# CHAPTER III

# MAIN RESULTS

This section derives an explicit formula for conditional expectations of a product of a P-EA transform of ECIR process with parameters $\gamma, \alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{N} \cup \{0\}$. Furthermore, the result is simplified to the standard CIR model with constants $\kappa, \theta$ and $\sigma$.

In this work, the assumption of Maghsoodi is assumed to guarantee that $r_t \geq 0$ for all $t \in [0, \infty)$, which is stated as follow.

**Assumption** The parameter functions $\theta(t), \kappa(t)$ and $\sigma(t)$ are positive and continuous on $[0, T]$ such that the dimension parameters $\delta(t) := \frac{4\theta(t)\kappa(t)}{\sigma^2(t)}$ of the ECIR process (1.3) is bounded and $\delta(t) \geq 2$ for all $t \in [0, T]$.

## 3.1 ECIR process

**Theorem 3.1.** *Suppose that $V_t$ follows the ECIR process (1.3) with $\gamma, \alpha, \beta \in \mathbb{R}$. Assume that the Assumption holds and let*

$$U_E^{(\gamma,\alpha,\beta)}(v, \tau) := \mathbb{E}^{\mathbb{P}}\left[V_T^{\gamma} e^{\alpha V_T + \beta} \mid V_t = v\right] \tag{3.1}$$

*for $v > 0$ and $\tau = T - t \geq 0$. Then,*

$$U_E^{(\gamma,\alpha,\beta)}(v, \tau) = \sum_{k=0}^{\infty} A_{\gamma-k}(\tau) v^{\gamma-k} e^{B(\tau)v+\beta}, \tag{3.2}$$

*given that the series converges, where*

$$A_\gamma(\tau) = \exp\left[\int_0^{\tau} \Big(\gamma\sigma^2(T-u)B(u) + \kappa(T-u)\theta(T-u)B(u) - \gamma\kappa(T-u)\Big)du\right] \tag{3.3}$$

*and for $k \in \mathbb{N}$,*

$$A_{\gamma-k}(\tau) = \exp\left[\int_0^{\tau} Q_{\gamma-k}(T-u)du\right] \times \int_0^{\tau} \exp\left[-\int_0^{s} Q_{\gamma-k}(T-u)du\right] P_{\gamma-k+1}(T-s)A_{\gamma-k+1}(s)ds \tag{3.4}$$

*where*

$$P_{\gamma-k+1}(\tau) = (\gamma-k+1)\left[\frac{1}{2}(\gamma-k)\sigma^2(\tau)+\kappa(\tau)\theta(\tau)\right], \quad (3.5)$$

$$Q_{\gamma-k}(\tau) = (\gamma-k)\sigma^2(\tau)B(T-\tau)+\kappa(\tau)\theta(\tau)B(T-\tau)-(\gamma-k)\kappa(\tau) \quad (3.6)$$

*and*

$$B(\tau) = \frac{\alpha\exp\left[-\int_0^{\tau}\kappa(T-u)du\right]}{1-\alpha\int_0^{\tau}\frac{1}{2}\sigma^2(T-s)\exp\left[-\int_0^{s}\kappa(T-u)du\right]ds}. \quad (3.7)$$

*Proof* By the definition of (3.1), $U_E^{(\gamma,\alpha,\beta)}(v,\tau)$ is the conditional expectations of a P-EA transform under the ECIR process of $V_t$. By applying the Feynman-Kac Theorem, we are seeking for the solution in the form

$$U := U_E^{(\gamma,\alpha,\beta)}(v,\tau) = \sum_{k=0}^{\infty} A_{\gamma-k}(\tau)v^{\gamma-k}e^{B(\tau)v+\beta}, \quad (3.8)$$

