# Application of Fractional Exponential Feature to GARCH Model Variants for Improvement in Value-at-Risk Prediction

Mr. Chanet Saisatian



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การประยุกต์ใช้คุณสมบัติการยกกำลังเศษส่วนในตัวแบบการ์ชที่มีการแปรเพื่อปรับปรุงการพยากร
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นายชะเนษฎ์ สายเสถียร



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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คณะพาณิชยศาสตร์และการบัญชี จุฬาลงกรณ์มหาวิทยาลัย
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Accepted by the FACULTY OF COMMERCE AND ACCOUNTANCY, Chulalongkorn University in Partial Fulfillment of the Requirement for the Master of Science

............................................ Dean of the FACULTY OF COMMERCE AND ACCOUNTANCY
(Associate Professor Wilert Puriwat, Ph.D.)

THESIS COMMITTEE

............................................ Chairman
(Associate Professor THAISIRI WATEWAI, Ph.D.)

............................................ Thesis Advisor
(Associate Professor SIRA SUCHINTABANDID, Ph.D.)

............................................ Examiner
(Tanawit Sae-Sue, Ph.D.)

............................................ External Examiner
(ANANT CHIARAWONGSE, Ph.D.)

ชะเนษฎ์ สายเสถียร :
การประยุกต์ใช้คุณสมบัติการยกกำลังเศษส่วนในตัวแบบการ์ชที่มีการแปรเพื่อปรับปรุงการพยากรณ์มูลค่าความเสี่ยง. ( Application of Fractional Exponential Feature to GARCH Model Variants for Improvement in Value-at-Risk Prediction) อ.ที่ปรึกษาหลัก : รศ. ดร.สิระ สุจินตะบัณฑิต

งานวิจัยนี้ ศึกษาเกี่ยวกับการใช้ตัวแบบการ์ชที่มีการแปร ในฐานะแนวทางอิงพารามิเตอร์ในการประมาณและพยากรณ์มูลค่าความเสี่ยงรายวัน ซึ่งเป็นหนึ่งในมาตรวัดความเสี่ยงที่แพร่หลายโดยเฉพาะในวงการการเงิน เพื่อให้รับกับข้อเท็จจริงหลายประการเกี่ยวกับความผันผวนของตลาด งานวิจัยนี้ได้นำเสนอตัวแบบการ์ชผสมสองตัว: ตัวแบบไฮ-จีเจอาร์-การ์ช ลูกผสมระหว่างตัวแบบไฮเพอร์โบลิกการ์ช (ไฮการ์ช) กับตัวแบบจีเจอาร์-การ์ช และตัวแบบไฮ-เอ็มเอส-การ์ช ที่ผสมระหว่างตัวแบบไฮการ์ชและตัวแบบการ์ชสับเปลี่ยนมาร์คอฟ (เอ็มเอสการ์ช) ตัวแบบผสมเหล่านี้ซึ่งมีการวางสูตรทางคณิตศาสตร์จำนวนมากและได้ประโยชน์จากตัวแบบฐาน ถูกคาดหวังว่าจะต้องมีความสามารถในการพยากรณ์มูลค่าความเสี่ยงรายวันที่ดีกว่าตัวแบบฐานและการจำลองแบบประวัติศาสตร์ ในการศึกษาเชิงประจักษ์จากผลตอบแทนรายวันของดัชนี S&P 500 ตั้งแต่ปี 1960 ถึง 1999 ได้นำตัวแบบผสมและตัวแบบฐานทั้งหมดมาใช้ปรับให้เหมาะกับข้อมูลและพยากรณ์มูลค่าความเสี่ยงรายวันล่วงหน้า ซึ่งจะถูกเปรียบเทียบด้วยการทดสอบหลายประการ ผลการศึกษาแสดงให้เห็นถึงความสามารถของตัวแบบผสม ซึ่งขึ้นอยู่กับสถานการณ์ของตลาด ว่าตัวแบบผสมใดและการแจกแจงใดที่ประกอบกันควรจะนำมาใช้ ผลการศึกษาชี้ให้เห็นด้วยว่าตัวแบบไฮ-จีเจอาร์-การ์ช กับการแจกแจงปกติ โดยทั่วไปแล้วมักดีกว่าตัวแบบอื่นโดยเปรียบเทียบ และตัวแบบไฮ-เอ็มเอส-การ์ช กับการแจกแจงสติวเดนท์ที เป็นตัวแบบที่เหมาะสมที่สุดในช่วงที่ตลาดตกหนักเนื่องจากวิกฤต



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This research studies about using GARCH model variants as a parametric way in estimation and prediction of daily Value-at-Risk (VaR), one of famous risk measurement especially in financial world. To cope with various stylized facts on market's volatility, two mixed GARCH models are proposed in this research: HY-GJR-GARCH model, the hybrid between hyperbolic GARCH (HYGARCH) and GJR-GARCH models, and HY-MS-GARCH model as the amalgam between HYGARCH and Markov switching GARCH (MSGARCH) models. These mixed models, along with rich mathematical formulations and benefits from their base models, are expected that their performance in predicting daily VaR is advanced against the performance of their base models and historical simulations. Using the empirical study of daily S&P 500 index return from 1960 to 1999, all mixed and base models are implemented in fitting data and forward prediction of daily VaR that are compared in various tests. The results indicate dependence of mixed model's performance on market situation that suggested which mixed model and supplementary probability distribution should be used. These results also suggest that the HY-GJR-GARCH mixed model with normal distribution generally advances other models in comparisons, and the HY-MS-GARCH mixed model with Student's t distribution is the most appropriate when a big crisis plunges the market.



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Due to the stressful COVID-19 pandemic, this has been the harsh time while every moments in the program were forced to comply with the "New Normal". Thus, I have spend both time and budget for more than 3 years to achieve the degree, while my beloved paternal grandmother passed away in 2020 with strong expectation to see my completion of Master's degree that I dare not tell the truth when she asked me one day before her peaceful death. In memory of my paternal grandmother, and in financial support from my family, I would like to acknowledge them as integral parts for my achievement of this thesis.

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Chanet Saisatian

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# Chapter 1 Introduction

## 1.1 Motivations and Rationales

Value-at-Risk (VaR), defined by the expected loss minimum at the worst scenarios (that are usually counted by some proportion of all possible scenarios called the level of significance) in a period (called holding period), as illustrated in Figure 1, is widely used as a risk metrics by the business because of its straightforward concept. Especially, the financial industry generally acclaims the VaR to be one of well-known financial risk measurements. Then, the reasonable and practical estimation is needed to trace historical loss risk, project loss probability in the future, and compare loss risk by time and across the industry. However, there are two major branches of VaR estimation methods: parametric and non-parametric.

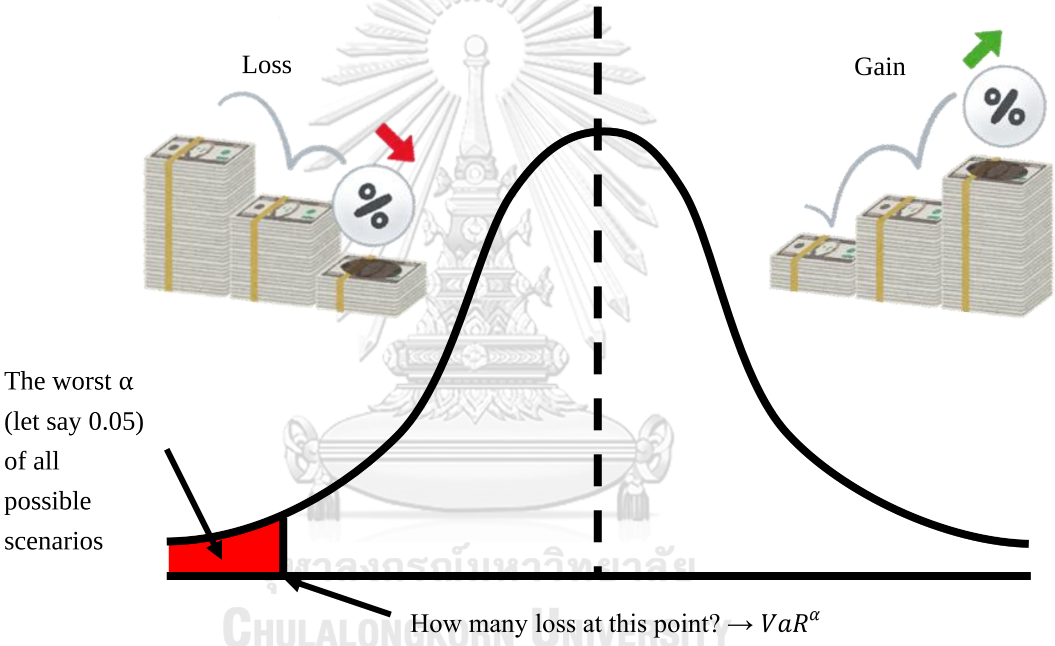


*Figure 1 The simple illustration for the concept of VaR*

Parametric VaR estimation methods, the more numerical and complicated branch, utilize time series models to describe behavior of financial returns under the assumption about their temporal relationship and their distribution of residuals. In conventional ways, the returns' conditional volatility is predicted first, and the VaR is estimated using the prediction and the assumed distribution of residuals. Assumed the conditional variance has linear relations with square of previous residuals, the autoregressive conditional heteroskedastic (ARCH) model, first introduced by Engle (1982), is one of renown models to parametrize the returns' volatility as a time series. Analogous to the combination of autoregression and moving averages as autoregressive moving average (ARMA) model, Bollerslev (1986) added the relation between present and past conditional volatilities to the ARCH model to form the generalized ARCH (GARCH) model that has been influential to practitioners and academics as discussed below.

Branched from the original simple GARCH model, econometrists have developed many variations of GARCH model to response specific stylized facts of volatility as found in financial time series, as the hierarchy illustrated in Figure 2.

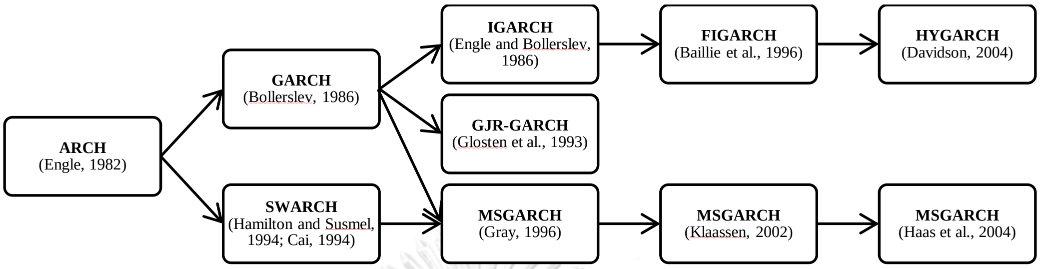


*Figure 2 The hierarchy of GARCH model variants, branched from the ARCH model of Engle (1982)*

One of the stylized facts is the leverage effect: the effect of residuals on the volatility depends on their signs, minus more than plus, as mentioned by Black (1976) and Nelson (1991). In order to reflect this fact in GARCH model, Glosten et al. (1993) proposed the addition of penalty terms for negative residuals, parallel to ARCH parameters, to modify the GARCH model variation that is commonly known as Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model.

Also, there is a stylized fact, mentioned by Engle and Mustafa (1992) and Friedman et al. (1989), about the inversely proportional persistence to shock size: the larger shock size, the shorter persistence of effect on volatility. Unfortunately, this fact cannot be covered by the simple GARCH model. Hence, using of Markov chain process was adapted to ARCH model to cope with inversely proportional persistence first by Hamilton and Susmel (1994) and Cai (1994) as Markov switching ARCH (SWARCH) model, which is adapted to the Markov switching GARCH (MSGARCH) model by Gray (1996). Furthermore, the concept from Gray (1996) MSGARCH model was also improved to the adjusted model of Klaassen (2002) and the simplified model of Haas et al. (2004).

The other stylized fact comes from the long memory properties of volatility in case that autocorrelation decreases hyperbolically by time, afterward, the sum of autocorrelation is infinite, as mentioned by Dacorogna et al. (1993) and Ding et al. (1993). Then, the variations of GARCH model with fractional exponent on residuals were developed. The prominent examples of fractional exponential GARCH models are the fractionally integrated GARCH (FIGARCH) model by Baillie et al. (1996) and the hyperbolic GARCH (HYGARCH) by Davidson (2004). Unlike other GARCH model variants, both FIGARCH and HYGARCH models allow fractional exponents that can expand to infinite series in the equations. The only difference between

FIGARCH and HYGARCH models is about the stationarity. The more flexible HYGARCH model can be stationary in some cases, but the FIGARCH cannot.

In contrast to parametric VaR estimation, non-parametric VaR estimation methods utilize the characteristics of returns to estimate the VaR directly without the assumption about returns' distribution or relationship with other factors. Historical simulation (HS), proposed by Hendricks (1996), is one of famous non-parametric estimation, simply assuming that the distribution of returns is always identical to history. Also, Barone-Adesi and Giannopoulos (1996) introduced the filtered historical simulation (FHS) that is semi-parametric, the combination between the use the GARCH model as a parametric way to predict volatility and the traditional historical simulation for standardized returns as a non-parametric way.

Consider the GJR-GARCH, MSGARCH, and HYGARCH models. As each model is developed to capture different stylized fact, unfortunately, a model cannot capture other stylized facts that the model is not designed to cope with. Consider especially when each model is used solely in the estimation of VaR. The GJR-GARCH is admired by many literatures from its better response to sensitive change in data, but fewer number of parameters makes the GJR-GARCH theoretically worse in fitting in-sample data in comparison to other GARCH model variants. For the MSGARCH model that emphasizes the better reflection of after-shock effect, especially in long-term data, the study from Zhang et al. (2019) found the weakness of MSGARCH model when fitting to daily data: the MSGARCH model is less-fitted to data than other models like simple GARCH or GJR-GARCH models, but this problem is not found for weekly data. Unlike other models, the fractional exponential models like FIGARCH and HYGARCH models have benefit from their flexibility to fit the data better. However, the study of Degiannakis et al. (2013) showed indifference between the FIGARCH and GARCH models when estimating the VaR.

Because using each of these GARCH model variants solely for estimation of VaR has its pros and cons, many literatures (Chen et al., 2012, Messaoud and Aloui, 2015, Zhang et al., 2018, Stavroyiannis, 2018) decided that the only the model is not enough: it needs more complexity on noise distribution to alternatively improve model's power on prediction.

This research, however, tries the direct way to improve these models: focusing on improvement on equation to be more complex, while remaining simple noise distribution like the Gaussian or Student's t distribution. One of interesting choices is combining between models to introduce a mixed model, following the prominent example of Hamilton and Susmel (1994) that the combination between the leverage of GJR-GARCH model and the Markov process was found.

Consider all possible combinations among the GJR-GARCH, MSGARCH, and HYGARCH models. If all three models are combined to produce a mixed model, its equation becomes too complicated in implementation sense. The combination between the GJR-GARCH and MSGARCH is identical to prior study of Hamilton and Susmel (1994). Consequently, there are two remaining combinations that is not found in previous studies, the HYGARCH model with GJR-GARCH model, and the HYGARCH model with MSGARCH model.

As mentioned before, the HYGARCH model seems to compensate the GJR-GARCH or MSGARCH model in problem of data fitting. Thus, this research decides to combine the feature of HYGARCH model with GJR-GARCH or MSGARCH model regarding two advantages: these model combinations could fit to data better due to the strength of HYGARCH model, and the problem of data overfitting in HYGARCH model could be avoided by the strength of other model (GJR-GARCH or MSGARCH) in these combinations.

However, the worthiness of combination is important to consider in this research. Firstly expected, these mixed models should predict the VaR better than their pair of base models. This expectation confirms the righteousness to combine these GARCH model variants together. Another expectation is about the competency versus simulation-based methods. If the model combination cannot advance the straightforward HS method, it is not economical to implement that mixed model for poorer prediction than the prediction from the simulation. Thus, the mixed model should prosper the HS and the semi-parametric FHS methods.

## 1.2 Research Question



Given three GARCH model variants (GJR-GARCH, Haas et al. (2004) MSGARCH, and HYGARCH) mentioned above, let two mixed models, proposed in this research, are formulated using the characteristics of each combinations between two base models as denoted in the following table:

*Table 1 Combination between GARCH model variants formulated in this study*

| | HYGARCH (Davidson, 2004) |
|---|---|
| GJR-GARCH (Glosten et al., 1993) | *Mixed Model 1:* ***HY-GJR-GARCH*** |
| MSGARCH (Haas et al., 2004) | *Mixed Model 2:* ***HY-MS-GARCH*** |

Are these mixed models better in estimation and prediction of 1-day VaR than their base models, the historical simulation, and the filtered historical simulation?

Considering prediction accuracy, autoregressive independence, and effect when the estimated VaR is violated by realized losses, how are these models better?

Consider the fractional exponential feature in HYGARCH model. The studies from Wu and Shieh (2007) and Charfeddine (2016) showed the superiority of fractional exponential feature in FIGARCH model that makes the model better than typical GARCH in prediction of VaR. Thus, it can be inferred that combining GJR-GARCH or MSGARCH model with HYGARCH model should boost prediction accuracy, boost autoregressive independence, and reduce excess losses under the VaR estimation to the mixed models against their own base models and, also, historical simulations.

Generally compared between GJR-GARCH and MSGARCH models, the GJR-GARCH model has higher prediction accuracy, higher autoregressive independence, and less negative difference whether the realized returns penetrate the VaR estimation, than the MSGARCH model. Thus, while comparing between two mixed models, the first mixed model with GJR-GARCH feature could be expected more effective than the second model.

# Chapter 2 Literature Review

## 2.1 GJR-GARCH model in the econometrics world

The GJR-GARCH model is well-known for its ability to capture the consequence from leverage effects. This benefit was first confirmed by Engle and Ng (1993) empirical results. In case study of Japanese TOPIX index since 1980 to 1988, the GJR-GARCH model was the best fitting model against prior GARCH model variants, i.e., the standard GARCH and exponential GARCH (EGARCH) model. In addition, the GJR-GARCH model can surpass the EGARCH model in parametrization of the impact on negative returns, as the EGARCH model overestimates this impact in extreme cases, but the GJR-GARCH model does not.

Furthermore, there are several studies on its application to estimate and forecast the VaR. One way to study is comparison between models. Su et al. (2011) used the GJR-GARCH model to forecast the VaR of simulated portfolios in comparison to the standard GARCH. Accompanied with conditional mean models and periodic updates in estimation, the GJR-GARCH is appropriate to calculate the VaR.

Also, the GJR-GARCH is also mentioned as a benchmark for further models. Louzis et al. (2014) chose the GJR-GARCH as a benchmark against the asymmetric heterogenous autoregressive (HAR) model to compete in estimation of VaR. After the study on various financial data represented each asset class, the results showed that the GJR-GARCH models with skewed Student’s t distribution, extreme value theory (EVT), and the FHS are accredited for their accuracy but not enough efficient for applications about capital requirements. Bams et al. (2017) placed the GJR-GARCH against the HS and implied volatility in the competition of VaR estimation. Based on American stock indices data, the VaR using GJR-GARCH model outperformed other rivals.

There are also the works that focused on the proposal of GJR-GARCH model variants or various distributions in use with the GJR-GARCH model. Chen et al. (2012) worked on the GJR-GARCH model with asymmetric Laplace distribution. With the empirical tests on data of market indices and exchange rates, the proposed GJR-GARCH model was suggested better than the GJR-GARCH model with Student’s t distribution and several conventional ways to estimate the VaR. Messaoud and Aloui (2015) proposed the GJR-GARCH model with the EVT and copula, along with empirical results on emerging market indices. Zhang et al. (2018) adapted the dynamic spatial panel to the GJR-GARCH model that resulted more accurate VaR prediction for world major market indices than the prediction from its base models. Stavroyiannis (2018) utilized the GJR-GARCH model with standardized Pearson type-IV distribution to the application on price of digital currency Bitcoin that proved how the GJR-GARCH model can reflect the higher risk of Bitcoin than typical financial assets.

## 2.2 Markov switching GARCH in competition with the single-regime one

Like the older GJR-GARCH model, there are many studies dedicated to the MSGARCH model, especially the MSGARCH model of Haas et al. (2004). In its specified application of predicting the VaR, Ardia et al. (2018) and Ardia et al. (2019) both used the MSGARCH model that is mentioned "the most natural and straightforward extension" to the GARCH model for the computation of VaR of different assets: stocks and exchange rates for Ardia et al. (2018), and digital currency rate of Bitcoin for Ardia et al. (2019). The following conclusion came that the MSGARCH model is more compatible with the equities, but the application of MSGARCH model usually benefits even when applied to other assets.

Both works applied the Markov process to the standard GARCH and GJR-GARCH models. However, some studies focused only on the Markov switching GJR-GARCH model. Sampid et al. (2018) proposed the specific use of the MSGARCH model with GJR-GARCH extension, skewed Student's t distribution, copula transformation, and EVT, on the VaR calculation based on stock prices. A similar study came from Liao et al. (2019), but the model with simpler symmetric Student's t distribution was preferred instead.

