มอดูลที่นิยามได้เหนือริงที่นิยามได้ซึ่งไม่มีตัวหารของศูนย์ในโครงสร้างแบบโอมินิมอล

นายจารุวัฒน์ รอดบรรจง

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DEFINABLE MODULES OVER DEFINABLE RINGS WITHOUT ZERO DIVISORS IN O-MINIMAL STRUCTURES

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ในโครงสร้างแบบโอมินิมอล ทุก ๆ กรุปที่นิยามได้จะเป็นแมนิโฟลด์แบบกรุปที่นิยามได้ ยิ่งกว่านั้น สำหรับริงที่นิยามได้ก็สามารถแสดงได้ในทำนองเดียวกัน ในงานวิจัยนี้ เราพิสูจน์ว่าทุก ๆ มอดูลที่นิยามได้เหนือริงที่นิยามได้ซึ่งไม่มีตัวหารของศูนย์จะเป็นแมนิโฟลด์แบบมอดูลที่นิยามได้ นอกจากนั้น เรายังจำแนกรูปแบบมอดูลที่นิยามได้เหนือริงที่นิยามได้ซึ่งไม่มีตัวหารของศูนย์ในโครงสร้างแบบโอมินิมอลทั้งหมดอีกด้วย

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In an o-minimal structure, every definable group admits a definable group manifold. Moreover, one can also show an analogue of this statement for definable rings. In this work, we prove that every definable module over definable ring without zero divisors admits a definable module manifold. In addition, we also give the classification of definable modules over definable rings without zero divisors in o-minimal structures.

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CHAPTER I
PRELIMINARIES

1.1 Notations

In this section, we introduce some notions that will be used.

Throughout, let m, n, k, p and q range over the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of all natural numbers. Let $\mathbb{P}$ denote the set of all prime numbers.

Let $X \subseteq Y$ and $\bar{a} = (a_1, \ldots, a_n) \in Y^n$. We denote by $X \cup \bar{a}$ the set $X \cup \{a_1, \ldots, a_n\}$ and $\bar{a} \subseteq X$ the statement $\{a_1, \ldots, a_n\} \subseteq X$.

Let $Q \subseteq M^{m+n}$. For each $a \in M^m$ and $b \in M^n$, we put

$$Q_a := \{y \in M^n : (a, y) \in Q\}, \text{ and}$$
$$Q^*_b := \{x \in M^m : (x, b) \in Q\}.$$

Let $(M, <)$ be a linearly ordered set with endpoints $-\infty$ and ∞. For $-\infty \leqslant a < b \leqslant \infty$, let

$$(a, b) := \{x \in M : a < x < b\},$$
$$(a, b] := \{x \in M : a < x \leqslant b\},$$
$$[a, b) := \{x \in M : a \leqslant x < b\}, \text{ and}$$
$$[a, b] := \{x \in M : a \leqslant x \leqslant b\}.$$

Observe that every linearly order set can be extended to a linearly order set with endpoints.

Let X be a set and $f, g : X \to M$. The notion $f < g$ means $f(x) < g(x)$ for all

$x \in X$.

Let (X, τ_X) and (Y, τ_Y) be topological spaces. For $Z \subseteq X$, we denote by $\operatorname{int} Z$ the **interior** of Z; $\operatorname{bd} Z$ the **boundary** of Z; $\operatorname{cl} Z$ the **closure** of Z; and ∂Z the **frontier** of Z. Let $(X \times Y, \tau_X \times \tau_Y)$ denote the topological space $X \times Y$ equipped with the canonical product topology.

Let $(A, \oplus, \ominus)$ be an abelian group, $a \in A$ and $k \in \mathbb{N}$. We write ka instead of $\underbrace{a \oplus \cdots \oplus a}_{k \text{ copies}}$. Let $(S, +, -, \cdot)$ be a ring and $s, t \in S$. We simply write st instead of $s \cdot t$.

1.2 Definability

1.2.1 Languages and structures

First, we recall some definitions from first-order logic. For more details, we refer to [9].

Definition 1.1. A **language** $\mathcal{L}$ is a disjoint union of $\mathcal{R}$ (called a set of **relation symbols**) and $\mathcal{F}$ (called a set of **function symbols**) where each symbol s is equipped with the associated natural number $a(s)$. For $r \in \mathcal{R}$ with $a(r) = n$, we call r an n-ary relation symbol. For $f \in \mathcal{F}$ with $a(f) = n$, we call f an n-ary function symbol.

0-ary function symbols are called **constant symbols**, and instead of saying "1-ary" and "2-ary", we say "unary" and "binary", respectively.

Definition 1.2. Let $\mathcal{L}$ be a language. An **$\mathcal{L}$-structure** $\mathfrak{M}$ consists of

1. a nonempty set M, called the **universe** or the **underlying set**;
2. the interpretation $r^{\mathfrak{M}} \subseteq M^{a(r)}$, for each $r \in \mathcal{R}$; and
3. the interpretation $f^{\mathfrak{M}} : M^{a(f)} \to M$, for each $f \in \mathcal{F}$.

The interpretation of constant symbols are referred to as constants. We also denote the structure $\mathfrak{M}$ by $(M, r^{\mathfrak{M}}, f^{\mathfrak{M}} : r \in \mathcal{R}, f \in \mathcal{F})$.

Definition 1.3. If $\mathcal{L}'$ is a language such that $\mathcal{L} \subseteq \mathcal{L}'$, we say that the structure $\mathfrak{M}'$ is an **expansion** of $\mathfrak{M}$ to $\mathcal{L}'$ if the universe of $\mathfrak{M}'$ is the same as the universe of $\mathfrak{M}$ and for every relation symbol r and function symbol f in $\mathcal{L}$, $r^{\mathfrak{M}} = r^{\mathfrak{M}'}$ and $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$.

Remark. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure and $C \subseteq M$. We could expand the language $\mathcal{L}$ to the language $\mathcal{L} \cup C$ by adding fresh constant symbols c_a to $\mathcal{L}$ corresponding to each $a \in C$, called c_a a **name** of a. Therefore, we can expand $\mathfrak{M}$ to the $\mathcal{L} \cup M$-structure $\mathfrak{M}_M$ by adding an interpretation $c_a^{\mathfrak{M}}$ as $a \in M$ to $\mathfrak{M}$.

Example.

1. For every nonempty set M, (M) is an $\varnothing$-structure.

2. The language of groups is $\mathcal{L}_{group} = \{e, ^{-1}, \cdot\}$ where e is a constant symbol, $^{-1}$ is a unary function symbol and $\cdot$ is a binary function symbol. For example, $(\mathbb{Z}, 0, -, +)$, where 0 is the interpretation of e, $-$ is the interpretation of $^{-1}$ and $+$ is the interpretation of $\cdot$, is an $\mathcal{L}_{group}$-structure.

3. The language of ordered rings is $\mathcal{L}_{or} = \{<, 0, 1, -, +, \cdot\}$ where $<$ is a binary relation symbol, $0, 1$ are constant symbols, $-$ is a unary function symbol and $+, \cdot$ are binary function symbols. For example, $(\mathbb{Z}, <, 0, 1, -, +, \cdot)$, where $<$ is the interpretation of $<$, 0 is the interpretation of 0, $-$ is the interpretation of $-$, $+$ is the interpretation of $+$ and $\cdot$ is the interpretation of $\cdot$, is an $\mathcal{L}_{or}$-structure.

Let $Var = \{v_1, v_2, \dots\}$ be the set of variables.

Definition 1.4. An **$\mathcal{L}$-term** is a a word on the alphabet $\mathcal{F} \cup Var$ obtained as follows:

1. Every variable is an $\mathcal{L}$-term.

2. For every n-ary function symbol $f \in \mathcal{F}$ and $\mathcal{L}$-terms $t_1, \dots, t_n$, $f(t_1, \dots, t_n)$ is an $\mathcal{L}$-term.

We write $t(x_1, \ldots, x_n)$ to indicate an $\mathcal{L}$-term where no variables other than $x_1, \ldots, x_n$ occur. Suppose $\mathfrak{M}$ is an $\mathcal{L}$-structure and $t(x_1, \ldots, x_m)$ is an $\mathcal{L}$-term. We interpret $t(x_1, \ldots, x_m)$ as a function $t^{\mathfrak{M}} : M^m \to M$ as follows:

1. If $t(x_1, \ldots, x_m) = x_i$, then $t^{\mathfrak{M}}(\bar{a}) = a_i$ for $\bar{a} = (a_1, \ldots, a_m)$.
2. If $t(x_1, \ldots, x_m) = f(t_1, \ldots, t_n)$ where f is an n-ary function symbol and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $t^{\mathfrak{M}}(\bar{a}) = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(\bar{a}), \ldots, t_n^{\mathfrak{M}}(\bar{a}))$ for $\bar{a} = (a_1, \ldots, a_m)$.

Definition 1.5. The **atomic $\mathcal{L}$-formulas** are words on the alphabet $\mathcal{L} \cup Var \cup \{\top, \bot, =\}$. We say that φ is an $\mathcal{L}$-atomic formula if φ is either

1. $\top$ or $\bot$;
2. $t_1 = t_2$ where t_1 and t_2 are $\mathcal{L}$-terms; or
3. $r(t_1, \ldots, t_n)$ where r is an n-ary relation symbol and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms.

Definition 1.6. The **$\mathcal{L}$-formula** are the words on the alphabet $\mathcal{L} \cup Var \cup \{\top, \bot, =, \neg, \vee, \wedge, \exists, \forall\}$ obtained as follows:

1. Every atomic $\mathcal{L}$-formula is an $\mathcal{L}$-formula.
2. If φ and ψ are $\mathcal{L}$-formulas, then so are $\neg\varphi$, $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$.
3. If φ is an $\mathcal{L}$-formula and x is a variable, then $\exists x\varphi$ and $\forall x\varphi$ are $\mathcal{L}$-formulas.

Definition 1.7. Let x be a variable and φ be an $\mathcal{L}$-formula. We say that an occurrence of x in φ is a **bound occurrence** if φ has a subformula which is of the form $\forall x\psi$ or $\exists x\psi$. We call φ an **$\mathcal{L}$-sentence** if all occurrences of variables in φ is bounded.

Definition 1.8. Let φ be an $\mathcal{L}$-formula. We say that φ is **quantifier free** if there is no occurences of $\forall$ and $\exists$ in φ.

Definition 1.9. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. A map $h : M \to N$ is an **$\mathcal{L}$-embedding** if

1. h is injective;

2. for every n-ary relation symbol r and $(a_1, \ldots, a_n) \in M^n$,

$$(a_1, \ldots, a_n) \in r^{\mathfrak{M}} \iff (h(a_1), \ldots, h(a_n)) \in r^{\mathfrak{N}}; \text{ and}$$

3. for every n-ary function symbol f and $(a_1, \ldots, a_n) \in M^n$,

$$h(f^{\mathfrak{M}}(a_1, \ldots, a_n)) = f^{\mathfrak{N}}(h(a_1), \ldots, h(a_n)).$$

Remark. We write $h : \mathfrak{M} \to \mathfrak{N}$ for an $\mathcal{L}$-embedding $h : M \to N$.

Example. Let $h : \mathbb{Z} \to \mathbb{R}$ be defined by $h(x) = e^x$. Then h is an $\mathcal{L}_{group}$-embedding of $(\mathbb{Z}, 0, -, +)$ into $(\mathbb{R}, 1, ^{-1}, \cdot)$.

Definition 1.10. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. We say that $\mathfrak{M}$ is a **substructure** of $\mathfrak{N}$ (or $\mathfrak{N}$ is an **extension** of $\mathfrak{M}$), denoted by $\mathfrak{M} \subseteq \mathfrak{N}$, if $M \subseteq N$ and the inclusion map $M \hookrightarrow N$ is an $\mathcal{L}$-embedding (equivalently, $M \subseteq N$ and for every $r \in \mathcal{R}$ and $f \in \mathcal{F}$, $r^{\mathfrak{M}} = r^{\mathfrak{N}} \cap M^{a(r)}$ and $f^{\mathfrak{M}}(a) = f^{\mathfrak{N}}(a)$ for any $a \in M^{a(f)}$).

Example. $(\mathbb{Z}, 0, -, +)$ is a substructure of $(\mathbb{R}, 0, -, +)$.

Definition 1.11. Let $\varphi(\bar{x}) = \varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $\bar{a} = (a_1, \ldots, a_n) \in M^n$. We define $\mathfrak{M} \models \varphi(a_1, \ldots, a_n)$ inductively as follows:

1. $\mathfrak{M} \models \top$ and $\mathfrak{M} \not\models \bot$.

2. If φ is $t_1 = t_2$ where t_1 and t_2 are $\mathcal{L}$-terms, then $\mathfrak{M} \models \varphi(\bar{a})$ if $t_1^{\mathfrak{M}}(\bar{a}) = t_2^{\mathfrak{M}}(\bar{a})$.

3. If φ is $r(t_1, \ldots, t_m)$ where r is an n-ary relation symbol and $t_1, \ldots, t_m$ are $\mathcal{L}$-terms, then $\mathfrak{M} \models \varphi(\bar{a})$ if $(t_1^{\mathfrak{M}}(\bar{a}), \ldots, t_m^{\mathfrak{M}}(\bar{a})) \in r^{\mathfrak{M}}$.

4. If φ is $\neg\psi$, then $\mathfrak{M} \models \varphi(\bar{a})$ if $\mathfrak{M} \not\models \psi(\bar{a})$.

5. If φ is $(\psi \vee \theta)$, then $\mathfrak{M} \models \varphi(\bar{a})$ if $\mathfrak{M} \models \psi(\bar{a})$ or $\mathfrak{M} \models \theta(\bar{a})$.

6. If φ is $(\psi \wedge \theta)$, then $\mathfrak{M} \models \varphi(\bar{a})$ if $\mathfrak{M} \models \psi(\bar{a})$ and $\mathfrak{M} \models \theta(\bar{a})$.

7. If φ is $\forall y \psi(\bar{x}, y)$, then $\mathfrak{M} \models \varphi(\bar{a})$ if $\mathfrak{M} \models \psi(\bar{a}, b)$ for all $b \in M$.

8. If φ is $\exists y \psi(\bar{x}, y)$, then $\mathfrak{M} \models \varphi(\bar{a})$ if there exists $b \in M$ such that $\mathfrak{M} \models \psi(\bar{a}, b)$.

We say that $\mathfrak{M}$ **satisfies** $\varphi(\bar{a})$ or $\varphi(\bar{a})$ **is true in** $\mathfrak{M}$ if $\mathfrak{M} \models \varphi(\bar{a})$.

Definition 1.12. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. We say that $\mathfrak{M}$ and $\mathfrak{N}$ are **elementary equivalent**, denoted by $\mathfrak{M} \equiv \mathfrak{N}$, if for every $\mathcal{L}$-sentence σ,

$$\mathfrak{M} \models \sigma \Longleftrightarrow \mathfrak{N} \models \sigma.$$

Definition 1.13. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $h : \mathfrak{M} \to \mathfrak{N}$ is said to be **elementary** if for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n) \in M^n$,

$$\mathfrak{M} \models \varphi(a_1, \ldots, a_n) \Longleftrightarrow \mathfrak{N} \models \varphi(h(a_1), \ldots, h(a_n)).$$

We say $\mathfrak{M}$ is an **elementary substructure** of $\mathfrak{N}$ (or $\mathfrak{N}$ is an **elementary extension** of $\mathfrak{M}$), denoted by $\mathfrak{M} \preccurlyeq \mathfrak{N}$, if $M \subseteq N$ and $\mathfrak{M} \hookrightarrow \mathfrak{N}$ is an elementary $\mathcal{L}$-embedding.

Example. In the language of ordered ring $\mathcal{L}_{or}$, the field of algebraic closure of the rationals is an elementary substructure of the field of complex numbers.

1.2.2 Definable sets and theories

Fix an $\mathcal{L}$-structure $\mathfrak{M}$.