which satisfies the corresponding PDE

$$\begin{aligned}
0 &= \frac{\partial U}{\partial t}+\frac{1}{2}\hat{\sigma}^2(t,v)\frac{\partial^2 U}{\partial v^2}+\hat{\mu}(t,v)\frac{\partial U}{\partial v}\\
&= -\frac{\partial U}{\partial \tau}+\frac{1}{2}\sigma^2(T-\tau)v\frac{\partial^2 U}{\partial v^2}+\kappa(T-\tau)\left[\theta(T-\tau)-v\right]\frac{\partial U}{\partial v}\\
&= -e^{B(\tau)v+\beta}\sum_{k=0}^{\infty}\left[\frac{d}{d\tau}A_{\gamma-k}(\tau)v^{\gamma-k}+\frac{d}{d\tau}B(\tau)A_{\gamma-k}(\tau)v^{\gamma-k+1}\right]\\
&\quad +\frac{1}{2}\sigma^2(T-\tau)ve^{B(\tau)v+\beta}\sum_{k=0}^{\infty}\left[A_{\gamma-k}(\tau)(\gamma-k)(\gamma-k-1)v^{\gamma-k-2}\right.\\
&\quad \left.+2B(\tau)A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1}+B^2(\tau)A_{\gamma-k}(\tau)v^{\gamma-k}\right]\\
&\quad +\kappa(T-\tau)\left[\theta(T-\tau)-v\right]\times\\
&\quad e^{B(\tau)v+\beta}\sum_{k=0}^{\infty}\left[A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1}+B(\tau)A_{\gamma-k}(\tau)v^{\gamma-k}\right]. \quad (3.9)
\end{aligned}$$

From (3.1) with condition at $\tau = 0$, we get the terminal condition $U_E^{(\gamma,\alpha,\beta)}(v,0) = v^{\gamma}e^{\alpha v+\beta}$. To solve (3.9), we need the conditions on $A$ and $B$, which are obtained via the terminal condition,

$$B(0)=\alpha, \quad A_{\gamma}(0)=1 \quad \text{and} \quad A_{\gamma-k}(0)=0, \quad (3.10)$$

when $k \in \mathbb{N}$.

Since $e^{B(\tau)v+\beta} > 0$, the PDE in (3.9) is simplified to

$$
\begin{aligned}
0 = &-\sum_{k=0}^{\infty}\left[\frac{d}{d\tau}A_{\gamma-k}(\tau)v^{\gamma-k} + \frac{d}{d\tau}B(\tau)A_{\gamma-k}(\tau)v^{\gamma-k+1}\right] \\
&+ \frac{1}{2}\sigma^2(T-\tau)v\sum_{k=0}^{\infty}\Big[A_{\gamma-k}(\tau)(\gamma-k)(\gamma-k-1)v^{\gamma-k-2} \\
&+ B(\tau)A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1} + B(\tau)A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1} + B^2(\tau)A_{\gamma-k}(\tau)v^{\gamma-k}\Big] \\
&+ \kappa(T-\tau)\left[\theta(T-\tau)-v\right]\sum_{k=0}^{\infty}\left[A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1} + B(\tau)A_{\gamma-k}(\tau)v^{\gamma-k}\right]
\end{aligned} \tag{3.11}
$$

Collecting the coefficients of power of $v$, we set

$$
\begin{aligned}
0 = &\left[-\frac{d}{d\tau}B(\tau)A_\gamma(\tau) + \frac{1}{2}\sigma^2(T-\tau)B^2(\tau)A_\gamma(\tau) - \kappa(T-\tau)B(\tau)A_\gamma(\tau)\right]v^{\gamma+1} \\
&+ \Big[-\frac{d}{d\tau}A_\gamma(\tau) - \frac{d}{d\tau}B(\tau)A_{\gamma-1}(\tau) + \frac{1}{2}\sigma^2(T-\tau)B(\tau)A_\gamma(\tau)(\gamma) \\
&+ \frac{1}{2}\sigma^2(T-\tau)B(\tau)A_\gamma(\tau)(\gamma) + \frac{1}{2}\sigma^2(T-\tau)B^2(\tau)A_{\gamma-1}(\tau) \\
&+ \kappa(T-\tau)\theta(T-\tau)B(\tau)A_\gamma(\tau) - \kappa(T-\tau)A_\gamma(\tau)(\gamma) - \kappa(T-\tau)B(\tau)A_{\gamma-1}(\tau)\Big]v^\gamma \\
&+ \sum_{k=2}^{\infty}\Big[-\frac{d}{d\tau}A_{\gamma-k+1}(\tau) - \frac{d}{d\tau}B(\tau)A_{\gamma-k}(\tau) \\
&+ \frac{1}{2}\sigma^2(T-\tau)A_{\gamma-k+2}(\tau)(\gamma-k+2)(\gamma-k+1) \\
&+ \sigma^2(T-\tau)B(\tau)A_{\gamma-k+1}(\tau)(\gamma-k+1) + \frac{1}{2}\sigma^2(T-\tau)B^2(\tau)A_{\gamma-k}(\tau) \\
&+ \kappa(T-\tau)\theta(T-\tau)A_{\gamma-k+2}(\tau)(\gamma-k+2) + \kappa(T-\tau)\theta(T-\tau)B(\tau)A_{\gamma-k+1}(\tau) \\
&- \kappa(T-\tau)A_{\gamma-k+1}(\tau)(\gamma-k+1) - \kappa(T-\tau)B(\tau)A_{\gamma-k}(\tau)\Big]v^{\gamma-k+1}.
\end{aligned} \tag{3.12}
$$