One of the characteristics of the MSGARCH model is about fittingness to the data. Mentioned in Charfeddine (2016), the 3-regime MSGARCH model with Student's t distribution is the most fitting model to the commodity future prices in comparison to other GARCH variants with the same distribution. However, the drawback of MSGARCH model comes from the highest number of parameters that compensate the fittingness. In overall, the MSGARCH model is not better than long-memory GARCH models.



## 2.3 The FIGARCH models in empirical use cases

Like priorly mentioned GJR-GARCH and MSGARCH models, the application of fractional exponential GARCH models for VaR estimation was studied due to the long memory in financial data. As the oldest fractional exponential GARCH model, the FIGARCH model is the most mentioned one. Wu and Shieh (2007) used T-bond interest rate future prices as a case study to prove the difference between the FIGARCH and GARCH models. Although the in-sample results showed the indifference, the out-sample results suggested that the FIGARCH model with skewed Student's t distribution was the most preferable model in comparison to the FIGARCH and GARCH models with normal, Student's t, and skewed Student's t distributions. This phenomenon was also confirmed in the study of Charfeddine (2016) with commodity future prices. The FIGARCH and the fractionally integrated exponential GARCH (FIEGARCH) models were the most preferable models according to both in-sample and out-sample results. Charfeddine (2016) also suggested that the weekly frequency is the appropriate one to observe long memory in commodity future prices.

As many studies preferred the FIGARCH model among other GARCH model variants, the FIGARCH model itself was used as a risk metric. Sukono et al. (2017) and Biage (2019) both used the FIGARCH model for VaR calculation to reflect the market risk in stock prices. Sukono et al. (2017) used the FIGARCH-modeled conditional variance, together with the ARMA-modeled conditional mean, for the market risk comparison between Indonesian stocks. In a different perspective, Biage (2019) used the sole FIGARCH model for the different comparison between constituents in American DJIA index and those in Brazilian Ibovespa index to compare the efficiency between markets.

## 2.4 FIAPARCH model: implication to some gap?

Also, there was much effort to adapt the fractional exponential characteristics to prior ARCH and GARCH models' variations after the FIGARCH model. The HYGARCH model of Davidson (2004), mentioned above, is one of prominent examples. However, there is the fractionally integrated APARCH (FIAPARCH) model of Tse (1998) that was proposed before the HYGARCH model. Tse (1998) applied fractional exponential feature to the asymmetric power ARCH (APARCH) model of Ding et al. (1993) to introduce the FIAPARCH model. Tse (1998) demonstrated indifference between prior APARCH and new FIAPARCH models when estimating the volatility parameter of daily yen-dollar exchange rate from 1978 to 1994 and the daily maintenance margin needed for trading the rate's future.

As seen from Figure 3, although the APARCH model equation with power parameter of two is considered equivalent to the GJR-GARCH model, their formations come from different perspectives: APARCH works on absolute value of residuals, but GJR-GARCH works on dummy variable on residuals. For the combination between FIGARCH and APARCH models, they become the FIAPARCH. If the FIGARCH model is replaced by the more flexible HYGARCH model, and the APARCH model is substituted by the more specified GJR-GARCH model, what will become from this combination? Hence, this research dedicates effort to adapt the fractional exponential characteristics to the case study of GJR-GARCH model, the mixed model 1 in the research question, that is difference from the prior study on the FIAPARCH model.

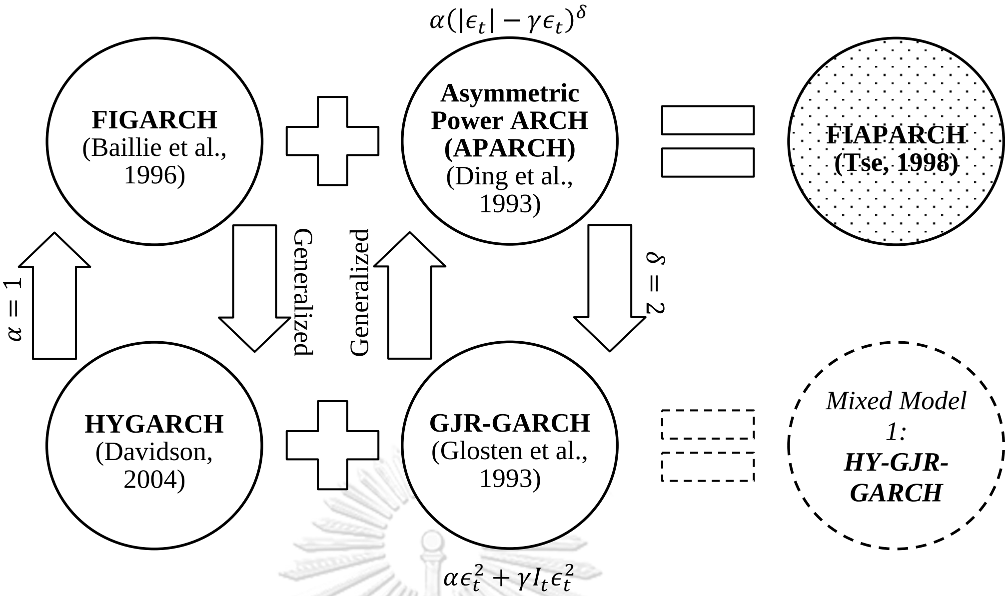


*Figure 3 Illustration of the literature gap as seen from the invention of FIAPARCH model*

## 2.5 FIGARCH, HYGARCH, and FIAPARCH models: who wins in comparison?

Afterward, the FIGARCH, HYGARCH, and FIAPARCH models were usually mentioned together in the comparison of fractional exponential GARCH models. For the application of VaR calculation, there are Tang and Shieh (2006), Aloui and Mabrouk (2010), Demiralay and Ulusoy (2014), Mabrouk (2016), and Buberkoku (2019). The skewed Student's t distribution was preferred by almost all these works, excluded Demiralay and Ulusoy (2014) that mentioned only symmetric distributions. In comparison between models, Aloui and Mabrouk (2010), Demiralay and Ulusoy (2014), and Mabrouk (2016) unanimously suggested the outperformance of the FIAPARCH model beyond the FIGARCH and HYGARCH models in various cases. However, Buberkoku (2019) disagreed with these results. After the test with various financial data, the HYGARCH model won all other models, the FHS, and the HS in the estimation of short-positioned VaR. Moreover, all models could not perform better than the FHS in long-positioned VaR estimation. However, in the pair comparison between the FIGARCH and HYGARCH models, Tang and Shieh (2006) did not incline to only one model due to the empirical results on stock index future prices that each model performed better on some data set.

## 2.6 When fractional exponential GARCH meets Markov process

Since this research concentrates on the combination between asymmetry GARCH and fractional exponential GARCH models, the combinations between models in literature are reviewed. For the combination between Markov switching process and fractional exponential GARCH models, Basatini and Rezakhah (2020) proposed the way to apply the Markov process to the smooth transition HYGARCH (ST-HYGARCH) model from Basatini and Rezakhah (2019) as the Markov switching smooth transition HYGARCH (MSST-HYGARCH) model. Basatini and Rezakhah (2020) also demonstrated how better the MSST-HYGARCH model can do than the ST-HYGARCH model and HYGARCH model from the case study of daily VaR prediction on S&P 500 and DJIA indices.

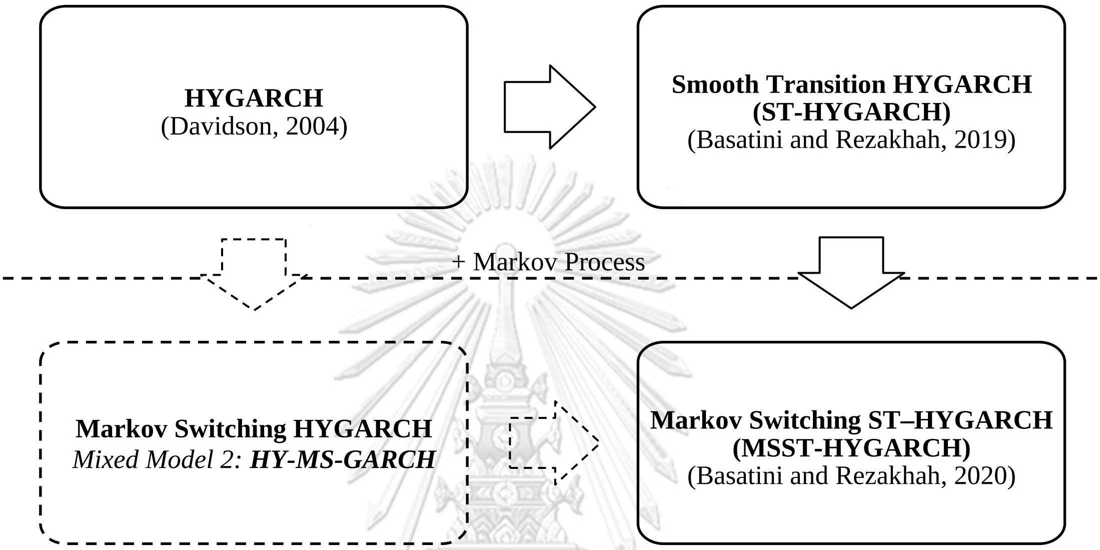


*Figure 4 The literature gap as seen from Basatini and Rezakhah (2019) to Basatini and Rezakhah (2020)*

As seen in Figure 4, the way to MSST-HYGARCH model in the literatures comes in order of smooth transition before the Markov process. However, there is an alternative way that the Markov process might apply to the HYGARCH model before smooth transition that Basatini and Rezakhah (2020) did not cover. What will happen if the Markov switching feature is applied to HYGARCH model? This question will be concerned in this research with the introduction of mixed model 2 in the research question.

Different from Basatini and Rezakhah (2020), Bildirici and Ersin (2014) made use of Markov switching process to the fractional exponential FIAPARCH model combining with the autoregressive moving average (ARMA) model. However, the MS-ARMA-FIAPARCH model was not the main player in the research. The MS-ARMA-FIAPARCH model was used as the noise model under the neural network, following the previous concept introduced by Spezia and Paroli (2008), to directly predict stock returns demonstrated by the Istanbul Stock Index ISE 100. This concept of using GARCH models under the more complicated neural network is interesting. However, when the neural network with GARCH noises is in use, the perspective is too close to the accuracy of the model, not the quantitative parameter estimated from the model to interpret for business. Hence, this research does not cover this complicated use of GARCH models and directly uses modified GARCH models as the main characteristics to predict the VaR.

# Chapter 3 Methodology

## 3.1 Models

### 3.1.1 Generalized autoregressive conditional heteroskedastic (GARCH) model

Assumed the returns $r_t$ is temporally constant around the mean $\mu_t$:

$$r_t = \mu_t + \epsilon_t, \tag{1}$$

and the residual $\epsilon_t$ is a random variable with the variance $\sigma_t^2$:

$$\epsilon_t = \eta_t \sigma_t, \tag{2}$$

so that $\eta_t$ is independent and identically distributed (i.i.d.) with zero mean and unit variance, conditionally on $\sigma_t$ and $\mathcal{I}_{t-1}$, the information as of time $t-1$, the generalized ARCH (GARCH) model for the variance, developed by Bollerslev (1986) on the base of Engle (1982) ARCH model, can be formulated as follows:

$$\boxed{\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2} \tag{3}$$

where $p > 0,\ q \geq 0,\ \omega > 0,\ \alpha_i \geq 0\ \forall i = 1, \dots, q,\ \beta_j \geq 0\ \forall j = 1, \dots, p.$

Mark that the terms $\omega, \alpha_i$ and $\beta_i$ are respectively called the constant, ARCH coefficients, and GARCH coefficients, $p$ is the number of lagged variances in use for the model, and $q$ is the number of lagged residuals. The numbers $p$ and $q$ are commonly denoted when referred to the GARCH model as the GARCH$(p, q)$ model, for example, the GARCH(1,2) model contains two lagged residuals and one lagged variance in the formulation. However, the equation (3) can be rewritten using the lag operators:

$$\begin{aligned} \sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2, & (3) \\ \sigma_t^2 - \beta_1 \sigma_{t-1}^2 - \cdots - \beta_p \sigma_{t-p}^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2, & \\ \left(1 - \beta_1 L - \cdots - \beta_p L^p\right) \sigma_t^2 &= \omega + \left(\alpha_1 L + \cdots + \alpha_q L^q\right) \epsilon_t^2. & (4) \end{aligned}$$

Let the coefficient functions be defined:

$$\alpha(x) = \sum_{i=1}^{q} \alpha_i x^i, \qquad \beta(x) = \sum_{j=1}^{p} \beta_j x^j.$$

The equation (4) can be rewritten as

$$\boxed{[1 - \beta(L)] \sigma_t^2 = \omega + \alpha(L) \epsilon_t^2.} \tag{5}$$

This formulation with lag operators and coefficient functions is convenient for further descriptions about GARCH model variations. The stationarity of the GARCH model occurs if and only if $\alpha(1)+\beta(1)<1$. Note that the i.i.d. $\eta_t$ needs the assumption about its distribution, commonly normal (Gaussian) or Student's t distribution, that define the likelihood function for the estimation step that commonly uses the maximum likelihood (ML) or quasi-maximum likelihood (QML) estimation process. Given the number of observations to estimate the model is $T$, if $\eta_t$ is assumed normal distributed, the log-likelihood function of the model is

$$LL_{\mathcal{N}} = -\frac{T\ln 2\pi}{2} - \frac{1}{2}\sum_{t=1}^{T}(\ln\sigma_t^2 + \eta_t^2). \tag{6}$$

If $\eta_t$ is assumed Student's t-distributed with the degree of freedom $\nu$ instead, the log-likelihood function is (Bollerslev, 1987)

$$LL_{t,\nu} = T\left(\ln\Gamma\left(\frac{\nu+1}{2}\right) - \ln\Gamma\left(\frac{\nu}{2}\right) - \frac{\ln(\nu-2)}{2}\right) - \frac{1}{2}\sum_{t=1}^{T}\left[\ln\sigma_t^2 + (1+\nu)\ln\left(1+\frac{\eta_t^2}{\nu-2}\right)\right] \tag{7}$$

where $\nu > 2$. Note that $\Gamma(\cdot)$ is the Gamma function.

In order to use the GARCH model to forecast the VaR at $1-\alpha$ (commonly 95 or 99 percent) confidence level and one-day holding period, denoted the $VaR_t^\alpha$, for the long position (that is only concerned in this research), first, the parameters in the model are estimated using historical data (practically recent return data of some sufficiently long period, i.e., 100 days). Next, the ahead conditional variance $E[\sigma_t|\mathcal{J}_{t-1}]$ is forecasted. For example of the GARCH(1,1) model,

$$\mathrm{E}[\sigma_t|\mathcal{J}_{t-1}] = (\mathrm{E}[\sigma_t^2|\mathcal{J}_{t-1}])^{\frac{1}{2}} = \left(\widehat{\omega} + \widehat{\alpha_1}\epsilon_{t-1}^2 + \widehat{\beta_1}\mathrm{E}[\sigma_{t-1}^2|\mathcal{J}_{t-1}]\right)^{\frac{1}{2}}. \tag{8}$$

The above assumption about the distribution of $\eta_t$ demands the final formula to calculate the VaR. If assumed normal, the formula is

$$VaR_t^\alpha = \mu_t + \Phi^{-1}(\alpha)\mathrm{E}[\sigma_t|\mathcal{J}_{t-1}] \tag{9}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the normal distribution. For the assumption of Student's t-distribution, the formula is

$$VaR_t^\alpha = \mu_t + t_\nu^{-1}(\alpha)\mathrm{E}[\sigma_t|\mathcal{J}_{t-1}] \tag{10}$$

where $t_\nu(\cdot)$ is the CDF of Student's t-distribution with degree of freedom $\nu$.

*3.1.1.1 Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model*

Given the returns $r_t$ be defined by the equations (1) and (2) and their additional assumptions, the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model for the variance is defined by the following formulation:

$$[1-\beta(L)]\sigma_t^2 = \omega + [\alpha(L) + \gamma(L)I_t]\epsilon_t^2 \tag{11}$$

where $I_t$ is a dummy variable defined as the following function:

$$I_t = \begin{cases} 1, & \epsilon_t < 0 \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

and the additional coefficient function $\gamma(x)$ is defined as

$$\gamma(x) = \sum_{i=1}^{q} \gamma_i x^i.$$

Glosten et al. (1993) studied the empirical observation on CRSP value-weighted index of NYSE stocks, and one of the conclusions was that the variance increases when the returns decrease below the expectation. Hence, the research proposed this modification for Bollerslev (1986) GARCH model. This model, named after the contributors as the GJR-GARCH model, allows difference responses on conditional volatility for positive and negative residuals.

Glosten et al. (1993) also suggested that the coefficient $\gamma_i$ is positive a priori. However, any $\alpha_i + \gamma_i$ might be negative, even $\sigma_t^2$ in some cases. For the specific case of the GJR-GARCH(1,1) model with a symmetric distribution, the stationarity of the model exists when $\alpha_1 + \beta_1 + \gamma_1/2 < 1$.

*3.1.1.2 Markov switching GARCH (MSGARCH) model*

After the Markov switching ARCH (SWARCH) model of Hamilton and Susmel (1994), Gray (1996), Klaassen (2002), and Haas et al. (2004) proposed the Markov switching GARCH (MSGARCH) models as the extension of SWARCH model. Although the MSGARCH models of Gray (1996) and Klaassen (2002) are the basement for further MSGARCH model by Haas et al. (2004), only the last models are considered to utilize in this research. However, the understanding about MSGARCH model of Gray (1996) and Klaassen (2002) is needed to describe the concept of the last MSGARCH model. Hence, the Gray (1996) and Klaassen (2002) MSGARCH model are explained briefly before.

3.1.1.2.1 Gray (1996) MSGARCH model

Given the returns $r_t$ have behaviors described by the equation (1) and (2) and the condition like the simple GARCH model's prerequisites, assumed that the volatility of return follows a Markov chain process with finite $K$ states, the conditional variance at time $t$ and state $k$, $\mathrm{E}[\sigma_t^2|S_t = k]$, as denoted afterward as $\sigma_{t,k}^2$, is described as the formulation below:

$$\begin{aligned}\sigma_{t,k}^2 &= \omega_k + \alpha_{1,k}\epsilon_{t-1}^2 + \beta_{1,k}\mathrm{E}[\sigma_{t-1}^2|\mathcal{I}_{t-2}] \\ &= \omega_k + \alpha_{1,k}\epsilon_{t-1}^2 + \beta_{1,k}\sum_{l=1}^{K}\mathrm{p}(S_{t-1} = l|\mathcal{I}_{t-2})\sigma_{t-1,l}^2\end{aligned} \tag{13}$$

As the descendant from SWARCH model, the Gray (1996) MSGARCH model used the expectation of lagged conditional variance to reduce the issue of path dependency found when SWARCH model is in use. Consider a Markov chain process with finite $K$ states and $T$ intervals of observations. According to the SWARCH model, each of $K^T$ possible paths in the overall process have their own conditional variances that depend on their paths. But for the Gray (1996) MSGARCH model, the final conditional variance for each path depends on only the final state, not the whole path.

3.1.1.2.2 Klaassen (2002) MSGARCH model

Given the same preconditions like Gray (1996) MSGARCH model's, the Klaassen (2002) formulation for the conditional variance $\sigma_{t,k}^2$ is

$$\begin{aligned}\sigma_{t,k}^2 &= \omega_k + \alpha_{1,k}\epsilon_{t-1}^2 + \beta_{1,k}\mathrm{E}[\sigma_{t-1}^2|\mathcal{I}_{t-1}, S_t = k] \\ &= \omega_k + \alpha_{1,k}\epsilon_{t-1}^2 + \beta_{1,k}\sum_{l=1}^{K}\mathrm{p}(S_{t-1} = l|\mathcal{I}_{t-1}, S_t = k)\sigma_{t-1,l}^2.\end{aligned} \tag{14}$$

In comparison to Gray (1996) MSGARCH model equation as shown in the equation (13), Klaassen (2002) decided to condition the expectation of lagged variance by the information to timestamp $t-1$ and the current state, which is slightly different from the equation (13) of Gray (1996).

However, to define the stationarity of Klaassen (2002) MSGARCH model is more complicated in comparison to the stationarity of a single-regime GARCH model. Consider the 2-regime MSGARCH model. Given the 2-by-2 matrix $A$ with elements $A_{ij} = (\alpha_{1,i} + \beta_{1,i})\mathrm{p}(S_{t+1} = j|S_t = i)$, and the 2-by-2 identity matrix $I_2$, the model is concerned stationary if the elements $A_{11}$ and $A_{22}$ are less than one, the determinant of $I_2 - A$ is positive, the stationarity for the GARCH models from each regime exists, and the corresponding Markov chain has long-run distribution.

3.1.1.2.3 Haas et al. (2004) MSGARCH model

Same preconditions as Gray (1996) and Klaassen (2002) MSGARCH models, the formulation for the conditional variance $\sigma_{t,k}^2$ proposed by Haas et al. (2004) is like the equation of standard GARCH model:

$$\sigma_{t,k}^2 = \omega_k + \alpha_{1,k}\epsilon_{t-1}^2 + \beta_{1,k}\sigma_{t-1,k}^2. \tag{15}$$

Haas et al. (2004) freed the lagged conditional variance term from its expectation and conditions to make the model simpler so that the GARCH model equation can be used independently across regimes.

