ลักษณะหลังการเควนช์สำหรับสมการเชิงอนุพันธ์ย่อยแบบพาราโบลิกกึ่งเชิงเส้นเอกฐานที่มีเงื่อนไขขอบแบบผสม

นายเบญจามิน ไทยถาวร

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ประยุกต์และวิทยาการคณนา
ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2565

BEYOND QUENCHING PROFILE FOR SINGULAR SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED BOUNDARY CONDITIONS

Mr. Benjamin Thaitavorn

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Applied Mathematics and
Computational Science

Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University

Academic Year 2022

Thesis Title	BEYOND QUENCHING PROFILE FOR SINGULAR SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED BOUNDARY CONDITIONS
By	Mr. Benjamin Thaitavorn
Field of Study	Applied Mathematics and Computational Science
Thesis Advisor	Associate Professor Ratinan Boonklurb, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Khamron Mekchay, Ph.D.)

. . . . Thesis Advisor

(Associate Professor Ratinan Boonklurb, Ph.D.)

. . . . Examiner

(Monchai Kooakachai, Ph.D.)

. . . . External Examiner

(Associate Professor Tawikan Treeyaprasert, Ph.D.)

เบญจามิน ไทยถาวร : ลักษณะหลังการเควนช์สำหรับสมการเชิงอนุพันธ์ย่อยแบบพาราโบลิกกึ่งเชิงเส้นเอกฐานที่มีเงื่อนไขขอบแบบผสม. (BEYOND QUENCHING PROFILE FOR SINGULAR SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED BOUNDARY CONDITIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร. รตินันท์ บุญเคลือบ, 54 หน้า.

กำหนดให้ $0 < r < 1$, $L > 0$, $T \leq \infty$, $D = (0, L)$, $\Omega = D \times (0, T]$ และ $\chi_{\mathbb{S}}$ เป็นฟังก์ชันลักษณะเฉพาะของเซต $\mathbb{S}$ วิทยานิพนธ์ฉบับนี้ ศึกษาลักษณะหลังการเควนช์ของผลเฉลยของปัญหาค่าเริ่มต้น และค่าขอบของปัญหาสองปัญหาต่อไปนี้

$$u_t - u_{xx} - \frac{r}{x} u_x = f(u)\chi_{\{u<c\}}(u) \text{ ใน } \Omega,$$
$$u(x, 0) = 0 \text{ บน } \bar{D}$$

ภายใต้เงื่อนไขค่าขอบแบบผสม

$$u(0, t) = 0 = u_x(L, t) \quad \text{สำหรับ } 0 < t < T$$

หรือ

$$u_x(0, t) = 0 = u(L, t) \quad \text{สำหรับ } 0 < t < T$$

โดยที่ f เป็นฟังก์ชันที่หาอนุพันธ์อย่างต่อเนื่องได้ถึงอันดับที่สองบนช่วง $[0, c)$ สำหรับค่าคงตัว c บางค่า และ $f(0) > 0$, $f' > 0$, $f'' \geq 0$ และ $\lim_{u \to c^-} f(u) = \infty$ เราสามารถแสดงได้ว่าในแต่ละปัญหา เมื่อ t เข้าสู่ ∞ ผลเฉลยอย่างอ่อนทุกตัวจะลู่เข้าสู่ผลเฉลยสภาวะคงตัว ซึ่งกำหนดโดยการแก้ปัญหาค่าขอบสองปัญหา โดยที่ปัญหาแรกบน $(0, \ell_s^*)$ และปัญหาที่เหลือบน (ℓ_s^*, L) ซึ่ง ℓ_s^* เป็นค่าคงตัวที่เป็นบวกที่คำนวณได้โดยวิธีเชิงตัวเลข ซึ่งขั้นตอนวิธีเชิงตัวเลข สำหรับการคำนวณค่าของ ℓ_s^* มีการสร้างไว้ให้สำหรับปัญหาแต่ละปัญหาด้วย

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต	....
สาขาวิชา	คณิตศาสตร์ประยุกต์และวิทยาการคณนา	ลายมือชื่อ อ.ที่ปรึกษาหลัก	....
ปีการศึกษา	2565	ลายมือชื่อ อ.ที่ปรึกษาร่วม	....

6470112223 : MAJOR APPLIED MATHEMATICS AND COMPUTATIONAL SCIENCE
KEYWORDS : QUENCHING / SINGULAR / CONVECTION-DIFFUSION / MIXED BOUNDARY CONDITIONS / NONLINEAR SOURCE

BENJAMIN THAITAVORN : BEYOND QUENCHING PROFILE FOR SINGULAR SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED BOUNDARY CONDITIONS. ADVISOR : ASSOC. PROF. RATINAN BOONKLURB, Ph.D., 54 pp.