Considering (3.12) as a power series of $v$, we obtain the followings.

(i) The coefficient function of $v^{\gamma+1}$ can be written as a deterministic PDE

$$
\frac{d}{d\tau}B(\tau) = \frac{1}{2}\sigma^2(T-\tau)B^2(\tau) - \kappa(T-\tau)B(\tau), \tag{3.13}
$$

whose solution according to the condition of $B$ in (3.10) is

$$
B(\tau) = \frac{\alpha\exp\left[-\int_0^\tau \kappa(T-u)du\right]}{1-\alpha\int_0^\tau \frac{1}{2}\sigma^2(T-s)\exp\left[-\int_0^s \kappa(T-u)du\right]ds}. \tag{3.14}
$$

(ii) Using the coefficient of $v^{\gamma}$, we obtain functional relationship between $A_{\gamma}(\tau)$, $A_{\gamma-1}(\tau)$ and $B(\tau)$ as

$$\begin{aligned}\frac{d}{d\tau}A_{\gamma}(\tau) &= -\frac{d}{d\tau}B(\tau)A_{\gamma-1}(\tau)\\ &+\frac{1}{2}\sigma^2(T-\tau)B(\tau)A_{\gamma}(\tau)(\gamma)+\frac{1}{2}\sigma^2(T-\tau)B(\tau)A_{\gamma}(\tau)(\gamma)\\ &+\frac{1}{2}\sigma^2(T-\tau)B^2(\tau)A_{\gamma-1}(\tau)+\kappa(T-\tau)\theta(T-\tau)B(\tau)A_{\gamma}(\tau)\\ &-\kappa(T-\tau)A_{\gamma}(\tau)(\gamma)-\kappa(T-\tau)B(\tau)A_{\gamma-1}(\tau). \end{aligned} \tag{3.15}$$

Using (3.14) and initial condition on $A_{\gamma}$ in (3.10) yields

$$A_{\gamma}(\tau)=\exp\left[\int_0^{\tau}\Big(\gamma\sigma^2(T-u)B(u)+\kappa(T-u)\theta(T-u)B(u)-\gamma\kappa(T-u)\Big)du\right]. \tag{3.16}$$

(iii) Similarly, using (3.12) and initial conditions on $A_{\gamma-k}$ in (3.10), the coefficients of $v^{\gamma-k+1}$ for $k\in\{2,3,4,\ldots\}$, give

$$\frac{d}{d\tau}A_{\gamma-k+1}(\tau)=Q_{\gamma-k+1}(T-\tau)A_{\gamma-k+1}(\tau)+P_{\gamma-k+2}(T-\tau)A_{\gamma-k+2}(\tau), \tag{3.17}$$

where

$$P_{\gamma-k+2}(\tau) = (\gamma-k+2)\left[\frac{1}{2}(\gamma-k+1)\sigma^2(\tau)+\kappa(\tau)\theta(\tau)\right] \text{ and} \tag{3.18}$$

$$\begin{aligned}Q_{\gamma-k+1}(\tau) &= (\gamma-k+1)\sigma^2(\tau)B(T-\tau)+\kappa(\tau)\theta(\tau)B(T-\tau)\\ &-(\gamma-k+1)\kappa(\tau).\end{aligned} \tag{3.19}$$

