CHAPTER I

INTRODUCTION

1.1 Mathematical concepts

For bond pricing, to improve the accuracy of calculations for the present value of all cash flows, equilibrium models are offered to be more suitable than models based on arbitrage arguments. Since equilibrium models come from price behavior, one of the interesting things of construction of the models is to predict a long term equilibrium. Vasicek [14] introduced equilibrium model for interest rate based on the idea of the Black-Scholes model [1] by considering arbitrage argument for option pricing, which is

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \tag{1.1}$$

where κ is the means reversion rate, θ is the equilibrium interest rate, σ is the volatility and W_t refers to the standard Brownian motion. The Vasicek model is the simplest model to illustrate the equilibrium as the mean and variance of r_t that converge to finite values when t approaches infinity. One of the earliest way to control positivity of r_t and to fix undesirable feature of (1.1) was introduced by Cox, Ingersoll and Ross [3], known as CIR model, which is suitable for describing the evolution and structure of interest rate, namely,

$$dr_t = \kappa(\theta - r_t)dt + \sqrt{r_t}\sigma dW_t. \tag{1.2}$$

It is true that CIR model can be used in a financial model, however, parameters for many examples may depend on time. To address the inconveniences of the CIR model, the extended Cox-Ingersoll-Ross (ECIR) model [7] is considered for time dependent parameters. In addition, the ECIR model is frequently used to explain behavior of structure term of interest rates, especially prices of zero-coupon bonds, where the price changes vary over time. The general form of ECIR is

$$dr_t = \kappa(t)(\theta(t) - r_t)dt + \sqrt{r_t}\sigma(t)dW_t, \tag{1.3}$$

where κ, θ and σ are functions depending on t.

1.2 Contributions and results

Onto the expected rate at time t for the maturity date T, for all $t \leq T$, $\mathbb{E}^{\mathbb{P}}[r_T|\mathcal{F}_t]$, can be useful in areas of finance, in particular, in estimating fair price, fitting interest rates structure, forecasting, as well as valuating of coupon bonds. In practice, Monte Carlo (MC) simulation is used to obtain the expectation which requires a lot of computational time. For the ECIR model (1.3) with positive real-valued, bounded, continuous functions, $\kappa(t), \theta(t), \sigma(t)$, Maghsoodi [9] derived the closed-form formula of the expectation of r_t based on the assumption to guarantee the positivity of r_t,

$$2\kappa(t)\theta(t) \geq \sigma^2(t). \tag{1.4}$$

Furthermore, the derived closed-form was applied to price the T-maturity discount bound,

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}. \tag{1.5}$$

In 2016, the same condition (1.4) of Maghsoodi were applied with Feynman-Kac theorem by Rujivan [11] to obtain the closed-form formula for the conditional expectation of moments, based on the ECIR model,

$$\mathbb{E}^{\mathbb{P}}[r_T^{\gamma} \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[r_T^{\gamma} \mid r_t = r], \tag{1.6}$$

for $0 \leq t \leq T$, and any real γ and positive r.

In this study, we extend the result of Rujivan using Feynman-Kac theorem with a suitable construct of transformation to obtain explicit formulas for conditional expectations of the product of polynomial and exponential function of an affine transform (P-EA transform),

$$\mathbb{E}^{\mathbb{P}}\left[r_T^{\gamma} e^{\alpha r_T + \beta} \,\middle|\, r_t = r\right]. \tag{1.7}$$

We immediately simplify the result to derive an explicit formula for the CIR model and compare to obtain the same result of Rujivan [11], when α, β are 0.

1.3 Structure of the thesis

The remainder of this thesis is structured as follows.

Chapter 2 gives an essential introduction to mathematical knowledge. Then, we collect some mathematical notions and ideas used during this research. We introduce the CIR process and focus more on ECIR process which we use for solving the main result. The problem of the research and its solutions are technically discussed in this chapter.

The explicit formulas for conditional expectations of the product of polynomial and exponential function of an affine transform are introduced in Chapter 3. We show that, by using Feynman-Kac theorem, the expectation problem under an assumption can be related to a partial differential equation (PDE) problem. Moreover, its consequences are explored, especially, in some cases, the explicit formula can be reduced to a closed-form formula.

In Chapter 4, we demonstrate advantages of the explicit formula compared with the MC simulations when computing the conditional expectations of the P-EA transform of the CIR and ECIR process.

The aim of this thesis are recapitulated and concluded in Chapter 5.