This proof is similar to Case 6.

Case 8. $x \neq a,\ y \neq a,\ z \neq a.$

$(x+y)z = xz + yz.$

Hence K is a seminear-field and we obtain (1) - (4).

Case II Either $F_L \subset F_R$ or $F_R \subset F_L$. We may assume that $F_L \subset F_R$. Extend + and · from D to K by

(1) $xa = xd$ and $ax = dx$ for all $x \in D$, $a^2 = d^2$,

(2) $x + a = a$ for all $x \in F_L$, $x + a = x + d$ for all $x \in D \setminus F_L$,

$a + x = a$ for all $x \in F_R$, $a + x = d + x$ for all $x \in D \setminus F_R$,

(3) $$a + a = \begin{cases} a & \text{if } (D,+) \text{ is a band and } d \in F_R, \\ d & \text{if } (D,+) \text{ is a band and } d \in D \setminus F_R, \\ d + d & \text{if } (D,+) \text{ is not a band.} \end{cases}$$

We shall first show that $x + (y+a) = (x+y) + a$ for all $x, y \in D$.

Case i. $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$.

$x + (y+a) = x + a = a = (x+y) + a$.

Case ii. $x + y \in D \setminus F_L$.

If $y \in F_L$ then $x \in D\setminus F_L$ and $d = y + d$ since $F_L \subseteq LI_D(d)$. Thus

$x + (y+a) = x + a = x + d = x + (y+d) = (x+y) + d = (x+y) + a$.

If $y \in D\setminus F_L$ then $x + (y+a) = x + (y+d) = (x+y) + d = (x+y) + a$.

Claim that $d \in D\setminus F_L$. Since $F_L \subset F_R$, there is an element t in $F_R\setminus F_L$. Thus $a + t = a$, $d + t = d$ and $t + a = t + d$. So $d + a = (d+t) + a = d + (t+a) = d + (t+d) = (d+t) + d = d + d \neq a$. Hence $d \in D\setminus F_L$.

To show that K is a seminear-field, we shall show that (a_5) $x(yz) = (xy)z$ for all $x,y,z \in K$, (b_5) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_5) $(x+y)z = xz + yz$ for all $x,y,z \in K$. The proof of (a_5) is the same as the proof of (a_1).

To prove (b_5), let $x,y,z \in K$. Consider the following cases.

Case 1. $x = y = z = a$.

Subcase 1.1. $a + a = a$.

$x + (y+z) = a + (a+a) = a + a = (a+a) + a = (x+y) + z$.

Subcase 1.2. $a + a = d$. Then $d \in D\setminus F_R$. Since $d \in D\setminus F_L$, $d + a = d + d$.

$x + (y+z) = a + (a+a) = a + d = d + d = d + a = (a+a) + a = (x+y) + z$.

Subcase 1.3. $a + a = d + d$. Since $(D,+)$ is not a band, $d \in D\setminus F_R$.

$x + (y+z) = a + (a+a) = a + (d+d) = d + (d+d) = (d+d) + d =$
$(d+d) + a = (a+a) + a = (x,y) + z$.

Case 2. $x = y = a$, $z \neq a$.

Subcase 2.1. $a + a = a$. Then $(D,+)$ is a band.

If $z \in F_R$ then $x + (y+z) = a + (a+z) = a + a = a = a + z =$
$(a+a) + z = (x+y) + z$.

If $z \in D\setminus F_R$ then $d + z \in D\setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z = a + z = (a+a) + z = (x+y) + z$.

<u>Subcase 2.2</u>. $a + a = d$. Then $(D,+)$ is a band.

If $z \in F_R$ then $d = d + z$ since $F_R \subseteq RI_D(d)$. Thus $x + (y+z) = a + (a+z) = a + a = d$, $(x+y) + z = (a+a) + z = d + z = d$.

If $z \in D\setminus F_R$ then $d + z \in D\setminus F_R$ which is an ideal of $(D,+)$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = d + z = (a+a) + z = (x+y) + z$.

<u>Subcase 2.3</u>. $a + a = d + d$.

If $z \in F_R$ then $d = d + z$ since $F_R \subseteq RI_D(d)$. Thus $x + (y+z) = a + (a+z) = a + a = d + d$, $(x+y) + z = (a+a) + z = (d+d) + z = d + (d+z) = d + d$.

If $z \in D\setminus F_R$ then $d + z \in D\setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = (a+a) + z = (x+y) + z$.

<u>Case 3</u>. $x = z = a$, $y \neq a$.

<u>Subcase 3.1</u>. $y \in F_L$. Then $y \in F_R$, so $a + y = y + a = a$. $x + (y+z) = a + (y+a) = a + a = (a+y) + a = (x+y) + z$.

<u>Subcase 3.2</u>. $y \in F_R\setminus F_L$. Since $F_R \subseteq RI_D(d)$, $d + y = d$.

<u>Subcase 3.2.1</u>. $(D,+)$ is a band, $d \in F_R$. Then $a + a = a$ and $y + d \in F_R$.

$x + (y+z) = a + (y+a) = a + (y+d) = a = a + a = (a+y) + a = (x+y) + z$.

<u>Subcase 3.2.2</u>. $(D,+)$ is a band, $d \in D\setminus F_R$. Then $a + a = d$ and $y + d \in D\setminus F_R$.

$x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = d + d = d = a + a = (a+y) + a = (x+y) + z$.

<u>Subcase 3.2.3</u>. $(D,+)$ is not a band. Then $a + a = d + d$ and $d \in D\setminus F_R$. Thus $y + d \in D\setminus F_R$, so $x + (y+z) =$

$a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = d + d = a + a =$

$(a+y) + a = (x+y) + z.$

<u>Subcase 3.3</u>. $y \in D \setminus F_R$. Since $D \setminus F_R \subset D \setminus F_L$, $y \in D \setminus F_L$. Thus $y + d \in D \setminus F_R$ and $d + y \in D \setminus F_L$, so $x + (y+z) = a + (y+a) =$

$a + (y+d) = d + (y+d) = (d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z.$

<u>Case 4</u>. $x \neq a$, $y = a$, $z = a$.

This proof is similar to Case 2.

<u>Case 5</u>. $x \neq a$, $y \neq a$, $z = a$.

By the first proof, we showed that $x + (y+a) = (x+y) + a$.

<u>Case 6</u>. $x \neq a$, $y = a$, $z \neq a$.

<u>Subcase 6.1</u>. $x \in F_L$, $z \in F_R$.

$x + (y+z) = x + (a+z) = x + a = a = a + z = (x+a) + z = (x+y) + z.$

<u>Subcase 6.2</u>. $x \in F_L$, $z \in D \setminus F_R$. Since $F_L \subseteq LI_D(d)$, $d = x + d$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = d + z = a + z =$

$(x+a) + z = (x+y) + z.$

<u>Subcase 6.3</u>. $x \in D \setminus F_L$, $z \in F_R$. Since $F_R \subseteq RI_D(d)$, $d = d + z$.

$x + (y+z) = x + (a+z) = x + a = x + d = x + (d+z) = (x+d) + z =$

$(x+a) + z = (x+y) + z.$

<u>Subcase 6.4</u>. $x \in D \setminus F_L$, $z \in D \setminus F_R$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = (x+a) + z = (x+y) + z.$

<u>Case 7</u>. $x = a$, $y \neq a$, $z \neq a$.

<u>Subcase 7.1</u>. $y + z \in F_R$. Since F_R is a filter in $(D,+)$, $y, z \in F_R$.

$x + (y+z) = a + (y+z) = a = a + z = (a+y) + z = (x+y) + z.$

<u>Subcase 7.2</u>. $y + z \in D \setminus F_R$.

If $y \in F_R$ then $z \in D \setminus F_R$. Since $F_R \subseteq RI_D(d)$, $d + y = d$. Thus

$x + (y+z) = a + (y+z) = d + (y+z) = (d+y) + z = d + z = a + z =$

$(a+y) + z = (x+y) + z$.

If $y \in D \setminus F_R$ then $x + (y+z) = a + (y+z) = d + (y+z) = (d+y) + z =$ $(a+y) + z = (x+y) + z$.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z$.

To prove (c_5), let $x,y,z \in K$. Consider the following cases :

Case 1. $x = y = z = a$.

This proof is the same as the proof of Case 1 in (c_1).

Case 2. $x = y = a$, $z \neq a$.

This proof is the same as the proof of Case 2 in (c_1).

Case 3. $x = z = a$, $y \neq a$.