This gives the solutions in the form

$$\begin{aligned}A_{\gamma-k+1}(\tau) &= \exp\left[\int_0^{\tau}Q_{\gamma-k+1}(T-u)du\right]\int_0^{\tau}\Bigg(\exp\left[-\int_0^{s}Q_{\gamma-k+1}(T-u)du\right]\times\\ &P_{\gamma-k+2}(T-s)A_{\gamma-k+2}(s)\Bigg)ds.\end{aligned} \tag{3.20}$$

as required. ■

**Remark 3.2.** Note that $B(\tau)$ is unbounded if

$$\mu(\tau):=\int_0^{\tau}\frac{1}{2}\sigma^2(T-s)\exp\left[-\int_0^{s}\kappa(T-u)du\right]ds=\frac{1}{\alpha}. \tag{3.21}$$

Since $\mu(\tau)$ is an increasing function in $\tau$ with $\mu(0)=0$, then $B(\tau)$ is bounded given that $\tau\in(0,T]$ where $\mu(T)<\frac{1}{\alpha}$. Therefore, if $\alpha<\mu(T)$, then it guarantees that $B(\tau)$ is bounded for all $\tau\in[0,T]$.

**Remark 3.3.** The result of the Theorem 3.1 can produce the same result of Rujivan [11] for $\mathbb{E}^{\mathbb{P}}\left[V_T^{\gamma} \mid V_t = v\right]$ when $\alpha$ and $\beta$ are set to be 0 in (3.1).

The following corollary describes a consequence that is deduced from Theorem 3.1 when $\gamma = 1 - \frac{2\kappa(\tau)\theta(\tau)}{\sigma^2(\tau)}$, the explicit form is reduced into the closed-form as shown in the following corollary.

**Corollary 3.4.** *Suppose that $V_t$ follows the ECIR process (1.3) where $\gamma, \alpha, \beta \in \mathbb{R}$ and $\gamma$ satisfies*

$$\gamma = 1 - \frac{2\kappa(\tau)\theta(\tau)}{\sigma^2(\tau)} \tag{3.22}$$

*for all $\tau \geq 0$. Then, (3.2) is reduced into the form*

$$U_E^{(\gamma,\alpha,\beta)}(v,\tau) = \tag{3.23}$$

$$\exp\left[B(\tau)v + \beta + \int_0^{\tau} \Big(\gamma\sigma^2(T-u)B(u) + \kappa(T-u)\theta(T-u)B(u) - \gamma\kappa(T-u)\Big)du\right] v^{\gamma}.$$

*Proof* It is obvious from (3.5) when $k = 1$ that $P_\gamma = 0$ if $\gamma = 1 - \frac{2\kappa(\tau)\theta(\tau)}{\sigma^2(\tau)}$ for all $\tau \geq 0$. Therefore, (3.4) implies $A_{\gamma-k}(\tau) = 0$ for all $k \in \mathbb{N}$, and the remaining term of (3.2) is $A_\gamma$. ■

The result of Theorem 3.1 can be simplified into a finite sum in the case when $\gamma$ is a non-negative integer, as stated in the following result.

**Theorem 3.5.** *Suppose that $V_t$ follows the ECIR process (1.3) with $\alpha, \beta \in \mathbb{R}$. Let $n$ be a non-negative integer. Then,*

$$U_E^{(n,\alpha,\beta)}(v,\tau) = e^{B(\tau)v+\beta} \sum_{j=0}^{n} A_j(\tau) v^j, \tag{3.24}$$

*where*

$$A_n(\tau) = \exp\left[\int_0^{\tau} \Big(n\sigma^2(T-u)B(u) + \kappa(T-u)\theta(T-u)B(u) - n\kappa(T-u)\Big)du\right], \tag{3.25}$$

$$A_j(\tau) = \exp\left[\int_0^{\tau} Q_j(T-u)du\right]\left(\int_0^{\tau}\left(\exp\left[-\int_0^{s} Q_j(T-u)du\right] \times P_{j+1}(T-s)A_{j+1}(s)\right)ds\right), \tag{3.26}$$

$$P_{j+1}(\tau) = (j+1)\left[\frac{1}{2}j\sigma^2(\tau) + \kappa(\tau)\theta(\tau)\right] \text{ and} \tag{3.27}$$

$$Q_j(\tau) = j\sigma^2(\tau)B(T-\tau) + \kappa(\tau)\theta(\tau)B(T-\tau) - j\kappa(\tau), \tag{3.28}$$

*for* $j \in \{1, 2, 3, ..., n-1\}$, *and* $B(\tau)$ *is given by (3.7). In addition,* $U_E^{(n,\alpha,\beta)}(v, \tau)$ *is strictly increasing with respect to* $v$ *for any* $\tau > 0$.

