CHAPTER II

PRELIMINARIES

In this chapter, we propose more importantly of mathematical backgrounds involved in the financial processes. We provide the processes based on Wiener process or Brownian motion, in particular of ECIR process and collect some properties for proving the main result. This chapter is divided into four sections, Stochastic processes, CIR process and ECIR process, Simulation issue and Feynman-Kac theorem.

2.1 Stochatic processes

In order to achieve the main results, we describe here the concepts of stochastic differential equation to understand the ECIR process.

Definition 2.1. A *stochastic process* $X_t = X(\cdot, t)$, $t \geq 0$ is a family of random variables $X : \Omega \times [0, \infty) \to \mathbb{R}$ with $t \mapsto X(\omega, t)$ continuous for all $\omega \in \Omega$.

Definition 2.2. A stochastic process $W = \{W_t : t \geq 0\}$ is called a *Brownian motion* or a *Wiener process* if the following conditions hold

(i) $W_0 = 0$,

(ii) W_t has independent increments, that is, if $s < t \leq u < v$, then $W_v - W_u$ and $W_t - W_s$ are independent stochastic variables,

(iii) W_t is normally distributed as $dW_t \sim N(0, t)$,

(iv) W_t is continuous in $t \geq 0$.

Remark 2.3. A Wiener process that is drawn from a standard normal distribution, $N(0, 1)$ could be called a *standard Wiener process.*

Levy [8] collected some properties of $dW_t := W_{t+dt} - W_t$ which are often used in the stochastic calculus.

(i) $\mathbb{E}[dW_t] = 0$,

(ii) $Var[dW_t] = dt$,

(iii) $\mathbb{E}[W_{t+dt} \mid \mathcal{F}_t] = W_t$,

(iv) $Cov[W_s, W_t] = \min(s, t)$.

From the second condition of the Wiener process, the increments dW_t are independent of past values, we can further state that a Brownian process is also a Markov process. Moreover, the third property implies that the Brownian process is also a martingale process.

Definition 2.4. A *stochastic differential equation* (*SDE*) is a differential equation in one or more stochastic process terms. A one factor SDE has the following form

$$dX_t = U_t dt + V_t dW_t, \tag{2.1}$$

where U is referred to the drift of the stochastic process, V is referred to the volatility and W_t is a Wiener process.

Definition 2.5. An *Itö process* or *stochastic integral* is a stochastic process X_t which can be written in the form

$$X_t = X_0 + \int_0^t U_s ds + \int_0^t V_s dW_s, \tag{2.2}$$

where $\int_0^t V_s dW_s$ is the Itö integral. When written in SDE form we get

$$dX_t = U_t dt + V_t dW_t. \tag{2.3}$$

Remark 2.6. From an Itö integral, for a Wiener process W_t, we have that W_t^2 is an Itö process

$$W_t^2 = \int_0^t ds + 2\int_0^t W_s dW_s, \tag{2.4}$$

and since W_t is not differentiable, its differentiation is different from the usual differentiation, namely,

$$d(W_t^2) = dt + 2W_t dW_t. \tag{2.5}$$

Theorem 2.7 (Itö lemma). *Let W_t be a Wiener process and S_t be an Itö process that satisfies the SDE*

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t. \tag{2.6}$$

Let $Y_t = G(S_t, t)$ be a new stochastic process such that $G(s, t)$ is a function of s and t. If $G(s, t)$ is a twice continuously differentiable scalar function, then the process Y_t is also an Itö process, given by

$$dY_t = \left[\frac{\partial G}{\partial t} + \mu(S_t, t)\frac{\partial G}{\partial s} + \frac{1}{2}\sigma^2(S_t, t)\frac{\partial^2 G}{\partial s^2}\right] dt + \sigma(S_t, t)\frac{\partial G}{\partial s} dW_t.$$