If $y \in F_R$ then $d = d + y$ since $F_R \subseteq RI_D(d)$. Thus $(x+y)z = (a+y)a$ $= a^2 = d^2 = (d+y)d = d^2 + yd = a^2 + ya = xz + yz$.

If $y \in D \setminus F_R$ then $(x+y)z = (a+y)a = (d+y)a = (d+y)d = d^2 + yd =$ $a^2 + ya = xz + yz$.

Case 4. $x \neq a$, $y = z = a$.

This proof is similar to Case 3.

Case 5. $x \neq a$, $y \neq a$, $z = a$.

$(x+y)z = (x+y)a = (x+y)d = xd + yd = xa + ya = xz + yz$.

Case 6. $x \neq a$, $y = a$, $z \neq a$.

If $x \in F_L$ then $x + d = d$. Thus $(x+y)z = (x+a)z = az = dz =$ $(x+d)z = xz + dz = xz + az = xz + yz$.

If $x \in D \setminus F_L$ then $(x+y)z = (x+a)z = (x+d)z = xz + dz = xz + az =$ $xz + yz$.

Case 7. $x = a$, $y \neq a$, $z \neq a$.

This proof is similar to Case 7.

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$(x+y)z = xz + yz$.

<u>Case III</u> $F_L \nsubseteq F_R$ and $F_R \nsubseteq F_L$.

Extend + and · from D to K by

(1) $xa = xd$ and $ax = dx$ for all $x \in D$, $a^2 = d^2$,

(2) $x + a = a$ for all $x \in F_L$, $x + a = x + d$ for all $x \in D\setminus F_L$,

$a + x = a$ for all $x \in F_R$, $a + x = d + x$ for all $x \in D\setminus F_R$ and

(3) $a + a = d + d$.

We shall first show that $x + (y+a) = (x+y) + a$ for all $x,y \in D$. Let $x,y \in D$.

<u>Case i</u> $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$. $x + (y+a) = x + a = a = (x+y) + a$.

<u>Case ii</u> $x + y \in D\setminus F_L$.

If $y \in F_L$, then $x \in D\setminus F_L$ and $y + d = d$. Thus $x + (y+a) = x + a = x + d = x + (y+d) = (x+y) + d = (x+y) + a$.

If $y \in D\setminus F_L$ then $x + (y+a) = x + (y+d) = (x+y) + d = (x+y) + a$.

Similarly, we can show that $a + (y+z) = (a+y) + z$ for all $y,z \in D$.

Claim that $d \in (D\setminus F_L) \cap (D\setminus F_R)$. Since $F_L \nsubseteq F_R$ and $F_R \nsubseteq F_L$, there are elements x_o and y_o in D such that $x_o \in F_L\setminus F_R$ and $y_o \in F_R\setminus F_L$. Thus $x_o + a = a$, $x_o + d = d$, $a + x_o = d + x_o$, $a + y_o = a$, $d + y_o = d$ and $y_o + a = y_o + d$. So $a + d = a + (x_o + d)$ $= (a + x_o) + d = (d + x_o) + d = d + (x_o + d) = d + d \neq a$. Hence $d \in D\setminus F_R$. And $d + a = (d + y_o) + a = d + (y_o + a) = d + (y_o + d) =$ $(d + y_o) + d = d + d \neq a$. Hence $d \in D\setminus F_L$. Therefore $d \in (D\setminus F_L) \cap (D\setminus F_R)$. Note that $D\setminus F_L$ and $D\setminus F_R$ are ideals of $(D,+)$.

To show that K is a seminear-field, we shall show that (a_6) $x(yz) = (xy)z$ for all $x,y,z \in K$, (b_6) $x + (y+z) = (x+y) + z$ for all $x,y,z \in K$ and (c_6) $(x+y)z = xz + yz$ for all $x,y,z \in K$.

The proof of (a_6) is the same as the proof of (a_1).

To prove (b_6), let $x,y,z \in K$. Consider the following cases.

Case 1. $x = y = z = a$. Since $d \in (D\setminus F_L) \cap (D\setminus F_R)$, $d + d \in (D\setminus F_L) \cap (D\setminus F_R)$.

$x + (y+z) = a + (a+a) = a + (d+d) = d + (d+d) = (d+d) + d = (d+d) + a = (a+a) + a = (x+y) + z$.

Case 2. $x = y = a$, $z \neq a$.

If $z \in F_R$ then $d = d + z$. Thus $x + (y+z) = a + (a+z) = a + a = d + d$, $(x+y) + z = (a+a) + z = (d+d) + z = d + (d+z) = d + d$.

If $z \in D\setminus F_R$ then $d + z \in D\setminus F_R$. Thus $x + (y+z) = a + (a+z) = a + (d+z) = d + (d+z) = (d+d) + z = (a+a) + z = (x+y) + z$.

Case 3. $x = z = a$, $y \neq a$.

Subcase 3.1. $F_L \cap F_R = \emptyset$.

Subcase 3.1.1. $y \in F_L$. Then $y \in D\setminus F_R$, $d + y \in D\setminus F_L$ and $d = y + d$.

$x + (y+z) = a + (y+a) = a + a = d + d = d + (y+d) = (d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z$.

Subcase 3.1.2. $y \in D\setminus F_L$. Then $y + d \in D\setminus F_R$ and $d + y \in D\setminus F_L$.

If $y \in F_R$ then $d + y = d$. Thus $x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = d + d = a + a = (a+y) + a = (x+y) + z$.

If $y \in D\setminus F_R$ then $x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = (d+y) + a = (a+y) + a = (x+y) + z$.

Subcase 3.2. $F_L \cap F_R \neq \emptyset$.

Subcase 3.2.1. $y \in F_L \cap F_R$.

$x + (y+z) = a + (y+a) = a + a = (a+y) + a = (x+y) + z$.

Subcase 3.2.2. $y \in F_L \cap (D\setminus F_R)$.

This proof is the same as the proof of Subcase 3.1.1.

<u>Subcase 3.2.3.</u> $y \in (D\setminus F_L) \cap F_R$. Then $y + d \in D\setminus F_R$, $d + y \in D\setminus F_L$ and $d = d + y$.

$x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d = d + d$
$= a + a = (a+y) + a = (x+y) + z.$

<u>Subcase 3.2.4.</u> $y \in (D\setminus F_L) \cap (D\setminus F_R)$. Then $y + d \in D\setminus F_R$ and $d + y \in D\setminus F_L$.

$x + (y+z) = a + (y+a) = a + (y+d) = d + (y+d) = (d+y) + d =$
$(d+y) + a = (a+y) + a = (x+y) + z.$

<u>Case 4.</u> $x \neq a$, $y = z = a$.

This proof is similar to Case 2.

<u>Case 5.</u> $x \neq a$, $y \neq a$, $z = a$.

By the first proof, we showed that $x + (y+a) = (x+y)+a$.

<u>Case 6.</u> $x \neq a$, $y = a$, $z \neq a$.

<u>Subcase 6.1.</u> $x \in F_L$, $z \in F_R$.

$x + (y+z) = x + (a+z) = x + a = a = a + z = (x+a) + z = (x+y) + z.$

<u>Subcase 6.2.</u> $x \in F_L$, $z \in D\setminus F_R$. Then $d = x + d$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = d + z = a + z =$
$(x+a) + z = (x+y) + z.$

<u>Subcase 6.3.</u> $x \in D\setminus F_L$, $z \in F_R$. Then $d = d + z$.

$x + (y+z) = x + (a+z) = x + a = x + d = x + (d+z) = (x+d) + z =$
$(x+a) + z = (x+y) + z.$

<u>Subcase 6.4.</u> $x \in D\setminus F_L$, $z \in D\setminus F_R$.

$x + (y+z) = x + (a+z) = x + (d+z) = (x+d) + z = (x+a) + z = (x+y) + z.$

<u>Case 7.</u> $x = a$, $y \neq a$, $z \neq a$.

We showed that $a + (y+z) = (a+y) + z$.

<u>Case 8.</u> $x \neq a$, $y \neq a$, $z \neq a$.

$x + (y+z) = (x+y) + z.$

To prove (c_6), let $x,y,z \in K$. Consider the following cases.

Case 1. $x = y = z = a$.

$(x+y)z = (a+a)a = (d+d)a = (d+d)d = d^2 + d^2 = a^2 + a^2 = xz + yz$.

Case 2. $x = y = a,\ z \neq a$.

$(x+y)z = (a+a)z = (d+d)z = dz + dz = az + az = xz + yz$.

Case 3. $x = z = a,\ y \neq a$.