*Proof* From the result of Theorem 3.1, let $\gamma = n$ be a non-negative integer when $k = n + 1$, (3.5) gives $P_0(\tau) = 0$. Therefore, from (3.4), we get $A_{-1}(\tau) = 0$. Similarly, by setting $k = n + 2, n + 3, n + 4, \ldots$, we obtain recursively $A_{-2}(\tau) = 0, A_{-2}(\tau) = 0, \ldots$, respectively. Thus, (3.2) is reduced to a finite sum in the form

$$U_E^{(n,\alpha,\beta)}(v, \tau) = e^{B(\tau)v+\beta} \sum_{k=0}^{n} A_{n-k}(\tau) v^{n-k}. \tag{3.29}$$

Setting $k = n - j$, the sum (3.29) can be rewritten as

$$U_E^{(n,\alpha,\beta)}(v, \tau) = e^{B(\tau)v+\beta} \sum_{j=0}^{n} A_j(\tau) v^j, \tag{3.30}$$

where the indexes of $A_\gamma(\tau)$, $A_{\gamma-k}(\tau)$, $P_{\gamma-k+1}(\tau)$ and $Q_{\gamma-k}$ in (3.3)-(3.6) become $A_n(\tau)$, $A_j(\tau)$, $P_{j+1}(\tau)$ and $Q_j$ as shown in (3.25)-(3.28), respectively.

Furthermore, since from (3.27) $P_{j+1}(\tau) > 0$ for all $\tau > 0$, and (3.25) and (3.26) guarantee that $A_j(\tau) > 0$ for $j \in \{0, 1, 2, \ldots, n\}$, we can conclude that, $U_E^{(n,\alpha,\beta)}(v, \tau)$ is strictly increasing with respect to $v$ for $\tau > 0$ and $v > 0$. ■

Calculations of the expectation (1.7) when $\kappa(t), \theta(t)$ and $\sigma(t)$ are constants for all $0 \le t \le T$, the ECIR model (1.3) reduces to the CIR model (1.2) as stated in Theorems 3.6 and 3.8.

## 3.2 CIR process

**Theorem 3.6.** *Suppose that $V_t$ follows the CIR process with $\kappa(t) = \kappa$, $\theta(t) = \theta$ and $\sigma(t) = \sigma$. Let $\gamma, \alpha, \beta \in \mathbb{R}$. Then,*

$$\begin{aligned} U_C^{(\gamma,\alpha,\beta)}(v,\tau) &:= \mathbb{E}^{\mathbb{P}}\left[V_T^{\gamma} e^{\alpha V_T+\beta} \mid V_t = v\right], \\ &= \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)} v + \beta + \gamma\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right] \times \\ &\quad \left(\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right)^{\frac{2}{\sigma^2}(\gamma\sigma^2+\kappa\theta)} v^{\gamma} \\ &\quad + \sum_{k=1}^{\infty}\left\{\exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)} v + \beta + (\gamma - k)\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right] \times \right. \\ &\quad \left. \left(\prod_{m=1}^{k} \bar{P}_{\gamma-m+1}\right)\left(\frac{e^{\kappa\tau} - 1}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right)^{k}\right\} v^{\gamma-k}, \end{aligned} \tag{3.31}$$

*where*

$$\bar{P}_{\gamma-m+1} = (\gamma + 1)\left(\frac{1}{2}(\gamma - m)\sigma^2 + \kappa\theta\right) \tag{3.32}$$