Although the Haas et al. (2004) GARCH equation is simpler in comparison to the GARCH equation of Klaassen (2002), the condition for its stationarity is more complex. Consider the case of 2-regime GARCH model previously mentioned to describe Klaassen (2002) MSGARCH model's stationarity. Let the elements in $A$ be substituted by the 2-by-2 matrix such that $A_{ij} = \left\{\mathrm{diag}\left(\beta_{1,1}, \beta_{1,2}\right) + \left(\alpha_{1,1}, \alpha_{1,2}\right)e_i'\right\}\mathrm{p}(S_{t+1} = i|S_t = j)$ where $e_i$ is the $i$th 2-by-1 unit vector, i.e., $e_1 = (1,0)$ and $e_2 = (0,1)$. Then, $A$ becomes a 4-by-4 matrix, and the 4-by-4 identity matrix $I_4$ is needed. To meet the stationary condition, all eigenvalues of $A$ are needed to lie in a unit circle, in the other word, all absolute values of eigenvalues of $A$ must be less than one.

*3.1.1.3 Hyperbolic GARCH (HYGARCH) model*

Given the same basements about returns $r_t$ as mentioned by equations (1) and (2), like the standard GARCH model's, the equation for the hyperbolic GARCH (HYGARCH) model is



$$\sigma_t^2 = \omega[1-\beta(L)]^{-1} + \left\{1 - [1-\beta(L)]^{-1}\phi(L)\left[1 + \alpha[(1-L)^d - 1]\right]\right\}\epsilon_t^2 \tag{16}$$

where $0 < \alpha < 1$, $0 < d < 1$, $\phi(x) = \sum_{i=0}^{p}\phi_i x^i$ is a polynomial function with degree $p$, and the exponential term $(1-L)^d$ is defined by the sum of infinite lag operator polynomials:

$$(1-L)^d = \sum_{u=0}^{\infty}\pi_u L^u = \sum_{u=0}^{\infty}\left[\prod_{v=1}^{u}\frac{v-1-d}{v}\right]L^u.$$

Note that all solutions for equations $\phi(x) = 0$ and $1 - \beta(x) = 0$ are needed to lie outside the unit circle, in the other words, all absolute values of roots of these equations are more than or equal one.

Before the HYGARCH model is described on details, the understanding on the fractionally integrated GARCH (FIGARCH) model is urgent regarding the fact that the HYGARCH model stems from the base of FIGARCH model. Thus, in this section, the FIGARCH model is also described to explain the early concept of fractional exponents on GARCH model variants, although the FIGARCH model has less role than the HYGARCH model in this research.

Baillie et al. (1996) introduced the FIGARCH model as the “long-memory” GARCH model after the integrated GARCH (IGARCH) by Engle and Bollerslev (1986). As prior empirical studies about financial asset price data had shown the long memory in autocorrelation of their volatility, the fractional exponential characteristic of autoregressive fractionally integrated moving average (ARFIMA) model of Granger and Joyeux (1980) was applied to the GARCH model, analogous to Engle and Bollerslev (1986) adaptation of autoregressive integrated moving average (ARIMA) model with the concept of GARCH model to introduce the IGARCH model. As the conditional variance in GARCH model is an analogy for the conditional mean in the autoregressive integrated moving average (ARMA) model, the FIGARCH model is also analogous to the ARFIMA model. Given $\nu_t = \epsilon_t^2 - \sigma_t^2$, the GARCH equation (5) can be rewritten as

$$[1 - \alpha(L) - \beta(L)]\epsilon_t^2 = \omega + [1 - \beta(L)]\nu_t \tag{17}$$

that is analogous to the ARMA$(m, q)$ model where $m = \max\{p, q\}$. In case that the polynomial $1 - \alpha(L) - \beta(L)$ has a unit root, $1 - \alpha(L) - \beta(L)$ can be defactorized as

$$\phi(L)(1 - L)\epsilon_t^2 = \omega + [1 - \beta(L)]\nu_t \tag{18}$$

and the corresponding GARCH model is defined as the IGARCH model. Analogous to the ARIMA model that the conditional difference of mean plays the role like conditional mean in the ARMA model, the conditional difference of volatility $(1 - L)\epsilon_t^2$ in the IGARCH model also plays the same role as $\epsilon_t^2$ in the standard GARCH model. As the ARFIMA model replaces $(1 - L)$ in the ARIMA model with the fractional exponential polynomial $(1 - L)^d$, providing the opportunity to formulate the long memory (in sense of autocorrelation) of returns, the equation for the FIGARCH model also comes by replacing $(1 - L)$ with $(1 - L)^d$:

$$\phi(L)(1 - L)^d\epsilon_t^2 = \omega + [1 - \beta(L)]\nu_t \tag{19}$$

that can be rearranged as

$$\sigma_t^2 = \omega[1 - \beta(L)]^{-1} + [1 - [1 - \beta(L)]^{-1}\phi(L)(1 - L)^d]\epsilon_t^2 \tag{20}$$

By the conditions that all above conditions for HYGARCH model exist, excepted the existence of $\alpha$, Baillie et al. (1996) claimed the strict stationarity of the model due to the characteristics of the IGARCH model that the FIGARCH model inherits. However, the unconditional variance of the model is infinite. Thus, the FIGARCH model is not weakly stationary.

Afterward, Davidson (2004) proposed the HYGARCH model like a weighing model between the FIGARCH model of Baillie et al. (1996) and the standard GARCH model. Thus, the HYGARCH model is more flexible in case that the decreasing rate of autocorrelation can vary. Given the GARCH model equation (5) is rearranged:

$$[1-\beta(L)]\sigma_t^2 = \omega + \alpha(L)\epsilon_t^2, \tag{5}$$

$$\begin{aligned} \sigma_t^2 &= \omega[1-\beta(L)]^{-1} + [\alpha(L)[1-\beta(L)]^{-1}]\epsilon_t^2 \\ &= \omega[1-\beta(L)]^{-1} + \Big[1 - [1-\beta(L)]^{-1}[1-\alpha(L)-\beta(L) \end{aligned} \tag{21}$$

and the FIGARCH model equation (20):

$$\sigma_t^2 = \omega[1-\beta(L)]^{-1} + [1-[1-\beta(L)]^{-1}\phi(L)(1-L)^d]\epsilon_t^2. \tag{20}$$

Let $1-\alpha(L)-\beta(L)$ be treated as same as $\phi(L)$, so the equations (20) and (21) are equivalent such that the weighing equation between equations (20) and (21) is simplified. Using the weighing parameter $\alpha$, defined to be zero when the combined equation is of GARCH model and one when the equation is of FIGARCH model, the equation of HYGARCH model is:

$$\begin{aligned} \sigma_t^2 &= (1-\alpha)\{\omega[1-\beta(L)]^{-1} + [1-[1-\beta(L)]^{-1}\phi(L)]\epsilon_t^2\} \\ &\qquad + \alpha\{\omega[1-\beta(L)]^{-1} + [1-[1-\beta(L)]^{-1}\phi(L)(1-L)^d]\epsilon_t^2\} \\ &= \omega[1-\beta(L)]^{-1} + \{1-[1-\beta(L)]^{-1}\phi(L)[1-\alpha+\alpha(1-L)^d]\} \\ &= \omega[1-\beta(L)]^{-1} + \Big\{1-[1-\beta(L)]^{-1}\phi(L)\Big[1+\alpha[(1-L)^d-1]\Big]\Big\} \end{aligned} \tag{16}$$

The main difference between FIGARCH and HYGARCH model is about the stationarity. As the FIGARCH is not weakly stationary, the HYGARCH model is weakly stationary if $[1-\beta(1)]^{-1}\phi(1)(1-\alpha)$ is positive.

*3.1.1.4 Mixed GARCH models*

As the GJR-GARCH, Haas et al. (2004) MSGARCH, and HYGARCH models are introduced above, and two combinations from these models are stated in the research question. This section is dedicated to the formulation of these two mixed models. For their simplicity in this research, these mixed models are all concerned first-ordered, in the other word, all degrees of lagged variables are one.

3.1.1.4.1 Mixed GARCH model 1: HY-GJR-GARCH model

Let the return $r_t$ be assumed by the equation (1), (2), and conditionally i.i.d. assumption as first mentioned in section about the standard GARCH model.

Combining the fractional exponents of the HYGARCH model and asymmetry in the GJR-GARCH model, the equation for the mixed GARCH model 1, the first-ordered HY-GJR-GARCH model, can be defined as

$$\boxed{\begin{aligned}\sigma_t^2 = \omega + (\beta_1 + \beta_1^I/2)\sigma_{t-1}^2 + [(k-\beta_1 L) - (k-\phi_1 L)\{1+\alpha[(1-L)^d - 1]\}]\epsilon_t^2 \\ + \Big\{[(2-2k) - \beta_1^I L] - [(2-2k) - \kappa_1 L]\Big\{1+\alpha^I\big[(1-L)^{d^I} - 1\big]\Big\}\Big\} I_t \epsilon_t^2 \qquad (22)\end{aligned}}$$

where $\omega > 0,\ 0 < d < 1,\ 0 < \alpha < 1,\ 0 < \alpha^I < 1,\ 0 < d^I < 1,\ -k < \beta_1 < k,$ $-k < \phi_1 < k, -|2-2k| < \beta_1^I < |2-2k|$ and $-|2-2k| < \kappa_1 < |2-2k|$.

To introduce the combination of equations of HYGARCH and GJR-GARCH models to formulate the equation (22), the derivations in this section start from the special case while all alphas equal one, said the combination between FIGARCH and GJR-GARCH models. Afterward, the equation expands to the general case like the extension from FIGARCH to HYGARCH model.

From the general GJR-GARCH model equation (11), the $\mathrm{GJR\text{-}GARCH}(1,1)$ model equation may be rearranged by using $\nu_t = \epsilon_t^2 - \sigma_t^2$ similarly to the formulation of equation (17) from equation (5) as

$$(1 - \alpha_1 L - \beta_1 L - \gamma_1 L I_t)\epsilon_t^2 = \omega + (1 - \beta_1 L)\nu_t. \qquad (23)$$

Consider the equation of $\mathrm{FIGARCH}(1, d, 1)$ model in the form of equation (19):

$$(1 - \phi_1 L)(1 - L)^d \epsilon_t^2 = \omega + (1 - \beta_1 L)\nu_t. \qquad (24)$$

The term for dummy variable $I_t$ in equation (23) is also needed in the left side of equation (24) to capture the residual's asymmetry like the GJR-GARCH model. Thus, let the variable $\kappa_1$ be defined like $\gamma_1$ in equation (23), and the term $\kappa_1 L I_t$ is added to the equation (24) as follows:

$$\begin{aligned}&(1 - \phi_1 L - \kappa_1 L I_t)(1 - L)^d \epsilon_t^2 = \omega + (1 - \beta_1 L)\nu_t, \\ &(1 - \beta_1 L)\sigma_t^2 = \omega + [1 - \beta_1 L - (1 - \phi_1 L)(1 - L)^d + \kappa_1 L (1 - L)^d I_t]\epsilon_t^2. \end{aligned} \qquad (25)$$

However, when the equation (25) expands to its $\mathrm{GJR\text{-}GARCH}(\infty, 0)$ form, and many constraints are added to ensure the positivity of conditional volatility when implemented, the coefficients on lagged dummy variables of $I_t \epsilon_t^2$ has a problem of sign inversion after several first-ordered lagged $I_t \epsilon_t^2$. This problem makes the model not consistent with the root assumption of GJR-GARCH model about asymmetry. To solve this problem, let $\nu_t^I = I_t \epsilon_t^2 - \sigma_t^2/2$ be introduced to help rearranging equation (11). For any symmetric distributed $\epsilon_t$, $\mathrm{E}[I_t \epsilon_t^2] = \sigma_t^2/2$, then the expectation of $\nu_t^I$ also equals zero like $\nu_t$. Instead of relying on only one $\nu_t$, the equation (11) can be rearranged like there are two separated components: one for $\epsilon_t^2$ and another one for $I_t\epsilon_t^2$. Concerning the left side of equation (11) of GJR-GARCH$(1,1)$ model:

$$(1-\beta_1 L)\sigma_t^2 = \omega + \alpha_1 L\epsilon_t^2 + \gamma_1 L(I_t\epsilon_t^2), \tag{26}$$

the conditional variance $\sigma_t^2$ can be substituted as either $\epsilon_t^2 - \nu_t$ or $2I_t\epsilon_t^2 - 2\nu_t^I$. Thus, the beta polynomial must be separated for each substitution. Let $k$ be a fraction of one assigned to $\nu_t$ part, $\beta_1^I/2$ be an additional beta coefficient corresponding to $\nu_t^I$, $\phi_1 = \alpha_1 + \beta_1$, and $\kappa_1 = \gamma_1 + \beta_1^I$. Rearrangement from equation (26) is demonstrated as follows.

$$\begin{aligned} [1-(\beta_1+\beta_1^I/2)L]\sigma_t^2 &= \omega + \alpha_1 L\epsilon_t^2 + \gamma_1 L(I_t\epsilon_t^2), \\ (k-\beta_1 L)\sigma_t^2 + [(1-k)-(\beta_1^I/2)L]\sigma_t^2 &= \omega + \alpha_1 L\epsilon_t^2 + \gamma_1 L(I_t\epsilon_t^2), \\ (k-\beta_1 L)(\epsilon_t^2-\nu_t) + [(1-k)-(\beta_1^I/2)L](2I_t\epsilon_t^2-2\nu_t^I) &= \omega + \alpha_1 L\epsilon_t^2 + \gamma_1 L(I_t\epsilon_t^2), \\ (k-\beta_1 L)(\epsilon_t^2-\nu_t) + [(2-2k)-\beta_1^I L](I_t\epsilon_t^2-\nu_t^I) &= \omega + \alpha_1 L\epsilon_t^2 + \gamma_1 L(I_t\epsilon_t^2), \\ (k-\phi_1 L)\epsilon_t^2 + [(2-2k)-\kappa_1 L]I_t\epsilon_t^2 &= \omega + (k-\beta_1 L)\nu_t + [(2-2k)-\beta_1^I L]\nu_t^I. \end{aligned} \tag{27}$$

Like the equation (25) as the mixture between equation (23) of GJR-GARCH$(1,1)$ model and equation (24) of FIGARCH$(1,d,1)$ model, each bundle on the left side of equation (27) is assigned with its own long-memory term $(1-L)^d$. Therefore, this introduced model equation becomes the composite between two FIGARCH elements:

$$\begin{aligned} (k-\phi_1 L)(1-L)^d\epsilon_t^2 + [(2-2k)-\kappa_1 L](1-L)^{d^I} I_t\epsilon_t^2 \\ = \omega + (k-\beta_1 L)\nu_t + [(2-2k)-\beta_1^I L]\nu_t^I \end{aligned} \tag{28}$$

that can rearrange as

$$\begin{aligned} \sigma_t^2 = \omega + (\beta_1+\beta_1^I/2)\sigma_{t-1}^2 + [(k-\beta_1 L)-(k-\phi_1 L)(1-L)^d]\epsilon_t^2 \\ + \left\{[(2-2k)-\beta_1^I L]-[(2-2k)-\kappa_1 L](1-L)^{d^I}\right\}I_t\epsilon_t^2. \end{aligned} \tag{29}$$

Note that there are two $d$'s in use, the first $d$ for the FIGARCH element corresponding to $\epsilon_t^2$, and the second, denoted differently as $d^I$, for the FIGARCH element on $I_t\epsilon_t^2$.

However, in the step of implementation to estimate parameters, the problem might occur that the conditional variance $\sigma_t^2$ becomes negative for some cases. To prevent this problem, some conditions are added to ensure $\sigma_t^2$ is always positive that is preferable. Consider the equation (29) again. The equation (29) can be rewritten in the form of GJR-GARCH$(\infty,0)$ model as

$$
\begin{aligned}
\sigma_t^2 = \omega &+ [k - \beta_1 L - (k - \phi_1 L)(1-L)^d]\epsilon_t^2 \\
&+ \left[2 - 2k - \beta_1^I L - (2 - 2k - \kappa_1 L)(1-L)^{d^I}\right] I_t \epsilon_t^2 \\
&+ (\beta_1 + \beta_1^I/2)\{\omega + [k - \beta_1 L - (k - \phi_1 L)(1-L)^d]\epsilon_{t-1}^2 \\
&+ \left[2 - 2k - \beta_1^I L - (2 - 2k - \kappa_1 L)(1-L)^{d^I}\right] I_{t-1}\epsilon_{t-1}^2\} \\
&+ (\beta_1 + \beta_1^I/2)^2\{\omega + [k - \beta_1 L - (k - \phi_1 L)(1-L)^d]\epsilon_{t-2}^2 \\
&+ \left[2 - 2k - \beta_1^I L - (2 - 2k - \kappa_1 L)(1-L)^{d^I}\right] I_{t-2}\epsilon_{t-2}^2\} + \cdots \\
= \omega(1 + (\beta_1 + \beta_1^I/2) &+ (\beta_1 + \beta_1^I/2)^2 + \cdots) + (\phi_1 - \beta_1 + kd)\epsilon_{t-1}^2 \\
&+ (\kappa_1 - \beta_1^I + (2-2k)d^I) I_{t-1}\epsilon_{t-1}^2 \\
&+ [(\beta_1 + \beta_1^I/2)(\phi_1 - \beta_1 + kd) + kd(1-d)/2 - \phi_1 d]\epsilon_{t-2}^2 \\
&+ [(\beta_1 + \beta_1^I/2)(\kappa_1 - \beta_1^I + (2-2k)d^I) + (2-2k)d^I(1-d^I)/2 \\
&- \kappa_1 d^I] I_{t-2}\epsilon_{t-2}^2 + \cdots \\
= \omega/[1 - (\beta_1 + \beta_1^I/2)] &+ \alpha_1'\epsilon_{t-1}^2 + \gamma_1' I_{t-1}\epsilon_{t-1}^2 + \alpha_2'\epsilon_{t-2}^2 + \gamma_2' I_{t-2}\epsilon_{t-2}^2 + \cdots
\end{aligned}
$$

where $\alpha_i' = (\beta_1 + \beta_1^I/2)\alpha_{i-1}' + \delta_i k - \delta_{i-1}\phi_1$, $\gamma_i' = (\beta_1 + \beta_1^I/2)\gamma_{i-1}' + \delta_i^I(2-2k) - \delta_{i-1}^I\kappa_1$, $\delta_i = [(i-1-d)/i]\delta_{i-1}$, and $\delta_i^I = [(i-1-d^I)/i]\delta_{i-1}^I$ for any integer $i > 1$, $\alpha_1' = \phi_1 - \beta_1 + kd$, $\gamma_1' = \kappa_1 - \beta_1^I + (2-2k)d^I$, $\delta_1 = d$, and $\delta_1^I = d^I$.

Let the fraction $k$ lie between zero and one, and both $\beta_1$ and $\beta_1^I$ be only positive. To confirm the positivity of all $\alpha_i'$s first, two constraints corresponding to $\beta_1$ and $\phi_1$ are defined in association with $\alpha_1'$ and other $\alpha_i'$. For the positivity of $\alpha_1'$, $\phi_1 > \beta_1 - kd$. The next constraint to confirm all $\alpha_i'$s are positive depends on $\delta_i k - \delta_{i-1}\phi_1$. Since $(\beta_1 + \beta_1^I/2)\alpha_{i-1}'$ is certainly positive as a result from constraints about $\beta_1$, $\beta_1^I$, and $\alpha_1'$. Then, $\delta_i k - \delta_{i-1}\phi_1 = \delta_{i-1}[k(i-1-d)/i - \phi_1]$ is the main factor to confirm the positivity, especially $k(i-1-d)/i - \phi_1$. As the possible minimum for $(i-1-d)/i$ is $(1-d)/2$, the second constraint for $\phi_1$ is that $\phi_1 < k(1-d)/2$. The strict range for $\phi_1$ also constraint back to $\beta_1$ to ensure that the range $(\beta_1 - kd, k(1-d)/2)$ exists, in other words, $\beta_1 < k(1-d)/2$. This condition restricts the range of $\beta_1$ without contradiction to prior constraints. In similar way to $\alpha_i'$s, the additional conditions for $\beta_1^I$ and $\kappa_1$, according to the confirmed positivity for $\gamma_i'$, are $\beta_1^I - (2-2k)d^I < \kappa_1 < (2-2k)(1-d^I)/2$ and $\beta_1^I < (2-2k)(1+d^I)/2$.

From this point, there exists the equation and additional conditions for HY-GJR-GARCH model in special case of unit alphas. Next, the alphas constraints are reclined to expand the equation to desired generalization. Like the adaptation from the FIGARCH model equation (20) to HYGARCH model equation (16), the equation (29) can be adapted to the equation of first-ordered HY-GJR-GARCH model using additional variables $\alpha$ and $\alpha^I$ and substitutions of $(1-L)^d$ and $(1-L)^{d^I}$ by $1 + \alpha[(1-L)^d - 1]$ and $1 + \alpha^I\left[(1-L)^{d^I} - 1\right]$ respectively, as seen in the equation (22).