If $y \in F_R$ then $d = d + y$. Thus $(x+y)z = (a+y)a = a^2 = d^2 =$ $(d+y)d = d^2 + yd = a^2 + ya = xz + yz$.

If $y \in D \setminus F_R$ then $(x+y)z = (a+y)a = (d+y)a = (d+y)d = d^2 + yd =$ $a^2 + ya = xz + yz$.

Case 4. $x \neq a,\ y = z = a$.

This proof is similar to Case 3.

Case 5. $x \neq a,\ y \neq a,\ z = a$.

$(x+y)z = (x+y)a = (x+y)d = xd + yd = xa + ya = xz + yz$.

Case 6. $x \neq a,\ y = a,\ z \neq a$.

If $x \in F_L$ then $d = x + d$. Thus $(x+y)z = (x+a)z = az = dz =$ $(x+d)z = xz + dz = xz + az = xz + yz$.

If $x \in D \setminus F_L$ then $(x+y)z = (x+a)z = (x+d)z = xz + dz = xz + az =$ $xz + yz$.

Case 7. $x = a,\ y \neq a,\ z \neq a$.

This proof is similar to Case 6.

Case 8. $x \neq a,\ y \neq a,\ z \neq a$.

$(x+y)z = xz + yz$.

Hence K is a seminear-field and we obtain (1) - (4).

By Theorem 3.14 and Proposition 3.22, if there exist extensions such that (1) and (2) hold then these are the only possible extensions of the binary operations on D to K.

#

We shall give an example where $(D,+)$ is a band and $F_R \subset F_L$.

Example 3.24. Q^+ with the usual multiplication is a group. Define + on Q^+ by $x + y = \max\{x,y\}$ for all $x,y \in Q^+$. Then $(Q^+,+,\cdot)$ is a ratio seminear-ring. Let $d \in Q^+$. Thus $LI_{Q^+}(d) = \{x \in Q^+ | x \leq d\} = RI_{Q^+}(d)$. Let $F_L = LI_{Q^+}(d)$ and $F_R = \{x \in Q^+ | x < \frac{d}{2}\}$. Then $F_R \subset F_L$ and $d \in F_L$. It is easy to show that F_L and F_R are filters in $(Q^+,+)$.

Let a be a symbol not representing any element of Q^+. Extend + and $\cdot$ from Q^+ to $Q^+ \cup \{a\}$ by

(1) $xa = xd$ and $ax = dx$ for all $x \in Q^+$, $a^2 = d^2$,

(2) $x + a = a$ for all $x \in F_L$, $x + a = x + d$ for all $x \in Q^+ \setminus F_L$,

$a + x = a$ for all $x \in F_R$, $a + x = d + x$ for all $x \in Q^+ \setminus F_R$ and

(3) $a + a = a$.

By Theorem 3.23, $(Q^+ \cup \{a\},+,\cdot)$ is a seminear-field with a as a category VI special element.

#

We shall give an example where $(D,+)$ is a band and $F_L \not\subseteq F_R$ and $F_R \not\subseteq F_L$.

Example 3.25. Let $Q_m^+ \times Q_M^+$ be the ratio seminear-ring given in Example 2.14. Let $d = (d_1,d_2) \in Q^+ \times Q^+$. Then $LI_{Q_m^+ \times Q_M^+}(d) = \{(x,y) \in Q^+ \times Q^+ | x \geq d_1, y \leq d_2\} = RI_{Q_m^+ \times Q_M^+}(d)$.

Let $F_L = \{(x,y) \in Q^+ \times Q^+ | x \geq d_1, y \leq \frac{d_2}{2}\}$ and

$F_R = \{(x,y) \in Q^+ \times Q^+ | x \geq 2d_1, y \leq d_2\}$.

Then $F_L \not\subseteq F_R$ and $F_R \not\subseteq F_L$. It is easy to show that F_L and F_R are Filters in $(Q_m^+ \times Q_M^+,+)$.

Let a be a symbol not representing any element of $\mathbb{Q}^+ \times \mathbb{Q}^+$. Extend + and $\cdot$ from $\mathbb{Q}^+ \times \mathbb{Q}^+$ to $(\mathbb{Q}^+ \times \mathbb{Q}^+) \cup \{a\}$ by

(1) $za = zd$ and $az = dz$ for all $z \in \mathbb{Q}^+ \times \mathbb{Q}^+$, $a^2 = d^2$,

(2) $z + a = a$ for all $z \in F_L$, $z + a = z + d$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Q}^+) \setminus F_L$,

$a + z = a$ for all $z \in F_R$, $a + z = d + z$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Q}^+) \setminus F_R$ and

(3) $a + a = d$.

By Theorem 3.23, $((\mathbb{Q}^+_m \times \mathbb{Q}^+_M) \cup \{a\}, +, \cdot)$ is a seminear-field with a as a category VI special element.

#

We shall give an example where (D,+) is not a band.

<u>Example 3.26</u>. From Example 2.16, $(\mathbb{Q}^+ \times \mathbb{Z}, \oplus, \odot)$ is a ratio seminear-ring. Let $d = (x_o, n_o) \in \mathbb{Q}^+ \times \mathbb{Z}$ be such that $n_o > 1$. Thus $LI_{\mathbb{Q}^+ \times \mathbb{Z}}(d) = \{(x,n) \in \mathbb{Q}^+ \times \mathbb{Z} \mid n > n_o\} = RI_{\mathbb{Q}^+ \times \mathbb{Z}}(d)$.

Let $F_L = \{(x,n) \in \mathbb{Q}^+ \times \mathbb{Z} \mid n > 2n_o\}$ and

$F_R = \{(x,n) \in \mathbb{Q}^+ \times \mathbb{Z} \mid n > n_o\}$.

It is easy to show that F_L and F_R are filters in $(\mathbb{Q}^+ \times \mathbb{Z}, \oplus)$.

Let a be a symbol not representing any element of $\mathbb{Q}^+ \times \mathbb{Z}$. Extend $\oplus$ and $\odot$ from $\mathbb{Q}^+ \times \mathbb{Z}$ to $(\mathbb{Q}^+ \times \mathbb{Z}) \cup \{a\}$ by

(1) $z \odot a = z \odot d$ and $a \odot z = d \odot z$ for all $z \in \mathbb{Q}^+ \times \mathbb{Z}$, $a^2 = d^2$,

(2) $z \oplus a = a$ for all $z \in F_L$, $z \oplus a = z \oplus d$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Z}) \setminus F_L$,

$a \oplus z = a$ for all $z \in F_R$, $a \oplus z = d \oplus z$ for all $z \in (\mathbb{Q}^+ \times \mathbb{Z}) \setminus F_R$ and

(3) $a \oplus a = d \oplus d$.

By Theorem 3.23, $((\mathbb{Q}^+ \times \mathbb{Z}) \cup \{a\}, \oplus, \odot)$ is a seminear-field with a as a category VI special element.

<u>Corollary 3.27</u>. Let D be a ratio seminear-ring. Let a be a symbol not representing any element of D and $d \in D$. Let F_L and F_R have the properties given in Theorem 3.23. Then $K = D \cup \{a\}$ is a distributive seminear-field with a as a category VI special element if and only if D is a distributive ratio seminear-ring.

<u>Proof</u>. By Theorem 3.23, we can construct K so that K is a seminear-field with a as a category VI special element, F_L is the left fundamental of a in K and F_R is the right fundamental of a in K. It is clear that if K is a distributive seminear-field with a as a category VI special element then D is a left ratio seminear-ring.

Conversely, assume that D is a left ratio seminear-ring. It is sufficient to show that $x(y+z) = xy + xz$ for all $x,y,z \in K$. Let $x,y,z \in K$. Note that $a + a = a$ or $a + a = d + d$.

<u>Case 1</u>. $x = y = z = a$.

<u>Subcase 1.1</u>. $a + a = a$. Then $(K,+)$ is a band.

$x(y+z) = a(a+a) = a^2 = d^2 = d^2+ d^2 = a^2+ a^2 = xy + xz.$

<u>Subcase 1.2</u>. $a + a = d + d$.

$x(y+z) = a(a+a) = a(d+d) = d(d+d) = d^2+ d^2 = a^2+ a^2 = xy + xz.$

<u>Case 2</u>. $x = y = a$, $z \neq a$.