*when $m \in \{1, 2, ..., k\}$.*

*Proof* From (3.3)-(3.7), when $\kappa(t)$, $\theta(t)$ and $\sigma(t)$ are constants, (3.7) can be written as

$$\bar{B}(\tau) = \alpha \exp\left[-\int_0^{\tau} \kappa du\right]\left[1 - \alpha\int_0^{\tau} \tfrac{1}{2}\sigma^2 \exp\left[-\int_0^{s} \kappa du\right] ds\right]^{-1} \tag{3.33}$$

$$= \frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}. \tag{3.34}$$

Thus, we have

$$\int_0^{\tau} \bar{B}(u)du = \int_0^{\tau} \frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa u}(2\kappa - \alpha\sigma^2)} du \tag{3.35}$$

$$= \frac{2}{\sigma^2}\left[\kappa\tau + \ln\left[\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right]\right]. \tag{3.36}$$

Consider $A_{\gamma}(\tau)$ from (3.3), we have

$$\bar{A}_{\gamma}(\tau) = \exp\left[-\gamma\kappa\tau + (\gamma\sigma^2 + \kappa\theta)\int_0^{\tau} \bar{B}(u)du\right] \tag{3.37}$$

$$= \exp\left[\gamma\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right]\left[\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right]^{\frac{2}{\sigma^2}(\gamma\sigma^2+\kappa\theta)}. \tag{3.38}$$

Letting

$$\bar{Q}_{\gamma-k}(\tau) = (\gamma - k)\sigma^2\bar{B}(T - \tau) + \kappa\theta\bar{B}(T - \tau) - (\gamma - k)\kappa \tag{3.39}$$

yields

$$\exp\left[\int_0^{\tau}\bar{Q}_{\gamma-k}(T-u)du\right] = \exp\left[-(\gamma-k)\kappa\tau+((\gamma-k)\sigma^2+\kappa\theta)\int_0^{\tau}\bar{B}(u)du\right] \quad (3.40)$$

$$= \exp\left[(\gamma-k)\kappa\tau+\frac{2\theta\kappa^2\tau}{\sigma^2}\right]\times \left[\frac{2\kappa}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}\right]^{\frac{2}{\sigma^2}((\gamma-k)\sigma^2+\kappa\theta)}. \quad (3.41)$$

From the result presented in (3.4), we obtain

$$\bar{A}_{\gamma-k}(\tau) = \bar{P}_{\gamma-k+1}e^{(\gamma-k)\kappa\tau+\frac{2\theta\kappa^2\tau}{\sigma^2}}\left[\frac{2\kappa}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}\right]^{\frac{2}{\sigma^2}((\gamma-k)\sigma^2+\kappa\theta)}\times \int_0^{\tau}e^{-(\gamma-k)\kappa\tau-\frac{2\theta\kappa^2\tau}{\sigma^2}}\left[\frac{2\kappa}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}\right]^{-\frac{2}{\sigma^2}((\gamma-k)\sigma^2+\kappa\theta)}A_{\gamma-k+1}(s)ds \quad (3.42)$$

for $k\in\mathbb{N}$. Using the inductive hypothesis

$$\bar{A}_{\gamma}(\tau) = e^{\gamma\kappa\tau+\frac{2\theta\kappa^2\tau}{\sigma^2}}\left[\frac{2\kappa}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}\right]^{\frac{2}{\sigma^2}(\gamma\sigma^2+\kappa\theta)} \quad (3.43)$$

with

$$\bar{A}_{\gamma-1}(\tau) = \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}v+\beta+(\gamma-1)\kappa\tau+\frac{2\theta\kappa^2\tau}{\sigma^2}\right]\times \bar{P}_{\gamma}\left(\frac{e^{\kappa\tau}-1}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}\right), \quad (3.44)$$

yields

$$\bar{A}_{\gamma-k}(\tau) = \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}v+\beta+(\gamma-k)\kappa\tau+\frac{2\theta\kappa^2\tau}{\sigma^2}\right]\times \left(\prod_{m=1}^{k}\bar{P}_{\gamma-m+1}\right)\left(\frac{e^{\kappa\tau}-1}{\alpha\sigma^2+e^{\kappa\tau}(2\kappa-\alpha\sigma^2)}\right)^{k} \quad (3.45)$$

for $k\in\mathbb{N}$. The formula (3.31) is obtained by inserting $A_{\gamma-k}(\tau)$, $k\in\{0,1,2,\ldots\}$ into (3.2). ■

Similarly, for ECIR case, the following corollary shows a consequence that is deduced from Theorem 3.6 when $\gamma = 1-\frac{2\kappa\theta}{\sigma^2}$. the explicit form is reduced to a closed-form as shown in the following corollary.

