CHAPTER I
INTRODUCTION

Laws of large numbers have been an important topic in probability theory having applications in many fields since it was introduced in the sixteenth century. Therefore, studies of laws of large numbers have brought interests from probabilists worldwide both in terms of relaxing independence assumption to different types of dependence structures and proposing variations of convergence concepts.

In the aspect of relaxing independence assumption, several relaxations have been proposed. For examples, Wang K. et al. [20] introduced a new dependence structure, widely orthant dependence, bringing interests from probabilists and statisticians.

In the direction of convergence variations, several extensions of convergence concepts have also been proposed in the last centuries. For example, Hsu and Robbins [9] proved that a sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to zero, provided the variance of summands is finite. Later in 1965, Buam and Katz [2] introduced an alternative form of complete convergence for a sequence of i.i.d. random variables with a weaker moment condition. The theorem is stated as follows.

Theorem 1.1. *([2]) Let r and α be real numbers such that $r > 1, \alpha > \frac{1}{2}$ and $\alpha r > 1$. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_n = 0$ for all $n \geq 1$. Then the following three statements are equivalent:*

1. $E|X_1|^r < \infty$,
2. $\sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\left|\sum_{i=1}^{n} X_i\right| > \epsilon n^{\alpha}\right) < \infty$ *for every* $\epsilon > 0$,
3. $\sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\sup_{k\geq n} \frac{1}{k^{\alpha}} \left|\sum_{i=1}^{k} X_i\right| > \epsilon\right) < \infty$ *for every* $\epsilon > 0$.

The result is then generalized to obtain complete convergence theorems for a sequence of random variables with weaker dependence structure such as negatively associated (NA) and widely orthant dependent (WOD) studied in Kuczmaszewska A. [15] and Ding Y. et al. [6], respectively.

The concept of the strong law of large numbers for random variables was extended to a sequence of random vectors taking values in $\mathbb{R}^d$ and in a real separable Hilbert space by several researchers. For example, in 1986, Burton et al. [3] introduced the concept and proved an invariance principle for NA random vectors taking values in $\mathbb{R}^d$ and in a real separable Hilbert space with the norm generated by an inner product. Later, Ko et al. [14] obtained the strong law of large numbers for NA random variables taking values in a real separable Hilbert space.

In 2014, Huan et al. [11] presented another concept of NA random vectors with values in a real separable Hilbert space, coordinatewise negatively associated (CNA), and pointed out that a sequence of NA random vectors implies CNA and gave a counterexample for the converse. Futhermore, they gave the complete convergence for a sequence of CNA random vectors which are weakly upper bounded by a random vector with finite moments.

In this study, we generalize the studies of Huan N.V. et al. [11] and Ding et al. [6] to construct a complete convergence for the sequence of random vectors taking values in a real separable Hilbert space with a weaker dependence structure called coordinatewise widely orthant dependence, defined in the following definitions.

Definition 1.2. Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of H-valued random vectors and $X_n^{(j)}$ be the inner product $\langle \mathbf{X}_n, \mathbf{e}_j \rangle$ where $\{\mathbf{e}_j, j \geq 1\}$ is an orthonormal basis of the Hilbert space H. Then the sequence $\{\mathbf{X}_n, n \geq 1\}$ is said to be *coordinatewise widely orthant dependent (CWOD)* with dominating coefficient $g(n)$ if the sequence $\left\{X_n^{(j)}, n \geq 1\right\}$ of random variables is WOD with dominating coefficient $g(n)$, for each $j \geq 1$.

Our result is stated as follows and the detailed proof is given in Chapter III.

Theorem A. Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of H-valued CWOD random vectors with mean zero and dominating coefficients $g(n)$ with $g(n) = O\left(n^{\alpha\left(1-\frac{r}{2}\right)} \log^{-2} n\right)$ where α and r are positive real numbers such that $\alpha r \geq 1$ and $0 < r < 2$. Assume that $\{\mathbf{X}_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector $\mathbf{X}$ with $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \log^2\left(1 + \left|X^{(j)}\right|\right) < \infty$. Then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} \mathbf{X}_i\right\| > \epsilon n^{\alpha}\right) < \infty.$$

Moreover, we are also interested in a stronger concept of convergence which is complete moment convergence, introduced in Chow [4]. It has been proved that the complete moment convergence implies complete convergence, see in Ding et al. [6], Ko [13] and Liu et al. [18], for discussion. Therefore, many autors are interested in obtaining complete moment convergence instead of complete convergence. For example, Chow [4] obtain complete moment convergence theorems for a sequence of i.i.d. random variables. In 2016, Ko [13] studied the complete moment convergence concept and generalized the result of Kuczmaszewska [15] for the complete moment convergence for a sequence of NA random variables. In 2017, Ding et al. [6] discussed the complete moment convergence for a sequence of WOD random variables with dominating coefficient $g(n)$ such that $g(n) = O\left(n^{\alpha t} \log^{-2} n\right)$ where $\alpha > \frac{1}{2}$ and $t > 0$ and Liu et al. [18] obtained the complete moment convergence for a sequence of WOD random variables with any dominating coefficient $g(n)$.