Let $0 < r < 1$, $L > 0$, $T \leq \infty$, $D = (0, L)$, $\Omega = D \times (0, T]$ and $\chi_{\mathbb{S}}$ be the characteristic function of the set $\mathbb{S}$. This thesis studies beyond quenching solution profiles of the two initial-boundary value problems:

$$u_t - u_{xx} - \frac{r}{x} u_x = f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega,$$
$$u(x, 0) = 0 \text{ on } \bar{D}$$

subject to the mixed boundary conditions:

$$u(0, t) = 0 = u_x(L, t) \quad \text{for } 0 < t < T$$

or

$$u_x(0, t) = 0 = u(L, t) \quad \text{for } 0 < t < T$$

where f is a twice continuously differentiable function on $[0, c)$, for some constant c, with $f(0) > 0$, $f' > 0$, $f'' \geq 0$ and $\lim_{u \to c^-} f(u) = \infty$. It is shown in each problem that, as t tends to ∞, all weak solutions tend to a unique steady-state solution determined by solving two boundary value problems: one on $(0, \ell_s^*)$ and the other on (ℓ_s^*, L), where ℓ_s^* is a positive constant to be obtained numerically. A numerical method for computing the value of ℓ_s^* is provided for each problem

Department :	Mathematics and Computer Science	Student's Signature ....
		Advisor's Signature ....
Field of Study :	Applied Mathematics and Computational Science	Co-advisor's Signature ....
Academic Year :	2022	

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to my thesis advisor, Associate Professor Dr. Ratinan Boonklurb, for his invaluable guidance, encouragement, and unwavering support throughout my graduate studies. His mentorship has been instrumental in shaping my research skills and nurturing my academic growth. I am also grateful to my thesis committee members, Associate Professor Dr. Khamron Mekchay, Dr. Monchai Kooakachai, and Associate Professor Dr. Tawikan Treeyaprasert, for their insightful feedback and constructive criticisms that greatly enhanced the quality of this work.

I would like to thank the Faculty of Science, Chulalongkorn University, for awarding me the "Vidyabundit" scholarship, which provided financial support for my studies. I also extend my gratitude to the Department of Mathematics and Computer Science for the opportunities to present my research and for funding other academic endeavors during my graduate studies.

My sincere thanks also go to my family and colleagues who have supported me through my studies, providing encouragement, motivation, and comfort during difficult times.

Lastly, I am grateful to OpenAI.com for providing access to their natural language processing service through ChatGPT API, which has greatly assisted in refining the wording and improving the readability of this thesis.

CONTENTS

Page

ABSTRACT IN THAI iv

ABSTRACT IN ENGLISH v

ACKNOWLEDGEMENTS vi

CONTENTS vii

LIST OF TABLES viii

LIST OF FIGURES ix

CHAPTER

1 INTRODUCTION **1**

1.1 Literature review 2

1.2 Research problem 4

1.3 Thesis organization 6

2 QUENCHING PROFILE OF LEFT IBVP **7**

2.1 Beyond quenching profile 8

2.2 Bound for ℓ_s^* 13

3 QUENCHING PROFILE OF RIGHT IBVP **16**

3.1 Beyond quenching profile 17

3.2 Bound for ℓ_s^* 21

4 NUMERICAL METHOD FOR QUENCHING PROFILES **24**

4.1 Boundary value problem discretization 25

4.2 Numerical method for the Left IBVP 26

4.3 Numerical method for the Right IBVP 29

4.4 Demonstrations 30

5 CONCLUSION **36**

REFERENCES 39

APPENDICES 41

BIOGRAPHY 54

LIST OF TABLES

Table Page

4.1 Numerical approximations of ℓ_s^* for $f(u) = (1-u)^{-\beta}$ 32

4.2 Numerical approximations of ℓ_s^* for $f(u) = \alpha - \beta \ln(1-u)$ 34

LIST OF FIGURES

Figure Page

4.1 Steady-state solution profiles for (a) Left IBVP and (b) Right IBVP where $r = 1/5$, $L = 5$ and $f(u) = (1-u)^{-1/3}$. 33

4.2 Steady-state solution profiles for (a) Left IBVP and (b) Right IBVP where $r = 1/5$, $L = 5$ and $f(u) = 1 - \ln(1-u)$. 35

CHAPTER I

INTRODUCTION

In the mid-twentieth century, studies on certain classes of differential equations that exhibit nonlinearity in their boundary conditions or equations revealed that their solutions can become unbounded and cease to exist after finite time intervals [13]. This occurrence is known as blow-up. Another related phenomenon is quenching, where a solution becomes discontinuous and loses its regularity.

Quenching was introduced by Kawarada [15] in 1975 when he studied the following reaction-diffusion equation with a nonlinear reaction term:

$$\left.\begin{aligned} &u_t - u_{xx} = (1-u)^{-1},\ x \in (0, L),\ t > 0, \\ &u(x, 0) = 0,\ x \in [0, L], \\ &u(0, t) = 0 = u(L, t),\ t > 0. \end{aligned}\right\} \tag{1.1}$$

The solution u is increasing with respect to t on $(0, L)$. If at some $t = t_q$, $u(L/2, t_q)$ reaches 1, then $u_t(L/2, t_q)$ becomes infinite due to the nonlinear term and (1.1) is not everywhere defined. As a consequence, u cannot be continued beyond t_q. Based on the previous observation, a solution u *quenches* if there exists $t_q \in (0, \infty]$ such that

$$\sup_{0 \le x \le L} \{u_t(x, t)\} \to \infty \text{ as } t \to t_q.$$

Chan and Kong [11] studied a similar problem and gave an equivalent definition as

$$\sup_{0 \le x \le L} \{u(x, t)\} \to c^- \text{ as } t \to t_q.$$

If t_q is finite, then u *quenches in a finite time*. Otherwise, u *quenches in infinite time*.