Proof See details in [8]. ■

Theorem 2.8 (Itö product rule). *The expressions for the product of two stochastic processes. In this case, $\phi \to \phi(X_1, X_2)$ with*

$$dX_1 = a_1 dt + b_1 dW_1 \qquad \text{and} \qquad dX_2 = a_2 dt + b_2 dW_2. \tag{2.7}$$

Then, the product rule is

$$d(X_1X_2) = X_2 dX_1 + X_1 dX_2 + \mathbb{E}[dX_1 dX_2]. \tag{2.8}$$

For the special case where X_1 is a Wiener process but X_2 has no random term, i.e.,

$$dX_1 = X_1\mu_1 dt + X_1\sigma_1 dW_1 \qquad \text{and} \qquad dX_2 = X_2\mu_2 dt. \tag{2.9}$$

We obtain $\mathbb{E}[dX_1 dX_2] = 0$ and we have

$$d(X_1X_2) = X_2 dX_1 + X_1 dX_2. \tag{2.10}$$

Proof See details in [8]. ■

2.2 CIR process and ECIR process

As introduced in the first chapter, the evolution of interest rates is an important part for explaining CIR process, and the movement of interest rates is described in the type of short-term form. The process is applied to estimate the interest rate of derivatives.

It was introduced in 1985 by Cox, Ingersoll and Ross as an improvement of Vasicek process. Its form is

$$dr_t = \kappa(\theta - r_t)dt + \sqrt{r_t}\sigma dW_t, \tag{2.11}$$

where κ is the parameter that corresponds to speed of adjustment, θ is the equilibrium interest rate, σ is the volatility and W_t is the standard Brownian motion.

In order to improve the inconveniences of the CIR process, the ECIR process [7] is considered for time dependent parameters. Its form was stated by Hull and White in 1990 as

$$dr_t = \kappa(t)(\theta(t) - r_t)dt + \sqrt{r_t}\sigma(t)dW_t, \tag{2.12}$$

where κ, θ and σ are generalized functions of time t.

2.3 Simulations of SDE

Although we already have the ECIR model, it will be difficult to work with when the closed-form solution is not known. One way to overcome this problem is by employing the MC simulation. The MC simulation is basically used to obtain the derivative prices. We present here one of the simplest MC approximation, known as the *Euler-Maruyama (EM) scheme.* The EM approximation is a method for approximating numerical solutions of SDE. Suppose that W is a Wiener process and a process S_t satisfies SDE

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \tag{2.13}$$

for all $t \in [0, T]$. If we discretize the interval $[0, T]$ into N partitions with equal time step $\Delta t_i = t_{i+1} - t_i$ and $\Delta W_i = W_{t_{i+1}} - W_{t_i}$ for $i \in \{0, 1, \ldots, N\}$. The EM approximation is defined as

$$S_{t_{i+1}} = S_{t_i} + \mu(S_t, t)\Delta t_i + \sigma(S_t, t)\Delta W_{t_i}, \tag{2.14}$$

where ΔW_{t_i} is approximated by $\sqrt{\Delta t_i}\ N(0, 1)$. The EM approximation converges to the explicit solution as $N \to \infty$ which, in practice, uses a lot of computational time. To reduce the computational time, one needs to go through the difficulty by finding an explicit solution where the idea is described in the next section.

2.4 Feynman-Kac theorem

In order to avoid costly computation using MC simulation as described above, one needs a way to obtain the exact formula for the ECIR process. One standard technique is the usage of the Feynman-Kac theorem, where the expectations of the process of SDE are obtained by solving PDEs.