Also, the conditions on its positivity when implemented are needed to adapt. Remaining the constraints about the value of $k$ between zero and one and the positivity of $\beta_1$ and $\beta_1^I$, the available ranges for $\phi_1$ and $\kappa_1$ are needed to modify. Note that when the equation (22) is rearranged to the GJR-GARCH$(\infty, 0)$ form, the coefficients $\alpha_i'$s and $\gamma_i'$s also depend on $\alpha$ and $\alpha^I$ such that $\alpha_i' = (\beta_1 + \beta_1^I/2)\alpha_{i-1}' + \alpha(k\delta_i - \phi_1\delta_{i-1})$ and $\gamma_i' = (\beta_1 + \beta_1^I/2)\gamma_{i-1}' + \alpha^I[(2-2k)\delta_i^I - \kappa_1\delta_{i-1}^I]$ for any integer $i > 1$, where $\alpha_1' = \phi_1 - \beta_1 + \alpha kd$ and $\gamma_1' = \kappa_1 - \beta_1^I + \alpha^I(2-2k)d^I$. Hence, the appropriate ranges for the value of $\phi_1$ and $\kappa_1$ become $(\beta_1 - \alpha kd, k(1-d)/2)$ and $(\beta_1^I - \alpha^I(2-2k)d^I, (2-2k)(1-d^I)/2)$ respectively, then the maximums of $\beta_1$ and $\beta_1^I$ shrink to $k[(1-d)/2 + \alpha d]$ and $(2-2k)[(1-d^I)/2 + \alpha^I d^I]$ respectively.

3.1.1.4.2 Mixed GARCH model 2: HY-MS-GARCH model

Under the same preconditions as mentioned in section about the mixed GARCH model 1, this combination between the HYGARCH model and MSGARCH model of Haas et al. (2004) uses the HYGARCH equation directly with satisfaction of all HYGARCH conditions for all regimes like the Haas et al. (2004) MSGARCH model that also utilize the GARCH model equation directly.

3.1.2 Historical simulation (HS)

As known as one of famous non-parametric way to describe the behavior of financial assets, Hendricks (1996) proposed the historical simulation (HS) as one of straightforward ways to estimate the VaR without any assumption about time series characteristics. The only key assumption is that the history repeats itself. Thus, the distribution of returns of any financial asset can be described by its history. In contrast to VaR prediction using GARCH models above, the $VaR_t^\alpha$ is simply predicted using the $\alpha$th quantile of historical return data (commonly recent data for long period to ensure that the estimated VaR appropriately reflects the market risk).

In this research, the HS method is chosen as one of baseline VaR estimation models to compare with GARCH models.

*3.1.2.1 Filtered historical simulation (FHS)*

Unlike Hendricks (1996), Barone-Adesi and Giannopoulos (1996) pioneered the alternative way to predict financial asset's VaR: using the historical simulation for standardized historical return data, and the Engle (1982) GARCH model to forecast return's volatility separately. By this method, the $\alpha$th quantile of standardized historical return data, denoted by $\text{quantile}_\alpha(z_t)$, is explored first and then replaced in typical VaR prediction from the estimated GARCH model to finally get the $VaR_t^\alpha$:

$$VaR_t^\alpha = \mu_t + \text{quantile}_\alpha(z_t)\mathrm{E}[\sigma_t|\mathcal{I}_{t-1}]. \tag{30}$$

Using both historical return distribution to find the standardized percentile at loss non-parametrically, and assumed return distribution to estimate the volatility parametrically, the FHS method is called a semi-parametric method to predict VaR due to its components.

The FHS method is also chosen to be one of baseline VaR estimation models along with the HS method in this research.

## 3.2 Measurements

In this research, four measurements are used to compare the efficiency of VaR estimation from each model mentioned in the last section.

The first two measurements, Kupiec's proportion of failure coverage test and dynamic quantile (DQ) test, are based on the frequency of violations when the realized returns penetrate the estimated line of VaR. The different between these tests is, while the Kupiec's test treats all violations in a specific period without recognition of temporal sequence between hits, the DQ test considers more on the sequence as the "conditional" version of the Kupiec's test itself. The description of quality the satiable VaR estimation should have by each test could expand this understanding. As the Kupiec's test defines the VaR line with closest ratio of violation to the preset as the best VaR line, many VaR lines can be satiable by this metric because they are set to the right level at the right time. However, this is not enough for the DQ test measurement that requires all satiable VaR lines to set to some dynamic level such that each violation through the VaR line is significantly random from time to time. This means the VaR line that tries to catch only one big "fall" once in a period is not preferable in sense of DQ test.



However, the "good" VaR line in views of both Kupiec's and DQ test might be set too optimistic. So, the realized returns penetrating this VaR line cause too much damage in real application. For example, when too optimistic VaR line defines the level of needed bad debt provision, and the bad debt is really default, the "too optimistic" provision will not be enough for covering the loss. Thus, the third measurement, loss function, is introduced to weigh the "good" VaR line not to be set too optimistic as the excess loss under the VaR line should be limited to some acceptable level.

The last measurement, the conditional predictive ability (CPA) test, is used only in comparison as its test statistic is derived from the loss function difference of pair of VaR lines. From the perspective of loss function, the less loss function, the better VaR line. The question from this point is whether the VaR line with less loss function in a comparison has "really" less loss function that does not come from luck. Thus, the CPA test comes to directly answer this question like the "conditional" confirmation on differences.

The detail in calculations of these four statistics, along with their critical values at various confidence levels, are provided below.

3.2.1 Kupiec's proportion of failures coverage test

Kupiec (1995) proposed the statistical test to measure whether the failures when the returns penetrate the VaR occurs proportionally with the confidence interval. Since its introduction, it has been widely used for VaR backtesting. By the null hypothesis of $\alpha = \alpha^*$, the likelihood ratio (LR) test statistic comes from the ratio between theoretical and empirical proportion of failure as the respective probability. Given there are $x$ failures out of $N$ observations, the LR test statistic is

$$-2\log\Lambda = -2\log[(\alpha^*)^x(1-\alpha^*)^{N-x}] + 2\log[(x/N)^x(1-x/N)^{N-x}] \sim \chi_1^2. \tag{31}$$

Since the goal of this test is to confirm the indifference between the realized ratio of failure and $\alpha$, the null hypothesis is needed to be accepted. To accept null hypothesis at 0.1 confidence level, the test statistic is needed to be less than 2.7055. For looser levels of 0.05 and 0.01, the thresholds are 3.8415 and 6.6349 respectively.

3.2.2 Dynamic quantile test

Engle and Manganelli (2004) proposed the dynamic quantile (DQ) test as one of the ways VaR estimations are backtested. According to the Engle and Manganelli (2004) conditional autoregressive VaR (CAViaR) model, the VaR is autoregressive on its own lags and its underlying return lags. However, in using the test, the VaR time series is converted by the hit function:

$$Hit_t(x) = I(r_t < VaR_t^{\alpha,x}) - \alpha = \begin{cases} 1-\alpha, & r_t < VaR_t^{\alpha,x} \\ -\alpha, & \text{otherwise} \end{cases}. \tag{32}$$

If $VaR_t^{\alpha,x}$ is estimated appropriately, the expectation of $Hit_t(x)$ equals zero. Then, the DQ test is performed to test whether $Hit_t(x)$ is independent from its regressors. For examples, Engle and Manganelli (2004) used its own lags to the fourth order as the regressors. However, Bams et al. (2017) also used the first-order lag VaR and squared return as the additional regressors to test its independence from market regimes. The lagged change in implied volatility was also used to test the independence from a shock that suddenly affects the volatility. In this research, $Hit_t(x)$ is DQ tested with its own four lags and one lagged VaR.

In this test, like Engle and Manganelli (2004), the typical linear regression coefficients are also calculated to describe the relationship between $Hit_t(x)$ and its regressors. Under the null hypothesis that all coefficients equal zero, the test statistic is calculated by the formula:

$$DQ_{oos} = \frac{Hit_t'(x)X_t(x)[X_t'(x)X_t(x)]^{-1}X_t'(x)Hit_t(x)}{\lambda(1-\lambda)} \sim \chi_w^2 \tag{33}$$

where $X_t$ is the $T \times w$ matrix of corresponding regressors and $w$ is the number of regressors.

Like Kupiec's test in Sub-section 3.2.1, the goal is accepting the null hypothesis to confirm autoregressive independence, indicated by less DQ test statistic than the critical values. For said specific case in this research, $w$ is 5. Thus, the critical values become 9.2364, 11.0705, and 15.0863 for 0.1, 0.05, and 0.01 confidence level respectively.

### 3.2.3 Loss function

Chen and Gerlach (2013) developed the loss function from Koenker and Bassett (1978) quantile regression objective function. It can be defined as

$$LF = \sum_{t=1}^{T} (r_t - VaR_t)\big(\alpha - I(r_t < VaR_t)\big). \tag{34}$$

However, the sole loss function has no information about its distribution. Thus, Chen and Gerlach (2013) proposed the estimation of loss function by Politis and Romano (1994) block bootstrap method to get information so that the loss function can be tested for statistically significant difference from each other. In this research, 1000 blocks of time with length $b$, suggested $T^{1/3}$ by Politis and Romano (1994), are generated randomly to calculate the loss function differences between each pair of VaR time series. Then, the distribution of each loss function can be implied by its block bootstrapped distribution.

In this research, however, the block bootstrapped distribution is not assumed to each loss function itself. But it is assumed to the differences between loss functions from a pair of VaR lines for benefit of comparison whether the difference is significant. Using the Student's t test with 999 degrees of freedom, the absolute value of t statistic is needed to be more than 2.5808 to indicate significance at 0.01 level. For looser levels of 0.05 and 0.1, the critical values are 1.9623 and 1.6464 respectively.

### 3.2.4 Conditional predictive ability test

Giacomini and White (2006) proposed the conditional predictive ability (CPA) test as a loss-function-based comparison between two predictive models to test whether the more accurate model is not more accurate because of luck. This test concentrates on the loss function difference between the estimation result from two models. If the time series of loss function difference has no autocorrelation, the implication comes that the comparative accuracy of said more accurate model might be lower in the future because of its uncertainty. Then, the autocorrelation of loss function difference can be used to confirm the superiority of the more accurate model against the other model. Based on the null hypothesis that the conditional expectation of loss function difference from two VaR estimations on the information at time $t$ is zero, the Wald-type test statistic is

$$CPA = T\left(T^{-1}\sum_{t=1}^{T-1} h_t \Delta L_{t+1}\right)'\left(T^{-1}\sum_{t=1}^{T-1} Z_{t+1}Z'_{t+1}\right)^{-1}\left(T^{-1}\sum_{t=1}^{T-1} h_t \Delta L_{t+1}\right) \sim \chi_q^2 \qquad (35)$$

where $Z_{t+1} = h_t \Delta L_{t+1}$, $h_t$ is a $q \times 1$ vector of test function, $\Delta L_t$ is the loss difference between models at time $t$. Like Bams et al. (2017), $h_t = (1, \Delta L_t)$ is used in this research as the test function following Giacomini and White (2006), and the test is based on the loss function of Chen and Gerlach (2013).

Like the loss function in Sub-section 3.2.3, this CPA test focuses on significance of loss function difference. Thus, the expected result is the rejection of null hypothesis. Due to the $\chi^2$ statistic with 2 degrees of freedom, the size of $h_t$ used in this research as mentioned in the last paragraph, the test statistic calculated from equation (35) must be greater than 9.2103 to mention a significant difference at 0.01 significance level. For more relaxing levels of 0.05 and 0.1, the critical values are respectively 5.9915 and 4.6052.

## 3.3 Data

In this research, the S&P 500 daily index data from January 12th, 1956, to December 31st, 1999, from http://finance.yahoo.com is selected as the sample for VaR estimations. Since the VaR is based on the returns, the daily index $P_t$ is converted to the daily index logarithmic return:

$$r_t = \log P_t - \log P_{t-1}.$$

Then, there are data points of $r_t$ from the second date, January 13th, 1956, to the last date of 1999. These $r_t$ are illustrated in the following diagrams and table of description:

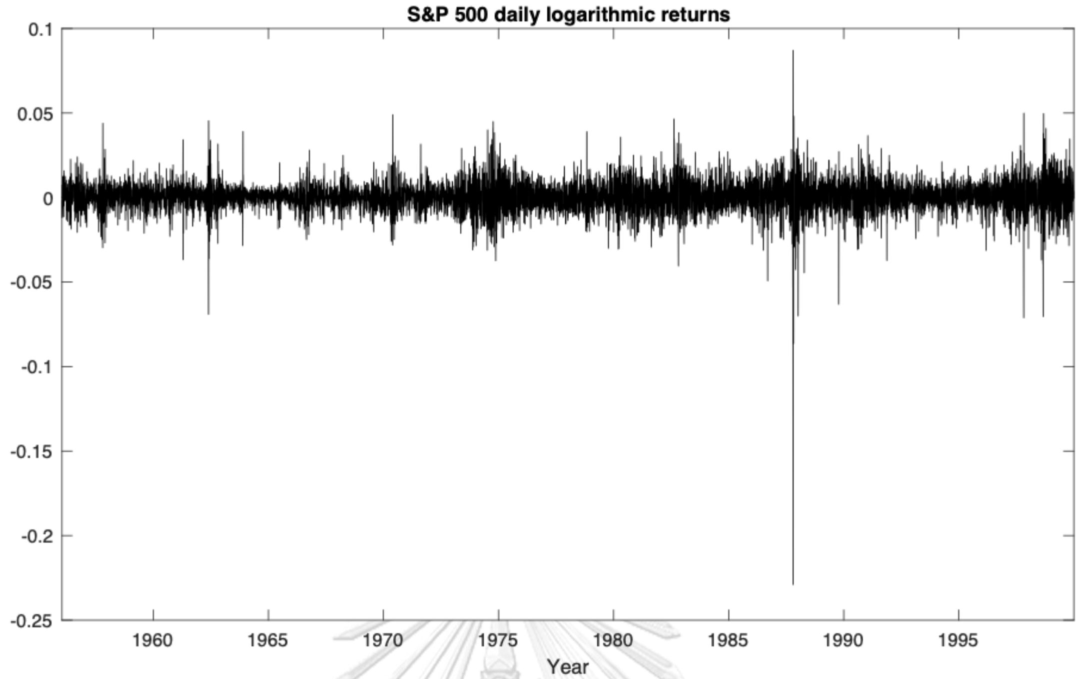


*Figure 5 S&P 500 daily logarithmic returns, January 13th, 1956 – December 31st, 1999*


*Figure 6 S&P 500 cumulative daily logarithmic returns, January 13th, 1956 – December 31st, 1999*

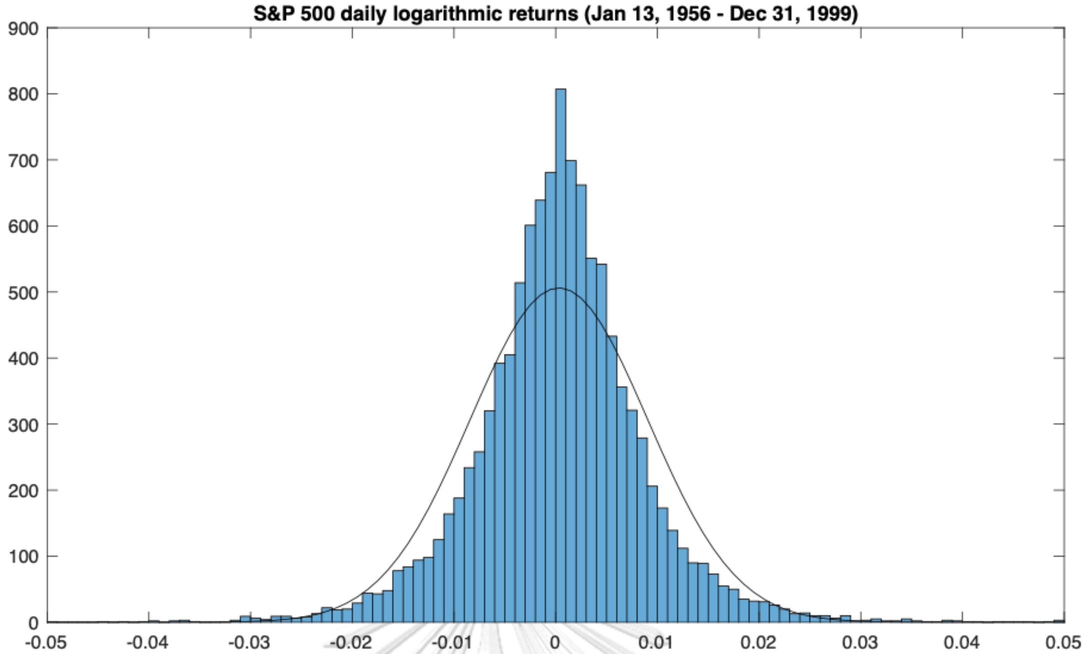


*Figure 7 Distribution of S&P 500 daily logarithmic returns, Gaussian normal distribution shown by the curved line*

*Table 2 Numerical characteristics of S&P 500 daily logarithmic returns*

| | |
|---|---|
| Number of data points | 11071 |
| Mean | 0.000315 |
| Standard deviation | 0.008720 |
| Skewness | –1.776612 |
| Excess kurtosis | 48.550896 |
| Minimum | –0.228997 |
| $1^{st}$ percentile | –0.021937 |
| $5^{th}$ percentile | –0.013127 |
| $1^{st}$ quartile | –0.003976 |
| Median | 0.000373 |
| $3^{rd}$ quartile | 0.004692 |
| $95^{th}$ percentile | 0.013561 |
| $99^{th}$ percentile | 0.022581 |
| Maximum | 0.087089 |

Priorly, the data points are divided as many subsamples containing 1,000 first points as training data and other points as test data to reflect various market environments that the hypothesis will be tested in this research. Afterward, the parameters of two mixed GARCH models, three base GARCH models, and GARCH part for filtered historical simulation, are estimated for each subsample. To predict 1-day-ahead VaR, historical returns of latest 1,000 days are used to calculate the VaR by each pre-defined model or simulation. For example, the historical returns $r_1, r_2, \dots, r_{1000}$ are the base for estimation of all models' parameters and the first 1-

day-ahead VaR prediction $VaR_{1001}$ by all models, then for $VaR_{1002}$, the historical returns $r_2, r_3, \dots r_{1001}$ are used as a moving window in prediction using the same model according to $r_1, r_2, \dots, r_{1000}$. When finished, there are VaR time series calculated by each model along with the original return series to be tested by above measurements.

In this research, the estimations of mentioned models are implemented using MATLAB. As its own Econometrics Toolbox provides the functions only for the simple GARCH and GJR-GARCH models, third-party code packages by developers are used. For the estimations of the MSGARCH model, Thomas Chuffart's MSGTools, influenced by Kevin Sheppard's MFE Toolbox, is modified in this research to make more stable estimations than the random result from the original codes. The concepts and structures of these two mentioned packages are also used to develop the tailor-made packages for the estimation of more complex models like the HYGARCH and all mixed models.

In concentration of computational costs, the cost of estimation for the mixed models with higher complexity is not significantly more in comparison to the estimation of less complex models. Although the time optimization of third-party MATLAB codes is not comparable to the proprietary codes that is more robust. Usage of these mixed models, estimated by third-party codes, is beneficial with only slightly more expense.

All models, excluded the historical simulation that has no additional assumption on return distribution, are estimated using assumption about two distributions: Gaussian and Student's t distribution. Hence, there are 13 VaR lines estimated in this research: one from historical simulation, and six for one distribution (one from FHS, three from base models, and two from mixed models.) All comparisons in this research are done among models with same distribution as described in the table below.

*Table 3 The comparison matrix between the mixed GARCH models and other models*

| **Mixed model No.** | **Base models** | | | **FHS** | **HS** |
|---|---|---|---|---|---|
| | **GJR-GARCH** | **MSGARCH** | **HYGARCH** | | |
| 1 (HY-GJR) | Yes | No | Yes | Yes | Yes |
| 2 (HY-MS) | No | Yes | Yes | Yes | Yes |

Thus, there are eight pairs of comparison for the models in one distribution, totally sixteen pairs.

# Chapter 4 Empirical Results

## 4.1 Overview of sub-periods

As mentioned in sub-section about data, the daily S&P 500 indices from 1960 to 1999 are converted to daily logarithmic returns and used as the case study in this research. However, the whole data set shows various schemes of United States' securities market when the uptrend was dominated in those four decades, and the use of whole period as training and test data set to produce one set of test statistics is inferior such that the result has no comparativeness to indicate which model is better in which situation.