Quenching can be interpreted in various physical contexts, such as in gas combustion processes where the gas is compressed, causing its temperature to increase. At a certain point, known as the autoignition point, the gas spontaneously ignites, further increasing its temperature significantly. In this case, the autoignition temperature can be considered the quenching point in the solution, where the solution undergoes an abrupt change in value due to the ignition. This phenomenon can result in a loss of regularity in the solution, as the sudden increase in temperature can cause a singularity in the equation being modeled.

1.1 Literature review

In 1976, Acker and Walter [2] studied the reaction-diffusion equation with a generalized reaction term as follows:

$$\left.\begin{aligned} &u_t - u_{xx} = f(u),\ x \in (-L, L),\ t > 0,\\ &u(x,0) = 0,\ x \in [-L, L],\\ &u(-L,t) = 0 = u(L,t),\ t > 0, \end{aligned}\right\} \tag{1.2}$$

where f is locally Lipchitz continuous in the range of u, $f(0) > 0$, and $\lim_{u\to c^-} f(u) = \infty$ for some $c > 0$. Let L^* be the supremum of all L such that a steady-state solution to (1.2) exists. They found that if $L > L^*$, the solution u quenches in a finite time; otherwise, u exists globally.

In 1994, Chan and Ke [10] expanded the previous works by studying what happens after a solution quenches in a finite time. To let the quenched solution continue past the first quenching time, they multiply the reaction term by the characteristic function $\chi_{\mathbb{S}}$ which acts as a switch to nullify f at the region where the solution has quenched. The resulting problem is as follows:

$$\left.\begin{aligned} &u_t - u_{xx} = f(u)\chi_{\{u<c\}}(u),\ x \in (0, L),\ t \in (0, T],\\ &u(x,0) = 0,\ x \in [0, L],\\ &u(0,t) = 0 = u(L,t),\ t \in (0, T], \end{aligned}\right\} \tag{1.3}$$

where $L > 0$, $T \leq \infty$, $\chi_{\mathbb{S}}(x) = \begin{cases} 1 \text{ if } x \in \mathbb{S}, \\ 0 \text{ if } x \notin \mathbb{S} \end{cases}$ is the characteristic function of the set $\mathbb{S}$, f is twice continuously differentiable on $[0, c)$ with $f(0) > 0$, $f' > 0$, $f'' \geq 0$ and $\lim_{u \to c^-} f(u) = \infty$. It was shown that, for L sufficiently large, weak solutions exist and tend to the unique solution of the following boundary value problems:

$$\left.\begin{aligned} W(x) &= c, \; x \in (b_s^*, B_s^*), \\ -W''(x) &= f(W(x)), \; x \in (0, b_s^*), \; W(0) = 0, W(b_s^*) = c, \\ -W''(x) &= f(W(x)), \; x \in (B_s^*, L), \; W(B_s^*) = c, W(L) = 0, \end{aligned}\right\} \tag{1.4}$$

where $b(t) = \inf\{x : u(x,t) = c\}$, $B(t) = \sup\{x : u(x,t) = c\}$, $b_s^* = \lim_{t \to \infty} b(t)$ and $B_s^* = \lim_{t \to \infty} B(t)$.

In 2018, Boonklurb et al. [6] studied the following singular convection-diffusion problem:

$$\left.\begin{aligned} \mathcal{L}u := u_t - u_{xx} - \frac{r}{x}u_x &= f(u)\chi_{\{u<c\}}(u), \; x \in (0, L), \; t \in (0, T], \\ u(x, 0) &= 0, \; x \in [0, L], \\ u(0, t) &= 0 = u(L, t), \; t \in (0, T], \end{aligned}\right\} \tag{1.5}$$

where $0 < r < 1$, and all other variables defined in the same manner as in (1.3). For L sufficiently large, they arrived at conclusions similar to (1.4); however, the added singular term ru_x/x destroys the symmetry of the beyond quenching profile of (1.5) [9].

A number of physical phenomena involve the linear operator $\mathcal{L}$. For instance, when $0 \leq r \leq 2$, the operator represents heat conduction in geometric bodies with r being their shape parameters [9]. When r is a positive integer, it corresponds to radially symmetric diffusion in $r + 1$ dimensions [4]. In probability theory, the same operator describes a singular diffusion as a part of the Fokker-Planck equation and a radial component of a Brownian motion as a part of the Kolmogorov equation [3].

1.2 Research problem

In this thesis, we extend the previous works by studying the following singular convection-diffusion initial-boundary value problems (IBVP):

$$\mathcal{L}u = f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega, \tag{1.6}$$

with the initial condition

$$u(x,0) = 0 \text{ on } \bar{D}, \tag{1.7}$$

where $D = (0, L)$, $\Omega = D \times (0, T]$ and all other variables and functions are defined in the same manner as in (1.3).