Definition 1.14. Let $B \subseteq M$. A set $X \subseteq M^n$ is called a B**-definable set** in $\mathfrak{M}$ if there exist an $\mathcal{L}$-formula $\varphi(x_1, ..., x_n, y_1, ..., y_k)$ and $b_1, ..., b_k \in B$ such that

$$X = \{(a_1, ..., a_n) \in M^n : \mathfrak{M} \models \varphi(a_1, ..., a_n, b_1, ..., b_k)\}.$$

We say a map $f : C \to D$ ($C \subseteq M^m, D \subseteq M^n$) is a B**-definable map** if the graph of f,

$$\Gamma(f) := \{(c, f(c)) \in M^{m+n} : c \in C\},$$

is B-definable.

Remark. We simply use "definable" instead of "M-definable"; and "0-definable" instead of "$\varnothing$-definable".

Intuitively, a B-definable set is a set that can be defined by an $\mathcal{L}$-formula using parameters from B.

Example. The unit circle on the real plane $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is definable in $(\mathbb{R}, 0, 1, -, +, \cdot)$ but is not definable in $(\mathbb{R}, 0, -, +)$.

Proposition 1.15. *If $X \subseteq M^n$ is B-definable, then every $\mathcal{L}$-automorphism of $\mathfrak{M}$ that fixes B pointwise fixes X setwise, i.e. if $h : \mathfrak{M} \to \mathfrak{M}$ such that $h(b) = b$ for all $b \in B$, then $h[X] = X$.*

Definition 1.16. An $\mathcal{L}$**-theory** is a set of $\mathcal{L}$-sentences. An $\mathcal{L}$-structure $\mathfrak{M}$ is a **model** of an $\mathcal{L}$-theory T if $\mathfrak{M} \models \sigma$ for every $\sigma \in T$. We write $T \models \sigma$ if $\mathfrak{M} \models \sigma$ for every model $\mathfrak{M}$ of T.

Theorem 1.17 (Compactness Theorem)**.** *Let T be an $\mathcal{L}$-theory. If every finite subset of T has a model, then T has a model.*

Theorem 1.18 (Löwenheim-Skolem Theorem)**.** *Let T be an $\mathcal{L}$-theory and assume that T has an infinite model. If κ is a cardinal such that $\kappa \geqslant |\mathcal{L}| + \aleph_0$, then T has a model of size exactly κ.*

Theorem 1.19 (Downward Löwenheim-Skolem Theorem)**.** *Suppose $\mathfrak{M}$ is an infinite $\mathcal{L}$-structure and $Q \subseteq M$. Then $\mathfrak{M}$ has an elementary substructure $\mathfrak{N}$ such that $Q \subseteq N$ and $|N| \leqslant |Q| + |\mathcal{L}| + \aleph_0$.*

Theorem 1.20 (Upward Löwenheim-Skolem Theorem)**.** *Suppose $\mathfrak{M}$ is an infinite $\mathcal{L}$-structure. Then for each $\kappa \geqslant |\mathcal{L}| + |M|$, there is an elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ with $|N| = \kappa$.*

Definition 1.21. An $\mathcal{L}$-theory T admits **quantifier elimination** if for every $\mathcal{L}$-formula $\varphi(\overline{x})$, there is a quantifier free $\mathcal{L}$-formula $\psi(\overline{x})$ such that

$$T \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x})).$$

Definition 1.22. An $\mathcal{L}$-theory T is **model complete** if for any models $\mathfrak{M}, \mathfrak{N}$ of T where $\mathfrak{M} \subseteq \mathfrak{N}$, $\mathfrak{M} \preccurlyeq \mathfrak{N}$.

Proposition 1.23. *If T admits quantifier elimination, then T is model complete.*

Definition 1.24. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. We defined the **theory of $\mathfrak{M}$** by

$$\mathrm{Th}(\mathfrak{M}) = \{\sigma : \sigma \text{ is an } \mathcal{L}\text{-sentence and } \mathfrak{M} \models \sigma\}.$$

For $\bar{a} = (a_1, \ldots, a_n) \in M^n$, we say that $c_{\bar{a}} := (c_{a_1}, \ldots, c_{a_n})$ is a **name** of $\bar{a}$ if c_{a_i} is a name of a_i for all $i \in \{1, \ldots, n\}$.

Definition 1.25. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure and $B \subseteq M$. The **theory of $\mathfrak{M}$ in $\mathcal{L} \cup B$** is the set of $\mathcal{L} \cup B$-formulas

$$\mathrm{Th}(\mathfrak{M}, B) = \{\varphi(c_{\bar{a}}) : \varphi(\bar{x}) \text{ is an } \mathcal{L}\text{-formula }, \bar{a} \in B^n \text{ and } \mathfrak{M} \models \varphi(\bar{a})\}.$$

Definition 1.26. Let $B \subseteq M$. A **partial n-type over** B is a set $p(\bar{x})$ of $\mathcal{L} \cup B$-formulas with free variable among $\bar{x} = (x_1, \ldots, x_n)$. Let $p(\bar{x})$ be a partial n-type over B. We say that $p(\bar{x})$ is **consistent** if $\mathrm{Th}(\mathfrak{M}, B) \cup p(\bar{c})$ does not prove a contradiction where $\bar{c}$ are fresh constant symbols. We say that $p(\bar{x})$ is **complete** if for every $\mathcal{L} \cup B$-formula $\varphi(\bar{x})$, either $\varphi(\bar{x}) \in p(\bar{x})$ or $\neg\varphi(\bar{x}) \in p(\bar{x})$.

We write $S_n^{\mathfrak{M}}(B)$ for the set of all complete consistent n-types over B.

Definition 1.27. Let $b \in M^n$ and $B \subseteq M$. We denote the **type of b over B** by

$$\mathrm{tp}^{\mathfrak{M}}(b|B) := \{\varphi(\bar{x}) : \varphi \text{ is an } \mathcal{L} \cup B\text{-formula and } \mathfrak{M} \models \varphi(b)\}.$$

Let $p(\bar{x})$ be a partial n-type over B. We say that $b \in M^n$ **realizes** p if $p(\bar{x}) \subseteq \mathrm{tp}^{\mathfrak{M}}(b|B)$. We say that $p(\bar{x})$ **is realized in $\mathfrak{M}$** if there exists $b \in M^n$ realizing $p(\bar{x})$.

Remark. For every $b \in M^n$ and $B \subseteq M$, $\mathrm{tp}^{\mathfrak{M}}(b|B) \in S_n^{\mathfrak{M}}(B)$.

Example. Let $\mathfrak{M} = (\mathbb{R}, <, 0, 1, -, +, \cdot)$, $p_1(x) = \{0 < x\}$ and $p_2(x) = \{n < x : n \in \mathbb{N}\}$. Then $p_1(x)$ is realized in $\mathfrak{M}$ while $p_2(x)$ is not.

Definition 1.28. Let κ be an infinite cardinal. We say that an $\mathcal{L}$-structure $\mathfrak{M}$ is κ-**saturated** if for every $X \subseteq M$ with $|X| < \kappa$, every $p(x) \in S_1^{\mathfrak{M}}(X)$ is realized in $\mathfrak{M}$.

Theorem 1.29. *For every infinite cardinal κ, every $\mathcal{L}$-structure has an elementary extension that is κ-saturated.*

Example.

1. If $\mathfrak{M}$ is a model with finite universe, then $\mathfrak{M}$ is κ-saturated for every cardinal $\kappa \geqslant \aleph_0$.
2. $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ are $\aleph_0$-saturated, but $(\mathbb{R}, <)$ is not $\aleph_1$-saturated.

1.3 On o-minimal structures

1.3.1 Historical timeline on o-minimality

Semialgebraic sets and subanalytic sets are studied in real algebraic geometry and real analytic geometry. These sets possess a lot of nice properties. For example, each semialgebraic set has only finitely many connected components and these connected components are also semialgebraic.

In 1986, L. Van den Dries, J. F. Knight, A. Pillay and C. Steinhorn developed the concept of "o-minimality" ("o" abbreviates the word "order") which generalizes the concepts of semialgebraic and subanalytic theories, see e.g. [6],[17] and [26]. The theory of o-minimal structures realizes the concept of A. Grothendieck's tame topology, see [20]. In addition, it also has many applications in various fields of mathematics such as number theory, algebraic geometry and group theory.

One of well-known applications in o-minimal theory on group theory is that "every definable group admits a definable group-manifold" (see [16] for more information). This result can be considered as a pioneer in researches of definable groups in o-minimal structures.

1.3.2 Monotonicity theorem and Cell Decomposition Theorem

Let $(M, <)$ be a linearly ordered set without endpoints.

Definition 1.30. We say that $(M, <)$ is **dense** if for any $a, b \in M$ with $a < b$, there exists $c \in M$ such that $a < c < b$.

Definition 1.31. An expansion of $(M, <)$ is **o-minimal** if every unary definable set is a finite union of intervals and points.

Results presented in this section are classical results, so we refer to [25] for more details.

Definition 1.32. Let A be a group with identity e and $a \in A$. We say that a is a **torsion point** (or a **torsion element)** of A if a has finite order and is called a p**-torsion point** if a is of order p. We say that a group A is **torsion-free** if for any $b \in A \smallsetminus \{e\}$, b is not a torsion point.

Definition 1.33. A group $(A, \oplus, \ominus)$ is **divisible** if for all positive integer n and $a \in A$, there exists $b \in A$ such that $nb = a$.

Proposition 1.34. *Let $(M, <, 0, -, +)$ be an ordered group. If $(M, <, 0, -, +)$ is o-minimal, then the group $(M, 0, -, +)$ is abelian, divisible and torsion-free.*

Definition 1.35. An ordered field F is said to be a **real closed field** if

1. -1 is not a sum of squares, in particular, $F \neq F(\sqrt{-1})$; and

2. $F(\sqrt{-1})$ is algebraically closed.

Proposition 1.36. *Let* $(M, <, 0, 1, -, +, \cdot)$ *be an ordered ring. If* $(M, <, 0, 1, -, +, \cdot)$ *is o-minimal, then* $(M, <, 0, 1, -, +, \cdot)$ *is a real closed field.*

Fix an o-minimal structure $\mathfrak{M}$. We first introduce the **Monotonicity Theorem**.

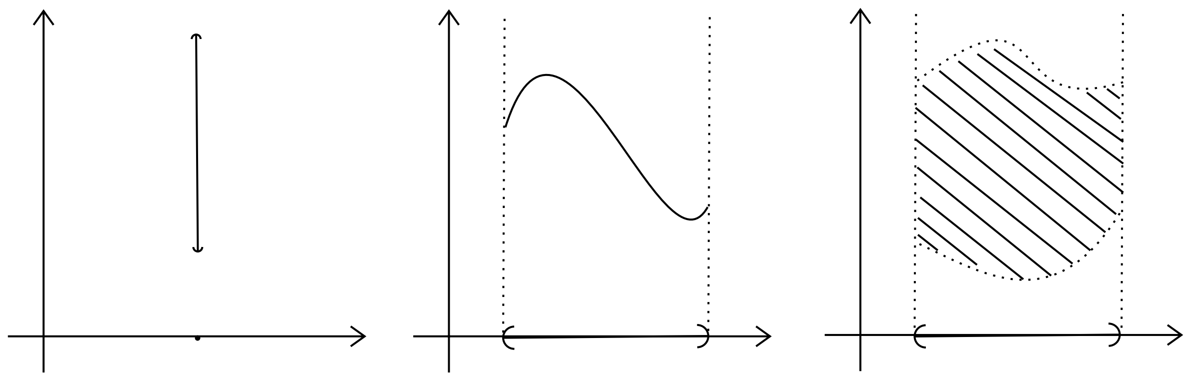
Theorem 1.37 (Monotonicity Theorem)**.** *Let* $-\infty \leqslant a < b \leqslant \infty$ *and* $f : (a, b) \to M$ *be a definable function. Then there are points* $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ *in* (a, b) *such that for each* $j \in \{0, \ldots, k\}$*, the function is either constant, or strictly monotone and continuous on* (a_j, a_{j+1})*.*

Moreover, the Monotonicity Theorem also implies every definable subset of M can be splitted into finitely many cells. Now, we recall the definition of cells.

Definition 1.38. Let $(i_1, \ldots, i_n)$ be a sequence of zeroes and ones of length n. An $(i_1, \ldots, i_n)$-**cell** is a definable subset of M^n defined inductively as follows:

1. A (0)-cell is a singleton $\{a\} \subseteq M$ and a (1)-cell is an open interval $(a, b) \subseteq M$.
2. Suppose $(i_1, \ldots, i_m)$-cells are defined. An $(i_1, \ldots, i_m, 0)$-cell is the graph of a definable and continuous function $f : X \to M$, and an $(i_1, \ldots, i_m, 1)$-cell is a set $\{(x, m) \in X \times M : f(x) < m < g(x)\}$, for some definable and continuous functions $f, g : X \to M \cup \{-\infty, +\infty\}$ such that $f < g$, where X is an $(i_1, \ldots, i_m)$-cell.

A cell in M^n is an $(i_1, \ldots, i_n)$-cell for some (unique) sequence $(i_1, \ldots, i_n)$. We call the $(1, \ldots, 1)$-cell an **open cell**.



(0,1)-cell (1,0)-cell (1,1)-cell

Figure 1 : Examples of cells in $\mathbb{R}^2$

Proposition 1.39. *Each cell in M^n is homeomorphic to an open cell in M^k for some $k \leqslant n$.*

Theorem 1.40 (Cell Decomposition Theorem)**.**

(I_n) *For any collection of definable sets $B_1, \ldots, B_k \subseteq M^n$, there is a collection of cells in M^n partitioning each of $B_1, \ldots, B_k$.*

(II_n) *For every definable function $f : M^n \to M$, there is a partition $\mathcal{D}$ of M^n into cells such that the restriction $f \restriction D : D \to M$ to each cell $D \in \mathcal{D}$ is continuous.*

Proposition 1.41. *Let $B \subseteq M^n$ be a closed definable set and $B_1, \ldots, B_k$ be definable subsets of B. Then there is a partition $\mathcal{D}$ of B into cells such that $\mathcal{D}$ partitions each $B_1, \ldots, B_k$ and for any $D \in \mathcal{D}$, ∂D is a union of elements in $\mathcal{D}$.*

Definition 1.42. A set $Y \subseteq M^{n+1}$ is **finite over** M^n if for each $x \in M^n$, Y_x is finite. We call Y is **uniformly finite over** M^n if there is $N \in \mathbb{N}$ such that $|Y_x| \leqslant N$ for all $x \in M^n$.

Proposition 1.43 (Uniform Finiteness Property)**.** *Suppose the definable set $Y \subseteq R^{n+1}$ is finite over R^n. Then Y is uniformly finite over R^n.*

1.3.3 More on geometry of definable sets

We now equip the universe M with the interval topology. In the higher dimension, the product M^n will be equipped with the corresponding product topology.

Proposition 1.44. *If $X \subseteq M^n$ is definable, then so are* $\mathrm{cl}(X)$ *and* $\mathrm{int}(X)$

Proposition 1.45. *If $X \subseteq Y \subseteq M^n$ are definable sets and X is open in Y, then there is a definable open set $U \subseteq M^n$ such that $U \cap Y = X$.*

Definition 1.46. We say that a definable set is **definably connected** if it cannot be written as a disjoint union of two nonempty definable open subsets.

Proposition 1.47. *The definably connected subsets of M are of the form $\varnothing$, $\{a\}$, (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ where $-\infty \leqslant a < b \leqslant \infty$.*

Proposition 1.48. *Every cell is definably connected.*

Proposition 1.49. *The image of a definably connected set $X \subseteq M^n$ under a definable continuous map from $X \to M^m$ is definably connected.*

Definition 1.50. A **definably connected component** of a nonempty definable set is a maximal definably connected subset of a definable set.