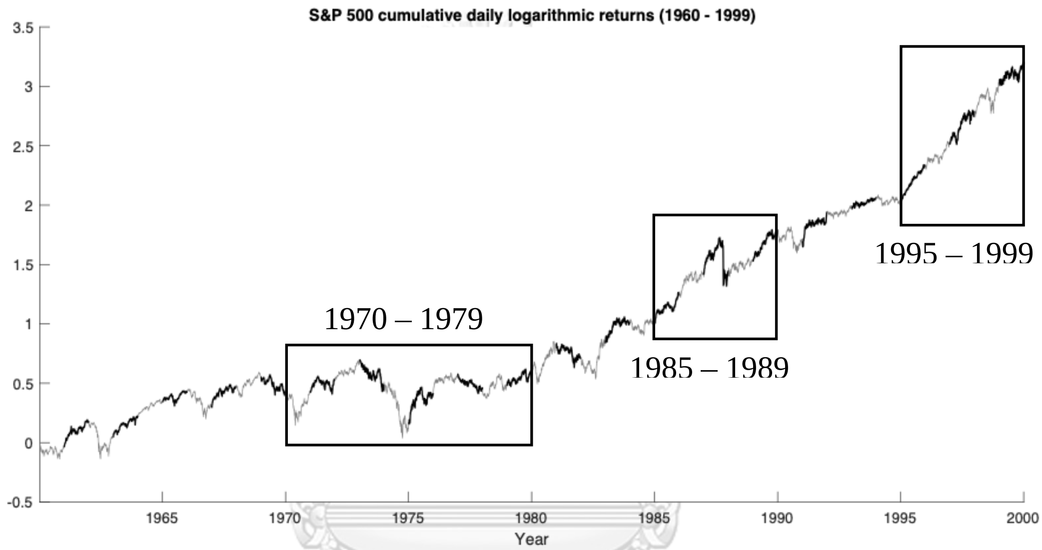


*Figure 8 Sub-period selections from the whole data set, illustrated in the cumulative return chart*



Fortunately, the whole data set can be portioned to many sub-periods which their themes can represent distinguish characteristics and schemes. Thus, the whole period of 40 years is separated, and some sub-periods with obvious characteristics are selected to produce various sets of test statistics: 1970 – 1979, 1985 – 1989, and 1995 – 1999. These selections are beneficial for the test results to distinguish relative models' performance among different market situations. However, there are more recent periods from 2000 that are not covered by this research. They might be interesting for further studies to reinforce the delineated results for future usages.

### 4.1.1 The 1970 – 1979 sub-period: the decade of "sideway"

Consider the daily S&P indices in the 1970's decade in Figure 8. Although the indices in whole period of 1960 – 1999 move slightly upward, the indices in the first two decades, 1960's and 1970's, move flatter than the indices in the late two decades.

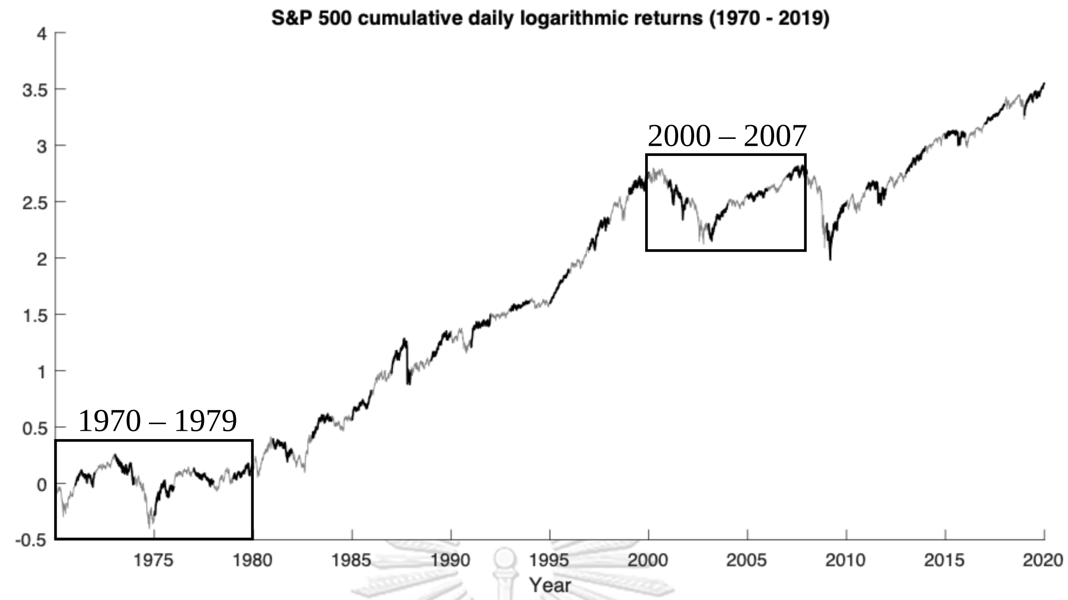


*Figure 9 1970 – 1979 sub-period in comparison with 2000 – 2007 sub-period, illustrated in the cumulative return chart*

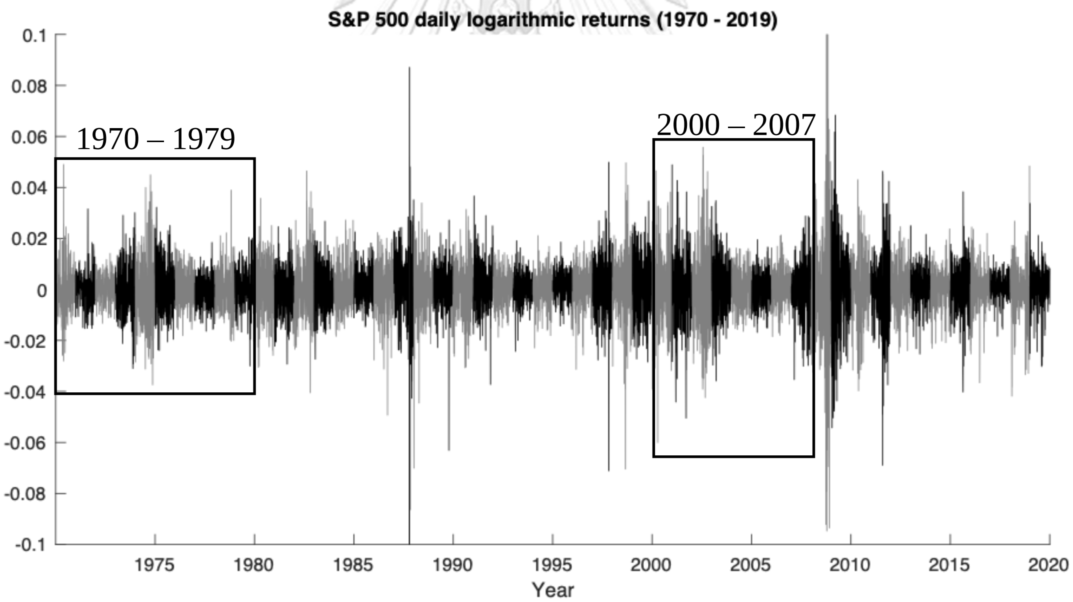


*Figure 10 1970 – 1979 sub-period in comparison with 2000 – 2007 sub-period, illustrated in the return chart*

regardless the 1980's and 1990's decades, the movement of indices in 1960's decade can be considered as an "upward" trend. Consequently, the remaining decade of 1970's can be seen as a relatively "sideway" period.

However, there is more recent period like 2000 – 2007 that can be decided as another "sideway" period. As can be seen from Figures 9 and 10, the 1970 – 1979 and 2000 – 2007 sub-periods are very similar since the market trends did not show obvious direction like “up” or “down” as clearly seen in other periods, and the volatilities of returns in both boxes in Figure 10 are slightly different as seen from naked eyes. Thus, both sub-periods may be considered samples of “sideway” trends in the market. And the results from 1970 – 1979 sub-period, in comparison to those from 1985 – 1989 and 1995 – 1999 sub-period, is believed to well represent the performance among models in relatively sideway period. But further tests with the more recent sideway sub-period can be carried out in future research for robustness check.

As seen in Figure 11, the story started from consecutively plunging of the indices from 1969, reaching the annual bottom in the middle 1970, following by the 2.5-year uptrend until 1972, then plunging again to the deeper nadir in late 1974 and bouncing back in the next year. Finally, the indices in remaining 4 years of 1976 – 1979 moved smoother in a 20-percent-wide range in comparison to the earlier movements that the range expanded to about 70 percent.


*Figure 11 S&P 500 cumulative daily logarithmic returns, 1970 – 1979, including first training period (leftmost black line), grey lines for even years’ returns*

Consider the complementary Figures 12 – 13, and Table 4. The Figure 12 shows the same data as in Figure 11, but the non-cumulative representation of indices neglects the movements “up” or “down” as seen from cumulative index returns, then the volatility is emphasized to observe more easily. The plunge of S&P 500 indices in 1974, as seen in Figure 11, is associated with a cluster with higher volatility shown in Figure 12. This stylized fact corresponds to the observations of Black (1976) and Nelson (1991) on the leverage that is parametrized in the equation of GJR-GARCH

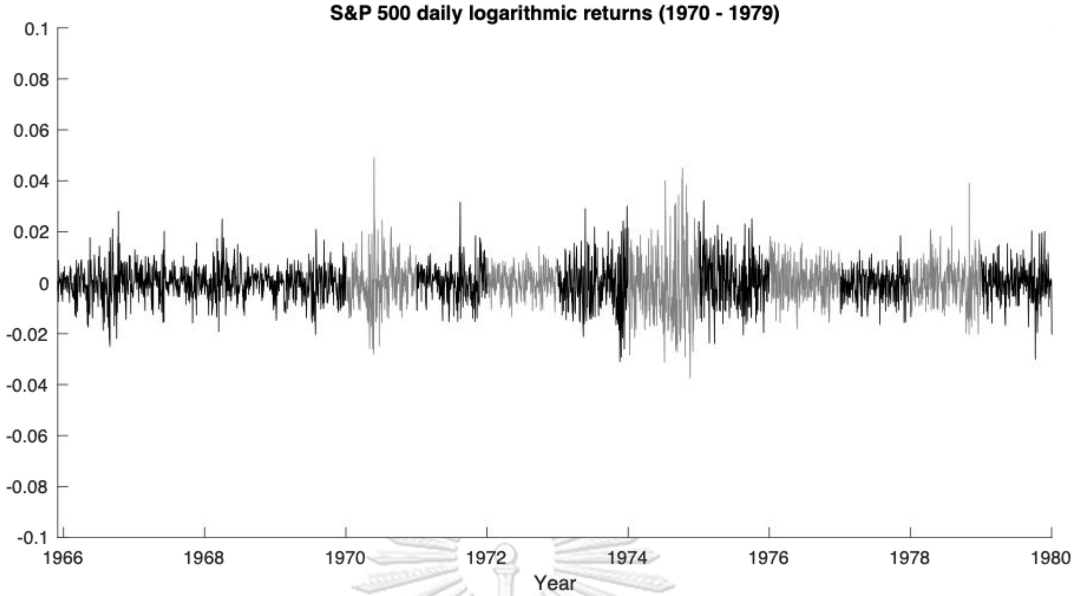


*Figure 12 S&P 500 daily logarithmic returns, 1970 – 1979, including first training period (leftmost black line), grey lines for even years' returns*

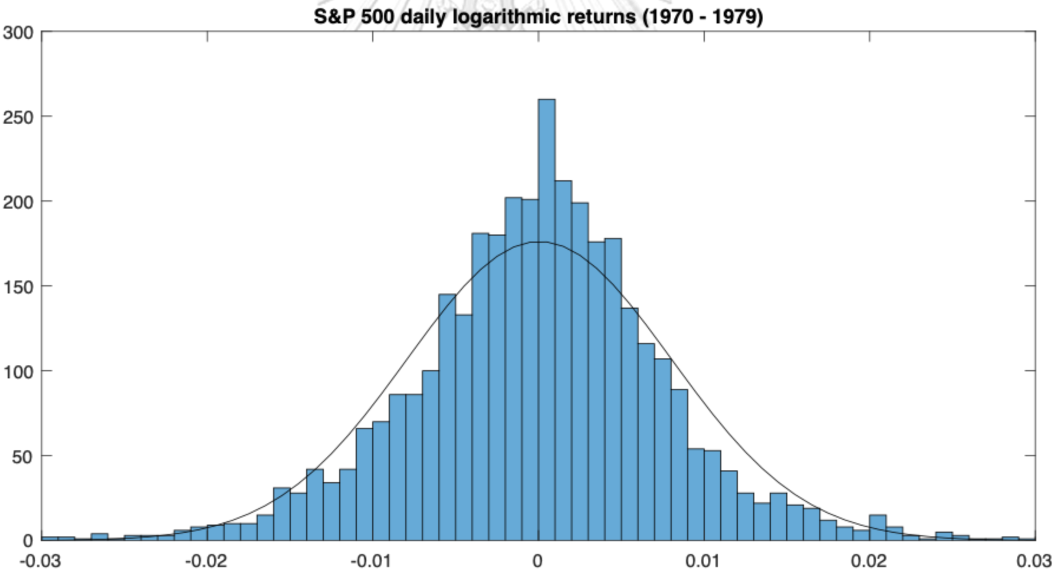


*Figure 13 Distribution of S&P 500 daily logarithmic returns, 1970 – 1979, including first training period, and Gaussian normal distribution shown by the curved line*

model. Thus, it can be assumed that the GJR-GARCH model should describe the market in this decade better than any other models. Furthermore, its derivative HY-GJR-GARCH model is introduced in this research with more flexibility than the old GJR-GARCH model. Given asymmetric structure from GJR-GARCH model and flexible hyperbolic decay from HYGARCH model, it can also be assumed that the

*Table 4 Numerical characteristics of S&P 500 daily logarithmic returns, 1970 – 1979, including first training period*

| | |
|---|---|
| Number of data points | 3526 |
| Mean | 0.000047 |
| Standard deviation | 0.007977 |
| Skewness | 0.217628 |
| Excess kurtosis | 2.406780 |
| Minimum | –0.037404 |
| $1^{\text{st}}$ percentile | –0.020111 |
| $5^{\text{th}}$ percentile | –0.013135 |
| $1^{\text{st}}$ quartile | –0.004465 |
| Median | 0.000195 |
| $3^{\text{rd}}$ quartile | 0.004486 |
| $95^{\text{th}}$ percentile | 0.012767 |
| $99^{\text{th}}$ percentile | 0.021130 |
| Maximum | 0.049003 |

HY-GJR-GARCH model should describe the market in this decade better than the original GJR-GARCH model can do.

Like the whole period data of 1960 – 1999, the data for this sub-period of 1970's shows leptokurtosis and fat tails for both sides as seen in Figure 13. But this comparison is distinguished when the numerical description of this sub-period, shown in Table 4, is compared to the description of the whole period in Table 2 (see Section 3.3). The degree of leptokurtosis for this selection, 2.406780, is extremely lower than the excess kurtosis of the whole data, 48.550896. Besides, the symmetry of the selection, represented by the skewness of 0.217628, is near to zero than the skewness of the whole data at –1.776612. This fact prefers the assumption of normal distribution to Student's t distribution to describe the residuals after GARCH models.

4.1.2 The 1985 – 1989 sub-period: the "big shock" of 1987

Consider the Figure 8 again. In the whole period of 40 years, the movement of S&P 500 indices in 1987 looks obviously unique from an extremely steep plunge near the end of that year. This phenomenon is well known as the "Black Monday" of October $19^{\text{th}}$, 1987, the greatest daily loss of S&P 500 index in history. From 282.70 when the market had been closed on Friday, October $16^{\text{th}}$, an index harshly dropped to 224.84 at the close time, or 20.47% down from the close index of previous day. Therefore, the sub-period with this Black Monday is also valuable for studying the effect persistence after various sizes of shock as stated by Engle and Mustafa (1992) and Friedman et al. (1989).

However, the more recent time of subprime crisis in late 2000's decade might be concerned a "nearer" and "harsher" event to the S&P 500 index due to a movement counted in index points. The similarity between these two times is seen from their movements of S&P 500 index since the daily return fell more than usual as recorded in the history. For the late 1980's decade, there were the Black Monday with –20.47% return and a week later, October 26th, with –8.28% return that are recorded as top 20 daily losses in the history of S&P 500 index. For the late 2000's decade, there were 4 daily returns in 2008: –8.79% on September 29th, –7.62% on October 9th, –9.04% on October 15th, and –8.93% on December 1st. Although these two sub-periods cannot be substitutable in the study of crisis in the market, the 1985 – 1989 sub-period is still chosen in comparison with other two sub-periods with an expectation that the test results from this sub-period could adequately represent the relative models' performance in the crisis against normal times that need further affirmation for robustness by testing with more recent data sets in future research.

Since the bigger size of shock, the lesser persistence of after-shock effect on volatility, the MSGARCH model should work better to describe the market in this situation due to its feature to formulate this inversely proportional persistence to shock size. Same as mentioned in Sub-section 4.1.1 about the HY-GJR-GARCH model, the HY-MS-GARCH model is introduced in this research with higher degree of flexibility as the decay rate can be more adjusted. Hence, by the assumption, the HY-MS-GARCH model, should excel the MSGARCH model in this case too.

In this research, the last half of 1980's decade from 1985 to 1989, which covers the Black Monday event, is selected as a sub-period to test all models in this study like the 1970's decade. Figures 14 – 16 and Table 5 below provides the overview of the returns and their characteristics that will be explained in the next paragraphs.

Like the distribution histogram of the whole data set in Figure 7, the histogram of this sub-period, as shown in Figure 16, is also leptokurtic. However, only the left-side tail in Figure 16 is fat. The numerical measurements in Table 5 also emphasize this fact: when compared to the Table 2 of whole data set (see Section 3.3), this sub-period comes with higher standard deviation (0.011065 to 0.008720), more negative skewness (–4.107120 to –1.776612), and more excess kurtosis (86.364856 to 48.550896). Thus, the assumption of Student's t distribution around the GARCH models' estimation should be considered more appropriate than the assumption with normal distribution.

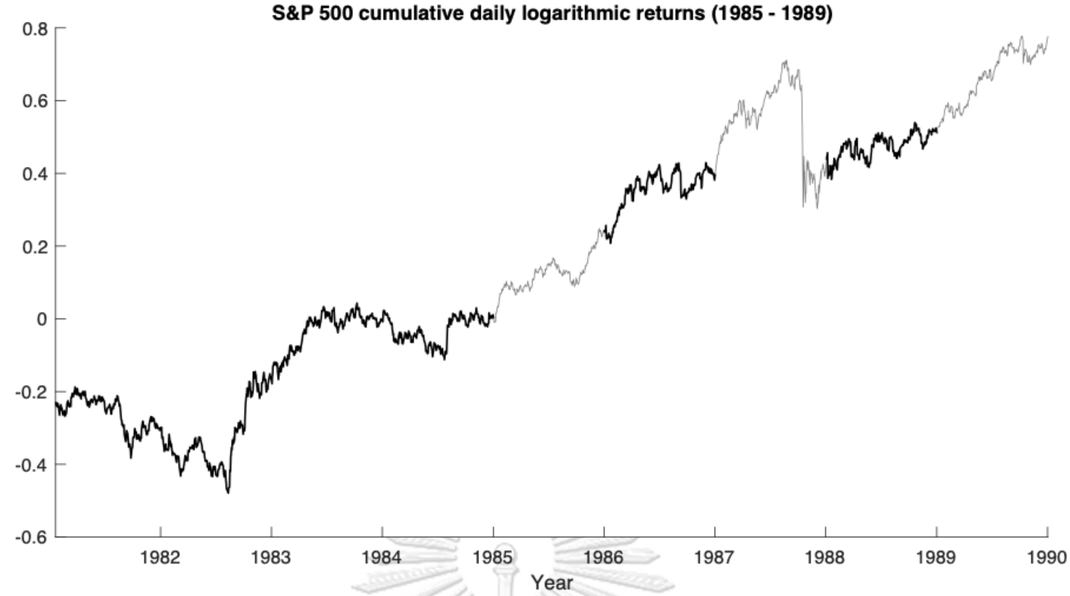


*Figure 14 S&P 500 cumulative daily logarithmic returns, 1985 – 1989, including first training period (leftmost black line), grey lines for odd years' returns*

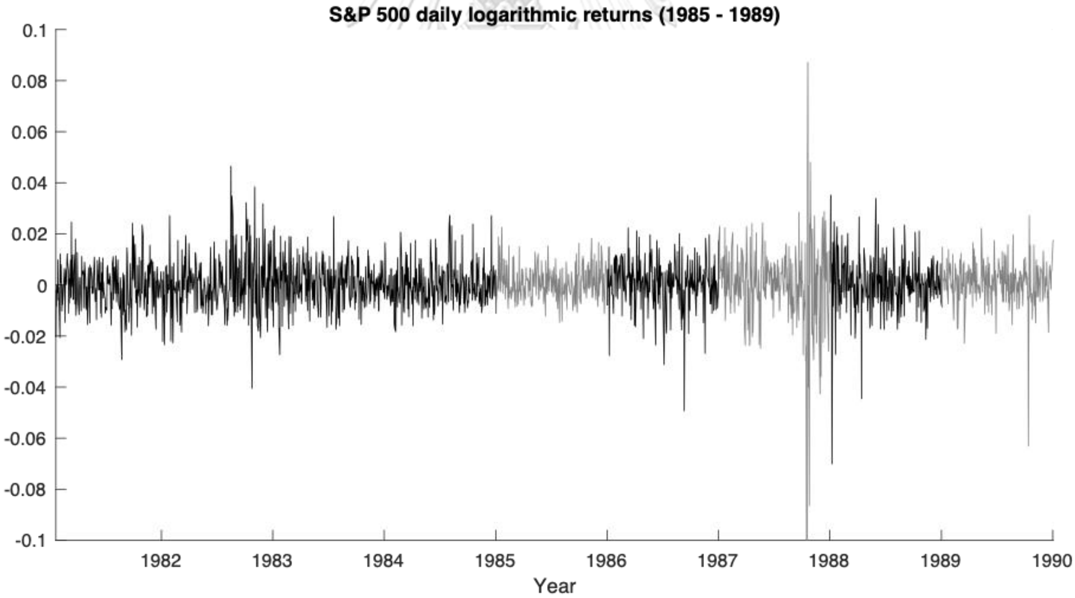


*Figure 15 S&P 500 daily logarithmic returns, 1985 – 1989, including first training period (leftmost black line), grey lines for odd years' returns*