We extend the previous work by Boonklurb et al. [6] to accommodate a wider range of radially symmetric systems by considering two other combinations of boundary conditions:

$$u(0,t) = 0 = u_x(L,t),\ t \in (0,T] \tag{1.8}$$

and

$$u_x(0,t) = 0 = u(L,t),\ t \in (0,T]. \tag{1.9}$$

We denote (1.6) – (1.8) as the *Left IBVP* and (1.6), (1.7), (1.9) as the *Right IBVP*. We multiply (1.6) by x^r to transform the singular term. The following formulation of (1.6) will be used for analysis throughout the thesis:

$$\mathcal{H}u := x^r u_t - (x^r u_x)_x = x^r f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega.$$

Since this thesis focuses mainly on the beyond quenching profiles, we assume the existence of the weak solution to both IBVPs. One can use techniques demonstrated in [8, 12] along with the following assumptions:

$$f'(u)\left(\frac{c-u}{f(u)}\right)^2 \leq K_1 \quad \text{and} \quad \int_u^c f(s)ds \leq \min\left\{K_2(c-u)f(u), K_3(c-u)^\gamma\right\}$$

for $0 \leq u < c$, some positive constants K_1, K_2, K_3, and $\gamma \in (0,2)$ to prove the existence of their weak solutions, which are defined as follows:

Definition 1.1. A function u is said to be a weak solution of either the Left IBVP or the Right IBVP if

(i) $u \in \boldsymbol{C}([0,t_0]; \boldsymbol{L}^1(D)) \cap \boldsymbol{L}^\infty(D \times (0,t_0))$ for each $t_0 > 0$;

(ii) for any $v \in \boldsymbol{C}^{2,1}(\bar{\Omega})$ such that v has a compact support with respect to t and $v(0,t) = 0 = v_x(L,t)$ for the Left IBVP or $v_x(0,t) = 0 = v(L,t)$ for the Right IBVP,

$$\int_0^\infty \int_0^L u\mathcal{H}^\dagger v dx dt + \int_0^\infty \int_0^L x^r f(u) \chi_{\{u<c\}}(u) v dx dt = 0$$

where $\mathcal{H}^\dagger := x^r \partial/\partial t + \partial/\partial x \, (x^r \partial/\partial x)$ is the adjoint operator of $\mathcal{H}$.

Remark 1.1. $\boldsymbol{C}([0,t_0]; \boldsymbol{L}^1(D))$ is the space of functions $g : [0,t_0] \to \boldsymbol{L}^1(D)$ such that g is continuous on $[0,t_0]$.

In addition, one can manipulate the results of [14] to deduce the following properties of the weak solutions:

Theorem 1.1. *Weak solutions u of the Left IBVP and the Right IBVP exist and possess the following properties:*

(i) $u(x,t) \in \boldsymbol{C}^{2,1}(\{(x,t)|u(x,t) < c\}) \cap \boldsymbol{C}^{1,0}(\Omega) \cap \boldsymbol{C}(\bar{\Omega})$;

(ii) $u \leq c$ *in* Ω;

(iii) if $u(x,t_0) = c$ *for some* $x \in D$ *and* $t_0 \in [\tau, \infty)$, *then* $u(x,t) = c$ *for* $t \in [t_0, \infty)$, *where* τ *is the first finite quenching time;*

(iv) u *is nondecreasing with respect to* t *in* $\{(x,t)|u(x,t) < c\}$;

(v) $u_x = 0$ *at the point* (x,t) *where* $u(x,t) = c$.

1.3 Thesis organization

This thesis is organized as follows. Chapter II and Chapter III derive the beyond quenching profiles for the Left and Right IBVPs, respectively. In Chapter IV, a numerical method is developed to calculate the parameters that characterize the solution profiles from the preceding sections. Finally, Chapter V presents a discussion of the results and the thesis concludes with a summary of the findings.

CHAPTER II

QUENCHING PROFILE OF LEFT IBVP

Let $\mathcal{L}u = u_t - u_{xx} - (r/x)u_x$ and $\mathcal{H}u = x^r u_t - (x^r u_x)_x$ where $0 < r < 1$. We restate the Left IBVP as

$$\begin{aligned}
&\mathcal{L}u = f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega,\\
&u(x,0) = 0,\ x \in \bar{D},\\
&u(0,t) = 0 = u_x(L,t),\ t \in (0,T]
\end{aligned}$$

or

$$\left.\begin{aligned}
&\mathcal{H}u = x^r f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega,\\
&u(x,0) = 0,\ x \in \bar{D},\\
&u(0,t) = 0 = u_x(L,t),\ t \in (0,T],
\end{aligned}\right\} \tag{2.1}$$

where $L > 0$, $T \leq \infty$, $D = (0,L)$, $\Omega = D \times (0,T]$, and f is twice continuously differentiable on $[0,c)$ for some constant c with $f(0) > 0$, $f' > 0$, $f'' \geq 0$ and $\lim_{u \to c^-} f(u) = \infty$.