If $z \in F_R$ then $d + z = d$. Thus $x(y+z) = a(a+z) = a^2 = d^2 = d(d+z) = d^2+ dz = a^2+ az = xy + xz.$

If $z \in D \setminus F_R$ then $x(y+z) = a(a+z) = a(d+z) = d(d+z) = d^2+ dz = a^2+ az = xy + yz.$

<u>Case 3</u>. $x = z = a$, $y \neq a$.

This proof is similar to Case 2.

Case 4. $x \neq a$, $y = z = a$.

Subcase 4.1. $a + a = a$. Then $(K,+)$ is a band.

$x(y+z) = x(a+a) = xa = xd = xd + xd = xa + xa = xy + xz.$

Subcase 4.2. $a + a = d + d$.

$x(y+z) = x(a+a) = x(d+d) = xd + xd = xa + xa = xy + xz.$

Case 5. $x \neq a$, $y \neq a$, $z = a$.

If $y \in F_L$ then $y + d = d$. Thus $x(y+z) = x(y+a) = xa = xd = x(y+d) = xy + xd = xy + xa = xy + xz.$

If $y \in D \setminus F_L$ then $x(y+z) = x(y+a) = x(y+d) = xy + xd = xy + xa = xy + xz.$

Case 6. $x \neq a$, $y = a$, $z \neq a$.

This proof is similar to Case 5.

Case 7. $x = a$, $y \neq a$, $z \neq a$.

$x(y+z) = a(y+z) = d(y+z) = dy + dz = ay + az = xy + xz.$

Case 8. $x \neq a$, $y \neq a$, $z \neq a$.

$x(y+z) = xy + xz.$

Hence K is a distributive seminear-field.

#

Theorem 3.28. Let K and K' be seminear-fields with a and a' as category VI special elements, respectively. Let $D = K \setminus \{a\}$ and $D' = K' \setminus \{a\}$ with e and e' as their multiplicative identities respectively. Let $d \in D$ and $d' \in D'$ be such that $xa = xd$ and $ax = dx$ for all $x \in K$ and $xa' = xd'$ and $a'x = d'x$ for all $x \in K'$. Let S_L and S_R be the left and right fundamental sets of a in K, respectively. Let S_L' and S_R' be the left and right fundamental sets of a' in K', respectively. Suppose that there exists an isomorphism $\eta : K \to K'$. Let $\varphi = \eta|_{S_L}$, $\psi = \eta|_{D \setminus S_L}$, $\varphi' = \eta|_{S_R}$ and $\psi' = \eta|_{D \setminus S_R}$. Then the following statements hold :

(1) $\eta(e) = e'$, $\eta(d) = \eta(d')$ and $\eta(a) = a'$.

(2) $S_L = \emptyset$ if and only if $S_L = \emptyset$ and
if $S_L \neq \emptyset$ then $S_L \cong S_L'$ as additive semigroups.

(3) $D \setminus S_L = \emptyset$ if and only if $D' \setminus S_L' = \emptyset$ and
if $D \setminus S_L \neq \emptyset$ then $D \setminus S_L \cong D' \setminus S_L'$ as additive semigroups.

(4) $S_R = \emptyset$ if and only if $S_R' = \emptyset$ and
if $S_R \neq \emptyset$ then $S_R \cong S_R'$ as additive semigroups.

(5) $D \setminus S_R = \emptyset$ if and only if $D \setminus S_R' = \emptyset$ and
if $D \setminus S_R \neq \emptyset$ then $D \setminus S_R \cong D \setminus S_R'$ as additive semigroups.

(6) If $a + a = a$ then $a' + a' = a'$.

(7) If $a + a = d + d$ then $a' + a' = d' + d'$.

(8) If $x \in S_L \cap S_R$ then $\varphi(x) = \varphi'(x)$.

(9) If $x \in S_L \cap (D \setminus S_R)$ then $\varphi(x) = \psi'(x)$.

(10) If $x \in (D \setminus S_L) \cap S_R$ then $\psi(x) = \varphi'(x)$.

(11) If $x \in (D \setminus S_L) \cap (D \setminus S_R)$ then $\psi(x) = \psi'(x)$.

(12) If $x \in S_L$ and $y \in D \setminus S_L$ then $\psi(x+y) = \varphi(x) + \psi(y)$.

(13) If $x \in D \setminus S_L$ and $y \in S_L$ then $\psi(x+y) = \psi(x) + \varphi(y)$.

(14) If $x,y \in S_L$ and $xy \in S_L$ then $\varphi(xy) = \varphi(x)\varphi(y)$.

(15) If $x \in S_L$, $y \in D \setminus S_L$ and $xy \in S_L$ then $\varphi(xy) = \varphi(x)\psi(y)$.

(16) If $x \in D \setminus S_L$, $y \in S_L$ and $xy \in S_L$ then $\varphi(xy) = \psi(x)\varphi(y)$.

(17) If $x,y \in D \setminus S_L$ and $xy \in S_L$ then $\varphi(xy) = \psi(x)\psi(y)$.

(18) If $x,y \in S_L$ and $xy \in D \setminus S_L$ then $\psi(xy) = \varphi(x)\varphi(y)$.

(19) If $x \in S_L$, $y \in D \setminus S_L$ and $xy \in D \setminus S_L$ then $\psi(xy) = \varphi(x)\psi(y)$.

(20) If $x \in D \setminus S_L$, $y \in S_L$ and $xy \in D \setminus S_L$ then $\psi(xy) = \psi(x)\varphi(y)$.

(21) If $x,y,xy \in D \setminus S_L$ then $\psi(xy) = \psi(x)\psi(y)$.

Proof. (1) Since e' is the only multiplicative idempotent of K' and $[\eta(e)]^2 = \eta(e)$, $\eta(e) = e'$. To show that $\eta(a) = a'$, suppose that $\eta(a) \neq a'$. Then $\eta(a) = e' \cdot \eta(a) = \eta(e) \cdot \eta(a) = \eta(e \cdot a) =$

$\eta(e \cdot d) = \eta(d)$. Since η is 1-1, $a = d$, a contradiction. Hence $\eta(a) = a'$. Now $\eta(d) = \eta(e \cdot d) = \eta(e \cdot a) = \eta(e) \cdot \eta(a) = e' \cdot a' = e' \cdot d' = d'$.

(2) Assume that $S_L = \emptyset$. Suppose that $S_L' \neq \emptyset$. Let $y \in S_L'$. Since η is onto, there exists an element x in D such that $\eta(x) = y$. Now $x + a = x + d$. $a' = y + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(x+d) = \eta(x) + \eta(d) = y + d' \in D'$, a contradiction. Hence $S_L' = \emptyset$.

Assume that $S_L \neq \emptyset$. Claim that $\varphi : S_L \longrightarrow S_L'$. Let $x \in S_L$. Then $x + a = a$, so $\varphi(x) + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(a) = a'$. Thus $\varphi(x) \in S_L'$. Hence $S_L' \neq \emptyset$. It is clear-that φ is a monomorphism. To show φ is onto, let $y \in S_L'$. Then $y + a' = a'$. Since η is onto, there exists an elements an element x in K such that $\eta(x) = y$. Now $x \neq a$ so $x \in D$. Claim that $x \in S_L$. Suppose that $x \in D \setminus S_L$. Then $x + a \neq a$, so $\eta(x+a) \in D'$. $a' = y + a' = \eta(x) + \eta(a) = \eta(x+a) \in D'$, a contradiction. Hence $x \in S_L$. So we get that $\varphi(x) = \eta(x) = y$. Thus φ is onto. Hence $S_L \cong S_L'$ as additive semigroups. Therefore we obtain (2).

(3) Assume that $D \setminus S_L = \emptyset$. Then $S_L = D$. To show that $S_L' = D'$, let $y \in D'$. Since η is onto, there exists an element x in S_L such that $\eta(x) = y$. Now $x + a = a$ so $y + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(a) = a'$. Hence $y \in S_L'$. Therefore $D' \setminus S_L' = \emptyset$.

Assume that $D \setminus S_L \neq \emptyset$. Claim that $\psi : D \setminus S_L \longrightarrow D' \setminus S_L'$. Let $x \in D \setminus S_L$. Then $\psi(x) + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(x+d) = \eta(x) + \eta(d) = \psi(x) + d'$ so $\psi(x) \in D' \setminus S_L'$. Thus $D' \setminus S_L' \neq \emptyset$. It is clear that ψ is a monomorphism. To show that ψ is onto, let $y \in D' \setminus S_L'$. Then $y + a' = y + d'$. Since η is onto, there exists an element x in K such that $\eta(x) = y$. Now $x \neq a$. If $x \in S_L$ then $x + a = a$. So $y + d' = y + a' = \eta(x) + \eta(a) = \eta(x+a) = \eta(a) = a'$, a contradiction. Hence $x \in D \setminus S_L$. Therefore ψ is onto and so $D \setminus S_L \cong D' \setminus S'_L$ as additive semigroups. We obtain (3).