In Chapter IV, we extend Theorem A to obtain two versions of the complete moment convergence for a sequence of $H-$valued CWOD random vectors which are generalizations of Ding et al. [6] and Liu et al. [18], stated in Theorem B and Theorem C, respectively.

Theorem B. Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of $H-$valued CWOD random vectors with mean zero and dominating coefficients $g(n)$ with $g(n) = O\left(n^{\alpha(1-\frac{r}{2})} \log^{-2} n\right)$ where $\alpha r \geq 1$ and $0 < r < 2$. Assume that $\{\mathbf{X}_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector $\mathbf{X}$ with $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \log^3\left(1 + \left|X^{(j)}\right|\right) < \infty$.

Then, for $0 < q < r$ and any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} E\left[\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| - \epsilon n^{\alpha}\right]_{+}^{q} < \infty,$$

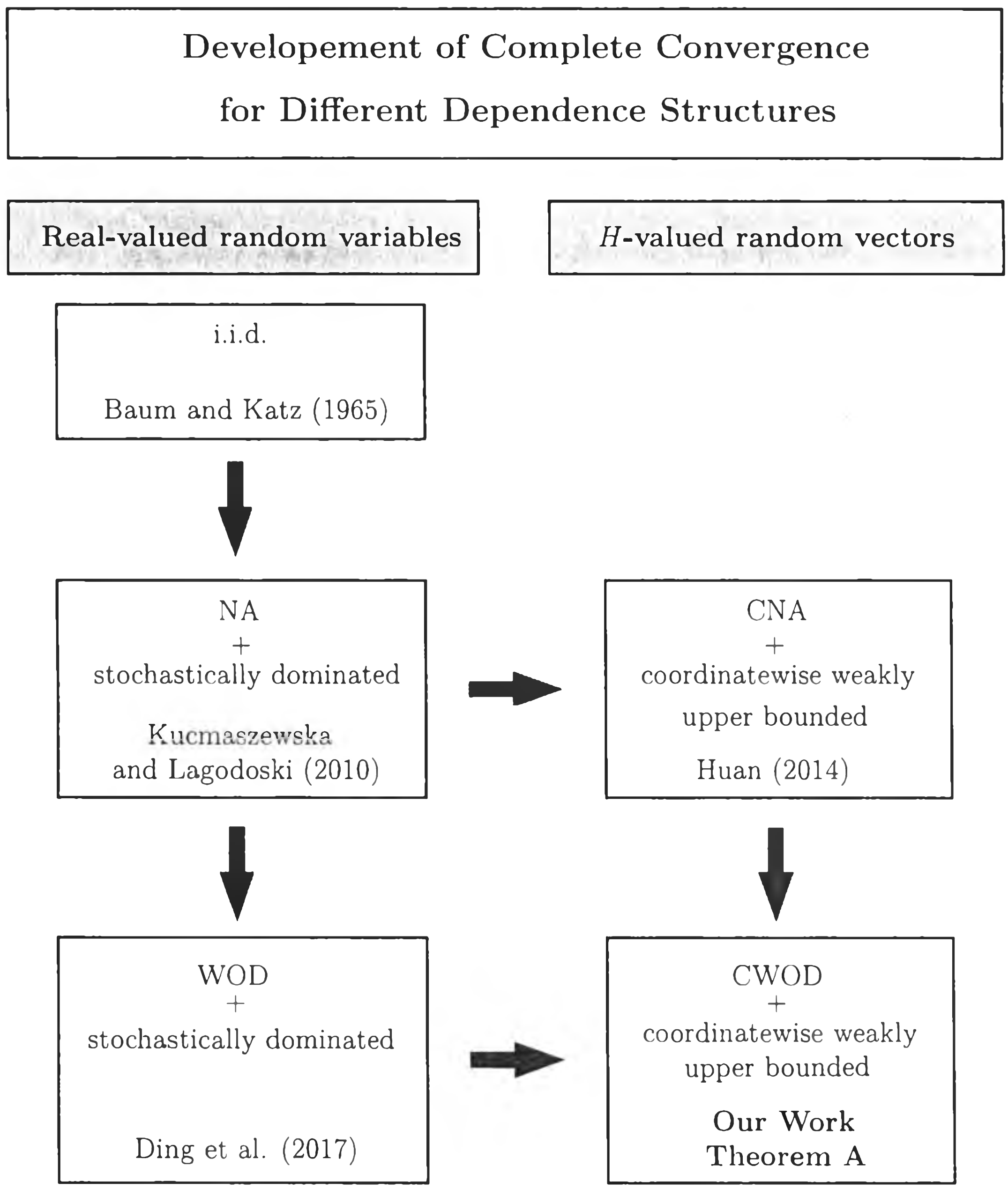
where $x_{+}^{q} = (x_{+})^{q}$ and $x_{+} = \max\{x, 0\}$.