Theorem 2.9 (Feynman-Kac theorem). *Suppose that S_t follows the Itö process*

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \tag{2.15}$$

where W_t is a Wiener process. Let $K \subseteq \mathbb{R}$ be a compact support and $u := u(s,t) \in C^{2,1}(K \times [0,T])$ follows the PDE driven by

$$\frac{\partial u}{\partial t} + \mu(s,t)\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s,t)\frac{\partial^2 u}{\partial s^2} - V(s,t)u + f(s,t) = 0, \tag{2.16}$$

subject to the terminal condition $u(s,T) = \Psi(s)$ for all s and V is bounded below. Then, the solutions, $u(s,t)$ satisfies

$$u(s,t) = \mathbb{E}\left[e^{-\int_t^T V(S_r,r)dr}\Psi(S_T) + \int_t^T e^{-\int_t^r V(S_q,q)dq} f(S_r,r)dr \,\middle|\, S_t = s \right]. \tag{2.17}$$

Proof The summaries of the proof based on [10] is as follows. Let $u(s,t)$ be a solution to the (2.16). Applying Itö lemma to the process

$$Y(\tau) = e^{-\int_t^\tau V(S_r,r)dr}u(S_\tau,\tau) + \int_t^\tau e^{-\int_t^r V(S_q,q)dq} f(S_r,r)dr, \tag{2.18}$$

yields

$$\begin{aligned}
dY(\tau) &= d\left(e^{-\int_t^\tau V(S_r,r)dr}\right)u(S_\tau,\tau) + e^{-\int_t^\tau V(S_r,r)dr}du(S_\tau,\tau) \\
&\quad + d\left(\int_t^\tau e^{-\int_t^r V(S_q,q)dq} f(S_r,r)dr\right), \\
&= \left(-V(S_\tau,\tau)e^{-\int_t^\tau V(S_r,r)dr}d\tau\right)u(S_\tau,\tau) \\
&\quad + e^{-\int_t^\tau V(S_r,r)dr}\left(\frac{\partial u}{\partial t} + \mu(s,t)\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s,t)\frac{\partial^2 u}{\partial s^2} + \sigma(S_\tau,\tau)\frac{\partial u}{\partial S}dW_\tau\right) \\
&\quad + e^{-\int_t^\tau V(S_q,q)dq} f(S_\tau,\tau)d\tau, \\
&= e^{-\int_t^\tau V(S_r,r)dr}\left(-V(S_\tau,\tau)u(S_\tau,\tau) + f(S_\tau,\tau) + \frac{\partial u}{\partial t} + \mu(s,t)\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s,t)\frac{\partial^2 u}{\partial s^2}\right) \\
&\quad + e^{-\int_t^\tau V(S_r,r)dr}\sigma(S_\tau,\tau)\frac{\partial u}{\partial S}dW_\tau.
\end{aligned} \tag{2.19}$$

By the assumption, the first term vanishes. Then

$$dY(\tau) = e^{-\int_t^\tau V(S_r,r)dr}\sigma(S_\tau,\tau)\frac{\partial u}{\partial S}dW_\tau. \tag{2.20}$$

Integrating (2.20) from t to T, we get

$$\begin{aligned}\mathbb{E}\left[Y(T) - Y(t) \mid S_t = s\right] &= \mathbb{E}\left[\int_t^T e^{-\int_t^\tau V(S_r,r)dr}\sigma(S_\tau,\tau)\frac{\partial u}{\partial S}dW_\tau \,\middle|\, S_t = s\right] \\ &= 0. \end{aligned} \tag{2.21}$$

Thus,

$$\mathbb{E}\left[Y(T) \mid S_t = s\right] = \mathbb{E}\left[Y(t) \mid S_t = s\right] = u(s,t) \tag{2.22}$$

as stated. ■

For the uniqueness of the solution, we refer to the theorem below.

Theorem 2.10. *If $w(s,t)$ is a bounded solution to (2.17) with the terminal condition $w(s,T) = \Psi(s)$ for $s \in K$, then $w(s,t) = u(s,t)$.*

Proof See details in [10]. ■

Next chapter, we take a closer look on our problem of explicit formulas for conditional expectations of the product of P-EA transform, $\mathbb{E}^{\mathbb{P}}\left[r_T^\gamma e^{\alpha r_T+\beta} \mid r_t = r\right]$. We start with an assumption to ensure that the conditional expectations of the P-EA transform of the ECIR process exists.