The proofs of (4) and (5) are similar to the proof of (2) and (3), respectively.

The proofs of (6) – (21) are straightforward and we will omit them.

#

Theorem 3.29. Let D and D' be ratio seminear-rings and let d and d' elements in D and D' respectively. Let a and a' be symbols not representing any element of D and D' respectively. Let $F_L \subseteq LI_D(d)$ be either $\emptyset$ or a filter in $(D,+)$ and let $F_R \subseteq RI_D(d)$ be either $\emptyset$ or a filter in $(D,+)$. Let $F'_L \subseteq LI_{D'}(d')$ be either $\emptyset$ or a filter in $(D',+)$. Suppose that there are bijections $\varphi : F_L \longrightarrow F'_L$ and $\psi : D \setminus F_L \longrightarrow D \setminus F'_L$ such that $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x,y \in F_L$, $\varphi(d) = d'$ if $d \in F_L$, $\psi(x+y) = \psi(x) + \psi(y)$ for all $x,y \in D \setminus F_L$ and $\psi(d) = d'$ if $d \in D \setminus F_L$. Suppose that there are bijections $\varphi' : F_R \longrightarrow F'_R$ and $\psi' : D \setminus F_R \longrightarrow D \setminus F'_R$ such that $\varphi'(x+y) = \varphi'(x) + \varphi'(y)$ for all $x,y \in F_R$, $\varphi'(d) = d'$ if $d \in F_R$, $\psi'(x+y) = \psi'(x) + \psi'(y)$ for all $x,y \in D \setminus F_R$ and $\psi'(d) = d'$ if $d \in D \setminus F_R$.

Suppose that the following conditions are satisfied :

(1) $F_L = \emptyset$ iff $F'_L = \emptyset$.

(2) $F_L = D$ iff $F'_L = D'$.

(3) $\emptyset \neq F_L \subset D$ iff $\emptyset \neq F'_L \subset D'$.

(4) $F_R = \emptyset$ iff $F'_R = \emptyset$.

(5) $F_R = D$ iff $F'_R = D'$.

(6) $\emptyset \neq F_R \subset D$ iff $\emptyset \neq F'_R \subset D$.

(7) If $a + a = a$ then $a' + a' = a'$.

(8) If $a + a = d + d$ then $a' + a' = d' + d'$.

(9) If $x \in F_L \cap F_R$ then $\varphi(x) = \varphi'(x)$.

(10) If $x \in F_L \cap (D \setminus F_R)$ then $\varphi(x) = \psi'(x)$.

(11) If $x \in (D \setminus F_L) \cap F_R$ then $\psi(x) = \varphi'(x)$.

(12) If $x \in (D \setminus F_L) \cap (D \setminus F_R)$ then $\psi(x) = \psi'(x)$.

(13) If $x \in F_L$ and $y \in D \setminus F_L$ then $\psi(x+y) = \varphi(x) + \psi(y)$.

(14) If $x \in D \setminus F_L$ and $y \in F_L$ then $\psi(x+y) = \psi(x) + \varphi(y)$.

(15) If $x,y \in F_L$ and $xy \in F_L$ then $\varphi(xy) = \varphi(x)\varphi(y)$.

(16) If $x \in F_L$, $y \in D \setminus F_L$ and $xy \in F_L$ then $\varphi(xy) = \varphi(x)\psi(y)$.

(17) If $x \in D \setminus F_L$, $y \in F_L$ and $xy \in F_L$ then $\varphi(xy) = \psi(x)\varphi(y)$.

(18) If $x,y \in D \setminus F_L$ and $xy \in F_L$ then $\varphi(xy) = \psi(x)\psi(y)$.

(19) If $x,y \in F_L$ and $xy \in D \setminus F_L$ then $\psi(xy) = \varphi(x)\varphi(y)$.

(20) If $x \in F_L$, $y \in D \setminus F_L$ and $xy \in D \setminus F_L$ then $\psi(xy) = \varphi(x)\psi(y)$.

(21) If $x \in D \setminus F_L$, $y \in F_L$ and $xy \in D \setminus F_L$ then $\psi(xy) = \psi(x)\varphi(y)$.

(22) If $x,y,xy \in D \setminus F_L$ then $\psi(xy) = \psi(x)\psi(y)$.

Then $\eta : K \to K'$ defined by

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in F_L, \\ \psi(x) & \text{if } x \in D \setminus F_L, \\ a' & \text{if } x = a, \end{cases}$$

is an isomorphism between K and K' where $K = D \cup \{a\}$ and $K' = D' \cup \{a'\}$ are seminear-fields with a and a' as category VI special elements, respectively.

<u>Proof</u>. By Theorem 3.23, we can construct K and K' so that K and K' are seminear-fields and a and a' are category VI special elements of K and K', respectively.

<u>Case I</u> $F_L = \emptyset$. Then $F_L' = \emptyset$. Define $\eta : K \to K'$ by

$$\eta(x) = \begin{cases} \psi(x) & \text{if } x \in D, \\ a' & \text{if } x = a. \end{cases}$$

It is clear that η is a bijection. We need only show that

(a_1) $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$ and (b_1) $\eta(x+y) = \eta(x) + \eta(y)$ for all $x,y \in K$. Note that, by (22), $\psi(xy) = \psi(x)\psi(y)$ for all $x,y \in D$.

To prove (a_1), let $x,y \in K$.

Case 1. $x = y = a$. Then $a^2 = d^2$ and $a'^2 = d'^2$.

$\eta(xy) = \eta(a^2) = \eta(d^2) = \psi(d^2) = \psi(d)\psi(d) = d'^2 = a'^2 = \eta(a)\eta(a) = \eta(x)\eta(y)$.

Case 2. $x = a$, $y \neq a$. Then $ay = dy$.

$\eta(xy) = \eta(ay) = \eta(dy) = \psi(dy) = \psi(d)\psi(y) = d'\psi(y) = a'\psi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Case 3. $x \neq a$, $y = a$.

This proof is similar to Case 2.

Case 4. $x \neq a$, $y \neq a$. Then $xy \in D$.

$\eta(xy) = \psi(xy) = \psi(x)\psi(y) = \eta(x)\eta(y)$.

To prove (b_1), let $x,y \in K$.

Case 1. $x = y = a$.

Subcase 1.1. $a + a = a$. Then $a' + a' = a'$.

$\eta(x+y) = \eta(a+a) = \eta(a) = a' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Subcase 1.2. $a + a = d + d$. Then $a' + a' = d' + d'$.

$\eta(x+y) = \eta(a+a) = \eta(d+d) = \psi(d+d) = \psi(d) + \psi(d) = d' + d' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Case 2. $x = a$, $y \neq a$.

If $y \in F_R$ then by (11) $\psi(y) = \varphi'(y)$. Thus $a' = a' + \varphi'(y) = a' + \psi(y)$.

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \psi(y) = \eta(a) + \eta(y) = \eta(x) + \eta(y)$.

If $y \in D \setminus F_R$ then by (12) $\psi(y) = \psi'(y)$. Thus $a + y = d + y$ and $a' + \psi(y) = a' + \psi'(y) = d' + \psi'(y)$.

$\eta(x+y) = \eta(a+y) = \eta(d+y) = \psi(d+y) = \psi(d) + \psi(y) = d' + \psi(y) = a' + \psi(y) = \eta(a) + \eta(y) = \eta(x) + \eta(y)$.

Case 3. $x \neq a$, $y = a$. Then $x + a = x + d$. $\eta(x+y) = \eta(x+a) = \eta(x+d) = \psi(x+d) = \psi(x) + \psi(d) = \psi(x) + d' = \psi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$.

Case 4. $x \neq a$, $y \neq a$.

$\eta(x+y) = \psi(x+y) = \psi(x) + \psi(y) = \eta(x) + \eta(y)$.

Hence η is an isomorphism.

<u>Case II</u> $F_L = D$. Then $F_L' = D'$. Define $\eta : K \to K'$ by

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in D, \\ a' & \text{if } x = a. \end{cases}$$

It is clear that η is a bijection. We need only show that (a_2) $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$ and (b_2) $\eta(x+y) = \eta(x) + \eta(y)$ for all $x,y \in K$. Note that, by (15), $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x,y \in D$.

The proof of (a_2) is the same as the proof of (a_1).