In this chapter, we show that under the properties in Theorem 1.1, weak solutions of (2.1) converge uniformly to the unique solution of the following steady-state problem as t tends to infinity:

$$W(x) = c,\ x \in [\ell_s^*, L], \tag{2.2}$$

$$-(x^r W'(x))' = x^r f(W(x)),\ x \in (0,\ell_s^*),\ W(0) = 0, W(\ell_s^*) = c, \tag{2.3}$$

where ℓ_s^* is a positive constant to be determined. Let u denote any weak solution of (2.1). We would like to study the behavior of u beyond the first quenching time.

2.1 Beyond quenching profile

Let τ be the first finite time when quenching occurs. For $t \geq \tau$, we define $\ell(t) = \inf\{x : u(x,t) = c\}$ and $\ell^* = \lim_{t\to\infty} \ell(t)$.

By manipulating the ideas of Lemma 2 and Lemma 6 of Chan and Ke [10], we arrive at Lemma 2.1.

Lemma 2.1. *The function $\ell(t)$ is nonincreasing and $\ell^* \geq \ell_s^* > 0$.*

One physical interpretation of Lemma 2.1 is that, as time approaches infinity, the region within D where the temperature u approaches c from below should expand. This is because the Neumann boundary condition at $x = L$ requires zero heat flux at that end, causing a buildup of heat that would eventually raise the temperature to the critical value c at that point. Meanwhile, the Dirichlet boundary condition at $x = 0$ constrains the temperature u to remain below c in a certain part of D, preventing the critical temperature from being reached in that region. Therefore, the solution profile after the first quenching has occurred should separate into two segments demarcated by ℓ_s^* as in Lemma 2.2 and Lemma 2.3.

From (ii) and (iv) in Theorem 1.1, we can use the Dini's Theorem ([19], p.143) to deduce that $u(x,t)$ converges uniformly to its continuous limit as $t \to \infty$ on $\bar{D}$ denoted by

$$U(x) = \lim_{t\to\infty} u(x,t).$$

Lemma 2.2. *For $x \in (0, \ell^*)$, $u(x,t)$ converges uniformly to a solution of (2.3) as $t \to \infty$ with $\ell^* = \ell_s^*$.*

Proof. We use the same idea presented in [6]. Consider u in the region $[0, \tilde{\ell}] \times (0,\infty)$ where $\tilde{\ell} \in [0, \ell^*]$. We define

$$F(x,t) = \int_0^{\tilde{\ell}} \xi^r G(x;\xi) u(\xi,t)\, d\xi \tag{2.4}$$

where $G(x;\xi)$ is the Green's function corresponding to (2.3) (see Appendix A1) with ℓ_s^* replaced by $\tilde{\ell}$:

$$G(x;\xi) = \begin{cases} \dfrac{x^{1-r}}{1-r}\left(1-\left(\dfrac{\xi}{\tilde{\ell}}\right)^{1-r}\right) & \text{for } 0 \le x < \xi, \\ \dfrac{\xi^{1-r}}{1-r}\left(1-\left(\dfrac{x}{\tilde{\ell}}\right)^{1-r}\right) & \text{for } \xi < x \le \tilde{\ell}. \end{cases}$$

We consider the time derivative of (2.4),

$$F_t(x,t) = \frac{\partial}{\partial t}\int_0^{\tilde{\ell}} \xi^r G(x;\xi)u(\xi,t)\,d\xi.$$

From (i) in Theorem 1.1 and the continuity of $G(x,\xi)$, we have, by the Leibniz integral rule ([16], p.422), that

$$\begin{aligned} F_t(x,t) &= \int_0^{\tilde{\ell}} \xi^r G(x;\xi)u_t(\xi,t)\,d\xi \\ &= \int_0^{\tilde{\ell}} G(x;\xi)(\xi^r u_\xi(\xi,t))_\xi\,d\xi + \int_0^{\tilde{\ell}} G(x;\xi)\xi^r f\left(u(\xi,t)\right)\,d\xi. \end{aligned} \tag{2.5}$$

Using Green's formula ([18], p.167) and the properties of the Green's function on the first term, we arrive at

$$F_t(x,t) = \tilde{\ell}^r u(\tilde{\ell},t)G_\xi(x;\tilde{\ell}) - u(x,t) + \int_0^{\tilde{\ell}} G(x;\xi)\xi^r f\left(u(\xi,t)\right)\,d\xi.$$

Since f is increasing, it follows from the Monotone Convergence Theorem ([17], p.87) and the continuity of f that

$$\lim_{t\to\infty} F_t(x,t) = \tilde{\ell}^r U(\tilde{\ell})G_\xi(x;\tilde{\ell}) - U(x) + \int_0^{\tilde{\ell}} G(x;\xi)\xi^r f\left(U(\xi)\right)\,d\xi.$$

According to (iv) in Theorem 1.1, we have from (2.5) that F is nondecreasing with respect to t. Furthermore, we have that

$$\lim_{t\to\infty} F_t(x,t) \ge 0.$$

Suppose that this limit were positive at some point x_0. $F(x_0,t)$ would be nondecreasing, and $\lim_{t\to\infty} F(x_0,t)=\infty$, which would contradict (ii) in Theorem 1.1. Therefore, the limit must be zero, and

$$\begin{aligned} U(x) &= \tilde{\ell}^r U(\tilde{\ell}) G_\xi(x;\tilde{\ell}) + \int_0^{\tilde{\ell}} G(x;\xi)\xi^r f\left(U(\xi)\right)\, d\xi \\ &= c\left(\frac{x}{\tilde{\ell}}\right)^{1-r} + \int_0^{\tilde{\ell}} G(x;\xi)\xi^r f\left(U(\xi)\right)\, d\xi. \end{aligned}$$