Proposition 1.51. *Every definable set has only finitely many definably connected components.*

Definition 1.52. The **dimension** of a set $X \subseteq M^m$, denoted by $\dim X$, is defined as follows: For an $(i_1, ..., i_m)$-cell C in M^m, let $\dim C := i_1 + \cdots + i_m$. For a definable set $X \subseteq M^m$, let $\dim X := \max\{\dim C : C \subseteq X \text{ and } C \text{ is a cell}\}$ where $\dim(\varnothing) = -\infty$.

Proposition 1.53. *Let* $m, n \in \mathbb{N}$*.*

1. *If* $X \subseteq Y \subseteq M^n$ *and* X, Y *are definable, then* $\dim X \leqslant \dim Y \leqslant n$*.*
2. *If* $X \subseteq M^m$ *and* $Y \subseteq M^n$ *are definable and there is a definable bijection between* X *and* Y*, then* $\dim X = \dim Y$*.*
3. *If* $X, Y \subseteq M^n$ *are definable, then* $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$*.*
4. *If* $X \subseteq M^m$ *and* $Y \subseteq M^n$ *are definable, then* $\dim(X \times Y) = \dim X + \dim Y$*.*

Theorem 1.54. *Let* $X \subseteq M^n$ *be a nonempty definable set. Then*

1. $\dim \partial X < \dim X$*;*
2. $\dim \operatorname{cl} X = \dim X$*; and*
3. $\dim \operatorname{bd} X < n$*.*

1.3.4 Examples of o-minimal structures

In this part, we introduce some classic examples of o-minimal structures.

Example. Dense linearly ordered sets without endpoints

In the language $\mathcal{L} = \{<\}$, a dense-linearly ordered set without endpoints is a trivial example of o-minimal structure such as $(\mathbb{R}, <)$. We can show the result by describing explicitly all definable sets, see [7] and [17].

Example. Divisible ordered abelian groups

In the language of ordered group $\mathcal{L}_{og} = \{<, 0, -, +\}$, due to the quantifier elimination for the theory of divisible ordered abelian groups, we have that such structures are o-minimal. For more details, see [17] and [19].

Definition 1.55. A subset $X \subseteq \mathbb{R}^n$ is a **basic semialgebraic** set if there exist polynomials with coefficients in $\mathbb{R}$ and in n indeterminates $f_1, \ldots, f_k$ and g such that

$$X = \{x \in \mathbb{R}^n : f_1(x) > 0, \ldots, f_k(x) > 0 \text{ and } g(x) = 0\}.$$

A subset of $\mathbb{R}^n$ is **semialgebraic** if it is a finite union of basic semialgebraic subsets of $\mathbb{R}^n$.

Note that every basic semialgebraic subset of $\mathbb{R}$ can be written as a finite union of intervals and points.

Example. Semialgebraic geometry

By Tarski-Seidenberg Theorem, the theory of the structure $\mathbb{R}_{alg} := (\mathbb{R}, <, 0, 1, -, +, ^{-1}, \cdot)$ admits quantifier elimination, see [23]. Therefore every definable set in $\mathbb{R}_{alg}$ is a semialgebraic set. It follows that every unary definable subset of $\mathbb{R}_{alg}$ is a finite union of intervals and points. Hence $\mathbb{R}_{alg}$ is o-minimal.

Moreover, every ordered ring $\mathfrak{M} = (M, <, 0, 1, -, +, \cdot)$ that is elementary equivalent to $\mathbb{R}_{alg}$ is o-minimal. Since every real closed field can be considered as an ordered ring, it follows that every real closed field is o-minimal.

Definition 1.56. Let I be an open subset of $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be **analytic** if for every $x_0 \in I$, there exist a neighborhood J of x_0 and a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$f(x) = \sum_{n\in\mathbb{N}} a_n(x - x_0)^n$$

for all $x \in J$. If U is an open subset of $\mathbb{R}^k$, then a function $f : U \to \mathbb{R}$ is called **analytic** if for every $x_0 \in U$, there exist a neighborhood V of x_0 and a sequence of homogeneous polynomials $\{P_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$f(x) = \sum_{n\in\mathbb{N}} P_n(x - x_0)^n$$

for all $x \in V$.

Definition 1.57. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is **restricted analytic** if f is 0 outside $[-1, 1]^n$ and there exist $U \subseteq \mathbb{R}^n$ which is an open neighborhood of $[-1, 1]^n$ and an analytic function $g : U \to \mathbb{R}$ such that $f = g$ on $[-1, 1]^n$.

Definition 1.58. Let M be an analytic manifold. A subset X of M is called **semianalytic** if for every $x \in X$, there exists an open neighborhood U of x such that $X \cap U$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f_1(x) > 0, \dots, f_k(x) > 0 \text{ and } g(x) = 0\}$$

where $f_1, \dots, f_k$ and g are analytic on U. A subset Y of M is called **subanalytic** if for every point in Y admits an open neighborhood V such that $Y \cap V$ is a projection of a relatively compact semianalytic set.

Example. Subanalytic geometry

Consider the structure

$$\mathbb{R}_{an} := \mathbb{R}_{alg} \cup \{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is restricted analytic and } n \in \mathbb{N}\}.$$

One can characterize definable subsets in this structure as follows: The definable subsets in $\mathbb{R}_{an}$ is exactly the subanalytic sets in its projective space, see [26]. As a result, $\mathbb{R}_{an}$ is model complete and o-minimal. For details, see [1] and [8].

Definition 1.59. An **exponential set** in $\mathbb{R}^n$ is a set of the form

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) = 0\}$$

where P is a polynomial with real coefficients in $2n$ indeterminates. A **subexponential set** in $\mathbb{R}^n$ is the image of an exponential set in $\mathbb{R}^{n+k}$ (for some k) under the natural projection from $\mathbb{R}^{n+k} \to \mathbb{R}^n$.

Example. Exponential geometry

The structure $\mathbb{R}_{exp} := \mathbb{R}_{alg} \cup \{\exp\}$, where exp is the exponential function on $\mathbb{R}$, is model complete, see [29]. As a consequence of this result, we have that every definable set in $\mathbb{R}_{exp}$ is a subexponential set. By the result of A. Khonvanskii in [5], every exponential set (hence every subexponential set) has only finitely many connected components. Combine the results, we obtain that the structure $\mathbb{R}_{exp}$ is o-minimal. Moreover, if we fix an irrational α and restrict the domain to $x > 0$, the function x^α and $\exp(\frac{-1}{x})$ are definable in $\mathbb{R}_{exp}$.

Consider the structure $\mathbb{R}_{an,exp} := \mathbb{R}_{an} \cup \{\exp\}$. In 1994, L. van den Dries and C. Miller extended A. Wilkie's method to prove that $\mathbb{R}_{an,exp}$ is also o-minimal, see [27] and [28].

Definition 1.60. A finite sequence $(f_1, \ldots, f_k)$ of C^1-function from $\mathbb{R}^n \to \mathbb{R}$ is a **Pfaffian chain** if there exist real polynomials $P_{ij}(X_1, \ldots, X_n, Y_1, \ldots, Y_j)$ such that

$$\frac{\partial f_j}{\partial x_i}(x) = P_{ij}(x, f_1(x), \ldots, f_j(x))$$

for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$ and $x \in \mathbb{R}^n$. A C^1-function $g : \mathbb{R}^n \to \mathbb{R}$ is a **Pfaffian function** if there exists a Pfaffian chain $(f_1, \ldots, f_k)$ such that for every $x \in \mathbb{R}^n$,

$$g(x) = P_{ij}(x, f_1(x), \ldots, f_j(x)).$$

Example. Pfaffian geometry

Let $f_1(x) = e^x$ and for each $n \geqslant 2$, $f_n(x) = e^{f_{n-1}(x)}$. It is easy to see that for each $n \geqslant 1$, $f_n' = f_{n-1}' \cdot f_n = \cdots = f_1 \cdot f_2 \cdot \ldots \cdot f_n$. This implies that $(f_1, \ldots, f_n)$ is a Pfaffian chain. The polynomials and exponential functions are Pfaffian functions.

Consider the structure

$$\mathbb{R}_{Pfaff} := \mathbb{R}_{an} \cup \{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is a Pfaffian function }\}.$$

In 1996, A. Wilkie showed the o-minimality of $\mathbb{R}_{Pfaff}$, see [29].

$$\begin{tikzcd}
 & & \mathbb{R}_{an} \arrow[r] & \mathbb{R}_{an,exp} \\
(\mathbb{R}, <) \arrow[r] & \mathbb{R}_{alg} \arrow[ur] \arrow[dr] & & \\
 & & \mathbb{R}_{exp} \arrow[uur] \arrow[r] & \mathbb{R}_{Pfaff}
\end{tikzcd}$$

Figure 2 : Examples of o-minimal structures over $\mathbb{R}$

$\mathfrak{M} \longrightarrow \mathfrak{N}$ means $\mathfrak{N}$ is an o-minimal expansion of $\mathfrak{M}$

CHAPTER II
LITERATURE REVIEW AND OUTLINE

2.1 Literature review

Throughout, we fix an o-minimal structure $\mathfrak{M}$.

Definition 2.1. We say that a group $(A, \oplus, \ominus)$ is a **definable group** if A is definable and $\oplus, \ominus$ are definable maps.

Definition 2.2. Let $(A, \oplus, \ominus)$ be a definable group with $A \subseteq M^k$ and $\dim A = n \leqslant k$. A topology on A is a **definable group manifold topology** if $\oplus$ and $\ominus$ are continuous and there is a finite set $\mathcal{D} := \{(D_i, \phi_i) : i \in I\}$ such that

1. for each $i \in I$, D_i is a definable open subset of A and $\phi_i : D_i \to M^n$ is a definable homeomorphism onto its image;
2. $A = \bigcup\limits_{i \in I} D_i$; and
3. for all $i, j \in I$, if $D_i \cap D_j \neq \varnothing$, then $D_{ij} := \phi_i[D_i \cap D_j]$ is a definable open subset of $\phi_i[D_i]$ and $(\phi_j \circ \phi_i^{-1}) \restriction D_{ij}$ is a definable homeomorphism onto its image.

We say that A admits a **definable group manifold** if there exists a definable group manifold topology on A.

Next, we review some important theorems on definable groups in o-minimal structures.

Theorem 2.3 (A. Pillay, [16])**.** *Every definable group admits a definable group manifold.*

We begin with reviews of results on definable groups in o-minimal structures with dimensions 1, 2 and 3.

In 1991, V. Razenj adapted the classification of one-dimensional topological Hausdorff manifolds to obtain that a definable group A has either one definably connected component (called S^1-type) or two definably connected components (called $\mathbb{R}$-type). Moreover, we obtain the classification of one-dimensional definably connected definable groups as shown in Theorem 2.5.

Definition 2.4. Let A be a group with identity e. We say that A has **bounded exponent** if there exists the least natural number n such that $a^n = e$ for all $a \in A$. Otherwise, we say A does not have bounded exponent.

Theorem 2.5 (V. Razenj, [18]). *Let A be a one-dimensional definably connected definable group. Then*

1. *A is abelian;*
2. *A does not have bounded exponent; and*
3. *A is either torsion-free or for any prime number p, the set of all p-torsion points has exactly p elements.*

Definition 2.6. Let A and G be definable groups. We say that A is **definably isomorphic** to G if there exists a group-isomorphism between A and G that is definable.

Theorem 2.7 (A.W. Strzebonski, [22]). *Let $\mathfrak{M}$ be an expansion of a real closed field. A one-dimensional definably connected definable group is definably isomorphic to either an abelian group on $(0,1)$ or S^1, where $S^1 := ([0,1), \oplus, 0)$ with*

$$x \oplus y = \begin{cases} x + y & : \textit{for } x + y < 1, \\ x + y - 1 & : \textit{for } x + y \geq 1. \end{cases}$$

.

Definable groups of dimensions 2 and 3 are studied by A. Nesin, A. Pillay and V. Razenj in [10]. To see the results, we first introduce some definitions.

Definition 2.8. Let A be a group with the identity e. A **subnormal series** of A is a finite sequence $G_0, G_1, \ldots, G_n$ of subgroups of A such that $G_0 = \{e\}$, $G_n = A$ and for all $0 \leqslant i \leqslant n-1$, G_i is normal in G_{i+1}. Each group G_{i+1}/G_i is called a **factor** associated to the series.

Definition 2.9. A group is said to be **solvable** if it has a subnormal series with abelian factors.

Definition 2.10. Let A be a group. The **center** of A, denoted by $Z(A)$, is the set of elements that commute with every element of A. A group is said to be **centerless** if the center of the group is trivial.

Definition 2.11. Let A be a group with an identity element e and G, H be subgroups of A. We say that A is a **semidirect product** of G and H if G is normal in A, $G \cap H = \{e\}$ and $A = GH$.

Definition 2.12. Let F be a field and $n \geqslant 1$. The **general linear group of degree** n **over field** F, denoted by $GL_n(F)$, is the set of $n \times n$ invertible matrices with entries from F equipped with matrix multiplication as the group operation, i.e.,

$$GL_n(F) = \{Q \in M_n(F) : \det Q \neq 0\}.$$

Definition 2.13. Let F be a field and $n \geqslant 1$. The **special linear group of degree** n **over field** F, denoted by $SL_n(F)$, is the set

$$SL_n(F) = \{Q \in GL_n(F) : \det Q = 1\}$$

equipped with matrix multiplication as the group operation. The **projective special linear group** (of degree n over field F), denoted by $PSL_n(F)$, is the quotient of $SL_n(F)$ by its center.

Definition 2.14. Let F be a field and $n \geqslant 1$. The **orthogonal group of degree n over field** F, denoted by $O_n(F)$, is the set

$$O_n(F) = \{Q \in GL_n(F) : QQ^t = Q^tQ = I_n\}$$

equipped with matrix multiplication as the group operation. The **special orthogonal group** (of degree n over field F), denoted by $SO_n(F)$, is the set

$$SO_n(F) = O_n(F) \cap SL_n(F)$$

equipped with matrix multiplication as the group operation.

Theorem 2.15 (A. Nesin, A. Pillay and V. Razenj, [10]).

1. *If a definable group A is two-dimensional definably-connected, then*
   - *A is solvable; and*
   - *A is either abelian or A is centerless and definably isomorphic to a semidirect product of M_a (the additive group of M) and M_m (the multiplicative group of positive elements of M) for some definable real closed field M.*
2. *If A is 3-dimensional nonsolvable definably-connected, $A/Z(A)$ is definably isomorphic to either $PSL_2(M)$ or $SO_3(M)$ for some definable real closed field M.*

Definition 2.16. A group is **semisimple** if it has no nontrivial infinite normal abelian subgroups.

Theorem 2.17 (Y. Peterzil, A. Pillay and S. Starchenko, [12]). *Every definably connected centerless semisimple definable group is definably isomorphic to a direct product of definably connected definably simple definable groups.*

Theorem 2.18 (Y. Peterzil, A. Pillay and S. Starchenko, [12]). *Suppose $\mathfrak{M}$ is an expansion of a real closed field. Then every definably connected centerless* *semisimple definable group is definably isomorphic to a semialgebraic linear group over M.*

Theorem 2.19 (Y. Peterzil, A. Pillay and S. Starchenko, [13]). *Suppose $\mathfrak{M}$ is an expansion of real closed field and A is a definably connected definable subgroup of $GL_n(M)$, for some $n \in \mathbb{N}$. Then there are a normal solvable definable subgroup G and a semialgebraic semisimple subgroup H such that $G \cap H$ is finite and $A = GH$.*

Theorem 2.20 (Y. Peterzil, A. Pillay and S. Starchenko, [13]). *Suppose $\mathfrak{M}$ is an expansion of real closed field and A is a definably connected definable subgroup of $GL_n(M)$, for some $n \in \mathbb{N}$. Then there are semialgebraic groups G_1, G_2 of $GL_n(M)$ such that $G_2 < A < G_1$, G_2 is normal in G_1 and G_1/G_2 is abelian. Moreover, there are abelian definably connected subgroups $A_1, \ldots, A_k$ of A such that $A = G_2 A_1 \cdots A_k$.*

Definition 2.21. We say that a group A is **abelian-by-finite** if there exists an abelian normal subgroup G of A such that A/G is finite.