To prove (b_2), let $x,y \in K$.

<u>Case 1.</u> $x = y = a$.

This proof is the same as the proof of Case 1 in (b_1).

<u>Case 2.</u> $x = a$, $y \neq a$.

If $y \in F_R$ then, by (9), $\varphi(y) = \varphi'(y)$. Thus $a' = a' + \varphi'(y) = a' + \varphi(y)$.
$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi(y) = \eta(a) + \eta(y) = \eta(x) + \eta(y)$.

If $y \in D \setminus F_R$ then, by (10), $\varphi(y) = \psi'(y)$. Thus $a + y = d + y$ and $a' + \varphi(y) = a' + \psi'(y) = d' + \psi'(y)$, so $\eta(x+y) = \eta(a+y) = \eta(d+y) = \varphi(d+y) = \varphi(d) + \varphi(y) = d' + \psi'(y) = a' + \varphi(y) = \eta(a) + \eta(y) = \eta(x) + \eta(y)$.

<u>Case 3.</u> $x \neq a$, $y = a$. Then $x + a = a$ and $\varphi(x) + a' = a'$.
$\eta(x+y) = \eta(x+a) = \eta(a) = a' = \varphi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$.

<u>Case 4.</u> $x \neq a$, $y \neq a$.

$\eta(x+y) = \varphi(x+y) = \varphi(x) + \varphi(y) = \eta(x) + \eta(y)$.

Hence η is an isomorphism.

<u>Case III</u> $\emptyset \neq F_L \subset D$. Then $\emptyset \neq F_L' \subset D$. Define $\eta : K \to K'$ by

$$\eta(x) = \begin{cases} \varphi(x) & \text{if } x \in F_L, \\ \psi(x) & \text{if } x \in D \setminus F_L, \\ a' & \text{if } x = a. \end{cases}$$

It is clear that η is a bijection. We need only show that (a_3) $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$ and (b_3) $\eta(x+y) = \eta(x) + \eta(y)$ for all $x,y \in K$.

To prove (a_3), let $x,y \in K$.

<u>Case 1.</u> $x = y = a$.

<u>Subcase 1.1.</u> $d, d^2 \in F_L$. By (15), $\varphi(d^2) = \varphi(d)\varphi(d)$.
$\eta(xy) = \eta(a^2) = \eta(d^2) = \varphi(d^2) = \varphi(d)\varphi(d) = d'^2 = a'^2 = \eta(a)^2 = \eta(x)\eta(y)$.

Subcase 1.2. $d \in F_L$, $d^2 \in D \setminus F_L$. By (19), $\psi(d^2) = \varphi(d)\varphi(d)$. $\eta(xy) = \eta(a^2) = \eta(d^2) = \psi(d^2) = \varphi(d)\varphi(d) = d'^2 = a'^2 = \eta(a)^2 = \eta(x)\eta(y)$.

Subcase 1.3. $d \in D \setminus F_L$, $d^2 \in F_L$. By (18), $\varphi(d^2) = \psi(d)\psi(d)$. $\eta(xy) = \eta(a^2) = \eta(d^2) = \varphi(d^2) = \psi(d)\psi(d) = d'^2 = a'^2 = \eta(a)^2 = \eta(x)\eta(y)$.

Subcase 1.4. $d, d^2 \in D \setminus F_L$. By (22), $\psi(d^2) = \psi(d)\psi(d)$. $\eta(xy) = \eta(a^2) = \eta(d^2) = \psi(d^2) = \psi(d)\psi(d) = d'^2 = a'^2 = \eta(a)^2 = \eta(x)\eta(y)$.

Case 2. $x = a$, $y \neq a$.

Subcase 2.1. $d, y, dy \in F_L$. By (15), $\varphi(dy) = \varphi(d)\varphi(y)$. $\eta(xy) = \eta(ay) = \eta(dy) = \varphi(dy) = \varphi(d)\varphi(y) = d'\varphi(y) = a'\varphi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.2. $d, y \in F_L$, $dy \in D \setminus F_L$. By (19), $\psi(dy) = \varphi(d)\varphi(y)$. $\eta(xy) = \eta(ay) = \eta(dy) = \psi(dy) = \varphi(d)\varphi(y) = d'\varphi(y) = a'\varphi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.3. $d, dy \in F_L$, $y \in D \setminus F_L$. By (16), $\varphi(dy) = \varphi(d)\psi(y)$. $\eta(xy) = \eta(ay) = \eta(dy) = \varphi(dy) = \varphi(d)\psi(y) = d'\psi(y) = a'\psi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.4. $d \in D \setminus F_L$, $y, dy \in F_L$. By (17), $\varphi(dy) = \psi(d)\varphi(y)$. $\eta(xy) = \eta(ay) = \eta(dy) = \varphi(dy) = \psi(d)\varphi(y) = d'\varphi(y) = a'\varphi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.5. $d, y \in D \setminus F_L$, $dy \in F_L$. By (18), $\varphi(dy) = \psi(d)\psi(y)$. $\eta(xy) = \eta(ay) = \eta(dy) = \varphi(dy) = \psi(d)\psi(y) = d'\psi(y) = a'\psi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.6. $d, dy \in D \setminus F_L$, $y \in F_L$. By (21), $\psi(dy) = \psi(d)\varphi(y)$. $\eta(xy) = \eta(ay) = \eta(dy) = \psi(dy) = \psi(d)\varphi(y) = d'\varphi(y) = a'\varphi(y) = \eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.7. $d \in F_L, y, dy \in D \sim F_L$. By (20), $\psi(dy) = \varphi(d)\psi(y)$.

$\eta(xy) = \eta(ay) = \eta(dy) = \psi(dy) = \varphi(d)\psi(y) = d'\varphi(y) = a'\varphi(y) =$

$\eta(a)\eta(y) = \eta(x)\eta(y)$.

Subcase 2.8. $d, y, dy \in D \sim F_L$. By (22), $\psi(dy) = \psi(d)\psi(y)$.

$\eta(xy) = \eta(ay) = \eta(dy) = \psi(dy) = \psi(d)\psi(y) = d'\psi(y) = a'\psi(y) =$

$\eta(a)\eta(y) = \eta(x)\eta(y)$.

Case 3. $x \neq a, y = a$.

This proof is similar to Case 2.

Case 4. $x \neq a, y \neq a$.

By (9)-(16), we can show that $\eta(xy) = \eta(x)\eta(y)$.

To prove (b_3), let $x, y \in K$. Note that $a + a = a$ or

$a + a = d + d$.

Case 1. $x = y = a$.

Subcase 1.1. $a + a = a$. Then $a' + a' = a'$.

$\eta(x+y) = \eta(a+a) = \eta(a) = a' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Subcase 1.2. $a + a = d + d$. Then $a' + a' = d' + d'$.

If $d \in F_L$ then $d + d \in F_L$. Thus $\eta(x+y) = \eta(a+a) = \eta(d+d) = \varphi(d+d)$

$= \varphi(d) + \varphi(d) = d' + d' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

If $d \in D \sim F_L$ then $d + d \in D \sim F_L$. Thus $\eta(x+y) = \eta(a+a) = \eta(d+d) =$

$\psi(d+d) = \psi(d) + \psi(d) = d' + d' = a' + a' = \eta(a) + \eta(a) = \eta(x) + \eta(y)$.

Case 2. $x = a, y \neq a$.

Subcase 2.1. $F_L \cap F_R = \emptyset$.

Subcase 2.1.1. $y \in F_R$. Then $y \in D \sim F_L$ and

$\varphi'(y) = \psi(y)$.

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi'(y) = \eta(a) + \psi(y) = \eta(x) + \eta(y)$.

Subcase 2.1.2. $y \in D \sim F_R$, $y \in F_L$. Then $\varphi(y) = \psi'(y)$.

If $d \in F_L$ then $d + y \in F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(d+y) =$

$\varphi(d+y) = \varphi(d) + \varphi(y) = d' + \psi'(y) = a' + \psi'(y) = \eta(a) + \varphi(y) =$

$\eta(x) + \eta(y)$.

If $d \in D \setminus F_L$ then $d + y \in D \setminus F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(d+y) = \psi(d+y) \overset{\text{by}(14)}{=} \psi(d) + \varphi(y) = d' + \psi'(y) = a' + \psi'(y) = \eta(a) + \varphi(y) = \eta(x) + \eta(y)$.

Subcase 2.1.3. $y \in D \setminus F_R$, $y \in D \setminus F_L$. Then $\psi(y) = \psi'(y)$.