Differentiating with respect to x, we have that

$$U'(x) = \left(\frac{1-r}{\tilde{\ell}^{1-r}}\right) c x^{-r} + \int_0^{\tilde{\ell}} G_x(x;\xi)\xi^r f\left(U(\xi)\right)\, d\xi. \tag{2.6}$$

By multiplying x^r and differentiating both sides of (2.6), we have

$$\begin{aligned} -\left(x^r U'(x)\right)' &= \int_0^{\tilde{\ell}} -\left(x^r G_x(x;\xi)\right)_x \xi^r f\left(U(\xi)\right)\, d\xi \\ &= \int_0^{\tilde{\ell}} \delta(x-\xi)\xi^r f\left(U(\xi)\right)\, d\xi \\ &= x^r f\left(U(x)\right). \end{aligned}$$

Since $\tilde{\ell}$ is arbitrary, and U is continuous on $\bar{D}$, Lemma 2.2 is proven. □

Lemma 2.3. *For $x\in[\ell^*,L]$, $U(x)=c$.*

Proof. Suppose that there exists $x_0\in[\ell^*,L]$ such that $U(x_0)<c$. By the continuity of U, there exists an interval (x_1,x_2) with $\ell^*<x_1<x_0<x_2<L$ such that $U(x_1)=c=U(x_2)$ and $U(x)<c$ for all $x\in(x_1,x_2)$. Since $u_t\geq 0$ for $u<c$, we have $u(x,t)<c$ in $\{(x,t): x\in(x_1,x_2)\,\text{and}\, t>0\}$. This implies that

$$\mathcal{H}u = x^r f(u),\ x\in(x_1,x_2),\ t>0.$$

Let

$$F(x,t) = \int_{x_1}^{x_2} \xi^r G(x;\xi) u(\xi,t)\, d\xi,$$

where

$$G(x;\xi) = \begin{cases} \left(\dfrac{x^{1-r} - x_1^{1-r}}{1-r}\right)\left(\dfrac{x_2^{1-r} - \xi^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) & \text{for } x_1 \leq x < \xi, \\ \left(\dfrac{x_2^{1-r} - x^{1-r}}{1-r}\right)\left(\dfrac{\xi^{1-r} - x_1^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) & \text{for } \xi < x \leq x_2 \end{cases}$$

is the Green's function corresponding to the operator $\mathcal{H}$ with vanishing boundary conditions (see Appendix A2).

We use the same idea as Lemma 2.2 to compute $F_t(x,t)$. Since $u(x,t)$ is nondecreasing with respect to t, it follows that $F_t(x,t) \geq 0$. By direct calculation, we get

$$\begin{aligned} F_t(x,t) = {} & \left(\frac{x_2^{1-r} - x^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) u(x_1,t) - \left(\frac{x_1^{1-r} - x^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) u(x_2,t) - u(x,t) \\ & + \int_{x_1}^{x_2} G(x;\xi)\xi^r f(u(\xi,t))\, d\xi. \end{aligned}$$

Because f is nondecreasing, it follows from the Monotone Convergence Theorem ([17], p.87), the continuity of f, and $U(x_1) = c = U(x_2)$ that

$$\begin{aligned} \lim_{t\to\infty} F_t(x,t) = {} & \left(\frac{x_2^{1-r} - x^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) U(x_1) - \left(\frac{x_1^{1-r} - x^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) U(x_2) - U(x) \\ & + \int_{x_1}^{x_2} G(x;\xi)\xi^r f(U(\xi))\, d\xi. \end{aligned}$$

Suppose that the limit were strictly positive at some $x_0 \in (x_1, x_2)$, since F is nondecreasing, as t tends to ∞, $F(x_0,t)$ would increase without bound, which would contradict the fact that F is bounded due to the boundedness of u as $t \to \infty$. Therefore, $\lim_{t\to\infty} F_t(x,t) = 0$, and

$$U(x) = c + \int_{x_1}^{x_2} G(x;\xi)\xi^r f(U(\xi))\, d\xi > c$$

for all $x \in (x_1, x_2)$. This contradicts Theorem 1.1 (ii). Therefore, Lemma 2.3 is proven. $\square$

Lemma 2.4. $U'(\ell_s^*) = 0$

Proof. Because $u(x,t)$ converge uniformly to $U(x)$ on $\bar{D}$, by applying Theorem 1.1 (i) and (v), we obtain

$$\lim_{(x,t)\to(\ell_s^*,\infty)} u_x(x,t) = 0.$$

Therefore, $U'(\ell_s^*) = 0$, and we prove Lemma 2.4. $\square$

We modify the proof of Lemma 3.4 of Chan and Boonklurb [8], to prove the uniqueness of (2.3) resulting in Lemma 2.5.