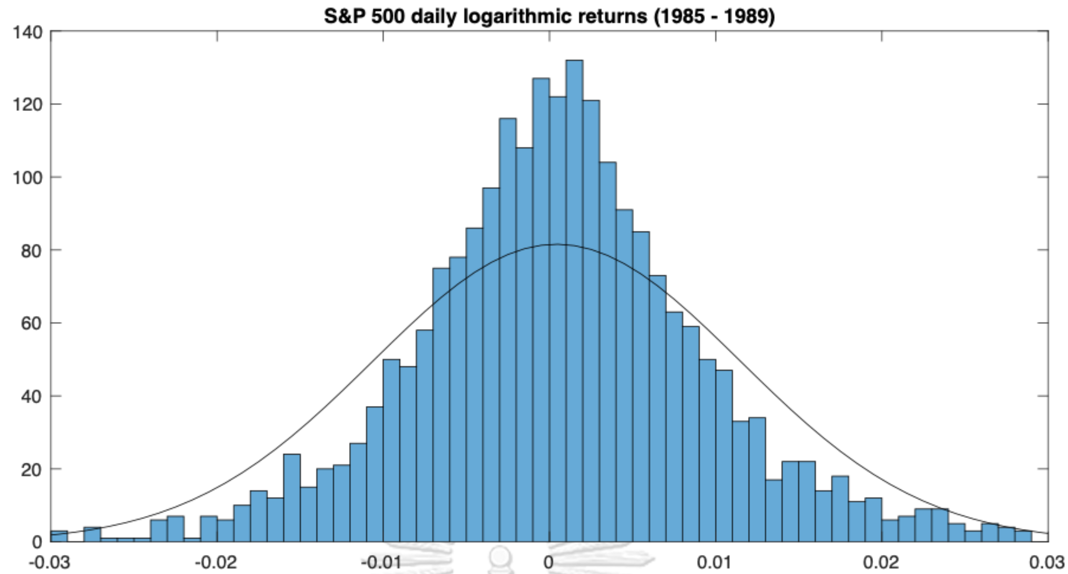


*Figure 16 Distribution of S&P 500 daily logarithmic returns, 1985 – 1989, including first training period, and Gaussian normal distribution shown by the curved line*

*Table 5 Numerical characteristics of S&P 500 daily logarithmic returns, 1985 – 1989, including first training period*

| | |
|---|---|
| Number of data points | 2263 |
| Mean | 0.000427 |
| Standard deviation | 0.011065 |
| Skewness | –4.107120 |
| Excess kurtosis | 86.364856 |
| Minimum | –0.228997 |
| 1st percentile | –0.023760 |
| 5th percentile | –0.014795 |
| 1st quartile | –0.004655 |
| Median | 0.000468 |
| 3rd quartile | 0.005649 |
| 95th percentile | 0.016233 |
| 99th percentile | 0.025678 |
| Maximum | 0.087089 |

4.1.3 The 1995 – 1999 sub-period: the smooth "bull" time

Unlike the 1970's decade of sideway and the last half of 1980's decade around the Black Monday, the remaining whole data of S&P 500 index returns from 1960 to 1999 illustrates the time of "bull" market, or when the market trend ascends consecutively. However, when seeing the Figure 8 again, the market trend for each sub-period gives different degrees of ascension along with several declines as the "average corrections" over the time.

From the last two sub-sections, there are three sub-periods that are still not selected: the 1960's decade, the first half of 1980's decade (1980 – 1984) and the 1990's decade. Consider the 1990's decade. Although the upside trend dominated most of the time, the whole trend obviously separated by half. The S&P 500 returns of last half of 1990's decade, 1995 – 1999, climbed up with a steeper upside trend than any other prior upside trends from 1960. Besides, there was a little "average correction" in middle 1998 that bounced to the same old level more quickly than any other declines from 1960.

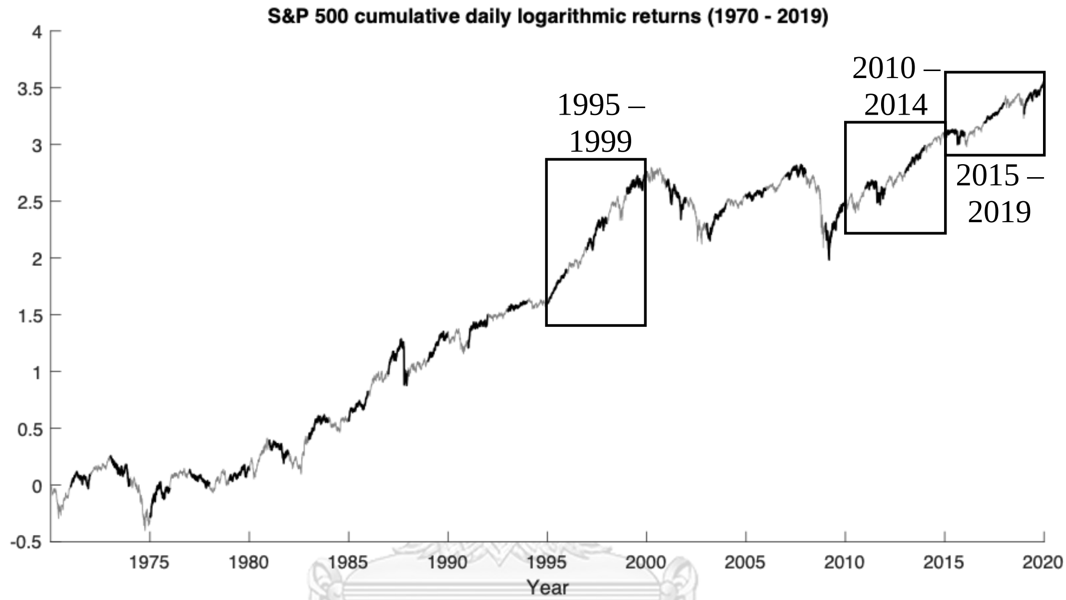


*Figure 17 1995 – 1999, 2010 – 2014, and 2015 – 2019 sub-periods in comparison, illustrated in the cumulative return chart*



However, the more recent period after 2010 is also considered an another "bull" time as seen in the S&P 500 index. When comparing these three 5-year sub-periods of 1995 – 1999, 2010 – 2014, and 2015 – 2019, the same characteristic on the first glance of Figure 17 when seeing the index chart of these sub-periods is the uptrend. Although these sub-periods might be considered as examples of "bull" market situation, the 1995 – 1999 sub-period is chosen in this research to perform the test and compare to the results from other sub-periods with different characteristics. It is hoped that the result comparison is worth enough to indicate the trend of relative model performance in another "bull" trend of the market in general. It remains for future study to assess the models' performance in more recent bull periods.

Hence, in this research, this last half of 1990's decade is also selected as a representation of "bull" market situation in contrast with the sideway situation of 1970's decade and the market with big decline in the last half of 1980's decade.

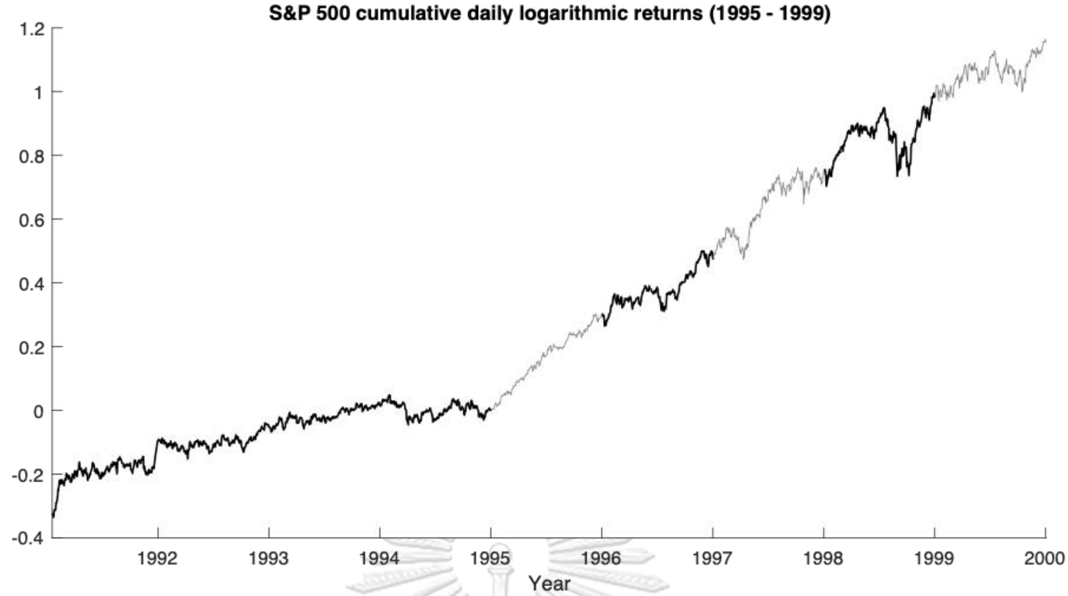


*Figure 18 S&P 500 cumulative daily logarithmic returns, 1995 – 1999, including first training period (leftmost black line), grey lines for odd years' returns*

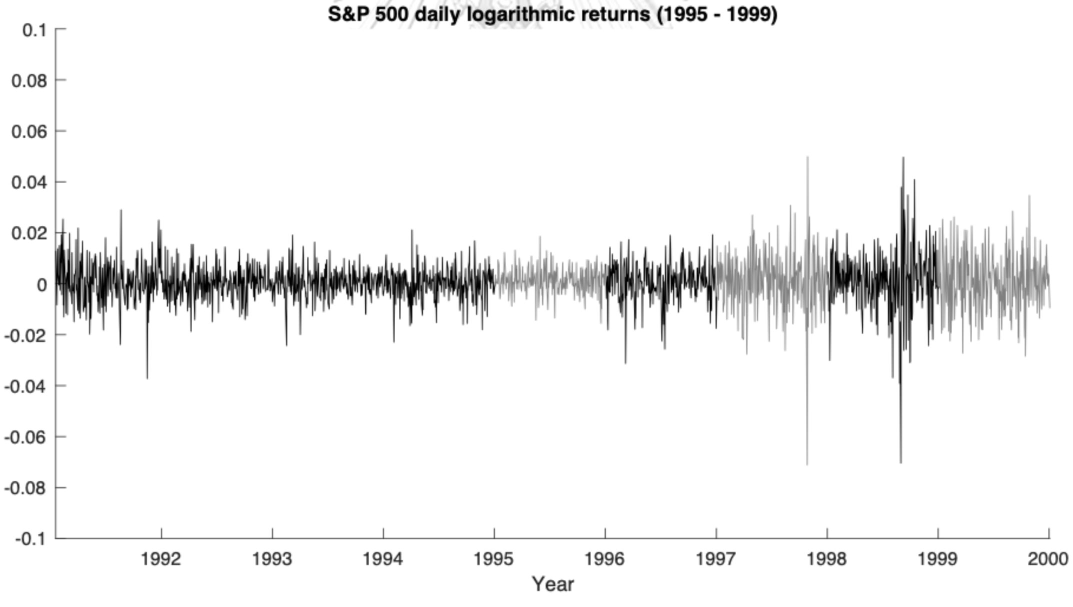


*Figure 19 S&P 500 daily logarithmic returns, 1995 – 1999, including first training period (leftmost black line), grey lines for odd years' returns*

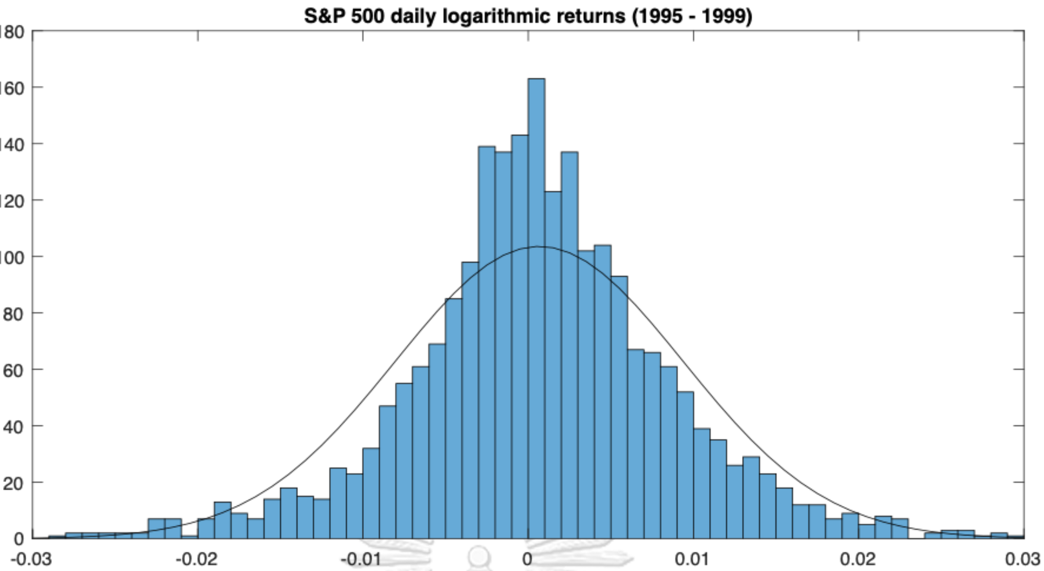


*Figure 20 Distribution of S&P 500 daily logarithmic returns, 1995 – 1999, including first training period, and Gaussian normal distribution shown by the curved line*

*Table 6 Numerical characteristics of S&P 500 daily logarithmic returns, 1995 – 1999, including first training period*

| | |
|---|---|
| Number of data points | 2263 |
| Mean | 0.000663 |
| Standard deviation | 0.008710 |
| Skewness | –0.392379 |
| Excess kurtosis | 6.134029 |
| Minimum | –0.071127 |
| 1st percentile | –0.022584 |
| 5th percentile | –0.013176 |
| 1st quartile | –0.003589 |
| Median | 0.000481 |
| 3rd quartile | 0.005220 |
| 95th percentile | 0.014372 |
| 99th percentile | 0.022647 |
| Maximum | 0.049887 |

Further information about this sub-period is provided by Figures 18 – 20, Table 6, and their explanations afterwards.

Both Figures 18 and 19 illustrate S&P 500 logarithmic returns closely in 1995 – 1999 sub-period with different perspective: Figure 15 shows the movement of index in logarithmic scale like a normal index chart, but Figure 19 separates each daily return regardless previous cumulative returns.

Consider the "average correction" in later 1998 as seen from Figure 18. In Figure 19, the returns at the same time exploded like happening in 1974 as mentioned in Sub-section 4.1.1, although the level of correction from prior peak to the adjacent bottom was less than the dropping in 1974. Also, there is a similar burst in the same Figure 19 near the end of 1997, but it seems like a little sideway when surveying Figure 18 at the same time with the naked eyes. These two bursts in Figure 19 resemble the Black Monday as shown in Figure 14 in Sub-section 4.1.2. From this intuition, the MSGARCH model should be considered an appropriate model to describe the market in this situation due to its regime switching that directly copes with stylized fact of inverse persistence to shock size.

Also, the “cylinder” form of daily returns in 1999, as seen from Figure 19, can indicate hyperbolically decrease of autocorrelation as mentioned by Dacorogna et al. (1993) and Ding et al. (1993). In this case, the HYGARCH model, which its formulation allows more adjustment in decrease rate of autocorrelation than the formulation of simple GARCH model, can be considered an appropriate model like the MSGARCH model. Thus, the proposed HY-MS-GARCH model, as a mixture between MSGARCH and HYGARCH models, is assumed the better choice to describe the market in this situation than both base models.

The Figure 20 and Table 6 indicate more about the distribution of returns in this sub-period. Leptokurtosis and fat tails are also seen in this sub-period like happening in the whole data. However, the degree of leptokurtosis decreases more from 48.550896 to 6.134029, and the level of left-skewness decreases too from –1.776612 to –0.392379. Like the 1970’s decade in Sub-section 4.1.1, the assumption of normal-distributed residuals around GARCH models’ estimation might be preferred to the assumption of Student’s t-distributed residuals. Although about one-and-half-time higher leptokurtosis than the excess kurtosis in Table 4 might prevent this too quick summarization, the histogram of Figure 20 has a sense of tendency to the Figure 13 of 1970’s decade, which the normal distribution is assumed, instead the Figure 16 of the time around the 1987 Black Monday. In conclusion, the residuals should be assumed normal-distributed.

Furthermore, the comparisons between Table 6 of this sub-period and Table 2 of the whole data (see Section 3.3) are confirmations of the visible uptrend in this sub-period. As the mean, all quartiles, and 95$^{th}$ and 99$^{th}$ percentiles in Table 6 are more than those in Table 2, there is a tendency that data from sub-period of 1995 – 1999 is generally higher than the whole data of 1960 – 1999.

## 4.2 Descriptive log-likelihood functions

Before describing the predictive test statistics, the log-likelihood function for each parametric model is calculated by using each model to fit the whole return data in each sub-period without any model parameter revision. There are 5 parametric models mentioned in Section 3.1 as selected models: GJR-GARCH, MSGARCH, HYGARCH, HY-GJR-GARCH, and HY-MS-GARCH models. Combined with 2 choices of assumed distribution, normal and Student's t, there are 10 combinations of parametric models and assumed distribution that are noted in this sub-section and afterwards by model name and a letter suffix in a parenthesis. The suffix (N) is used for a model with normal distribution. For a model with Student's t distribution, its suffix notation is (t). The calculated log-likelihood functions are shown in Table 7 below.

*Table 7 Descriptive log-likelihood functions of base and mixed parametric models*

| Sub-period | | 1970 – 1979 | 1985 – 1989 | 1995 – 1999 |
|---|---|---|---|---|
| **Log-likelihood function** | **GJR-GARCH(N)** | 3731.27 | 3300.67 | 3608.69 |
| | **MSGARCH(N)** | 3719.04 | 3309.27 | 3643.50 |
| | **HYGARCH(N)** | 3714.30 | 3301.88 | 3615.72 |
| | **HY-GJR-GARCH(N)** | 3737.35 | 3304.70 | 3618.81 |
| | **HY-MS-GARCH(N)** | 3720.67 | 3311.58 | 3643.23 |
| | **GJR-GARCH(t)** | 3732.35 | 3313.86 | 3638.77 |
| | **MSGARCH(t)** | 3718.34 | 3314.38 | 3644.01 |
| | **HYGARCH(t)** | 3717.81 | 3314.05 | 3644.02 |
| | **HY-GJR-GARCH(t)** | 3739.17 | 3315.09 | 3645.99 |
| | **HY-MS-GARCH(t)** | 3722.74 | 3316.25 | 3643.70 |

## 4.3 Test statistics by sub-period and discussions

The Section 3.2 is all mentioned about four test statistics in use for comparison of daily 95% VaR lines produced by models: the Kupiec's proportion of failure test, dynamic quantile (DQ) test, loss function, and conditional predictive ability (CPA) test. Both the Kupiec's and DQ tests are concentrated on frequency when the VaR is violated by realized returns, but the DQ test measures VaR hits conditionally as a time sequence, while the Kupiec's test measures them unconditionally. Differently from first two test statistics, the loss function and CPA test measures the degree of effects after the violations of VaR by realized returns, in the other words, "how far" the violations come beyond the VaR line estimation. Unlike the loss function that is plainly measured with no more context, the CPA test complements the loss function whether two loss functions are "really" different temporally.

To calculate all test statistics for each sub-period, each model is used solely to calculate the daily VaR at 95% confidence interval for each sub-period with quarterly model parameter revisions. (See Section 3.1 for model descriptions and Section 3.3 for the description of training and test process.) Afterward, all test statistics are derived for each 95% VaR line as sources for comparisons between proposed mixed models and their benchmarks as follows.

However, each test statistic has different goal for its null hypothesis. For both Kupiec's and DQ tests, it is more satisfiable when the null hypothesis is accepted, or the test statistics are close to zero, that means indifference between realized proportion of failure and pre-defined $\alpha$ in case of Kupiec's test, or random time of VaR's penetration for DQ test. Otherwise, the rejection of null hypothesis, as the test statistics are away from zero, is preferrable to exhibit the significance of last two test statistics: loss functions (as derived to the difference between the loss function for a mixed model and the loss function for each benchmark) and corresponding CPA test statistic (as the confirmation of sign in a loss function difference). Note that loss function difference is the only one test statistic in this research with sign, and the minus sign of loss function difference, in other words, less loss function for a mixed model than loss function for a benchmark, is preferrable.