Theorem 2.22 (Y. Peterzil and S. Starchenko,[14]). *Let A be a definably compact definable group. Then either A is abelian-by-finite or $A/Z(A)$ is semisimple. In particular, if A is solvable, then A is abelian-by-finite.*

We now review some results on definable rings.

Definition 2.23. We say that a ring $(S, +, -, \cdot)$ is a **definable ring** if S is definable and $-, +, \cdot$ are definable maps.

Definition 2.24. Let $(S, +, -, \cdot)$ be a definable ring with $S \subseteq M^k$ and $\dim S = n \leqslant k$. A topology on A is a **definable ring manifold topology** if $+, -$ and $\cdot$ are continuous and there is a finite set $\mathcal{P} := \{(P_i, \phi_i) : i \in I\}$ such that

1. for each $i \in I$, P_i is a definable open subset of S and $\phi_i : P_i \to M^n$ is a definable homeomorphism onto its image;

2. $S = \bigcup_{i \in I} P_i$; and

3. for all $i, j \in I$, if $P_i \cap P_j \neq \varnothing$, then $P_{ij} := \phi_i[P_i \cap P_j]$ is a definable open subset of $\phi_i[P_i]$ and $(\phi_j \circ \phi_i^{-1}) \upharpoonright P_{ij}$ is a definable homeomorphism onto its image.

We say that S admits a **definable ring manifold** if there exists a definable ring manifold topology on S.

Definition 2.25. Let R and S be definable rings. We say that R is **definably isomorphic** to S if there exists a ring-isomorphism between R and S that is definable.

Theorem 2.26 (M. Otero, Y. Peterzil and A. Pillay, [11])**.** *Every definable ring admits a definable ring manifold.*

Theorem 2.27 (Y. Peterzil and C. Steinhorn, [15])**.** *If S is an infinite definable ring without zero divisors, then S is a division ring and there is a one-dimensional definable ring I which is a subring of S such that I is a real closed field and S is definably isomorphic to either I, $I(\sqrt{-1})$, or the ring of quaternions over I.*

These give rise to the question that;

"can we classify all definable modules over definable rings?"

2.2 Outline

In this dissertation, we show that every definable module over definable ring without zero divisors in the o-minimal structures admits a definable module manifold. Moreover, we characterize all definable modules over definable rings without zero divisors. This thesis is organized as follows.

In Section 3.1, we give definitions and notations concerning modules. In Section 3.2, basic knowledges and some important results concerning generic elements are introduced here. Section 3.3 is devoted to show that every definable module over definable ring without zero divisors admits a definable module manifold, i.e., a definable group admits a definable group manifold, a definable ring admits a definable ring manifold and the scalar multiplication of a module is continuous.

In Chapter IV, we discuss the forms of definable modules over definable rings without zero divisors. First, we show that every definable module over definable ring without zero divisors is a free module with a finite basis. We also discuss about the Frobenius Theorem and its consequences in the context of o-minimality. Finally, we give a complete characterization of definable infinite rings without zero divisors in an o-minimal structure.

CHAPTER III
DEFINABLE MODULES OVER DEFINABL E RINGS WITHOUT ZERO DIVISORS

Throughout this chapter, let $\mathfrak{M}$ be an o-minimal expansion of a dense linear ordering $(M,<)$, $(S,+,-,\cdot,^{-1})$ be a ring without zero divisors and $(A,\oplus,\ominus)$ be an S-module.

Definition 3.1. We say that A is a **definable S-module** if S is a definable ring, A is a definable abelian group and the scalar multiplication from $S\times A$ to A is a definable map.

Definition 3.2. Let S be a definable ring and A a definable S-module. We say that A admits a **definable S-module manifold** (or simply a **definable module manifold**) if

1. A admits a definable group manifold;
2. S admits a definable ring manifold; and
3. the scalar multiplication from $S\times A$ to A is continuous with respect to the corresponding topologies on S and A.

The main goal of this chapter is the following:

Theorem 3.3. *Suppose A is a definable S-module in $\mathfrak{M}$. Then A admits a definable S-module manifold.*

3.1 Modules

First, we recall the definition of modules.

Definition 3.4. Suppose that $(S, +, -, \cdot)$ is a ring and 1_S is its multiplicative identity. A **(left)** S**-module** A consists of an abelian group $(A, \oplus, \ominus)$ and an operation $\rho : S \times A \to A$, we call it the **scalar multiplication**, such that for all $s, t \in S$ and $a, b \in A$,

1. $\rho(s, a \oplus b) = \rho(s, a) \oplus \rho(a, b)$,
2. $\rho(s + t, a) = \rho(s, a) \oplus \rho(t, a)$,
3. $\rho(s \cdot t, a) = \rho(s, \rho(t, a))$, and
4. $\rho(1_S, a) = a$.

Definition 3.5. Let S be a ring. We say that G is an S**-submodule** of an S-module A if G is a subgroup of A as an additive group and the scalar multiplications $S \times A \to A$ and $S \times G \to G$ agree on $S \times G$.

Definition 3.6. Let $X \subseteq A$ and G be an S-submodule of A. We say that X **spans** M if for any $a \in G$, there are $s_1, \ldots, s_k \in S$ and $x_1, \ldots, x_k \in X$ such that $a = \rho(s_1, x_1) \oplus \cdots \oplus \rho(s_k, x_k)$. We say that M is **finitely generated** if there is a finite subset of M that spans M.

Definition 3.7. We say that a finite set $\{a_1, \ldots, a_k\} \subseteq A$ is **linearly independent (over** S**)** if for any $s_1, \ldots, s_k \in S$ with $\rho(s_1, a_1) \oplus \cdots \oplus \rho(s_k, a_k) = 0_A$, we have $s_1 = \cdots = s_k = 0_S$. A subset X of A is linearly independent (over S) if every finite subset of X is linearly independent (over S).

Definition 3.8. Let $X \subseteq A$. We say that A is a **free** S**-module on a set** X if X spans A and X is linearly independent over S. Such X is called a **basis** for A. We say that a module A is **free** if there exists a basis for A.

The following two theorems are well-known theorems in algebra.

Theorem 3.9 (P.A. Grillet, [3]). *Every module over a division ring is free.*

Theorem 3.10 (A. Tarcan and C.C. Yücel, [24]). *Every finitely generated free module has a finite basis.*

3.2 Generic elements

In this section, we introduced definitions and properties concerning generic elements. For more details, we refer to [16].

Definition 3.11. Let $B \subseteq M$. The **definable closure** of B, denoted by $\mathrm{dcl}(B)$, is the set

$$\{a \in M : \{a\} \text{ is } B\text{-definable}\}.$$

Definition 3.12. Let $\bar{a} \in M^n$ and $B \subseteq M$. The **dimension of $\bar{a}$ over** B, denoted by $\dim(\bar{a}/B)$, is defined to be the least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \subseteq \mathrm{dcl}(B \cup \bar{a}')$.

Definition 3.13. Let $B, I \subseteq M$. We say I is **independent over** B if for all $x \in I$, we have $x \notin \mathrm{dcl}(B \cup (I \setminus \{x\}))$.

Remark. Any two points $\bar{a}$ and $\bar{b}$ are independent over B if $\dim(\bar{a}/B \cup \bar{b}) = \dim(\bar{a}/B)$ (equivalently, $\dim(\bar{b}/B \cup \bar{a}) = \dim(\bar{b}/B)$).

Lemma 3.14. *The dimension of $\bar{a}$ over B is equal to the cardinality of a maximal independent subtuple of $\bar{a}$ over B.*

Proposition 3.15. *Suppose $\mathfrak{M}$ is κ-saturated. Let $B \subseteq M$ with $|B| < \kappa$ and $X \subseteq M^n$ be B-definable. Then*

$$\dim(X) = \max\{\dim(\bar{a}/B) : \bar{a} \in X\}.$$

Definition 3.16. Let $B \subseteq M$ and $X \subseteq M^n$ be B-definable. We say a point $\overline{a}$ is a **generic point** (or simply a **generic**) of X over B if $\overline{a} \in X$ and $\dim(\overline{a}/B) = \dim(X)$.

Example. Consider the real field $\mathbb{R}$ in the language $\{<\}$. Let $X = [3,4] \subseteq \mathbb{R}$. We can see that X is $\{3,4\}$-definable and $\dim X = 1$. Since $\pi \in X$ cannot be not defined by $\mathcal{L}_{or}$-formulas with parameters from $\{3,4\}$, $\dim(\pi/\{3,4\}) = 1 = \dim X$. So π is a generic point of X over $\{3,4\}$.

Lemma 3.17 (Exchange Lemma)**.** *Let $B \subseteq M$ and $a, b \in M$. If $b \in \operatorname{dcl}(\{a\} \cup B)$ and $b \notin \operatorname{dcl}(B)$, then $a \in \operatorname{dcl}(\{b\} \cup B)$.*

Definition 3.18. Let $X \subseteq M^n$. We say $\overline{a}, \overline{b} \in X$ are **mutually generic** (over $\varnothing$) if they are generics of X and independent (over $\varnothing$).

Definition 3.19. Let $Y \subseteq X \subseteq M^n$ be definable. We say that Y is **large** in X if $\dim(X \smallsetminus Y) < \dim(X)$.

Theorem 3.20. *Let $Y \subseteq X$ be definable. Then Y is large in X if and only if for every $B \subseteq M$ over which X and Y are defined, every generic point of X over B is in Y.*

Proposition 3.21. *Let $X \subseteq M^n$ be B-definable. Let $\varphi(\overline{x}, \overline{y})$ be a formula without parameters, where $\overline{x} = (x_1, \ldots, x_n)$. Then the set $\{\overline{b} : \varphi(\overline{x}, \overline{b})^{\mathfrak{M}} \cap X$ is large in $X\}$ is B-definable, where $\varphi(\overline{x}, \overline{b})^{\mathfrak{M}} := \{\overline{a} \in M^n : \mathfrak{M} \models \varphi(\overline{a}, \overline{b})\}$.*

Remark. By Proposition 3.21, the set

$$\{\bar{a} \in M^n : \text{ If } b \text{ is a generic of } A \text{ over } \bar{a}, \text{ then } \mathfrak{M} \models \varphi(\bar{a}, b)\}$$

is a definable set, where $\varphi(\bar{x}, y)$ is a formula in our language.

Lemma 3.22. *Let $b \in A$ and let a be a generic of A over b. Then $b \oplus a$ is a generic of A over b.*

Lemma 3.23. *For $g \in A$, there are generics g_1 and g_2 of A over g such that $g = g_1 \oplus g_2$.*

Definition 3.24. Let X be a subset of A and $a \in A$. The set

$$a \oplus X := \{a \oplus b : b \in X\}$$

is a **translation** of X by a.

Lemma 3.25. *Let X be a large definable subset of A. Then finitely many translates of X cover A, i.e. there are $a_1, \ldots, a_k \in A$ such that $A = \bigcup\{a_i \oplus X : i = 1, \ldots, k\}$.*

Theorem 3.26. *Let B be a definable subgroup of A. Then B has a finite index in A if and only if $\dim B = \dim A$.*

Theorem 3.27 (Descending Chain Condition (DCC))**.** *Let $\{A_n\}_{n\in\mathbb{N}}$ be a descending chain of definable subgroups of A. Then there is $k \in \mathbb{N}$ such that $A_n = A_k$ for all $n \geq k$.*

3.3 Topologies on S and A

We devote this section to prove Theorem 3.3.

Throughout, assume A is a definable S-module. We will assume that $\mathfrak{M}$ is $\aleph_0$-saturated. However, our main results in this section hold for arbitrary $\mathfrak{M}$. Without loss of generality, we suppose the ring S and the S-module A are 0-definable in the structure $\mathfrak{M}$ and the scalar multiplication ρ from $S \times A \to A$ is 0-definable. By Theorem 2.27, S is a division ring. Assume $S \subseteq M^m$ with $\dim S = m' \leqslant m$ and $A \subseteq M^n$ with $\dim A = n' \leqslant n$.

Throughout the rest of this chapter, by Proposition 1.41, we may assume that there exist $E_1, \dots, E_p \subseteq A$ such that

$$(\dagger) \quad \begin{cases} 1.\ E_1, \dots, E_p \text{ are the 0-definable pairwise disjoint cells of dimension } n'; \\ 2.\ \text{for each } i \neq j, \operatorname{cl} E_i \cap \operatorname{cl} E_j = \varnothing; \text{ and} \\ 3.\ V_0 := E_1 \cup \dots \cup E_p \text{ is open and large in } A. \end{cases}$$

Remark. For each $i = 1, \dots, p$, if $U \subseteq M^n$ is open in A, then $E_i \cap U$ is open in U.

Similarly, for a ring S, we may assume that there exist $T_1, \dots, T_q \subseteq S$ such that

$$(\dagger\dagger) \quad \begin{cases} 1.\ T_1, \dots, T_q \text{ are the 0-definable pairwise disjoint cells of dimension } m'; \\ 2.\ \text{for each } i \neq j, \operatorname{cl} T_i \cap \operatorname{cl} T_j = \varnothing; \text{ and} \\ 3.\ X_0 := T_1 \cup \dots \cup T_q \text{ is open and large in } S. \end{cases}$$

Definition 3.28. We say that a 5-tuple (V, W, X, Y, P) of 0-definable sets has the **property** $(\star)$ if

1. V is open and large in A, and $\ominus : V \to V$ is a 0-definable continuous bijection;
2. W is large and open in $A \times A$, and $\oplus : W \to V$ is 0-definable and continuous;
3. X is open and large in S, and $- : X \to X$ and $^{-1} : X \to X$ are 0-definable continuous bijections;
4. Y is large and open in $S \times S$, and $+ : Y \to X$ and $\cdot : Y \to X$ are 0-definable and continuous;
5. P is large and open in $S \times A$, and $\rho : P \to V$ is 0-definable and continuous;
6. for any $v_1 \in V$, if v_2 is a generic of A over v_1, then $(v_2, v_1) \in W$ and $(\ominus v_2, v_1 \oplus v_2) \in W$;
7. for any $x_1 \in X$, if x_2 is a generic of S over x_1, then $(x_2, x_1) \in Y$, $(-x_2, x_1 + x_2) \in Y$ and $(x_2^{-1}, x_2 \cdot x_1) \in Y$; and

8. for any $x \in X$, if v is a generic of A over x, then $(x, v) \in P$ and $(x^{-1}, \rho(x, v)) \in P$.

First, we shall construct a 5-tuple (V, W, X, Y, P) of 0-definable sets that satisfies the property $(\star)$.

Lemma 3.29.

1. *There exists a* 0*-definable large open subset* V_1 *of* A *such that the restriction of the additive inversion to* V_1 *is a* 0*-definable continuous bijection from* V_1 *onto* V_1*.*

2. *There exists a* 0*-definable large open subset* W_1 *of* $A \times A$ *such that the restriction of the addition to* W_1 *is a* 0*-definable continuous map from* W_1 *into* V_0*.*

Proof. *1.* By the Cell Decomposition Theorem, we obtain a 0-definable large open dense subset $\tilde{V_0}$ of V_0 such that $\ominus \restriction \tilde{V_0}$ is continuous. Set $V_1 = \tilde{V_0} \cap (\ominus \tilde{V_0})$. It is clear that V_1 is open in V_0. To show that V_1 is large in A, let v be a generic of A over $\varnothing$. Since $\tilde{V_0}$ is large in A, by Theorem 3.20, $v \in \tilde{V_0}$. Note that $\ominus v$ is also a generic of A over $\varnothing$. So $\ominus v = w$ for some $w \in \tilde{V_0}$, i.e. $v \in \ominus \tilde{V_0}$. Therefore, $v \in V_1$ and so we have that V_1 is large in A. Since $V_1 = \ominus V_1$, the additive inversion from V_1 into V_1 is a 0-definable continuous bijection.