Theorem C. Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of $H-$valued CWOD random vectors with mean zero and dominating coefficients $g(n)$. Assume that $\{\mathbf{X}_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector $\mathbf{X}$ with $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^{rp}\log^{2}\left(1+\left|X^{(j)}\right|\right) < \infty$ where $r > 1$ and $1 \leq p < \frac{2}{r}$. Then, any $\epsilon > 0$,

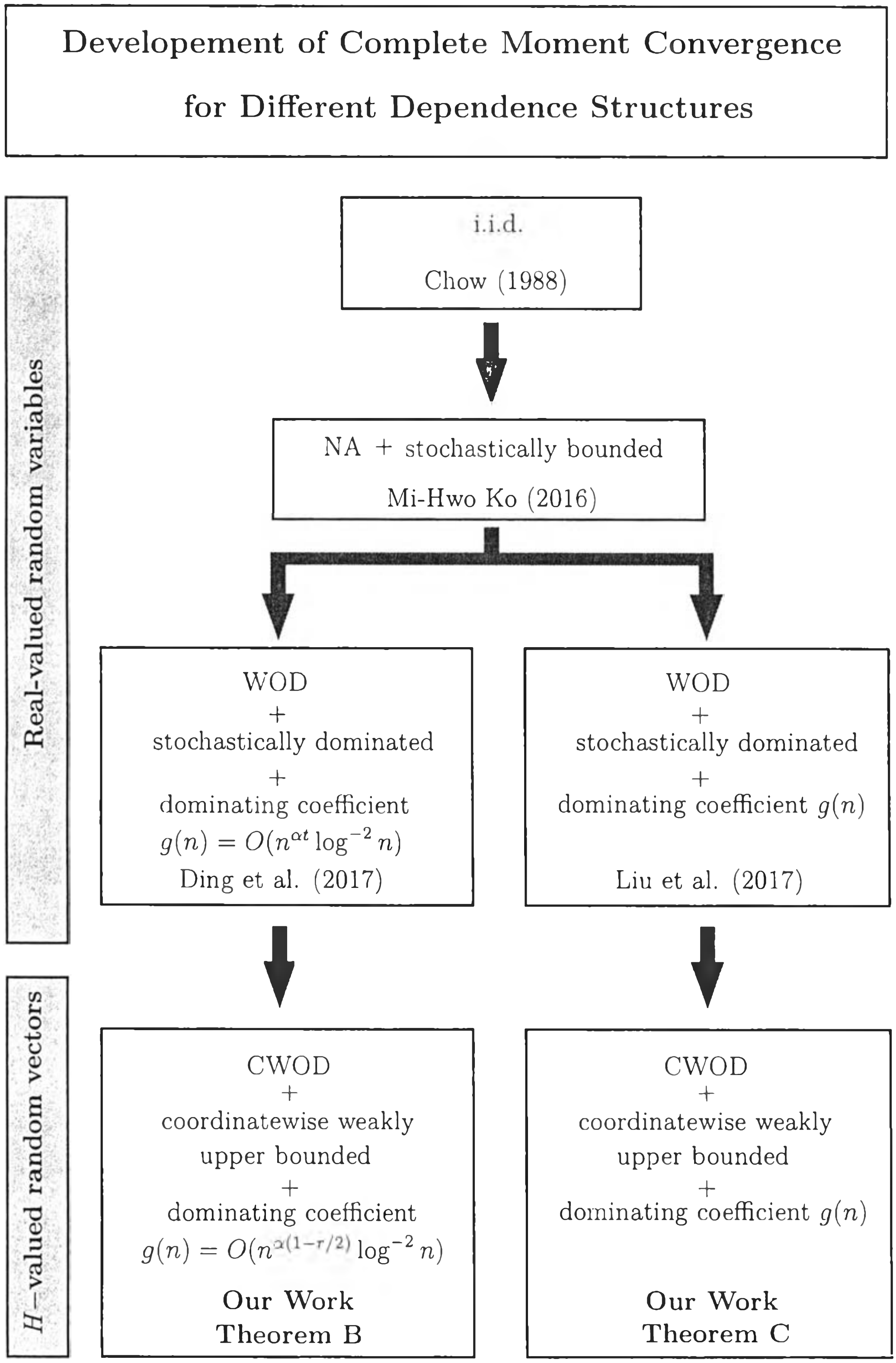
$$\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E\left[\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| - \epsilon\left(1+g(n)\right)n^{\frac{1}{p}}\right]_{+} < \infty,$$

where $x_{+} = \max\{x, 0\}$.

The flowcharts 1.1 and 1.2 shown below explain a literature review of the complete convergence and complete moment convergence concepts, respectively.



Flowchart 1.1: Complete Convergence



Flowchart 1.2: Complete Moment Convergence