### 4.3.1 The 1970 – 1979 sub-period

#### *4.3.1.1 Overall: accurate proportion of failure but clustered hits*

*Table 8 Proportions of failure, Kupiec's test statistics, DQ test statistics, and loss function for each model using data set of 1970 – 1979 sub-period*

| Model | Proportion of Failure[1] | Kupiec's test[2] | DQ test[2] | Loss function[2] |
|---|---|---|---|---|
| **HS** | 5.9778% | 4.7983* | 158.3566 | 2.3759 |
| **FHS(N)** | 5.1465% | **0.1131*** ** | 20.1933 | 2.0819 |
| **GJR-GARCH(N)** | 5.0277% | **0.0041*** ** | 17.0410 | 2.0347 |
| **MSGARCH(N)** | 5.8987% | 4.0707* | 31.0080 | 2.1221 |
| **HYGARCH(N)** | 5.5424% | **1.5134*** ** | 32.3924 | 2.1032 |
| **HY-GJR-GARCH(N)** | 5.0673% | **0.0240*** ** | 31.4422 | 2.0500 |
| **HY-MS-GARCH(N)** | 6.1758% | 6.8623 | 37.2622 | 2.1212 |
| **FHS(t)** | 5.2652% | **0.3680*** ** | 32.2767 | 2.0879 |
| **GJR-GARCH(t)** | 4.2755% | 2.9290** | 12.3257* | 2.0388 |
| **MSGARCH(t)** | 6.0174% | 5.1833* | 27.7101 | 2.1393 |
| **HYGARCH(t)** | 4.9881% | **0.0008*** ** | 32.7556 | 2.1041 |
| **HY-GJR-GARCH(t)** | 4.6318% | **0.7382*** ** | 29.6697 | 2.0536 |
| **HY-MS-GARCH(t)** | 6.1362% | 6.4218* | 39.3540 | 2.1318 |

[1] The closer proportion of failure to the preset (5%), the better VaR estimation.
[2] The less test statistic, the better VaR estimation.
*** = null hypothesis accepted at 0.1 significance
** = null hypothesis accepted at 0.05 significance
* = null hypothesis accepted at 0.01 significance

Tables 8 – 10 show the performance of proposed mixed models and their benchmark models in prediction of one-day 95% VaR in situation of 1970's decades. As seen from Table 8, the VaR lines from 7 out of 13 models can pass the Kupiec's proportion of failure test at strictest confidence level of 0.1, but almost all VaR lines cannot pass the DQ test even at loosest level of 0.01. The predicted VaR line by the GJR-GARCH model with Student's t distribution is only one VaR line that can pass the DQ test at the loosest confidence level of 0.01 but pass the Kupiec's test at looser

*Table 9 Dynamic quantile linear regression model parameters for each model using data set of 1970 – 1979 sub-period*

| | **HS** | **FHS(N)** | **GJR-GARCH(N)** | **MSGARCH(N)** | **HYGARCH(N)** | **HY-GJR-GARCH(N)** | **HY-MS-GARCH(N)** | **FHS(t)** | **GJR-GARCH(t)** | **MSGARCH(t)** | **HYGARCH(t)** | **HY-GJR-GARCH(t)** | **HY-MS-GARCH(t)** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Intercept** | **0.0796** | **0.0279** | 0.0215 | **0.0283** | **0.0339** | 0.0226 | 0.0282 | **0.0314** | 0.0135 | **0.0285** | 0.0234 | 0.0150 | **0.0293** |
| p-value | **0.0047*** ** | **0.0197** ** | 0.0925* | **0.0476** ** | **0.0111** ** | 0.0844* | 0.0627* | **0.0104** ** | 0.2670 | **0.0426** ** | 0.0617* | 0.2422 | **0.0483** ** |
| **Hit_lag_1** | **0.1446** | **0.0590** | **0.0633** | **0.0661** | **0.0669** | **0.0953** | **0.0715** | **0.0778** | **0.0522** | **0.0545** | **0.0879** | **0.0951** | **0.0793** |
| p-value | **0.0000*** ** | **0.0031*** ** | **0.0015*** ** | **0.0009*** ** | **0.0008*** ** | **0.0000*** ** | **0.0003*** ** | **0.0001*** ** | **0.0088*** ** | **0.0063*** ** | **0.0000*** ** | **0.0000*** ** | **0.0001*** ** |
| **Hit_lag_2** | 0.0141 | –0.0338 | –0.0159 | –0.0303 | –0.0206 | –0.0170 | –0.0247 | –0.0255 | –0.0173 | –0.0240 | –0.0149 | –0.0163 | –0.0251 |
| p-value | 0.4793 | 0.0905* | 0.4267 | 0.1314 | 0.3016 | 0.3958 | 0.2180 | 0.2002 | 0.3867 | 0.2308 | 0.4556 | 0.4159 | 0.2105 |
| **Hit_lag_3** | **0.0871** | **0.0466** | **0.0507** | **0.0544** | **0.0621** | **0.0567** | **0.0667** | **0.0656** | 0.0274 | **0.0571** | **0.0647** | **0.0519** | **0.0606** |
| p-value | **0.0000*** ** | **0.0194** ** | **0.0114** ** | **0.0064*** ** | **0.0018*** ** | **0.0046*** ** | **0.0008*** ** | **0.0010*** ** | 0.1716 | **0.0042*** ** | **0.0012*** ** | **0.0096*** ** | **0.0024*** ** |
| **Hit_lag_4** | **0.1046** | 0.0301 | 0.0097 | 0.0377 | **0.0472** | 0.0044 | 0.0210 | 0.0303 | 0.0033 | 0.0354 | 0.0207 | 0.0155 | 0.0288 |
| p-value | **0.0000*** ** | 0.1311 | 0.6270 | 0.0587* | **0.0177** ** | 0.8273 | 0.2930 | 0.1285 | 0.8707 | 0.0760* | 0.2981 | 0.4369 | 0.1481 |
| **VaR_lag_1** | **5.4909** | **2.0484** | 1.6463 | 1.6291 | **2.2320** | 1.6906 | 1.4365 | **2.2337** | 1.5100 | 1.5362 | **1.7223** | 1.3668 | 1.5570 |
| p-value | **0.0083*** ** | **0.0172** ** | 0.0781* | 0.1316 | **0.0199** ** | 0.0757* | 0.2131 | **0.0109** ** | 0.0774* | 0.1418 | **0.0462** ** | 0.1361 | 0.1664 |
| **Adjusted $R^2$** | **0.0524** | **0.0079** | **0.0059** | **0.0086** | **0.0116** | **0.0115** | **0.0091** | **0.0126** | **0.0029** | **0.0070** | **0.0124** | **0.0109** | **0.0101** |
| p-value | **0.0000*** ** | **0.0001*** ** | **0.0014*** ** | **0.0001*** ** | **0.0000*** ** | **0.0000*** ** | **0.0000*** ** | **0.0000*** ** | **0.0303** ** | **0.0004*** ** | **0.0000*** ** | **0.0000*** ** | **0.0000*** ** |

*** = null hypothesis rejected at 0.01 significance; ** = null hypothesis rejected at 0.05 significance; * = null hypothesis rejected at 0.1 significance

*Table 10 Loss function differences and CPA test statistics between each mixed model and its benchmarks using data set of 1970 – 1979 sub-period*

| HY-GJR-GARCH(N) vs | GJR-GARCH(N) | HYGARCH (N) | FHS(N) | HS |
|---|---|---|---|---|
| **LF difference**[1] | 0.0154 | **–0.0531** | –0.0318 | **–0.3259** |
| **LF diff. test**[2] | 1.4428 | **–2.9999**** | –1.7930 | **–4.5483**** |
| **CPA test**[2] | 2.1889 | **10.8913**** | 4.6272 | **38.4610**** |
| **HY-GJR-GARCH(t) vs** | **GJR-GARCH(t)** | **HYGARCH (t)** | **FHS(t)** | **HS** |
| **LF difference**[1] | 0.0149 | **–0.0505** | –0.0343 | **–0.3223** |
| **LF diff. test**[2] | 1.2890 | **–3.0860**** | –1.9916* | **–4.4096**** |
| **CPA test**[2] | 1.9644 | **10.5351**** | 4.1605 | **36.4162**** |
| **HY-MS-GARCH(N) vs** | **MSGARCH (N)** | **HYGARCH (N)** | **FHS(N)** | **HS** |
| **LF difference**[1] | –0.0009 | 0.0181 | 0.0394 | **–0.2547** |
| **LF diff. test**[2] | –0.1022 | 1.2247 | 1.7919 | **–4.1705**** |
| **CPA test**[2] | 0.0536 | 2.1573 | 4.2414 | **32.4857**** |
| **HY-MS-GARCH(t) vs** | **MSGARCH (t)** | **HYGARCH (t)** | **FHS(t)** | **HS** |
| **LF difference**[1] | –0.0075 | 0.0277 | 0.0439 | **–0.2441** |
| **LF diff. test**[2] | –0.6038 | 1.2767 | 2.2122* | **–3.9899**** |
| **CPA test**[2] | 0.4337 | 2.6450 | 5.4044 | **29.9139**** |

[1] The negativity shows that the mixed model has less loss function than the benchmark i.e. the VaR estimation from mixed model is better. Otherwise, the VaR estimation from mixed model is worse.
[2] The greater magnitude, the higher significance of the corresponding difference.
** = null hypothesis rejected at 0.01 significance; * = null hypothesis rejected at 0.05 significance

level of 0.05 unlike the VaR line from GJR-GARCH model with Gaussian normal distribution. This phenomenon that the VaR line well pass the Kupiec's test while struggling to pass the DQ test is because of the characteristics of the market returns in this sub-period. See the Figure 11 in Sub-section 4.1.1. The time of big plunges in 1970 and 1974 that has the most chance of violation of VaR line shows consecutive daily losses for a week at least. This fact is also reflected by the Table 9 that the first and third lagged hit variables are influential for almost all VaR lines (excluded the VaR line from GJR-GARCH(t) model that the third lagged hit is not significant). Descriptively said, when the VaR line is penetrated by the realized daily return, the VaR line usually adjusts itself completely on the next day. However, in case of a big plunge, one adjustment is not enough. Then, the VaR line will usually be penetrated again for two consecutive days after the first adjustment and then adjust again. This means the parametric VaR lines normally have "double adjustments" for a big plunge of realized daily return, but this mechanism is not found in the non-parametric VaR line from historical simulation that is indisputably beaten by all parametric VaR lines in this sub-period

*4.3.1.2 HY-GJR-GARCH(N) model: winner by the influence of GJR-GARCH model and normal distribution*

Assumed in Sub-section 4.1.1 about properness of GJR-GARCH and HY-GJR-GARCH models in this sub-period, both models with normal distribution can undoubtedly pass Kupiec's test with strictest confidence level of 0.1. Focusing on the HY-GJR-GARCH(N) model, it seems like the GJR-GARCH(N) model gets slightly closer proportion of failure (just 0.0396% nearer or 1 time less out of 2526) to the preset of 0.05 than the HY-GJR-GARCH(N) model. Inferred from the Kupiec's test, however, the proportion of failure given from VaR line from HY-GJR-GARCH(N) model is closer than the proportion from VaR line of other three benchmark models: HYGARCH(N) model, filtered historical simulation with normal distribution, and historical simulation. Considering the loss function difference in Table 10, the comparison results resemble the comparison results in Table 8 that the VaR line from HY-GJR-GARCH(N) model gets less loss function than the VaR line from all benchmark models excepting the GJR-GARCH(N) model that its loss function is insignificantly less. These results verify the assumption of improvement after mixture between GJR-GARCH and HYGARCH models. While the asymmetry feature of GJR-GARCH model is added to the inferior HYGARCH model, the HY-GJR-GARCH model is improved in prediction of VaR that the sole HYGARCH model does not excel.

*4.3.1.3 HY-GJR-GARCH(t) model: misspecification in distribution*

However, this result is not identical when the assumed distribution is Student's t instead. The VaR line from GJR-GARCH(t) model has farther proportion of failure from the preset as seen from itself in the first column and the associate Kupiec's test statistics in the second column of Table 8. The clue from intercept terms in Table 9 that the intercept for the GJR-GARCH(t) model is the least suggests that the GJR-GARCH(t) model predicts "too low" VaR line so that the penetration rate is too low than the preset as seen 4.2755% in the Table 8. This problem is solved when the fractional exponential feature of HYGARCH model is applied to the sole GJR-GARCH(t) model because the proportion of failure is closer to the preset as expected. But this solution is not enough because its penetration rate is farther in comparison with the VaR from filtered historical simulation with Student's t distribution, one of benchmark models, although the comparison on loss function is indifferent from that of the HY-GJR-GARCH(N) model. The assumption about normality in Sub-section 4.1.1 might be an accurate diagnosis for this phenomenon since a misspecification in distribution might diminish the performance of the well-selected model. Hence, both model and distribution specifications are important to produce an accurate parametric VaR line prediction.

#### *4.3.1.4 HY-MS-GARCH model: the right model but the wrong time*

In controversy to the success of HY-GJR-GARCH mixed model in this sub-period, the HY-MS-GARCH model, as the combination between fractional exponents of HYGARCH model and regime switching of MSGARCH model, does not show any obvious sign of success in competition with benchmarks. Like the mentioned case of HY-GJR-GARCH(t) model, this problem comes from a misspecification of model by using the variant of MSGARCH model, designed for a market situation with market losses in diverse harshness, in a market situation with no obvious difference among the plunges of market returns. However, the failure of HY-MS-GARCH model in comparison to the minor fault of HY-GJR-GARCH(t) model indicates that a misspecification in model selection is dominant over a misspecification in residuals' distribution.

### 4.3.2 The 1985 – 1989 sub-period

#### *4.3.2.1 Overall: more challenging time for GARCH variants*

*Table 11 Proportions of failure, Kupiec's test statistics, DQ test statistics, and loss function for each model using data set of 1985 – 1989 sub-period*

| **Model** | **Proportion of Failure[1]** | **Kupiec's test[2]** | **DQ test[2]** | **Loss function[2]** |
|---|---|---|---|---|
| **HS** | 6.0966% | 2.9975** | 37.1449 | 1.8889 |
| **FHS(N)** | 8.0760% | 21.3816 | 31.2304 | 1.8731 |
| **GJR-GARCH(N)** | 5.7007% | **1.2515*** ** | **10.2536** ** | 1.8096 |
| **MSGARCH(N)** | 6.5717% | 6.0026* | 22.9064 | 1.7780 |
| **HYGARCH(N)** | 4.1964% | **1.8122*** ** | 24.9944 | 1.7201 |
| **HY-GJR-GARCH(N)** | 4.5131% | **0.6508*** ** | 13.4593* | 1.6897 |
| **HY-MS-GARCH(N)** | 6.4133% | 4.8946* | 19.6128 | 1.7719 |
| **FHS(t)** | 6.4133% | 4.8946* | 18.3523 | 1.7381 |
| **GJR-GARCH(t)** | 3.2462% | 9.2873 | 22.1101 | 1.7542 |
| **MSGARCH(t)** | 4.3547% | **1.1555*** ** | 26.0664 | 1.7881 |
| **HYGARCH(t)** | 3.4046% | 7.5856 | 20.5158 | 1.7832 |
| **HY-GJR-GARCH(t)** | 3.3254% | 8.4118 | 27.8604 | 1.7542 |
| **HY-MS-GARCH(t)** | 4.6714% | **0.2932*** ** | 31.9210 | 1.7837 |

[1] The closer proportion of failure to the preset (5%), the better VaR estimation.
[2] The less test statistic, the better VaR estimation.
*** = null hypothesis accepted at 0.1 significance
** = null hypothesis accepted at 0.05 significance
* = null hypothesis accepted at 0.01 significance

Tables 11 – 13, same format as Tables 8 – 10 in Sub-section 4.3.1, show test results of VaR lines from all mentioned models with different data set of S&P 500 index return in 1985 – 1989 sub-period. A different situation makes a different result,

*Table 12 Dynamic quantile linear regression model parameters for each model using data set of 1985 – 1989 sub-period*

| | **HS** | **FHS(N)** | **GJR-GARCH(N)** | **MSGARCH(N)** | **HYGARCH(N)** | **HY-GJR-GARCH(N)** | **HY-MS-GARCH(N)** | **FHS(t)** | **GJR-GARCH(t)** | **MSGARCH(t)** | **HYGARCH(t)** | **HY-GJR-GARCH(t)** | **HY-MS-GARCH(t)** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Intercept** | **0.1300** | 0.0429 | 0.0249 | –0.0112 | 0.0034 | –0.0048 | 0.0134 | –0.0013 | –0.0040 | 0.0162 | –0.0043 | 0.0014 | 0.0115 |
| p-value | **0.0086*** ** | 0.1311 | 0.2374 | 0.7124 | 0.7794 | 0.6842 | 0.5698 | 0.9574 | 0.7360 | 0.4774 | 0.7244 | 0.8994 | 0.5793 |
| **Hit_lag_1** | **0.0807** | **0.0593** | **0.0676** | 0.0514 | **0.1027** | **0.0763** | 0.0312 | **0.0726** | **0.1127** | 0.0534 | **0.1024** | **0.1287** | **0.0760** |
| p-value | **0.0043*** ** | **0.0362** ** | **0.0170** ** | 0.0690* | **0.0003*** ** | **0.0070*** ** | 0.2699 | **0.0103** ** | **0.0001*** ** | 0.0588* | **0.0003*** ** | **0.0000*** ** | **0.0073*** ** |
| **Hit_lag_2** | **0.0975** | 0.0289 | 0.0437 | **0.0639** | **0.0852** | **0.0571** | **0.0745** | 0.0269 | 0.0276 | **0.1198** | 0.0480 | 0.0446 | **0.1191** |
| p-value | **0.0006*** ** | 0.3174 | 0.1344 | **0.0247** ** | **0.0028*** ** | **0.0451** ** | **0.0085*** ** | 0.3445 | 0.3332 | **0.0000*** ** | 0.0910* | 0.1174 | **0.0000*** ** |
| **Hit_lag_3** | 0.0350 | –0.0041 | 0.0131 | 0.0350 | 0.0209 | 0.0194 | 0.0454 | 0.0298 | 0.0397 | **0.0560** | 0.0277 | 0.0301 | **0.0581** |
| p-value | 0.2157 | 0.8873 | 0.6506 | 0.2203 | 0.4611 | 0.4949 | 0.1096 | 0.2953 | 0.1650 | **0.0482** ** | 0.3293 | 0.2907 | **0.0409** ** |
| **Hit_lag_4** | 0.0079 | 0.0102 | 0.0003 | –0.0052 | –0.0363 | –0.0364 | –0.0116 | 0.0023 | –0.0140 | –0.0097 | –0.0202 | –0.0176 | –0.0206 |
| p-value | 0.7793 | 0.7211 | 0.9906 | 0.8546 | 0.1994 | 0.2003 | 0.6834 | 0.9357 | 0.6245 | 0.7312 | 0.4740 | 0.5344 | 0.4672 |
| **VaR_lag_1** | **8.5866** | 1.2094 | 1.2832 | –1.8696 | 0.6039 | –0.0404 | 0.0754 | –1.0511 | 0.5718 | 1.3205 | 0.4807 | 0.7912 | 0.8713 |
| p-value | **0.0132** ** | 0.5937 | 0.3560 | 0.4037 | 0.3541 | 0.9494 | 0.9641 | 0.5649 | 0.3175 | 0.3379 | 0.4094 | 0.1225 | 0.4839 |
| **Adjusted $R^2$** | **0.0237** | 0.0007 | 0.0034 | **0.0061** | **0.0180** | **0.0074** | **0.0052** | 0.0041 | **0.0130** | **0.0191** | **0.0120** | **0.0201** | **0.0231** |
| p-value | **0.0000*** ** | 0.3130 | 0.1010 | **0.0270** ** | **0.0000*** ** | **0.0136** ** | **0.0421** ** | 0.0700* | **0.0007*** ** | **0.0000*** ** | **0.0012*** ** | **0.0000*** ** | **0.0000*** ** |

*** = null hypothesis rejected at 0.01 significance; ** = null hypothesis rejected at 0.05 significance; * = null hypothesis rejected at 0.1 significance

*Table 13 Loss function differences and CPA test statistics between each mixed model and its benchmarks using data set of 1985 – 1989 sub-period*

| **HY-GJR-GARCH(N) vs** | **GJR-GARCH(N)** | **HYGARCH (N)** | **FHS(N)** | **HS** |
|---|---|---|---|---|
| **LF difference**[1] | –0.1199 | –0.0303 | –0.1834 | –0.1992 |
| **LF diff. test**[2] | –1.6766 | –2.0613* | –1.9485 | –1.9771* |
| **CPA test**[2] | 3.4104 | 4.6158 | 4.2757 | 5.2398 |
| **HY-GJR-GARCH(t) vs** | **GJR-GARCH(t)** | **HYGARCH (t)** | **FHS(t)** | **HS** |
| **LF difference**[1] | 0.0000 | –0.0289 | 0.0161 | –0.1347 |
| **LF diff. test**[2] | 0.0005 | –1.4969 | 0.2138 | –1.2196 |
| **CPA test**[2] | 1.0911 | 1.8725 | 0.6337 | 2.4282 |
| **HY-MS-GARCH(N) vs** | **MSGARCH (N)** | **HYGARCH (N)** | **FHS(N)** | **HS** |
| **LF difference**[1] | –0.0061 | 0.0519 | –0.1012 | **–0.1170** |
| **LF diff. test**[2] | –0.3988 | 0.8735 | –1.7908 | **–2.6222**** |
| **CPA test**[2] | 0.2752 | 1.7500 | 5.7163 | 9.0041* |
| **HY-MS-GARCH(t) vs** | **MSGARCH (t)** | **HYGARCH (t)** | **FHS(t)** | **HS** |
| **LF difference**[1] | –0.0043 | 0.0005 | 0.0456 | –0.1052 |
| **LF diff. test**[2] | –0.2829 | 0.0100 | 1.4174 | –1.9551 |
| **CPA test**[2] | 0.5409 | 0.4666 | 2.1961 | 5.7087 |

[1] The negativity shows that the mixed model has less loss function than the benchmark i.e. the VaR estimation from mixed model is better. Otherwise, the VaR estimation from mixed model is worse.
[2] The greater magnitude, the higher significance of the corresponding difference.
** = null hypothesis rejected at 0.01 significance; * = null hypothesis rejected at 0.05 significance

especially the dynamic quantile test statistics in Table 11. As the VaR line from GJR-GARCH(N) model pass the DQ test with stronger confidence level of 0.05, and most of VaR lines' DQ test statistics decrease, so the message is the VaR violations come more scattered in this sub-period unlike 1970's.