2. By the Cell Decomposition Theorem, we obtain a 0-definable large open dense subset W_1 of $V_0 \times V_0$ such that $\oplus \restriction W_1$ is continuous. □

Lemma 3.30.

1. *There exists a* 0*-definable large open subset* X_1 *of* S *such that the restrictions of both additive inversion and multiplicative inversion to* X_1 *are* 0*-definable continuous bijections from* X_1 *onto* X_1*.*

2. *There exists a* 0*-definable large open subset* Y_1 *of* $S \times S$ *such that the restriction of both addition and multiplication to* Y_1 *are* 0*-definable continuous maps from* Y_1 *into* V_0*.*

Proof. 1. By the Cell Decomposition Theorem, we obtain 0-definable large open dense subsets Z_1 and Z_2 of X_0 such that $- \restriction Z_1$ and ${}^{-1} \restriction Z_2$ are continuous. Set $\tilde{X_0} = Z_1 \cap Z_2$ and $X_1 = \tilde{X_0} \cap (-\tilde{X_0}) \cap \tilde{X_0}^{-1} \cap (-\tilde{X_0})^{-1}$. It is clear that X_1 is open in X_0. To show that X_1 is large in S, let x be a generic of S over $\varnothing$. Since Z_1 and Z_2 are large in S, by Theorem 3.20, $x \in \tilde{X_0}$. Note that $-x$, x^{-1} and $(-x)^{-1}$ are also generics of S over $\varnothing$. By the same argument as in Lemma 3.29, we obtain that $x \in X_1$. Therefore, X_1 is large in S. Since $X_1 = -X_1 = X_1^{-1}$, the additive inversion and the multiplicative inversion from X_1 into X_1 are 0-definable continuous bijections.

2. By the Cell Decomposition Theorem, we obtain a 0-definable large open dense subset Y_1 of $X_0 \times X_0$ such that $+ \restriction Y_1$ and $\cdot \restriction Y_1$ are continuous. $\square$

Lemma 3.31. *There exists a* 0*-definable large open subset* P_1 *of* $X_0 \times V_0$ *such that the restriction of the scalar multiplication to* P_1 *is a* 0*-definable continuous map from* P_1 *into* V_0.

Proof. By the Cell Decomposition Theorem, we obtain a 0-definable large open dense subset P_1 of $X_0 \times V_0$ such that $\rho \restriction P_1$ is continuous. $\square$

Throughout, we fix sets V_1, W_1 as in Lemma 3.29, X_1, Y_1 as in Lemma 3.30 and P_1 as in Lemma 3.31.

Lemma 3.32. *Let* $\bar{a}, \bar{b} \in A$. *If* $\bar{a}$ *is a generic of* A *over* $\bar{b}$ *and* $\bar{b}$ *is a generic of* A *over* $\varnothing$, *then* $\bar{b}$ *is also a generic of* A *over* $\bar{a}$.

Proof. Assume $\bar{a}$ is a generic of A over $\bar{b}$ and $\bar{b}$ is a generic of A over $\varnothing$. Suppose to the contrary that $\bar{b}$ is not a generic of A over $\bar{a}$. Without loss of generality, we may assume that $\dim A = n$ and $\bar{a} = (a_0, \bar{a'})$, $\bar{b} = (b_0, \bar{b'})$ where $b_0 \in \operatorname{dcl}(\bar{a} \cup \bar{b'})$ but $b_0 \notin \operatorname{dcl}(\bar{a'} \cup \bar{b'})$. By the Exchange Lemma, $a_0 \in \operatorname{dcl}(\bar{a'} \cup \bar{b})$, which is absurd. $\square$

We now construct a 5-tuple (V, W, X, Y, P) of 0-definable sets that satisfies the property $(\star)$.

Lemma 3.33. *There exists a* 0*-definable large open subset* V *of* A *such that*

1. $V \subseteq V_1$;

2. *the restriction of* $\ominus$ *to* V *is a* 0*-definable continuous bijection from* V *onto* V*; and*

3. *for* $b \in V$*, if* a *is a generic of* A *over* b*, then* $(a, b) \in W_1$ *and* $(\ominus a, a \oplus b) \in W_1$.

Proof. Let V_2 be the subset of A such that $b \in V_2$ if and only if :

(i) $b \in V_1$; and

(ii) for every generic a of A over b, $(a, b) \in W_1$ and $(\ominus a, a \oplus b) \in W_1$.

Since W_1 is 0-definable, by the remark after Proposition 3.21, V_2 is 0-definable. To show that V_2 is large in A, let b be a generic of A over $\varnothing$. Since V_1 is 0-definable and large in A, $b \in V_1$. Let a be a generic of A over b. By Lemma 3.32, b is a generic of A over a, so a and b are mutually generics over $\varnothing$, i.e. (a, b) is a generic point of $A \times A$ over $\varnothing$. Since W_1 is large in $A \times A$, $(a, b) \in W_1$. By Lemma 3.22, $a \oplus b$ is a generic of A over a. By Lemma 3.32 again, a is a generic of A over $a \oplus b$ and hence $(\ominus a, a \oplus b) \in W_1$. Therefore, $b \in V_2$. Hence V_2 is large in A (by Theorem 3.20). By the Cell Decomposition Theorem, we obtain a 0-definable subset $V_3 \subseteq V_2$ such that V_3 is large in A and open in V_0. Set $V = V_3 \cap (\ominus V_3)$. It is clear that V is also large in A and open in V_0. Since $V = (\ominus V)$, the restriction $\ominus : V \to V$ is a 0-definable continuous bijection. $\square$

Remark. $V \times V$ is large in $A \times A$ and open in $V_1 \times V_1$.

Lemma 3.34. *There exists a* 0*-definable large open subset* X *of* S *such that*

1. $X \subseteq X_1$;

2. *the restrictions of* $-$ *and* $^{-1}$ *to* X *are* 0*-definable continuous bijections from* X *onto* X*;*

3. *for* $t \in X$*, if* s *is a generic of* S *over* t*, then* $(s,t) \in Y_1, (-s, s+t) \in Y_1$*, and* $(s^{-1}, st) \in Y_1$*; and*

4. *for* $t \in X$*, if* a *is a generic of* A *over* t*, then* $(t,a) \in P_1$ *and* $(t^{-1}, \rho(t,a)) \in P_1$*.*

Proof. Let X_2 be the subset of S such that $t \in X_2$ if and only if :

(i) $t \in X_1$;

(ii) for every generic s of S over t, $(s,t) \in Y_1, (-s, s+t) \in Y_1$, and $(s^{-1}, st) \in Y_1$; and

(iii) for every generic a of A over t, $(t,a) \in P_1$ and $(t^{-1}, \rho(t,a)) \in P_1$.

By the same argument as in Lemma 3.33, X_2 is large in S. By the Cell Decomposition Theorem, there is a 0-definable subset $X_3 \subseteq X_2$ such that X_3 is large in S and open in X_0. Set $X = X_2 \cap (-X_2) \cap X_2^{-1} \cap (-X_2)^{-1}$. Since $X = -X = X^{-1}$, the additive inversion and the multiplicative inversion from $X \to X$ are 0-definable continuous bijections. □

Remark. $X \times X$ is large in $S \times S$ and open in $X_1 \times X_1$.

Lemma 3.35. *There exists a* 0*-definable large open subset* W *of* $A \times A$ *such that*

1. $W \subseteq (V \times V) \cap W_1$*;*

2. *the restriction of* $\oplus$ *to* W *is a* 0*-definable continuous map from* W *into* V*; and*

3. *for* $b \in V$*, if* a *is a generic of* A *over* b*, then* $(a,b) \in W$ *and* $(\ominus a, a \oplus b) \in W$*.*

Proof. Define $W = (V \times V) \cap \{(a,b) \in W_1 : a \oplus b \in V\}$. By Lemma 3.29, since V is open in V_0, the restriction of $\oplus$ to W is a 0-definable continuous map from W into V and W is open in $V_0 \times V_0$. To verify that W is large in $A \times A$, let (a_1, a_2) be a generic of $A \times A$ over $\varnothing$. Then a_1 and a_2 are mutually generic (over $\varnothing$), so $a_1 \oplus a_2$ is a generic of A over $\varnothing$. We have (a_1, a_2) is in $(V \times V) \cap W_1$ and $a_1 \oplus a_2 \in V$, i.e. $(a_1, a_2) \in W$. Therefore, W is large in $A \times A$.

To complete this proof, let $b \in V$ and let a be a generic of A over b. We will show that $(a, b) \in W$ and $(\ominus a, a \oplus b) \in W$. Since $b \in V$, by Lemma 3.33, $(a, b) \in W_1$ and $(\ominus a, a \oplus b) \in W_1$. By Lemma 3.22, $a \oplus b \in V$, i.e. $(a, b) \in W$. Since $(\ominus a) \oplus (a \oplus b) = b \in V$, $(\ominus a, a \oplus b) \in W$. $\square$

Lemma 3.36. *There exists a* 0*-definable large open subset* Y *of* $S \times S$ *such that*

1. $Y \subseteq (X \times X) \cap Y_1$;
2. *the restrictions of* $+$ *and* $\cdot$ *to* Y *are* 0*-definable continuous maps from* Y *into* X*; and*
3. *for* $t \in X$*, if* s *is a generic of* S *over* t*, then* $(s, t) \in Y, (-s, s + t) \in Y$*, and* $(s^{-1}, st) \in Y$.

Proof. Define $Y = (X \times X) \cap \{(s, t) \in Y_1 : s + t \in X \text{ and } st \in X\}$. By Lemma 3.30, since X is open in X_1, we have the restrictions of both $+$ and $\cdot$ to Y are 0-definable continuous maps from Y into X and Y is open in $V_0 \times V_0$. To verify that Y is large in $A \times A$, let (s_1, s_2) be a generic of $S \times S$ over $\varnothing$. Then s_1 and s_2 are mutually generic (over $\varnothing$). So $s_1 \oplus s_2$ and $s_1 s_2$ are generics of S over $\varnothing$. We have (s_1, s_2) is in $(X \times X) \cap Y_1$, $s_1 \oplus s_2 \in X$ and $s_1 s_2 \in X$, i.e. $(s_1, s_2) \in Y$. Therefore, Y is large in $S \times S$.

To complete this proof, let $t \in X$ and let s be a generic of S over t. We will show that $(s, t) \in Y, (-s, s + t) \in Y$, and $(s^{-1}, st) \in Y$. Since $t \in X$, by Lemma 3.34, $(s, t) \in Y_1$, $(-s, s + t) \in Y_1$, and $(s^{-1}, st) \in Y_1$. Since $t \neq 0_S$, by Lemma 3.22, $s + t \in X$ and $st \in X$, that is $(s, t) \in Y$. Since $(-s) + (s + t) = t \in X$, $(-s)(s + t) \in X$, and s is a generic of S over t, $(-s, s + t) \in Y$. Similarly, $(s^{-1}, st) \in Y$. $\square$

The following lemma was proved in [16].

Lemma 3.37 (A. Pillay, [16])**.** *Let* $f : A \to A$ *be a* B*-definable endomorphism of* A *with finite kernel. Then* $\operatorname{Im} f$ *has finite index in* A*. In particular, if* a *is a generic of* A *over* B*, then* $f(a)$ *is also a generic of* A *over* B*.*

Lemma 3.38. *Let $s \in S$ be nonzero and let a be a generic of A over s. Then $\rho(s,a)$ is a generic of A over s.*

Proof. Let $f : A \to A$ be a function defined by $f(x) = \rho(s,x)$. Then f is a $\{s\}$-definable endomorphism (since ρ is 0-definable). Since A is a free module and $s \neq 0$, f is injective. Therefore, $\ker(f) = 0$. Hence, f has a finite kernel. Since a is a generic of A over s, by Lemma 3.37, $\rho(s,a)$ is a generic of A over s. □

Lemma 3.39. *There exists a 0-definable large open subset P of $S \times A$ such that*

1. *$P \subseteq (X \times V) \cap P_1$;*
2. *the restriction of ρ to P is a 0-definable continuous map from P into V; and*
3. *for $t \in X$, if a is a generic of A over t, then $(t,a) \in P$ and $(t^{-1}, \rho(t,a)) \in P$.*

Proof. Define $P = (X \times V) \cap \{(s,a) \in P_1 : \rho(s,a) \in V\}$. By Lemma 3.31, since $X \times V$ is open in $X_0 \times V_0$, the restriction of ρ to P is a 0-definable continuous map from P into V and P is open in $X_0 \times V_0$. To verify that P is large in $S \times A$, let (s,a) be a generic of $S \times A$ over $\varnothing$. Then s and a are mutually generic (over $\varnothing$). By Lemma 3.38, $\rho(s,a)$ is a generic of A over s. So we have $(s,a) \in (X \times V) \cap P_1$ and $\rho(s,a) \in V$, i.e. $(s,a) \in P$. Hence, P is large in $S \times A$.

To complete this proof, let $t \in X$ and a be a generic of A over t. We will show that $(t,a) \in P$ and $(t^{-1}, \rho(t,a)) \in P$. Since $t \in X$, by Lemma 3.34, $(t,a) \in P_1$ and $(t^{-1}, \rho(t,a)) \in P_1$. By Lemma 3.38, $\rho(t,a) \in V$, i.e. $(t,a) \in P$. Since $\rho(t^{-1}, \rho(t,a) = a \in V)$, $(t^{-1}, \rho(t,a)) \in P$. □

By Lemmas 3.33, 3.34, 3.35, 3.36 and 3.39, we now obtain a 5-tuple (V, W, X, Y, P) satisfying the property $(\star)$ as desired.

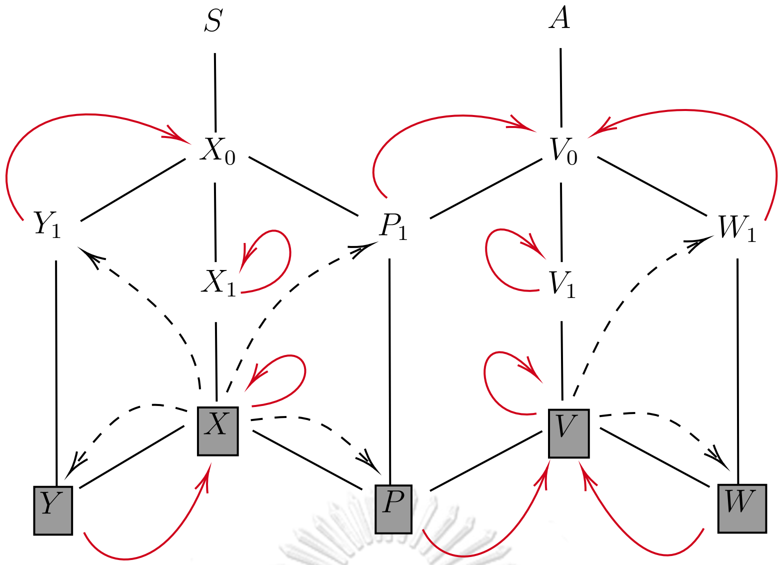


Figure 3 : a 5-tuple (V, W, X, Y, P) satisfying the property $(\star)$

$\longrightarrow$ the corresponding maps

$\dashrightarrow$ the properties of a set concerning generic points

To define the topology τ_A on A and the topology τ_S on S, we first recall lemmas from [16]. For convenience of readers, we include the proofs of these lemmas.

Lemma 3.40. *Let $a \in A$. Then the set $T_a := \{b \in V : a \oplus b \in V\}$ is open in V. Moreover, the map $b \mapsto a \oplus b$ is a homeomorphism from $T_a \to a \oplus T_a$.*

Proof. To show that T_a is open in V, it is enough to show that for every $b \in T_a$, there is an open neighborhood of b in T_a. Let $b \in T_a$. Then $b \in V$ and $a \oplus b \in V$. Let c be a generic of A over $\{a, b\}$. Set

$$U_0 = \{v \in V : (c \oplus a, v) \in W\}, \text{ and}$$

$$U_1 = \{v \in U_0 : (\ominus c, c \oplus a \oplus v) \in W\}.$$

Observe that the constructions of U_0 and U_1 depend on b.