If $d \in F_L$ then $d + y \in D \setminus F_L$. Thus $\eta(x+y) = \eta(a+y) = \eta(d+y) = \psi(d+y) \overset{\text{by}(13)}{=} \varphi(d) + \psi(y) = d' + \psi'(y) = a' + \psi'(y) = \eta(a) + \psi(y) = \eta(x) + \eta(y)$.

If $d \in D \setminus F_L$ then $\eta(x+y) = \eta(a+y) = \eta(d+y) = \psi(d+y) = \psi(d) + \psi(y) = d' + \psi'(y) = a' + \psi'(y) = \eta(a) + \psi(y) = \eta(x) + \eta(y)$.

Subcase 2.2. $F_L \cap F_R \neq \emptyset$.

Subcase 2.2.1. $y \in F_L \cap F_R$. Then $\varphi(y) = \varphi'(y)$ and $a' = a' + \varphi'(y)$.

$\eta(x+y) = \eta(a+y) = \eta(a) = a' = a' + \varphi'(y) = \eta(a) + \varphi(y) = \eta(x) + \eta(y)$.

Subcase 2.2.2. $y \in F_L \cap (D \setminus F_R)$.

This proof is the same as the proof of Subcase 2.1.2.

Subcase 2.2.3. $y \in (D \setminus F_L) \cap F_R$.

This proof is the same as the proof of Subcase 2.1.1.

Subcase 2.2.4. $y \in (D \setminus F_L) \cap (D \setminus F_R)$.

This proof is the same as the proof of Subcase 2.1.3.

Case 3. $x \neq a$, $y = a$.

Subcase 3.1. $x \in F_L$. Then $a' = \varphi(x) + a'$.

$\eta(x+y) = \eta(x+a) = \eta(a) = a' = \varphi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$.

Subcase 3.2. $x \in D \setminus F_L$. Thus $x + d \in D \setminus F_L$ and $\psi(x) + a' = \psi(x) + d'$. If $d \in D \setminus F_L$ then $\eta(x+y) = \eta(x+a) = \eta(x+d) = \psi(x+d) = \psi(x) + \psi(d) = \psi(x) + d' = \psi(x) + a' = \eta(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$. If $d \in F_L$ then $\eta(x+y) = \eta(x+a) = \eta(x+d) = \psi(x+d) \overset{\text{by}14}{=} \psi(x) + \varphi(d) = \psi(x) + d' = \psi(x) + a' = \eta(x) + \eta(a) = \eta(x) + \eta(y)$.

Case 4. $x \neq a$, $y \neq a$.

Subcase 4.1. $x + y \in F_L$. Since F_L is a filter in $(D,+)$, $x,y \in F_L$.

$\eta(x+y) = \varphi(x+y) = \varphi(x) + \varphi(y) = \eta(x) + \eta(y)$.

Subcase 4.2. $x + y \in D \setminus F_L$.

Subcase 4.2.1. $x \in F_L$. Then $y \in D\setminus F_L$ which is an ideal of $(D,+)$.

$\eta(x+y) = \psi(x+y) \overset{\text{by}(13)}{=} \varphi(x) + \psi(y) = \eta(x) + \eta(y)$.

Subcase 4.2.2. $x \in D\setminus F_L$.

If $y \in F_L$ then $\eta(x+y) = \psi(x+y) \overset{\text{by}(14)}{=} \psi(x) + \varphi(y) = \eta(x) + \eta(y)$.

If $y \in D\setminus F_L$ then $\eta(x+y) = \psi(x+y) = \psi(x) + \psi(y) = \eta(x) + \eta(y)$.

Hence η is an isomorphism.

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Remark : $\eta : K \longrightarrow K'$ may be defined by

$$\eta(x) = \begin{cases} \varphi'(x) & \text{if } x \in F_R, \\ \psi'(x) & \text{if } x \in D\setminus F_R, \\ a' & \text{if } x = a. \end{cases}$$

The proof is straighforward but very long.

We shall now give an example where $F_L \cong F_L'$ as additive semigroups and $D\setminus F_L \cong D\setminus F_L'$ as additive semigroups but $K \not\cong K'$.

Example 3.30. $(\mathbb{Q}^+,+,\cdot)$ is a ratio seminear-ring where $+$ is defined by $x + y = \min\{x,y\}$ and $\cdot$ is the usual multiplication. Let d, $d' \in \mathbb{Q}^+$. Then $d = rd'$ where $r \in \mathbb{Q}^+$. Let $S_R = S_L = LI_{\mathbb{Q}^+}(d) = \{x \in \mathbb{Q}^+ | x \geq d\} = RI_{\mathbb{Q}^+}(d)$. Let $F_R' = F_L' = LI_{\mathbb{Q}^+}(d') = \{x \in \mathbb{Q}^+ | x \geq 2d'\}$ $= \{x \in \mathbb{Q}^+ | x \geq 2\frac{d}{r}\} = RI_{\mathbb{Q}^+}(d')$. It is clear that F_L and F_L' are filters in $(\mathbb{Q}^+,\min)$.

Define $\varphi : F_L \longrightarrow F_L'$ by $\varphi(x) = \frac{2x}{r}$ for all $x \in F_L$. Then φ is clearly a bijection. To show that φ is homomorphism, let $x,y \in F_L$. We may assume $x \geq y$, so $x + y = y$. Thus $\varphi(x+y) = \varphi(y)$

$= \frac{2y}{r} = \frac{2x}{r} + \frac{2y}{r} = \varphi(x) + \varphi(y)$. Hence $F_L \cong F_L'$ as additive semigroups.

$Q^+ \setminus F_L = \{x \in Q^+ | x < d\}$ and $Q^+ \setminus F_L' = \{x \in Q^+ | x < \frac{2d}{r}\}$.

Thus $Q^+ \setminus F_L$ and $Q^+ \setminus F_L'$ are ideals of $(Q^+, +)$.

Define $\psi : Q^+ \setminus F_L \to Q^+ \setminus F_L'$ by $\psi(x) = \frac{2x}{r}$ for all $x \in Q^+ \setminus F_L$.

Using the same proof as was used for φ we can show that $Q^+ \setminus F_L \cong Q^+ \setminus F_L'$ as additive semigroups.

Let a and a' be symbols not representing any element of Q^+. Extend $+$ and $\cdot$ from Q^+ to $Q^+ \cup \{a\}$ and $\cdot$ and $+$ from Q^+ to $Q^+ \cup \{a'\}$ by

(1) $xa = xd$ and $ax = dx$ for all $x \in Q^+$, $a^2 = d^2$,

(2) $x + a = a + x = a$ for all $x \in F_L$, $x + a = x + d$ and $a + x = d + x$ for all $x \in Q^+ \setminus F_L$,

(3) $a + a = a$ and

(1') $ya' = yd'$ and $a'y = d'y$ for all $y \in Q^+$, $a'^2 = d'^2$

(2') $y + a' = a' + y = a'$ for all $y \in F_L'$, $y + a' = y + d'$ and $a' + y = d' + y$ for all $y \in Q^+ \setminus F_L'$,

(3') $a' + a' = a'$.

By Theorem 3.23, $(Q^+ \cup \{a\}, +, \cdot)$ and $(Q^+ \cup \{a'\}, +, \cdot)$ are seminear-fields and a and a' are category VI special elements of $Q^+ \cup \{a\}$ and $Q^+ \cup \{a'\}$, respectively. We shall show that $Q^+ \cup \{a\} \not\cong Q^+ \cup \{a'\}$.

Suppose that $Q^+ \cup \{a\} \cong Q^+ \cup \{a'\}$. Let η be an isomorphism from $Q^+ \cup \{a\}$ to $Q^+ \cup \{a'\}$. By Theorem 3.28 (1), $\eta(a) = a'$ and $\eta(d) = d'$. Since $d \in F_L$, $d + a = a$. Thus $a' = \eta(a) = \eta(d+a) = \eta(d) + \eta(a) = d' + a' = d' + d'$, a contradiction. Hence $Q^+ \cup \{a\} \not\cong Q^+ \cup \{a'\}$.

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Now we shall compute all finite seminear-fields with a category VI special element.