Lemma 2.5. *(2.3) has a unique solution.*

Proof. Let $W_1(x)$ and $W_2(x)$ be two distinct solutions of (2.3) and $Z(x) = W_1(x) - W_2(x)$. Consider

$$-\left(x^r Z'(x)\right)' = x^r \left(f\left(W_1(x)\right) - f\left(W_2(x)\right)\right).$$

By the Mean Value Theorem ([1], p.155),

$$-\left(x^r Z'(x)\right)' = x^r f'(\theta) Z(x)$$

for some θ between $W_1(x)$ and $W_2(x)$. Multiplying by $x^r Z'(x)$ and integrating both sides from x to ℓ_s^*, we have

$$\frac{\left(x^r Z'(x)\right)^2}{2} - \frac{\left((\ell_s^*)^r Z'(\ell_s^*)\right)^2}{2} = \int_x^{\ell_s^*} \rho^{2r} f'(\theta) Z(\rho) Z'(\rho)\, d\rho.$$

Since $W_1'(\ell_s^*) = 0 = W_2'(\ell_s^*)$, $Z'(\ell_s^*) = 0$ and we have that

$$\frac{\left(x^r Z'(x)\right)^2}{2} = \int_x^{\ell_s^*} \rho^{2r} f'(\theta) Z(\rho) Z'(\rho)\, d\rho. \tag{2.7}$$

Since $Z(0) = 0 = Z(\ell_s^*)$, it follows from Rolle's Theorem ([5], p.196) that there exists $x_0 = \max\{x \in (0, \ell_s^*) | Z'(x) = 0\}$. Without loss of generality, we assume $W_1(x) > W_2(x)$

for all $x \in (x_0, \ell_s^*)$. Then, $Z(x) > 0$ and $Z'(x) < 0$ for all $x \in (x_0, \ell_s^*)$. Thus, from (2.7),

$$\frac{\left(x_0^r Z'(x_0)\right)^2}{2} = \int_{x_0}^{\ell_s^*} \rho^{2r} f'(\theta) Z(\rho) Z'(\rho)\, d\rho. \tag{2.8}$$

Since the left-hand side evaluates to zero, while the integrand is strictly negative, both sides of (2.8) are not equal. This contradiction proves Lemma 2.5. □

From Lemma 2.2 to Lemma 2.4, we conclude the following

Theorem 2.1. *As $t \to \infty$, all weak solutions of (2.1) with the properties (i) and (iv) of Theorem 1.1 tend to the unique steady-state solution given by (2.2) and (2.3).*

2.2 Bound for ℓ_s^*

We want to find an integral representation of ℓ_s^*. First, we multiply (2.3) by $x^r W'(x)$ and integrate from x to ℓ_s^*. Since $W'(\ell_s^*) = 0$, we have

$$\frac{1}{2}\left(x^r W'(x)\right)^2 = \int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho.$$

From $W'(x) \geq 0$ for all $x \in [0, \ell_s^*]$, we have

$$\frac{1}{x^r} = \frac{1}{\sqrt{2}} \left(\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \right)^{-\frac{1}{2}} W'(x).$$

Hence,

$$\int_0^{\ell_s^*} \frac{1}{x^r}\, dx = \frac{1}{\sqrt{2}} \int_0^{\ell_s^*} \left(\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \right)^{-\frac{1}{2}} W'(x)\, dx,$$

which gives

$$(\ell_s^*)^{1-r} = \frac{1-r}{\sqrt{2}} \int_0^{\ell_s^*} \left(\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \right)^{-\frac{1}{2}} W'(x)\, dx.$$

Therefore,

$$\ell_s^* = \left(\frac{1-r}{\sqrt{2}} \int_0^{\ell_s^*} \left(\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \right)^{-\frac{1}{2}} W'(x)\, dx \right)^{\frac{1}{1-r}}. \tag{2.9}$$

We want to find a lower bound of ℓ_s^*. Since $\rho^{2r} \leq (\ell_s^*)^{2r}$, where $\rho \in [x, \ell_s^*]$ and $f\left(W(\rho)\right) W'(\rho) > 0$, we have

$$\begin{aligned}\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho &\leq (\ell_s^*)^{2r} \int_x^{\ell_s^*} f\left(W(\rho)\right) W'(\rho)\, d\rho \\ &= (\ell_s^*)^{2r} \int_{W(x)}^{c} f(\eta)\, d\eta\end{aligned}$$

which gives

$$\left(\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \right)^{-\frac{1}{2}} \geq (\ell_s^*)^{-r} \left(\int_{W(x)}^{c} f(\eta)\, d\eta \right)^{-\frac{1}{2}}.$$

Since $W'(x) > 0$, we obtain

$$\begin{aligned}&\frac{1-r}{\sqrt{2}} \int_0^{\ell_s^*} \left(\int_x^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \right)^{-\frac{1}{2}} W'(x)\, dx \\ &\qquad \geq \frac{1-r}{\sqrt{2}(\ell_s^*)^r} \int_0^{\ell_s^*} \left(\int_{W(x)}^{c} f(\eta)\, d\eta \right)^{-\frac{1}{2}} W'(x)\, dx.\end{aligned}$$