**Corollary 3.7.** *Suppose that $V_t$ follows the CIR process with $\kappa(t) = \kappa$, $\theta(t) = \theta$ and $\gamma$ satisfies*

$$\gamma = 1 - \frac{2\kappa\theta}{\sigma^2}. \tag{3.46}$$

*Then,*

$$\begin{aligned} U_C^{(\gamma,\alpha,\beta)}(v,\tau) &= \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}v + \beta + \kappa\tau\right] \times \\ &\left(\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right)^{\frac{2}{\sigma^2}(\sigma^2 - \kappa\theta)} v^{1-\frac{2\kappa\theta}{\sigma^2}}. \end{aligned} \tag{3.47}$$

*Proof* Similar to the proof of Corollary 3.4, from (3.32) when $k = 1$, that $\bar{P}_\gamma = 0$ if $\gamma = 1 - \frac{2\kappa\theta}{\sigma^2}$ for all $\tau \geq 0$. Therefore, (3.45) implies $\bar{A}_{\gamma-k}(\tau) = 0$ for all $k \in \mathbb{N}$ and the remaining term of (3.31) is $\bar{A}_\gamma$. ■

Similarly, the result of Theorem 3.6 can be simplified into a finite sum in the case when $\gamma$ is a non-negative integer, as stated in the following result.

**Theorem 3.8.** *Suppose that $V_t$ follows the CIR process with $\kappa(t) = \kappa$, $\theta(t) = \theta$ and $\sigma(t) = \sigma$. Let $n$ be a non-negative integer. Then,*

$$\begin{aligned} U_C^{(n,\alpha,\beta)}(v,\tau) &:= \mathbb{E}^{\mathbb{P}}\left[V_T^n e^{\alpha V_T + \beta} \mid V_t = v\right], \\ &= \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}v + \beta + n\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right] \times \\ &\left(\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right)^{\frac{2}{\sigma^2}(n\sigma^2 + \kappa\theta)} v^n \\ &+ \sum_{j=0}^{n-1} \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}v + \beta + j\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right] \times \\ &\prod_{m=1}^{n-j} \bar{P}_{n-m+1} \frac{2^{n-j}}{(n-j)!}\left(\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right)^{\frac{2}{\sigma^2}(j\sigma^2 + \kappa\theta)} \times \\ &\left(\frac{e^{\kappa\tau} - 1}{\alpha\sigma^2 + e^{\kappa\tau}(2\kappa - \alpha\sigma^2)}\right)^{n-j} v^j \end{aligned} \tag{3.48}$$

*for all $v > 0$ and $\tau = T - t \geq 0$ where $\bar{P}_{n-m+1} = (n-m+1)\left(\frac{1}{2}(n-m)\sigma^2 + \kappa\theta\right)$ when $m \in \{1, 2, 3, ..., n-j\}$, for $j \in \{1, 2, 3, ..., n-1\}$.*

*Proof* From the result of Theorem 3.6, for $\gamma = n$ be a non-negative integer and $k = n + 1$, (3.32) gives $\bar{P}_0(\tau) = 0$. Therefore, from (3.44), we get $\bar{A}_{-1}(\tau) = 0$. Similarly, by setting $k = n+2, n+3, n+4, \ldots$, we obtain recursively $\bar{A}_{-2}(\tau) = 0, \bar{A}_{-3}(\tau) = 0, \bar{A}_{-4}(\tau) = 0, \ldots$, respectively. Thus, (3.2) is reduced to a finite sum in the form

$$U_C^{(n,\alpha,\beta)}(v,\tau) = e^{\bar{B}(\tau)v+\beta} \sum_{k=0}^{n} \bar{A}_{n-k}(\tau) v^{n-k}. \tag{3.49}$$

Setting $k = n - j$, the sum (3.49) can be rewritten in the form

$$U_C^{(n,\alpha,\beta)}(v,\tau) = e^{\bar{B}(\tau)v+\beta} \sum_{j=0}^{n} \bar{A}_j(\tau) v^j, \tag{3.50}$$

as required. ∎