We will show that U_1 is a subset of T_a containing b. Since $\oplus[W] \subseteq V$, for every $v \in U_1$, $a \oplus v = (\ominus c) \oplus (c \oplus a \oplus v) \in V$. Then $U_1 \subseteq T_a$. Since $b \in V$ and c is a generic of A over $\{a, b\}$, $c \oplus a$ is a generic of A over b and $(c \oplus a, b) \in W$. So

$b \in U_0$. Note that $a \oplus b = (\ominus c) \oplus (c \oplus a \oplus b)$ and c is a generic of A over $\{a, b\}$. Then $(\ominus c, c \oplus a \oplus b) \in W$. It follows that $b \in U_1$.

Next, we shall show that U_0 and U_1 are open in V. Observe that $U_0 = V \cap W_{c \oplus a}$. Since W is open in $A \times A$, we have that U_0 is open in V. Define $g_1 : V \to A \times A$ by $g_1(v) = (\ominus c, c \oplus a \oplus v)$. Since $\oplus$ is continuous on W, g_1 is continuous on U_0. Observe that $U_1 = U_0 \cap g_1^{-1}[W]$. Since $U_1 \subseteq U_0$, W is open in $A \times A$ and g_1 is continuous on U_0, we obtain that U_1 is open in V. Therefore, T_a is open in V.

Consider the map $g : v \mapsto a \oplus v$. Since $b \in T_a$, g_1 is continuous on $U_1 \subseteq U_0$, $\oplus$ is continuous on W and $g = \oplus \circ g_1$ on U_1, we obtain that that g is a continuous map on a neighborhood of b. Therefore, we obtain that the map $g : v \mapsto a \oplus v$ is a continuous map from $T_a \to a \oplus T_a$. Since $a \oplus T_a = T_{\ominus a}$, by the same argument, the map g is a homeomorphism. $\square$

Lemma 3.41. *Let $a \in A$. Then the set $\Sigma_a = \{(b_1, b_2) \in V \times V : a \oplus b_1 \oplus b_2 \in V\}$ is open in $V \times V$.*

Proof. Fix $(b_1, b_2) \in \Sigma_a$. Then $(b_1, b_2) \in V \times V$ and $a \oplus b_1 \oplus b_2 \in V$. To show that Σ_a is open in $V \times V$, it is enough to find an open neighborhood of (b_1, b_2) contained in Σ_a. Let c be a generic of A over $\{a, b_1, b_2\}$. Let

$$\begin{aligned} U_0 =& \{(x, y) \in V \times V : (c \oplus a, x) \in W\}, \\ U_1 =& \{(x, y) \in U_1 : (c \oplus a \oplus x, y) \in W\}, \text{ and} \\ U_2 =& \{(x, y) \in U_2 : (\ominus c, c \oplus a \oplus x \oplus y) \in W\}. \end{aligned}$$

First, we will show that U_2 is a subset of Σ_a containing (b_1, b_2). Since $\oplus[W] \subseteq V$, for every $(x, y) \in U_2$, $a \oplus x \oplus y = (\ominus c) \oplus (c \oplus a \oplus x \oplus y) \in V$. Then $U_2 \subseteq \Sigma_a$. Since $b_1 \in V$ and c is a generic of A over $\{a, b_1\}$, $c \oplus a$ is a generic of A over b_1 and $(c \oplus a, b_1) \in W$. So $(b_1, b_2) \in U_0$. Similarly, we have $(c \oplus a \oplus b_1, b_2) \in U_1$. Note that $a \oplus b_1 \oplus b_2 = (\ominus c) \oplus (c \oplus a \oplus b_1 \oplus b_2)$ and c is a generic of A over $\{a, b_1, b_2\}$. Then $(\ominus c, c \oplus a \oplus b_1 \oplus b_2) \in W$, it follows that $(b_1, b_2) \in U_2$.

Next, we shall show that U_0, U_1 and U_2 are open in $V \times V$. Observe that $U_0 = W_{c \oplus a} \times V$. Since W is open in $A \times A$, we have that U_0 is open in $V \times V$.

Define $g_1, g_2 : V \to A \times A$ by

$$g_1(x,y) = (c \oplus a \oplus x, y), \text{ and}$$
$$g_2(x,y) = (\ominus c, c \oplus a \oplus x \oplus y).$$

Since $\oplus$ is continuous on W, g_1 is continuous on U_0 and so g_2 is continuous on U_1. Observe that $U_1 = U_0 \cap g_1^{-1}[W]$ and $U_2 = U_1 \cap g_2^{-1}[W]$. Since each g_i is continuous on U_{i-1}, we obtain that each U_i is open in $V \times V$. In particular, we have that U_2 is an open subset of Σ_a containing (b_1, b_2). Therefore, Σ_a is open in $V \times V$. It follows that the set Σ_a is open in $V \times V$. □

Define the topology τ_A on A by

$U \subseteq A$ is τ_A-open if and only if
for any $a \in A$, $(a \oplus U) \cap V$ is open in V.

Lemma 3.42. *Let $U \subseteq V$ and $a \in A$. Then $a \oplus U$ is τ_A-open if and only if U is open in V.*

Proof. Assume $a \oplus U$ is τ_A-open. Then for any $b \in A$, $(b \oplus (a \oplus U)) \cap V$ is open in V. Therefore, $U = U \cap V = ((\ominus a) \oplus (a \oplus U)) \cap V$ is open in V. Conversely, assume U is open in V. To show that $a \oplus U$ is τ_A-open, let $b \in A$. Define $f_{\ominus a \ominus b} : V \to V$ by $f_{\ominus a \ominus b}(v) = \ominus a \ominus b \oplus v$. By Lemma 3.40, $f_{\ominus a \ominus b}$ is continuous on $T_{\ominus a \ominus b}$. Note that

$$(b \oplus (a \oplus U)) \cap V = ((b \oplus a) \oplus U) \cap V = f^{-1}_{\ominus a \ominus b}[U] \cap T_{\ominus a \ominus b}.$$

Since U and $T_{\ominus a \ominus b}$ are open in V and $f_{\ominus a \ominus b}$ is continuous on $T_{\ominus a \ominus b}$, we obtain that $(b \oplus (a \oplus U)) \cap V$ is open in V. Therefore, $a \oplus U$ is τ_A-open □

Proposition 3.43. *The additive inversion $\ominus$ is τ_A-continuous on A.*

Proof. Let U be τ_A-open in A. We shall show that the pre-image $\ominus^{-1}[U]$ is τ_A-open. Observe that $\ominus^{-1}[U] = \ominus U$. By Lemma 3.25, we may assume that $U \subseteq c \oplus V$ for some $c \in A$. By Lemma 3.42, $(\ominus c) \oplus U$ is open in V. Since $\ominus$ is continuous on V, $\ominus^{-1}[(\ominus c) \oplus U] = c \oplus (\ominus U) = c \ominus U \subseteq V$ is open in V. To verify that $\ominus U$ is τ_A-open, let $a \in A$. Define $f_{c\ominus a} : V \to V$ by $f_{c\ominus a}(v) = c \ominus a \oplus v$. By Lemma 3.40, $f_{c\ominus a}$ is continuous on $T_{c\ominus a}$. Note that

$$(a \oplus (\ominus U)) \cap V = ((a \ominus c) \oplus (c \ominus U)) \cap V = f_{c\ominus a}^{-1}[c \ominus U] \cap T_{c\ominus a}.$$

Since $(\ominus c) \oplus U$ and $T_{c\ominus a}$ are open in V and $f_{c\ominus a}$ is continuous on $T_{c\ominus b}$, we obtain that $(a \oplus (\ominus U)) \cap V$ is open in V. Therefore, $\ominus U$ is τ_A-open. □

Proposition 3.44. *The addition $\oplus$ is τ_A-continuous on A.*

Proof. Let U be τ_A-open in A. By Lemma 3.25, we may assume that $U \subseteq c \oplus V$ for some $c \in A$. By Lemma 3.42, $(\ominus c) \oplus U$ is open in V. We shall show that $\oplus^{-1}[U] = \{(a, b) \in A \times A : a \oplus b \in U\}$ is τ_A -open in $A \times A$. By Lemma 3.25 again, it suffices to prove that for any $p, q \in A$, $\Sigma^\dagger := \{(a, b) \in (p \oplus V) \times (q \oplus V) : a \oplus b \in U\}$ is τ_A-open in $A \times A$. Notice that

$$\begin{aligned} \Sigma^\dagger &= \{(a \oplus p, b \oplus q) : (a, b) \in V \times V, (a \oplus p) \oplus (b \oplus q) \in U\} \\ &= \{(a \oplus p, b \oplus q) : (a, b) \in V \times V, (\ominus c) \oplus a \oplus p \oplus b \oplus q \in (\ominus c) \oplus U\}. \end{aligned}$$

Since $(\ominus c) \oplus U \subseteq V$, by Lemmas 3.41 and 3.42, we obtain that $\Sigma^\dagger$ is τ_A-open in $A \times A$. Therefore, $\oplus^{-1}[U]$ is τ_A-open. □

By the same argument as in the proof of Lemmas 3.40 and 3.41, we obtain the following lemmas:

Lemma 3.45. *Let $s \in S$. Then the set $\{t \in X : a + t \in X\}$ is open in S.*

Lemma 3.46. *Let $s \in S$. Then the set $\{(t_1, t_2) \in X \times X : s + t_1 + t_2 \in X\}$ is open in $X \times X$.*

Lemma 3.47. *Let $s \in S$. Then the set $\{t \in X : st \in X\}$ is open in S.*

Define the topology τ_S on S by

$$U \subseteq S \text{ is } \tau_S\text{-open if and only if}$$
$$\text{for any } s \in S,\ (s+U) \cap X \text{ is open in } X$$

Immediately, from the proofs of Lemma 3.42, and Propositions 3.43 and 3.44, we obtain the following lemma and propositions:

Lemma 3.48. *Let $U \subseteq X$ and $s \in S$. Then $s + U$ is τ_S-open if and only if U is open in X.*

Proposition 3.49. *The additive inversion $-$ is τ_S-continuous on S.*

Proposition 3.50. *The addition $+$ is τ_S-continuous on S.*

Next, we consider the multiplication on S.

Lemma 3.51. *Let $s, t, u \in S$. Then the set $K = \{(x_1, x_2) \in X \times X : u + x_1 s + tx_2 + x_1 x_2 \in X\}$ is open in $X \times X$.*

Proof. Fix $(x_1, x_2) \in K$. Then $(x_1, x_2) \in X \times X$ and $u + x_1 s + tx_2 + x_1 x_2 \in X$. Since X is large in S, by Lemma 3.23, $s = s_1 + s_2$ where $s_1, s_2 \in X$. Without loss of generality, assume that $t \neq 0_S$. To show that K is open neighborhood of (x_1, x_2) contained in K. Let z be a generic of S over $\{s_1, s_2, t, u, x_1, x_2\}$. Let

$$\begin{aligned}
U_0 =& \{(x, y) \in X \times X : (z, x) \in Y, (zt, y) \in Y\},\\
U_1 =& \{(x, y) \in U_0 : (zx, s_1) \in Y, (zx, s_2) \in Y, (zx, y) \in Y\},\\
U_2 =& \{(x, y) \in U_1 : zu + zxs_1 + zxs_2 + zty + zxy \in X\}, \text{ and}\\
U_3 =& \{(x, y) \in U_2 : (z^{-1}, zu + zxs_1 + zxs_2 + zty + zxy) \in Y\}.
\end{aligned}$$

We will show that U_3 is a subset of K containing (x_1, x_2). Since the image of Y under $\cdot$ is a subset of X, we get that for every $(x, y) \in U_2$,

$$\begin{aligned}
u + xs + ty + xy &= u + xs_1 + xs_2 + ty + xy\\
&= (z^{-1}) \cdot (zu + zxs_1 + zxs_2 + zty + zxy) \in X.
\end{aligned}$$

Then $U_3 \subseteq K$. Since z is a generic of S over $\{t, x_1, x_2\}$, we get zt is a generic element of S and it follows that $(z, x_1) \in Y$ and $(zt, x_2) \in Y$. So $(x_1, x_2) \in U_0$. Since $s_1, s_2, x_2 \in X$ and z is a generic of S over $\{s_1, s_2, x_2\}$, we obtain that zx_1 is a generic of S and $(zx_1, s_1) \in Y, (zx_1, s_2) \in Y$ and $(zx_1, x_2) \in Y$, i.e. $(x_1, x_2) \in U_1$. Since $u + x_1 s + tx_2 + x_1 x_2 \in X$, $u + x_1 s + tx_2 + x_1 x_2 \neq 0_S$. It follows that

$$zu + zx_1 s_1 + zx_1 s_2 + ztx_2 + zx_1 x_2 = z(u + x_1 s + tx_2 + x_1 x_2) \in X,$$

i.e. $(x_1, x_2) \in U_2$. Similarly, we have $(z^{-1}, zu + zx_1 s_1 + zx_1 s_2 + ztx_2 + zx_1 x_2) \in Y$. Therefore $(x_1, x_2) \in U_3$.

Next, we shall show that U_0, U_1, U_2 and U_3 are open in $X \times X$. Observe that $U_0 = (X \cap Y_z) \times (X \cap Y_{zt})$. Since $X \times X$ is open in $S \times S$ and Y is open in $X \times X$, it follows that K_0 is open in $X \times X$. Consider $g_{11}, g_{12}, g_{13}, g_3 : X \times X \to S \times S$ and $g_2 : X \times X \to S$ defined by

$$\begin{aligned}
g_{11}(x, y) &= (zx, s_1), \\
g_{12}(x, y) &= (zx, s_2), \\
g_{13}(x, y) &= (zx, y), \\
g_2(x, y) &= zu + zxs_1 + zxs_2 + ztx + zxy, \text{ and} \\
g_3(x, y) &= (z^{-1}, zu + zxs_1 + zxs_2 + ztx + zxy).
\end{aligned}$$

Since $\cdot$ is continuous on Y, we obtain that g_{11}, g_{12} and g_{13} are continuous on U_0. Note that

(i) $+$ is τ_S-continuous; and

(ii) τ_S-topology coincides with the topology by R^m on the large open subset X.

Then we have g_2 is continuous on U_1. Since $K_2 \subseteq K_1$, g_3 is continuous on U_2.

Observe that $U_1 = U_0 \cap g_{11}^{-1}[Y] \cap g_{12}^{-1}[Y] \cap g_{13}^{-1}[Y]$, $U_2 = U_1 \cap g_2^{-1}[X]$ and $U_3 = U_2 \cap g_3^{-1}[Y]$. Since g_{11}, g_{12} and g_{13} are continuous on U_0, g_2 is continuous on U_1 and g_3 is continuous on U_2, we obtain that each U_i is open in $X \times X$. In particular, we obtain that U_3 is an open subset of K containing (x_1, x_2). Therefore the set K is open in $X \times V$. □

As immediate consequences, we have:

Proposition 3.52. *The multiplicative inversion* $^{-1}$ *is* τ_S*-continuous on* $S^\times$.

Proposition 3.53. *The multiplication* $\cdot$ *is* τ_S*-continuous on* S.

It remains to show that the scalar multiplication $\rho : (S \times A, \tau_S \times \tau_A) \to (A, \tau_A)$ is continuous. To begin, we introduce the following lemmas.

Lemma 3.54. *Let* $U \subseteq X \times V$ *and* $(s, a) \in S \times A$. *Then* $\{(s + s', a \oplus a') \in S \times A : (s', a') \in U\}$ *is* $(\tau_S \times \tau_A)$*-open if and only if* U *is open in* $X \times V$.