As in 1970's decade, 7 VaR lines pass the Kupiec's proportion of failure test with tightest confidence level of 0.1. The last half of 1980's decade, with the Black Monday of 1987, is more challenging for chosen parametric models to produce a VaR line estimation that can pass the Kupiec's test with 0.1 significance level because the number of passing VaR lines reduce to 5 out of 13: GJR-GARCH(N), HYGARCH(N), HY-GJR-GARCH(N), MSGARCH(t), and HY-MS-GARCH(t) models. One interesting point in Table 11 is about the historical simulations. As in Table 8, the VaR line from plain historical simulation is obviously inferior from both perspectives of Kupiec's and DQ test. Conversely in Table 11, both estimated VaR lines from filtered historical simulation (with GARCH model implementation) have farther penetration rates from preset of 0.05 than the VaR line from historical simulation. But the loss function of VaR line from historical simulation is also the greatest like happening in Table 8.

*4.3.2.2 HY-MS-GARCH models: clustering the violations is their strategy*

Assumed in Sub-section 4.1.2, the MSGARCH model is preferred to cope with a time of crisis like this sub-period of 1985 – 1989. Consequently, the first parametric models concerned in this sub-period are the MSGARCH model and its derivative, the HY-MS-GARCH model. Consider both models with Gaussian normal and Student's t distributions in comparison. According to Table 11, the VaR lines from both models with Student's t distribution are outstanding in proximity of proportion of failure against the VaR lines from same models with normal distribution, while the loss functions of VaR lines from both models with Student's t distribution are a little bit greater by 0.01 that could be concerned unsignificant. From these facts, the assumption of "fat tails" in Sub-section 4.1.2 is confirmed. However, the columns of MSGARCH(t) and HY-MS-GARCH(t) models in Table 12 give insight about the strategy both models use to estimate the VaR lines with close penetration rate to the preset: clustering the penetration. Because both models' DQ regressions emphasize significance of first three lagged hits unlike most DQ regressions, it could be interpreted that the VaR lines from both models allow a big fall of returns to penetrate themselves consecutively with slight adjustment. This strategy is proved useful in this market situation but might be useless in many "normal" market situations, and the expenditure for this strategy is failing the DQ test as this test admires the autoregressive independence of VaR hits that directly contradicts this strategy.

According to the main assumption about mixed model's superiority to its based models, the performance of HY-MS-GARCH model in this sub-period is one of supporting evidence. Although the comparisons of loss functions in the bottom half of Table 13 do not show significant decrease of loss function of VaR from HY-MS-GARCH model in comparison with listed benchmarks (excepting the historical simulation in case of HY-MS-GARCH with normal distribution). The closeness of proportion of failure as improvement from MSGARCH model's VaR line is evident in any cases, as comparisons of Kupiec's test statistics between HY-MS-GARCH(N) and MSGARCH(N) models and between HY-MS-GARCH(t) and MSGARCH(t) models in Table 11 show less test statistic of the mixed model. This means the MSGARCH model is improved by the addition of fractional exponential feature from HYGARCH model as the proposed HY-MS-GARCH model.

*4.3.2.3 HY-GJR-GARCH(N) model: the surprisingly outstanding model*

Contrast to the MSGARCH and HY-MS-GARCH model pair, the GJR-GARCH and HY-GJR-GARCH model pair can be assumed as a misspecification for the market situation in this sub-period. Surprisingly, the GJR-GARCH(N) and HY-GJR-GARCH(N) models are the only two models in this sub-period that their estimated VaR lines can pass both Kupiec's test with 0.1 significance and DQ test with 0.01 significance at least. Moreover, the upper part of Table 13 indicates that the HY-GJR-GARCH(N) model is the only one model to make the least loss function among the loss functions of VaR line from its own benchmarks. The description of these surprising results can refer to the details in market situation. See the Figures 14 and 15 in Sub-section 4.1.2. The Figure 14 indicates how the uptrend is dominated most of the time in this sub-period, excluding the Black Monday that is represented by almost vertical line in the chart of cumulative returns. But the Figure 15 illustrates the Black Monday as by the burst in volatility that would be pacified to normal level in one or two quarters. This burst is contrast to the burst after a downtrend end in middle 1982 as the smaller "shocks" causes higher volatility that spends about a year or more to tame down. This evidence well follows the stylized fact stated by Engle and Mustafa (1992) and Friedman et al. (1989). Thus, the GJR-GARCH model can also describe this situation well due to its formulation to support this stylized fact that the downside returns also cause an increase in volatility. As this stylized fact is more common and frequently observed in any market situations even the time of crisis, the GJR-GARCH and its derivative, the proposed HY-GJR-GARCH model, should be generally effective than the HY-MS-GARCH model that appears to be more specified to some situation, as assumed in Section 1.2. Then, this assumption is confirmed by the test statistics of HY-GJR-GARCH(N) model in this sub-section. However, this superiority does not apply to the HY-GJR-GARCH model when the Student's t distribution is assumed. Although this sub-period is assumed Student's t distributed as mentioned from Sub-section 4.1.2. In this case, the possibility of model misspecification might be used again as the general description like in Sub-section 4.3.1.



### 4.3.3 The 1995 – 1999 sub-period

#### *4.3.3.1 Overall: the easy time for all models*

The 1995 – 1999 sub-period, unlike the first two mentioned sub-periods before, comes with smoother "bull" market. Thus, it seems to be easier for any VaR line estimations to pass the qualification metrics as seen from Tables 14 – 16. Same format as Tables 8 – 10 in Sub-section 4.3.1 and Tables 11 – 13 in Sub-section 4.3.2, these tables display the test statistics of corresponding VaR lines estimation to listed models with the data set of S&P 500 returns in 1995 – 1999. At the first glance of Table 14, the VaR estimation from historical simulation is unquestionably the underdog due to the highest Kupiec's test statistic, DQ test statistic, and loss function. Then, it is useless to mention the simple historical simulation again in this sub-section. Another interestingness of Table 14 is the number of passing VaR lines for the tests. 8 VaR lines pass the Kupiec's test with strongest confidence level of 0.1,

*Table 14 Proportions of failure, Kupiec's test statistics, DQ test statistics, and loss function for each model using data set of 1995 – 1999 sub-period*

| Model | Proportion of Failure[1] | Kupiec's test[2] | DQ test[2] | Loss function[2] |
|---|---|---|---|---|
| **HS** | 7.9968% | 20.3675 | 43.1851 | 1.5224 |
| **FHS(N)** | 5.6215% | **0.9892*** ** | 14.0935* | 1.4544 |
| **GJR-GARCH(N)** | 5.8591% | **1.8641*** ** | 13.4199* | 1.4429 |
| **MSGARCH(N)** | 7.6010% | 15.6241 | 33.5391 | 1.4650 |
| **HYGARCH(N)** | 4.9881% | **0.0004*** ** | 21.4102 | 1.4364 |
| **HY-GJR-GARCH(N)** | 5.5424% | **0.7567*** ** | **10.6629** ** | 1.3878 |
| **HY-MS-GARCH(N)** | 6.3341% | 4.3800* | 17.5415 | 1.4414 |
| **FHS(t)** | 5.8591% | **1.8641*** ** | 43.1851 | 1.4505 |
| **GJR-GARCH(t)** | 4.3547% | **1.1555*** ** | 27.0107 | 1.4569 |
| **MSGARCH(t)** | 4.4339% | **0.8845*** ** | **10.8955** ** | 1.4576 |
| **HYGARCH(t)** | 3.4838% | 6.8076 | 27.7798 | 1.4861 |
| **HY-GJR-GARCH(t)** | 3.8005% | 4.1566* | 18.8343 | 1.4257 |
| **HY-MS-GARCH(t)** | 4.3547% | **1.1555*** ** | 14.3079* | 1.4501 |

[1] The closer proportion of failure to the preset (5%), the better VaR estimation.
[2] The less test statistic, the better VaR estimation.
*** = null hypothesis accepted at 0.1 significance
** = null hypothesis accepted at 0.05 significance
* = null hypothesis accepted at 0.01 significance

and 5 lines pass the DQ test with confidence level of 0.01 (with 2 lines at stronger level of 0.05). Unlike two previous sub-sections, The Table 15 indicates indifference in reaction to penetrations among VaR lines. Since all VaR lines, even the "worst" VaR line from historical simulation, show the same "double adjustment" after the first penetration like happening interpreted from Table 9 but in softer situation as only the second lagged hit variable is emphasized for every VaR lines.

*4.3.3.2 Two mixed models: the usefulness of HYGARCH model component*

In Sub-section 4.1.3 of 1995 – 1999 sub-period, the HYGARCH model is directly assumed as the appropriate model to describe the market situation in this sub-period, unlike in other sub-periods that the HYGARCH is mentioned as a subdominant feature in accompanied with GJR-GARCH or MSGARCH model. Thus, the HYGARCH model should be concerned as a main factor for success in VaR testing for the mixed models. A foolproof for this assumption, and the main assumption of improvement by HYGARCH component, is also in the first column of Table 14. Unlike first columns in Tables 8 and 11, VaR lines from all four mixed models can pass the Kupiec's test with at least 0.01 confidence level, and two of them can also pass the DQ test with at least 0.01 confidence level.

*Table 15 Dynamic quantile linear regression model parameters for each model using data set of 1995 – 1999 sub-period*

| | **HS** | **FHS(N)** | **GJR-GARCH(N)** | **MSGARCH(N)** | **HYGARCH(N)** | **HY-GJR-GARCH(N)** | **HY-MS-GARCH(N)** | **FHS(t)** | **GJR-GARCH(t)** | **MSGARCH(t)** | **HYGARCH(t)** | **HY-GJR-GARCH(t)** | **HY-MS-GARCH(t)** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Intercept** | 0.0064 | 0.0302 | 0.0218 | 0.0348 | 0.0136 | 0.0194 | **0.0428** | **0.0367** | 0.0236 | 0.0211 | 0.0068 | 0.0053 | 0.0137 |
| p-value | 0.8403 | 0.0768* | 0.1822 | 0.0838* | 0.4048 | 0.2486 | **0.0159**** | **0.0354**** | 0.0973* | 0.1523 | 0.6338 | 0.7192 | 0.3662 |
| **Hit_lag_1** | –0.0042 | 0.0104 | –0.0186 | 0.0060 | 0.0012 | –0.0126 | 0.0216 | 0.0045 | –0.0085 | –0.0061 | 0.0117 | 0.0059 | –0.0040 |
| p-value | 0.8827 | 0.7140 | 0.5095 | 0.8318 | 0.9652 | 0.6555 | 0.4457 | 0.8729 | 0.7644 | 0.8280 | 0.6778 | 0.8359 | 0.8864 |
| **Hit_lag_2** | **0.0860** | **0.0945** | **0.0878** | **0.1008** | **0.1186** | **0.0835** | **0.0933** | **0.1313** | **0.0747** | **0.1443** | **0.1053** | **0.0959** | **0.1279** |
| p-value | **0.0024***** | **0.0008***** | **0.0021***** | **0.0004***** | **0.0000***** | **0.0035***** | **0.0010***** | **0.0000***** | **0.0083***** | **0.0000***** | **0.0002***** | **0.0008***** | **0.0000***** |
| **Hit_lag_3** | 0.0417 | 0.0151 | 0.0011 | 0.0081 | –0.0510 | –0.0057 | –0.0171 | 0.0096 | –0.0390 | –0.0441 | –0.0382 | –0.0349 | –0.0434 |
| p-value | 0.1403 | 0.5941 | 0.9687 | 0.7752 | 0.0712* | 0.8433 | 0.5436 | 0.7329 | 0.1678 | 0.1176 | 0.1764 | 0.2222 | 0.1245 |
| **Hit_lag_4** | –0.0197 | –0.0355 | –0.0365 | –0.0234 | –0.0306 | –0.0303 | –0.0207 | –0.0472 | –0.0058 | –0.0281 | 0.0027 | 0.0018 | –0.0225 |
| p-value | 0.4856 | 0.2095 | 0.2017 | 0.4095 | 0.2807 | 0.2916 | 0.4625 | 0.0947* | 0.8381 | 0.3201 | 0.9246 | 0.9502 | 0.4256 |
| **VaR_lag_1** | –1.7956 | 1.7266 | 0.9554 | 0.8970 | 0.9270 | 0.9853 | 2.3415 | 2.0455 | **1.7957** | 1.6472 | 1.1633 | 0.9662 | 1.2425 |
| p-value | 0.5004 | 0.1233 | 0.3755 | 0.5594 | 0.3701 | 0.3700 | 0.0639* | 0.0749* | **0.0219**** | 0.0530* | 0.1202 | 0.2263 | 0.1620 |
| **Adjusted $R^2$** | **0.0058** | **0.0078** | **0.0052** | **0.0066** | **0.0136** | 0.0042 | **0.0087** | **0.0166** | **0.0071** | **0.0216** | **0.0106** | **0.0070** | **0.0159** |
| p-value | **0.0312**** | **0.0114**** | **0.0414**** | **0.0207**** | **0.0005***** | 0.0691* | **0.0070***** | **0.0001***** | **0.0159**** | **0.0000***** | **0.0025***** | **0.0166**** | **0.0001***** |

*** = null hypothesis rejected at 0.01 significance; ** = null hypothesis rejected at 0.05 significance; * = null hypothesis rejected at 0.1 significance

*Table 16 Loss function differences and CPA test statistics between each mixed model and its benchmarks using data set of 1995 – 1999 sub-period*

| **HY-GJR-GARCH(N) vs** | **GJR-GARCH(N)** | **HYGARCH (N)** | **FHS(N)** | **HS** |
|---|---|---|---|---|
| **LF difference**[1] | **–0.0551** | –0.0486 | **–0.0666** | **–0.1346** |
| **LF diff. test**[2] | **–3.1353**** | –2.5289* | **–3.1711**** | **–2.7278**** |
| **CPA test**[2] | 8.7417* | 6.5179* | 8.6886* | 8.6493* |
| **HY-GJR-GARCH(t) vs** | **GJR-GARCH(t)** | **HYGARCH (t)** | **FHS(t)** | **HS** |
| **LF difference**[1] | –0.0312 | **–0.0604** | –0.0248 | –0.0967 |
| **LF diff. test**[2] | –2.4667* | **–4.0445**** | –0.8986 | –1.7089 |
| **CPA test**[2] | 5.5627 | **12.2702**** | 7.6634* | 3.1497 |
| **HY-MS-GARCH(N) vs** | **MSGARCH (N)** | **HYGARCH (N)** | **FHS(N)** | **HS** |
| **LF difference**[1] | –0.0236 | 0.0050 | –0.0130 | –0.0810 |
| **LF diff. test**[2] | –1.8800 | 0.3593 | –0.9100 | –2.1723* |
| **CPA test**[2] | 5.1794 | 0.3611 | 2.0587 | 5.9909 |
| **HY-MS-GARCH(t) vs** | **MSGARCH (t)** | **HYGARCH (t)** | **FHS(t)** | **HS** |
| **LF difference**[1] | –0.0075 | **–0.0360** | –0.0004 | –0.0723 |
| **LF diff. test**[2] | –0.9288 | **–2.8004**** | –0.0193 | –1.4546 |
| **CPA test**[2] | 4.1376 | 8.4467* | 4.3545 | 2.5492 |

[1] The negativity shows that the mixed model has less loss function than the benchmark i.e. the VaR estimation from mixed model is better. Otherwise, the VaR estimation from mixed model is worse.
[2] The greater magnitude, the higher significance of the corresponding difference.
** = null hypothesis rejected at 0.01 significance; * = null hypothesis rejected at 0.05 significance

*4.3.3.3 The comparison among combinations: HY-GJR-GARCH(N) the winner and the others as misspecification*



Although all four mentioned mixed models consist of the HYGARCH model component, they all need to be compared. Then, the assumed distribution is concerned next. The Sub-section 4.1.3 prefers the normal to Student's t distribution to assumed about residuals' distribution in this 1995 – 1999 sub-period. Thus, the VaR lines from two mixed models with normal distribution are examined. The VaR line from HY-GJR-GARCH(N) model, like in previous two sub-sections, is concerned outstanding in passing Kupiec's and DQ tests, as one of two VaR lines that pass both tests with strictest possible confidence level, and giving the least loss function like happening in Table 11. The top part of Table 16 also confirms this superiority that the VaR line from HY-GJR-GARCH(N) model can beat all baselines with all significant loss function different and CPA test statistics that underline how really the loss function differences are.

However, the VaR line from HY-MS-GARCH(N) model is not outstanding although the Sub-section 4.1.3 is assumed about HY-MS-GARCH model as the proper model to describe the market in this sub-period. Concern the two “big bursts” in Figure 18 again. Mentioned as resemblance to the “Black Monday” of 1987 in Sub-section 4.1.3, they are not harsh like the Black Monday but comparable to the return swings in 1970’s decade. The market difference before and after late 1997, especially when the market is extremely peaceful before 1997, might lead understanding about the market from late 1997 like the outlier. But they are common like the market in 1970’s decade sub-period when excluding odd daily returns showing in Figure 18 like long single lines. Thus, the MSGARCH model is said not specific to this situation.

Due to more normality of market situation, the results from mixed models with Student’s t distribution is not satiable like the result when these models combining with normal distribution as seen in Table 14. However, the VaR line from HYGARCH(t) model does the tests with poor results. If the HYGARCH(t) model itself is not good in this situation, it will lower the competency of two mixed models in the same tests too as the HYGARCH model is concerned dominated in this sub-period.

# Chapter 5 Conclusion

In attempt to develop two mixed models, HY-GJR-GARCH and HY-MS-GARCH model, these models can pick advantages from both of their base models that are specific to different characteristics of the financial market. However, in empirical tests using the samples from S&P 500 index returns in different period of situations, there is some mixed model that cannot perform better than its benchmark models, specifically in some situation that its base models are misused to capture market features. That means mixing characteristics between two GARCH model variants cannot boost performance of a new mixed GARCH model variant to satiable level in every cases. Nevertheless, as seen from the results, market conditions are important in decision about which GARCH model variants are used for the mixed model and which probability distribution is assumed to be the distribution of residuals after the model.

From four combinations of two mixed models by two different probability distributions, the HY-GJR-GARCH model with normal distribution seems to be the combination with the best performance among all combinations as measured from tests in different market situations. Additionally, in the situation of big crisis causing sudden loss in the market like the Black Monday of 1987, the HY-MS-GARCH model with Student's t distribution also works well in such situation.

However, this research studies these two mixed models when working with only limited number of simple symmetric distributions like normal and Student's t distributions, limited number of parameters to the minimum requirement that each model can keep its identity, and limited test data sets from only one market index in distant time. Afterwards, these mixed models should be combined with more complex probability distributions or modified to some extent, i.e. increasing the number of regimes for MSGARCH and HY-MS-GARCH models, for further studies about their performance. And the mathematics beneath the mixed models should be developed more to reduce complexity that might consume resource in modeling and application both in this specific case of calculation for the VaR and other cases that need the knowledge about volatility. Also, using various data sets like the prices from different kinds of assets, with more recent time frame to better reflect the contemporary market situation, should be considered too for further studies after this research.

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# VITA

| | |
|---|---|
| **NAME** | Chanet Saisatian |
| **DATE OF BIRTH** | 29 January 1993 |
| **PLACE OF BIRTH** | Bangkok, Thailand |
| **INSTITUTIONS ATTENDED** | Chulalongkorn University (2011-2014)<br>Bachelor of Engineering (Second Class Honours) in Computer Engineering |