At first, we shall compute all finite seminear-fields of order 2. Let K = {a,e} be a seminear-field with a as a category VI special element. Since {e} is a ratio seminear-ring, e + e = e. Now a + a = e or a + a = a, a + e = e or a + e = a and e + a = e or e + a = a. So we have 8 cases to consider. They are :

(1)

+	e	a
e	e	e
a	e	e

(2)

+	e	a
e	e	e
a	e	a

(3)

+	e	a
e	e	e
a	a	e

(4)

+	e	a
e	e	a
a	e	e

(5)

+	e	a
e	e	a
a	a	e

(6)

+	e	a
e	e	a
a	e	a

(7)

+	e	a
e	e	e
a	a	a

(8)

+	e	a
e	e	a
a	a	a

K with tables(3) and (4) are not additive semigroups since a + (a+a) $\neq$ (a+a) + a. And we can verify that K with tables (1), (2),(5),(6),(7) and (8) are additive semigroups. By defining f(e) = a and f(a) = e, we have that semigroup with table (2) is isomorphic to semigroup with table (8). Therefore up to isomorphism there are 5 seminear-field with a as a category VI special element.

Finally we shall compute all seminear-field of order greater than 2.

<u>Theorem 3.31</u>. Let K be a finite seminear-field of order greater than 2 and let a be category VI special element of K. Let D = K$\setminus${a}, let e be the identity of (D,·) and let d be an element of D such that ax = dx and xa = xd for all x $\in$ K. Then

(1) x + a = x + d, a + x = d + x for all x $\in$ D and a + a = a

or (2) x + a = x + d, a + x = d + x for all x $\in$ D and a + a = d

or (3) x + a = a, a + x = d + x for all x $\in$ D and a + a = a or

(4) $x + a = x + d$, $a + x = a$ for all $x \in D$ and $a + a = a$.

Proof. By Theorem 3.12, D is a ratio seminear-ring. By Theorem 1.15, $D_1 = \{x \in D | x + e = x\}$ and $D_2 = \{x \in D | x + e = e\}$ are the unique ratio subseminear-rings of D such that (1) $x + y = x$ for all $x,y \in D_1$, (2) $x + y = y$ for all $x,y \in D_2$, (3) $(D,+) \cong (D_1,+) \times (D_2,+)$ and (4) $D_2 + D_1 = \{e\}$. It is clear that $D_2 = LI_D(e)$ and so $D_2 \cdot d = LI_D(e) \cdot d = LI_D(d)$ by Proposition 1.26 (4.1). Claim that $D_1 = RI_D(e)$. Let $x \in D_1$. Then $x^{-1} + e = x^{-1}$. Multiply on the right by x, we obtain that $e + x = e$. Hence $x \in RI_D(e)$. Therefore $D_1 \subseteq RI_D(e)$. Let $y \in RI_D(e)$. Then $e + y = e$, so $y^{-1} = ey^{-1} = (e+y)y^{-1} = ey^{-1} + yy^{-1} = y^{-1} + e$. Thus $y^{-1} \in D_2$. Since $(D_1,\cdot)$ is a group, $y \in D_1$. Hence $RI_D(e) \subseteq D_1$. Therefore $D_1 = RI_D(e)$. By Proposition 1.26 (4.2), $RI_D(d) = RI_D(e) \cdot d = D_1 \cdot d$.

Let $S_L = \{x \in D | x + a = a\}$ and $S_R = \{x \in D | a + x = a\}$. By Proposition 3.21 (1), $S_L \subseteq LI_D(d)$ and $S_R \subseteq RI_D(d)$.

Claim that (1) if S_L is nonempty then $S_L = D_2 \cdot d$,

(2) if S_R is nonempty then $S_R = D_1 \cdot d$.

To prove claim (1), assume that S_L is nonempty. To show that $D_2 \cdot d \subseteq S_L$, let $x \in D_2 \cdot d$. Then $xd^{-1} \in D_2$. Since $S_L \neq \emptyset$, there exists an element y in S_L. Thus $y + a = a$, so $e = dd^{-1} = ad^{-1} = (y+a)d^{-1} = yd^{-1} + ad^{-1} = yd^{-1} + dd^{-1} = yd^{-1} + e$. Hence $yd^{-1} \in D_2$. Now $xd^{-1} + yd^{-1} = yd^{-1}$. Multiply this equation by d, we obtain that $x + y = y \in S_L$. Since S_L is a filter in $(D,+)$, $x \in S_L$. Hence $D_2 \cdot d \subseteq S_L$. Therefore $S_L = D_2 \cdot d$.

The proof of claim (2) is similar to the proof of claim (1).

Consider D_1 and D_2.

Case 1. $D_1 \neq \{e\}$, $D_2 \neq \{e\}$. Claim that $S_L = S_R = \emptyset$.

Suppose that S_L is a filter in $(D,+)$. Then, by Claim (1), $S_L = D_2 \cdot d$. Let $d_1 \in D_1 \setminus \{e\}$, $d_2 \in D_2 \setminus \{e\}$. Then $(d_1 d + d_2 d) + d = d_1 d + (d_2 d + d) = d_1 d + (d_2 + e)d = d_1 d + ed = (d_1 + e)d = d_1 d \neq d$

(since if $d_1 d = d$ then $d_1 = e$, a contradiction). Hence $d_1 d + d_2 d \notin LI_D(d) = D_2 \cdot d$. Now $d_2 \cdot d \in D_2 \cdot d$. $d_2 d + (d_1 d + d_2 d) = (d_2 d + d_1 d) + d_2 d = (d_2 + d_1)d + d_2 d = ed + d_2 d = (e + d_2)d = d_2 d \in D_2 \cdot d = S_L$. Since S_L is a filter in $(D,+)$, $d_1 d + d_2 d \in S_L = D_2 \cdot d$ contradicting the fact that $d_1 d + d_2 d \notin D_2 \cdot d$. Hence $S_L = \emptyset$.

Similarly, if S_R is a filter in $(D,+)$ then we get a contradiction. Hence $S_R = \emptyset$. Therefore $x + a = x + d$, $a + x = d + x$ for all $x \in D$. By Theorem 3.14 (2), we obtain the $a + a = a$ or $a + a = d + d = d$. Hence we get (1) and (2).

Case 2. $D_1 = \{e\}$, $D_2 \neq \{e\}$. Claim that (3) $S_R = \emptyset$, (4) $S_L = \emptyset$ or $S_L = D$.

To prove claim (3), suppose that S_R is a filter in $(D,+)$. By Claim (1), $S_R = D_1 \cdot d = \{e\} \cdot d = \{d\}$. Let $d_2 \in D_2 \setminus \{e\}$. Then $d_2 + e = e$, so $d_2 d + d = d \in D_1 \cdot d = S_R$. Since S_R is a filter in $(D,+)$, $d_2 d = d$. It follows that $d_2 = e$, a contradiction. Hence $S_R = \emptyset$.

To prove claim (4), suppose that $S_L \neq \emptyset$. Then S_L is a filter in $(D,+)$. To show that $S_L = D$, let $x \in D$. Then $xd^{-1} \in D$. Since $(D,+) \cong (D_1,+) \times (D_2,+)$, there exists an element d_2^* in D_2 such that $xd^{-1} = e + d_2^*$. Thus $xd^{-1} = e + d_2^* = d_2^* \in D_2$, so $x = d_2^* d \in D_2 \cdot d$. Let $y \in S_L$. Since $S_L \subseteq LI_D(d) = D_2 \cdot d$, $y = d_2' \cdot d$ for some $d_2' \in D_2$. Thus $x + y = d_2^* d + d_2' d = (d_2^* + d_2')d = d_2' d = y \in S_L$. Since S_L is a filter in $(D,+)$, $x \in S_L$. Hence $D = S_L$. Therefore $S_L = \emptyset$ or $S_L = D$. If $S_L = S_R = \emptyset$ then we obtain (1) or (2). If $S_L = D$ and $S_R = \emptyset$ then $x + a = a$ and $a + x = d + x$ for all $x \in D$. By Proposition 3.22 (2), we get that $a + a = a$. Hence we obtain (3).

Case 3. $D_1 \neq \{e\}$, $D_2 = \{e\}$.

Using a proof similar to the one in Case 2, we can show that $S_L = \emptyset$ and ($S_R = \emptyset$ or $S_R = D$).

If $S_L = S_R = \emptyset$ then we obtain (1) or (2). If $S_L = \emptyset$ and $S_R = D$ then $x + a = x + d$ and $a + x = a$ for all $x \in D$. By Proposition 3.22 (1), we get that $a + a = a$. Hence we obtain (4).

<u>Case 4</u>. $D_1 = D_2 = \{e\}$. Since $(D,+) \cong (D_1,+) \times (D_2,+)$, $D = \{e\}$. Hence $|K| = 2$. This is a contradiction. Therefore this case cannot occur.

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