Proof. Assume $\Gamma := \{(s + s', a \oplus a') \in S \times A : (s', a') \in U\}$ is $(\tau_S \times \tau_A)$-open. Then there exists a collection $\{(S_i, A_i) : S_i \in \tau_S, A_i \in \tau_A, i \in I\}$, for some index set I, such that $\Gamma = \bigcup_{i \in I}(S_i \times A_i)$. Observe that $U = \bigcup_{i \in I}\big(((-s) + S_i) \times ((\ominus a) \oplus A_i)\big)$. Since each S_i is τ_S-open and $(-s) + S_i \subseteq X$, by Lemma 3.48, we have that each $(-s) + S_i$ is open in X. Similarly, by Lemma 3.42, we obtain that each $(\ominus a) \oplus A_i$ is open in V. Therefore, $U = \bigcup_{i \in I}((-s) + S_i) \times ((\ominus a) \oplus A_i)$ is open in $X \times V$. Conversely, suppose U is open in $X \times V$. Since X is open in S and V is open in A, there exists a collection $\{(X_j, V_j) : X_j \subseteq X$ is open in $S, V_j \subseteq V$ is open in $A, j \in J\}$, for some index set J, such that $U = \bigcup_{j \in J}(X_j \times V_j)$. By Lemmas 3.42 and 3.48, $s + X_j$ is τ_S-open and $a \oplus V_j$ is τ_A-open. So Γ is $(\tau_S \times \tau_A)$-open. □

Remark. For $U \subseteq X \times V$, U is $(\tau_S \times \tau_A)$-open if and only if U is open in $X \times V$.

Lemma 3.55. *Let* $a, b \in A$ *and* $s \in S$. *Then the set* $D = \{(x_0, v_0) \in X \times V : b \oplus \rho(x_0 + s, v_0 \oplus a) \in V\}$ *is open in* $X \times V$.

Proof. Fix $(x_0, v_0) \in D$. Then $(x_0, v_0) \in X \times V$ and $b \oplus \rho(x_0 + s, v_0 \oplus a) \in V$. Since S is infinite, there exists $n_0 \in S$ such that $x_0 + s + n_0 \neq 0_S$ and $x_0 + s + n_0 + 1_S \neq 0_S$. To show that D is open in $X \times V$, it is enough to find an open neighborhood of

(x_0, v_0) contained in D. Let t be a generic of S over $\{x_0, s, n_0\}$ and c be a generic of A over $\{a, b, s, t, n_0, x_0, v_0\}$. Let

$$\begin{aligned}
U_0 &= \{(x, v) \in X \times V : (t, x) \in Y, tx + ts + tn_0 \in X, (c \oplus a, v) \in W\},\\
U_1 &= \{(x, v) \in U_0 : (tx + ts + tn_0, c \oplus a \oplus v) \in P\},\\
U_2 &= \{(x, v) \in U_1 : \rho(t, c \oplus b) \oplus \rho(tx + ts + tn_0, c \oplus a \oplus v) \in V\},\\
U_3 &= \{(x, v) \in U_2 : (t^{-1}, \rho(t, c \oplus b) \oplus \rho(tx + ts + tn_0, c \oplus a \oplus v) \in P\}, \text{ and}\\
U_4 &= \{(x, v) \in U_3 : (\ominus\rho(x + s + n_0 + 1_S, c), c \oplus b \oplus \rho(x + s + n_0, c \oplus a \oplus v))\\
&\qquad\qquad \in W\}.
\end{aligned}$$

We will show that U_4 is a subset of D containing (x_0, v_0). Since $\oplus[W] \subseteq V$, for every $(x, v) \in U_4$,

$$b \oplus \rho(x+s, v \oplus a) = (\ominus\rho(x+s+n_0+1_S, c)) \oplus (c \oplus b \oplus \rho(x+s+n_0, c \oplus a \oplus v)) \in V.$$

Then $U_4 \subseteq D$. Since $x_0 \in X$, $x_0+s+n_0 \neq 0_S$ and t is a generic of S over $\{x_0, s, n_0\}$, $(t, x_0) \in Y$ and $tx_0 + ts + tn_0 = t(x_0 + s + n_0) \in X$. Note that $c \oplus a \in V$. Since $v_0 \in V$ and $c \oplus a$ is a generic of A over v_0, $(c \oplus a, v_0) \in W$. So now, we have $(x_0, v_0) \in U_0$. From $tx_0 + ts + tn_0 \in X$ and $c \oplus a \oplus v$ is a generic of A over $\{t, n_0, x_0\}$, $(tx_0 + ts + tn_0, c \oplus a \oplus v_0) \in P$, i.e. $(x_0, v_0) \in U_1$. Since c is a generic of A over $\{a, b, s, t, n_0, x_0, v_0\}$, we have (x_0, v_0) lies in both U_2 and U_3. Note that

$$c \oplus b \oplus \rho(x_0 + s + n_0, c \oplus a \oplus v_0) = \rho(x + s + n_0 + 1_S, c) \oplus [b \oplus \rho(x + s, v \oplus a)]$$

and $\rho(x + s + n_0 + 1_S, c)$ is a generic of A (use the property $x_0 + s + n_0 + 1 \neq 0_S$). It follows that $(x_0, v_0) \in U_4$.

Next, we shall show that U_0, U_1, U_2, U_3 and U_4 are open in $X \times V$. Observe that $U_0 = ((X \cap Y_t \cap \cdot^{-1}[\oplus^{-1}[X]_{ts+tn_0}]) \times (V \cap W^*_{c\oplus a}))$. Note that

(i) $X \times V$ is open in $S \times A$;

(ii) Y is open in $S \times S$;

(iii) W is open in $A \times A$; and

(iv) $+$ and $\cdot$ are τ_S-continuous.

Therefore, U_0 is $(\tau_S \times \tau_A)$-open. By Lemma 3.54, U_0 is open in $X \times V$. Consider $g_1, g_3 : X \times V \to S \times A$, $g_2 : X \times V \to A$ and $g_4 : X \times V \to A \times A$ defined by

$$
\begin{aligned}
g_1(x, v) &= (tx + ts + tn_0, c \oplus a \oplus v), \\
g_2(x, v) &= \rho(t, c \oplus b) \oplus \rho(tx + ts + tn_0, c \oplus a \oplus v), \\
g_3(x, v) &= (t^{-1}, \rho(t, c \oplus b) \oplus \rho(tx + ts + tn_0, c \oplus a \oplus v), \text{ and} \\
g_4(x, v) &= (\ominus\rho(x + s + n_0 + 1_S, c), c \oplus b \oplus \rho(x + s + n_0, c \oplus a \oplus v)).
\end{aligned}
$$

Since $\oplus$ is continuous on W, g_1 is continuous on U_0. Since ρ is continuous on P, $\oplus$ is τ_A-continuous and $U_1 \subseteq U_0$, g_2 is continuous on U_1. From $U_2 \subseteq U_1$, it is clear that g_3 is continuous on U_2. Again, since ρ is continuous on P, we have g_4 is continuous on U_3.

Observe that $U_1 = U_0 \cap g_1^{-1}[P]$, $U_2 = U_1 \cap g_2^{-1}[V]$, $U_3 = U_2 \cap g_3^{-1}[P]$ and $U_4 = U_3 \cap g_4^{-1}[W]$. Since each g_i is continuous on U_{i-1}, we obtain that each U_i is open in $X \times V$. Therefore, we obtain that U_4 is an open subset of D containing (x_0, v_0), it follows that the set D is open in $X \times V$. $\square$

Remark. By Lemma 3.55, the definable map $(x, v) \mapsto b \oplus \rho(x + s, v \oplus a)$ is continuous from $D \to V$.

Proposition 3.56. *The scalar multiplication ρ is continuous.*

Proof. Let $U \subseteq A$ be τ_A-open. We shall show that $D^\dagger := \{(s, a) \in S \times A : \rho(s, a) \in U\}$ is $(\tau_S \times \tau_A)$-open. By Lemma 3.25, we may assume that $U \subseteq c \oplus V$ for some $c \in A$. By Lemma 3.42, $(\ominus c) \oplus U$ is open in V.

To show $D^\dagger$ is $(\tau_S \times \tau_A)$-open, it suffices to show that for any $t \in S$ and $b \in A$, $\{(s, a) \in (t + X) \times (b \oplus V) : \rho(s, a) \in U\}$ is $(\tau_S \times \tau_A)$-open.

Consider a set

$$\begin{aligned}\tilde{D} &= \{(t+s, b\oplus a) \in S\times A : (s,a)\in X\times V, \rho(t+s, b\oplus a)\in U\}\\ &= \{(t+s, b\oplus a)\in S\times A : (s,a)\in X\times V,\\ &\qquad\qquad (\ominus c)\oplus\rho(t+s,b\oplus a)\in(\ominus c)\oplus U\}.\end{aligned}$$

By Lemmas 3.54 and 3.55, $\tilde{D}$ is $(\tau_S\times\tau_A)$-open. It follows that $D^\dagger$ is also $(\tau_S\times\tau_A)$-open. Therefore, the scalar multiplication ρ is continuous from $S\times A\to A$. $\square$

Next, we shall prove that A admits a definable S-module manifold. By Proposition 3.56, the scalar multiplication from $S\times A$ to A is continuous with respect to the corresponding topology τ_S and τ_A on S and A, respectively. So it remains to show that A admits a definable group manifold and S admits a definable ring manifold.

Theorem 3.57. *A admits a definable group manifold.*

Proof. By Propositions 3.44 and 3.43, the maps $\oplus$ and $\ominus$ are τ_A-continuous on A. Since V is large in A, by Lemma 3.25, there exists a finite collection $\{a_0, a_1 \dots, a_k\}\subseteq A$ such that $A=\bigcup\{a_i\oplus V : i=0,1,\dots,k\}$. Assume that $0\in I$ with $a_0=0_A$. So $V=a_0\oplus V$. Note that each $a_i\oplus V$ is τ_A-open.

Recall (†) in page 30, we have $V_0=E_1\cup\dots\cup E_p$ where each E_j is a cell of dimension n'. Since $V\subseteq V_0$, $V=(E_1\cap V)\cup\dots\cup(E_p\cap V)$. By Proposition 1.39, each E_j is homeomorphic to an open cell in $M^{n'}$, say by the homeomorphism (and so τ_A-homeomorphism) $\xi_j : E_j\to\xi_j(E_j)\subseteq M^{n'}$.

For each $b\in A$, let $\ell_b : A\to A$ defined by $\ell_b(x)=x\ominus b$. It is clear that each ℓ_b is a τ_A-homeomorphism. For each $i=1,\dots,k$ and $j=1,\dots,p$, let

$$\begin{aligned}&\xi_j^0=\xi_j\restriction V : E_j\cap V\to\xi_j(E_j\cap V)\subseteq M^{n'};\text{and}\\ &\xi_j^i=\xi_j\circ\ell_{\ominus a_i} : a_i\oplus(E_j\cap V)\to\xi_j(a_i\oplus(E_j\cap V))\subseteq M^{n'}.\end{aligned}$$

Consider the finite collection

$$\mathcal{D} := \{(a_i \oplus (E_j \cap V), \xi_j^i) : i = 0, \ldots, k \text{ and } j = 1, \ldots, p\}.$$

It remains to verify that the above collection $\mathcal{D}$ makes τ_A to be a definable group manifold topology (Definition 2.2).

1. By Proposition 3.44 and Proposition 3.43, the maps $\oplus$ and $\ominus$ are τ_A-continuous on A.

2. Let $i = 0, \ldots, k$ and $j = 1, \ldots, p$. From the remark of (†), $E_j \cap V$ is open in V. So it is a definable τ_A-open subset of A. Since ξ_j is a homeomorphism and $\ell_{\ominus a_i}$ is a τ_A-homeomorphism, the map $\xi_j^i : a_i \oplus (E_j \cap V) \to \xi_j(a_i \oplus (E_j \cap V)) \subseteq M^{n'}$ is a definable τ_A-homeomorphism onto its image.

3. Since $A = \bigcup\{a_i \oplus V : i = 0, 1, \ldots, k\}$ and $V = (E_1 \cap V) \cup \cdots \cup (E_p \cap V)$, we have that $A = \bigcup\{a_i \oplus (E_j \cap V) : i = 0, \ldots, k \text{ and } j = 1, \ldots, p\}$.

4. For $i_1, i_2 = 0, \ldots, k$ and $j_1, j_2 = 1, \ldots, p$, let $U_1 = a_{i_1} \oplus (E_{j_1} \cap V)$ and $U_2 = a_{i_2} \oplus (E_{j_2} \cap V)$. Suppose that $U_1 \cap U_2 \neq \varnothing$. Let $Z = \xi_{j_1}^{i_1}[U_1 \cap U_2]$. Since U_1, U_2 are definable τ_A-open and $\xi_{j_1}^{i_1}$ is a definable homeomorphism onto its image, we have that Z is a definable open subset of $\xi_{j_1}^{i_1}[U_1]$ and $\xi_{j_2}^{i_2} \circ \xi_{j_1}^{i_1} \restriction Z$ is a definable homeomorphism onto $\xi_{j_2}^{i_2}[U_1 \cap U_2]$.

Therefore, the collection $\mathcal{D}$ makes τ_A to be a definable group manifold topology. Hence, A admits a definable group manifold. $\square$

By the same argument as in the proof of Theorem 3.57, we obtain the following theorem.

Theorem 3.58. *S admits a definable ring manifold.*

By Proposition 3.56 and Theorems 3.57 and 3.58, this complete the proof of Theorem 3.3.

CHAPTER IV
CHARACTERIZATIONS OF DEFINABLE MODULES

Throughout this chapter, we fix a definable ring S without zero divisors.

Suppose A is a definable S-module. By Theorem 3.3, A admits a definable S-module manifold, say by the definable group manifold topology τ_A on A and the definable ring manifold topology τ_S on S. Let $\rho : S \times A \to A$ be the scalar multiplication. By Proposition 3.56, ρ is continuous with respect to the corresponding topologies τ_S and τ_A.

The main goal of this chapter is the following:

Theorem 4.1. *There exists a one-dimensional definable subring I of S which is a real closed field such that A is either definably isomorphic to I^k, $I(\sqrt{-1})^k$ or $\mathbb{H}(I)^k$ for some $k \in \mathbb{N}$.*

First, we recall the following theorem concerning the characterizations of definable rings without zero divisors.

The Frobenius Theorem, which characterizes all finite dimensional (in the sense of vector spaces) associative division algebras over the $\mathbb{R}$, was proved in 1878, see [2]. In fact, this theorem also holds in any expansion of a real closed field.

Theorem 4.2 (Frobenius Theorem)**.** *Suppose $\mathfrak{M}$ is an expansion of a real closed field. Then every finite dimensional associative division algebra over M is isomorphic to either M, $M(\sqrt{-1})$, or $\mathbb{H}(M)$ (the ring of quaternions over M).*

The following theorem was proved by Y. Peterzil and C. Steinhorn in [15]. It gives a complete characterization of definable infinite rings without zero divisors in an o-minimal structure.

Theorem 4.3 (Y. Peterzil and C. Steinhorn, [15])**.** *S is a division ring and there is a one-dimensional definable subring I of S which is a real closed field such that S is either definably isomorphic to I, $I(\sqrt{-1})$ or $\mathbb{H}(I)$.*

Corollary 4.4 (M. Otero, Y. Peterzil and A. Pillay, [11])**.** *Suppose $\mathfrak{M}$ is an expansion of a real closed field and S is an infinite definable ring without zero divisors. Then S is either definably isomorphic to M, $M(\sqrt{-1})$ or $\mathbb{H}(M)$.*

The following propositions were proved by A. Pillay in [16].

Proposition 4.5 (A. Pillay, [16])**.** *Let G be a definable subgroup of A. Let E be the definably τ_A-connected component of the identity in G. Then E is the smallest definable subgroup of G of finite index in G.*

Proposition 4.6 (A. Pillay, [16])**.** *S is definably τ_S-connected.*

We denote the definably τ_A-connected component of the identity in A by A^0.

Proposition 4.7. *A^0 is an S-submodule of A.*

Proof. By Proposition 4.5, $(A^0, \oplus)$ is a subgroup of $(A, \oplus)$. Let $C = \rho[S \times A^0]$. Since S is τ_S-connected and A^0 is τ_A-connected, $S \times A^0$ is also $(\tau_S \times \tau_A)$-connected. Since $\rho : (S \times A, \tau_S \times \tau_A) \to (A, \tau_A)$ is continuous, we have C is τ_A-connected. Since $(1_S, 0_A) \in S \times A^0$, $0_A = \rho(1_S \times 0_A) \in C$. We now obtain that A^0 and C are τ_A-connected sets containing 0_A. It follows that $C \subseteq A^0$ and hence A^0 is an S-submodule of A. □

The following theorem was proved in 1994 by A. Strzebonski, see [21].

Theorem 4.8 (A. Strzebonski, [21])**.** *If A is infinite and abelian, then it has an unbounded exponent. Moreover, if $\mathfrak{M}$ is $\aleph_0$-saturated, then A has an element of infinite order.*

Lemma 4.9. *Suppose $\mathfrak{M}$ is $\aleph_0$-saturated. Let G be a definable subgroup of A^0. Assume that for any $a \in A^0$, there exists a natural number $k \geqslant 1$ such that $ka \in G$. Then $G = A^0$.*

Proof. First, we will show that there exists $k_0 \in \mathbb{N} \setminus \{0\}$ such that $k_0 a \in G$ for all $a \in A^0$. Since A^0 and G are definable, there exist $d \in \mathbb{N}$ and $\mathcal{L}$-formulas $\varphi(x,y)$, $\psi(x,y)$ and $b \in M^d$ such that $A^0 = \{a \in M^n : \mathfrak{M} \models \varphi(a,b)\}$ and $G = \{a \in M^n : \mathfrak{M} \models \psi(a,b)\}$. Let $p(x) = \{\varphi(x,b)\} \cup \{\neg\psi(kx,b) : k \in \mathbb{N} \setminus \{0\}\}$.

Suppose to the contrary that for all $k \in \mathbb{N} \setminus \{0\}$, there is $a \in A^0$ such that $ka \notin G$. By Compactness Theorem, $p(x)$ is consistent. Since $\mathfrak{M}$ is $\aleph_0$-saturated and $\{b\}$ is finite, there exists $a \in A^0$ such that for all $k \in \mathbb{N} \setminus \{0\}$, $ka \notin G$, which is absurd.

Hence there exists $k_0 \in \mathbb{N} \setminus \{0\}$ such that $k_0 a \in G$ for all $a \in A^0$. Consider the quotient group A^0/G. For any $a \in A^0, k_0(a \oplus G) = k_0 a \oplus G = G$, it follows that A^0/G is of bounded exponent. By Theorem 4.8, A^0/G must be finite, that is $[A^0 : G] < \infty$. Since $[A : G] = [A : A^0][A^0 : G] < \infty$, by Proposition 4.5, we have $G = A^0$. $\square$

Lemma 4.10. *Let G be a definable subgroup of A^0. Assume that there is $b \in A^0$ such that $kb \notin G$ for all natural number $k \geqslant 1$. Then there exists the smallest definable subgroup G' of A^0 containing $G \cup \{b\}$. In addition, we have* $\dim G < \dim G' \leqslant \dim A^0$.

Proof. Firstly, we will show that there exists a smallest definable subgroup of A^0 containing $G \cup \{b\}$. Suppose not. Then for any definable subgroup H of A^0 containing $G \cup \{b\}$, there exists a definable subgroup H' of A^0 containing $G \cup \{b\}$ such that $H \nleqslant H'$. Next, we shall define a sequence of subgroups $(A_i)_{i \in \mathbb{N} \setminus \{0\}}$ of A^0 as follows:

Set $A_1 = A^0$. Suppose that $A_1, \ldots, A_i$ have been constructed. By the above assumption, there exists a definable subgroup A_i' of A^0 containing $G \cup \{b\}$ such that $A_i \nleqslant A_i'$. Set $A_{i+1} = A_i \cap A_i'$. So A_{i+1} is a definable subgroup of A^0 containing $G \cup \{b\}$ such that $A_i > A_{i+1}$.

Therefore, we obtain an infinite proper descending chain of definable subgroups $A^0 = A_1 > A_2 > A_3 > \ldots$, which contradicts Theorem 3.27. Therefore, there exists a smallest definable subgroup of A^0 containing $G \cup \{b\}$, say G'.

Consider $D = \{kb \oplus G : k \in \mathbb{N} \text{ and } k \geqslant 1\}$. To show that D is infinite, it is enough to show that if $k_1 b \oplus G = k_2 b \oplus G$, then $k_1 = k_2$. Suppose $k_1 b \oplus G = k_2 b \oplus G$. Then $(k_1 - k_2)b = k_1 b \ominus k_2 b \in G$. Hence $k_1 - k_2 = 0$, that is $k_1 = k_2$.

Note that $kb \in G'$ for all $k \in \mathbb{N}$. Since D infinite, G is a definable subgroup of G' such that $[G' : G] = \infty$. By Theorem 3.26, $\dim G < \dim G'$. Since $G' \subseteq A^0$, $\dim G < \dim G' \leqslant \dim A^0$. $\square$

Theorem 4.11. *Suppose $\mathfrak{M}$ is $\aleph_0$-saturated. Then every definable S-module A is a free module with a finite basis.*

Proof. By Theorems 3.9 and 3.10, it suffices to show that A is finitely generated. Since A is an infinite abelian group, so is A^0.

We recursively construct a sequence $(a_i, A_i)_{i \in \mathbb{N} \setminus \{0\}}$ such that for every $i \in \mathbb{N} \setminus \{0\}$, A_i is a definable subgroup of A^0, $a_i \in A_{i+1}$ and $A_i \subseteq \mathrm{Span}_S\{a_1, \ldots, a_i\}$ as follows:

Set $(a_1, A_1) = (0_A, \{0_A\})$. Suppose that $(a_1, A_1), \ldots, (a_i, A_i)$ have been constructed. If $A_i = A^0$, then set $(a_{i+1}, A_{i+1}) = (0_A, A^0)$. Suppose $A_i \neq A^0$. By Lemmas 4.9 and 4.10, there exist $a_{i+1} \in A^0$ and the smallest definable subgroup A_{i+1} of A^0 containing $A_i \cup \{a_{i+1}\}$. By the minimality of A_{i+1}, we have $A_{i+1} \subseteq \mathrm{Span}_S\{a_1, \ldots, a_{i+1}\}$. Observe that by Lemma 4.10, if $A_i \neq A^0$, then $\dim A_i < \dim A_{i+1}$.

Since each $A_i \subseteq A^0$ and $\dim A^0 = n$, we have $\dim A_j = n$ for every $j \geqslant n+1$. Therefore $A_{n+1} = A^0$. Since $A_{n+1} \subseteq \mathrm{Span}_S\{a_1, \ldots, a_{n+1}\}$ and $a_1, \ldots, a_{n+1} \in A^0$, by Proposition 4.7, we get $A^0 = \mathrm{Span}_S\{a_1, \ldots, a_{n+1}\}$.

Since A^0 is of finite index in A, we can say $b_1 \oplus A^0, \ldots, b_p \oplus A^0$ are all distinct (left) coset of A^0 in A. If $a \in A$, then $a = b_j \oplus a^0$ for some $j \in \{1, \ldots, p\}$ and $a^0 \in A^0$. Since $A^0 = \mathrm{Span}_S\{a_1, \ldots, a_{n+1}\}$, there are $s_1, \ldots, s_{n+1} \in S$ such that $a = b_j \oplus \rho(s_1, a_1) \oplus \cdots \oplus \rho(s_{n+1}, a_{n+1})$. Therefore we have that $A = \mathrm{Span}_S\{a_1, \ldots, a_{n+1}, b_1, \ldots, b_p\}$ is finitely generated. $\square$

Theorem 4.12. *Every definable S-module A is a free module with a finite basis.*

Proof. By Theorem 1.29, let $\mathfrak{N}$ be an $\aleph_0$-saturated elementary extension of $\mathfrak{M}$ with the universe N. Since $A, \oplus, S, \rho$ are definable, there exist $d \in \mathbb{N}$, $e \in M^d$ and $\mathcal{L} \cup \{e\}$-formula $\varphi(x)$, $\oplus_\varphi(x, y, z)$, $\psi(w)$ and $\chi(w, x, y)$ such that

$$\begin{aligned} A &= \{a \in M^n : \mathfrak{M} \models \varphi(a)\}, \\ \Gamma(\oplus) &= \{(a, b, c) \in M^{n+n+n} : \mathfrak{M} \models \oplus_\varphi(a, b, c)\}, \\ S &= \{a \in M^m : \mathfrak{M} \models \psi(a)\}, \text{ and} \\ \Gamma(\rho) &= \{(a, b, c) \in M^{m+n+n} : \mathfrak{M} \models \chi(a, b, c)\}. \end{aligned}$$

Let

$$\begin{aligned} A^{\mathfrak{N}} &= \{a \in N^n : \mathfrak{N} \models \varphi(a)\}, \\ \Gamma(\oplus)^{\mathfrak{N}} &= \{(a, b, c) \in N^{n+n+n} : \mathfrak{N} \models \oplus_\varphi(a, b, c)\}, \\ S^{\mathfrak{N}} &= \{a \in N^m : \mathfrak{N} \models \psi(a)\}, \text{ and} \\ \Gamma(\rho)^{\mathfrak{N}} &= \{(a, b, c) \in N^{m+n+n} : \mathfrak{N} \models \chi(a, b, c)\}. \end{aligned}$$

Note that $A^{\mathfrak{N}}$ is a definable $S^{\mathfrak{N}}$-module in $\mathfrak{N}$ and $S^{\mathfrak{N}}$ is a definable division ring. Since $\mathfrak{N}$ is $\aleph_0$-saturated, by Theorem 4.11, $A^{\mathfrak{N}}$ is a free $S^{\mathfrak{N}}$-module with a finite basis. Let

$$\begin{aligned} \sigma = &\exists a_1 \ldots \exists a_p \Big(\bigwedge_{i=1}^{p} \varphi(a_i) \wedge \forall a \big(\varphi(a) \to \exists s_1 \ldots \exists s_p \exists y_1 \ldots \exists y_p \exists z_1 \ldots \exists z_{p-1} \\ &\Big(\bigwedge_{i=1}^{p} \psi(s_i) \wedge \bigwedge_{i=1}^{p} \varphi(y_i) \wedge \bigwedge_{i=1}^{p-1} \varphi(z_i) \wedge \bigwedge_{i=1}^{p} \chi(s_i, a_i, y_i) \wedge z_1 = y_1 \wedge \\ &\bigwedge_{i=1}^{p-1} \oplus_\varphi(z_i, y_{i+1}, z_{i+1}) \wedge z_p = a \Big) \big) \Big). \end{aligned}$$

We can see that σ represents the sentence "A is of rank at most p" and $\mathfrak{N} \models \sigma$. Since σ is a sentence and $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$, $\mathfrak{M} \models \sigma$. That is an S-module A is finitely-generated. By Theorem 3.10, A is an S-module with a finite basis. □

Since A is a definable S-module with a finite basis, we have that A is definably isomorphic (as S-module) to S^k for some $k \in \mathbb{N}$. By Theorem 4.3 and Corollary 4.4, this completes the proof of Theorem 4.1.

Corollary 4.13. *Suppose $\mathfrak{M}$ is an expansion of a real closed field. Then A is either definably isomorphic to M^k, $M(\sqrt{-1})^k$ or $\mathbb{H}(M)^k$. In particular, if $M = \mathbb{R}$, then A is either definably isomorphic to $\mathbb{R}^k$, $\mathbb{C}^k$ or $\mathbb{H}(\mathbb{R})^k$ for some $k \in \mathbb{N}$.*

We now completely characterize definable modules over definable ring without zero divisors in o-minimal structures.

The following theorems were proved by Y. Peterzil and C. Steinhorn in [15].

Theorem 4.14 (Y. Peterzil and C. Steinhorn, [15].). *Every infinite definable ring without zero divisors is not definably compact.*

Theorem 4.15 (Y. Peterzil and C. Steinhorn, [15].). *Let A be an infinite definable group. If A is not definably compact in its topology, then A has a definable torsion-free subgroup of dimension 1.*

Since S is τ_S-connected, so are S^k for all $k \in \mathbb{N}$. By Theorem 4.1, A is τ_A-connected, that is $A = A^0$. By Theorem 4.14, S is not definably compact, which implies that A is also not definably compact. By Theorem 4.15, A has a definable torsion-free subgroup of dimension 1.

By Theorem Theorem 4.1 and 4.3, suppose $f : A \to S^k$ is a definable bijection. Then the definable bijection f induces a definable ring structure on $(A, \oplus, \odot)$, where $\odot$ is defined as follows : $a \odot b = f^{-1}(f(a) \cdot f(b))$. By Lemma 4.1 in [11], (A, τ_A) is a definable topological ring.

The following theorem was proved by J. Johns in [4].

Theorem 4.16 (J. John, [4]). *Let $\mathcal{O} \subseteq M^k$ be open and definable. Suppose that $f : \mathcal{O} \to M^k$ is a continuous definable injection. Then f is open, that is, $f[U]$ is open whenever U is an open subset of $\mathcal{O}$.*

Theorem 4.17. *Suppose $S^k \subseteq M^n$. Let $f : A \to S^k$ be a continuous module-isomorphism. Define a τ_A^*-topology on A by*

$$U \subseteq A \text{ is } \tau_A^*\text{-open if and only if } f[U] \text{ is } \tau_S\text{-open.}$$

Then the τ_A^*-topology is finer than τ_A-topology on A.

Proof. We will show that if $U \subseteq A$ is τ_A-open, then U is τ_A^*-open. Assume $U \subseteq A$ is τ_A-open. By the definition of τ_A-topology, for any $a \in A$, $(a \oplus U) \cap V$ is open in V. Since V is large in A, we may assume that $U \subseteq p \oplus V$ for some $p \in A$, that is $(\oplus p) \oplus U \subseteq V$. Since X^k is large in S^k, we may assume again that $f(\oplus p) + f[U] = f[(\oplus p) \oplus U] \subseteq q + X^k$ for some $q \in S^k$. That is $(-q) + f(\oplus p) + f[U] \subseteq X^k$.

Let $y \in (-q) + f(\ominus p) + f[U]$. Then $y = (-q) + f(\ominus p) + f(u)$ for some $u \in U$, and so $y + q = f(\ominus p) + f(u) = f(\ominus p \oplus u)$. Since U is τ_A-open, by the definition, there is an open set $B_p \subseteq A$ such that $B_p \cap V = (\ominus p \oplus U) \cap V = \ominus p \oplus U$ (since $(\ominus p) \oplus U \subseteq V$). Since f is a module-isomorphism,

$$f[B_p] \cap f[V] = f[B_p \cap V] = f[\ominus p \oplus U] = f(\ominus p) + f[U].$$

Since $y + q = f(\ominus p \oplus u) \in f[B_p] \cap f[V]$, $y \in (-q) + f[B_p] \cap f[V]$.

Set $Z = ((-q) + f[B_p]) \cap ((-q) + f[V])$. Since $f[B_p] \cap f[V] = f(\ominus p) + f[U]$, we get

$$Z = ((-q) + f[B_p]) \cap ((-q) + f[V]) = (-q) + f[B_p] \cap f[V] = (-q) + f(\ominus p) + f[U].$$

Since $(-q) + f(\ominus p) + f[U] \subseteq X^k$, we have $Z \subseteq X^k$. So

$$Z \cap X^k = Z = (-q) + f(\ominus p) + f[U] = ((-q) + f(\ominus p) + f[U]) \cap X^k.$$

Therefore $(-q) + f(\ominus p) + f[U] \subseteq X^k$ is open in X^k.

By Lemma 3.48, choose $s = -((-q) + f(\ominus p))$, we obtain that $f[U]$ is τ_S-open. Therefore U is τ_A^*-open. $\square$

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