From (2.9),

$$\begin{aligned}\ell_s^* &\geq \left(\frac{1-r}{\sqrt{2}(\ell_s^*)^r} \int_0^{\ell_s^*} \left(\int_{W(x)}^{c} f(\eta)\, d\eta \right)^{-\frac{1}{2}} W'(x)\, dx \right)^{\frac{1}{1-r}} \\ &\geq \left(\frac{1-r}{\sqrt{2}(\ell_s^*)^r} \int_0^{c} \left(\int_{\zeta}^{c} f(\eta)\, d\eta \right)^{-\frac{1}{2}} d\zeta \right)^{\frac{1}{1-r}}.\end{aligned} \tag{2.10}$$

The value of ℓ_s^* is critical in determining the beyond quenching steady-state solution profile, and is obtained by solving the nonlinear inequality (2.10). However, an exact value of ℓ_s^* is often difficult to obtain analytically, and thus we provide a numerical method in

Chapter IV for computing its value for any given problem. This numerical method allows for a more accurate and efficient determination of ℓ_s^* than analytical methods alone, and is a crucial step in analyzing the long-term behavior of solutions to the semilinear parabolic partial differential equation under consideration.

CHAPTER III

QUENCHING PROFILE OF RIGHT IBVP

Similar to Chapter II, we let $\mathcal{L}u = u_t - u_{xx} - (r/x)u_x$ and $\mathcal{H}u = x^r u_t - (x^r u_x)_x$ where $0 < r < 1$. We restate the Right IBVP as

$$\begin{aligned} &\mathcal{L}u = f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega, \\ &u(x,0) = 0,\ x \in \bar{D}, \\ &u_x(0,t) = 0 = u(L,t),\ t \in (0,T], \end{aligned}$$

or

$$\left.\begin{aligned} &\mathcal{H}u = x^r f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega, \\ &u(x,0) = 0,\ x \in \bar{D}, \\ &u_x(0,t) = 0 = u(L,t),\ t \in (0,T], \end{aligned}\right\} \tag{3.1}$$

where $L > 0$, $T \leq \infty$, $D = (0, L)$, $\Omega = D \times (0, T]$, and f is twice continuously differentiable on $[0, c)$ for some constant c with $f(0) > 0$, $f' > 0$, $f'' \geq 0$ and $\lim_{u \to c^-} f(u) = \infty$.

With similar techniques used in Chapter II, we show that under the properties in Theorem 1.1, weak solutions of (3.1) converge uniformly to the unique solution of the following steady-state problem as t tends to infinity:

$$W(x) = c,\ x \in [0, \ell_s^*], \tag{3.2}$$

$$-(x^r W'(x))' = x^r f(W(x)),\ x \in (\ell_s^*, L),\ W(\ell_s^*) = c, W(L) = 0, \tag{3.3}$$

where ℓ_s^* is a positive constant to be determined. Let u denote any weak solution of (3.1). We would like to study the behavior of u beyond the first quenching time.

3.1 Beyond quenching profile

Let τ be the first finite time when quenching occurs. For $t \geq \tau$, we change the definition of $\ell(t)$ to be $\sup\{x : u(x,t) = c\}$ and $\ell^* = \lim_{t\to\infty} \ell(t)$.

By manipulating the ideas of Lemma 2 and Lemma 6 of Chan and Ke [10], we arrive at Lemma 3.1.

Lemma 3.1. *The function $\ell(t)$ is nondecreasing and $\ell^* \leq \ell_s^* < L$.*

Lemma 3.1 has a physical interpretation similar to that of Lemma 2.1; however, the switch in the boundary conditions at $x = 0$ and $x = L$ causes the solution to behave differently. In this case, the temperature u would reach the critical value c at $x = 0$, while the boundary condition at $x = L$ would prevent the temperature from reaching c over the entire length of D. This results in a quenching profile that is distinct from that of (2.1), as the critical point now occurs at the opposite end of the domain. Hence, the solution profile after the first quenching has occurred should separate into two segments demarcated by ℓ_s^* as in Lemma 3.2 and Lemma 3.3.

From (ii) and (iv) in Theorem 1.1, we can use the Dini's Theorem ([19], p.143) to deduce that $u(x,t)$ converges uniformly to its continuous limit as $t \to \infty$ on $\bar{D}$ denoted by

$$U(x) = \lim_{t\to\infty} u(x,t).$$

Lemma 3.2. *For $x \in (\ell^*, L)$, $u(x,t)$ converges uniformly to a solution of (3.3) as $t \to \infty$ with $\ell^* = \ell_s^*$.*

Proof. We modify the idea used in Lemma 2.2 to accommodate the changes in the boundary conditions. Consider u in the region $[\tilde{\ell}, L] \times (0, \infty)$ where $\tilde{\ell} \in [\ell^*, L]$. We define

$$F(x,t) = \int_{\tilde{\ell}}^{L} \xi^r G(x;\xi) u(\xi,t)\, d\xi \tag{3.4}$$

where $G(x;\xi)$ is the Green's function corresponding to (3.3) (see Appendix A3) with ℓ_s^*

replaced by $\tilde{\ell}$:

$$G(x;\xi) = \begin{cases} \left(\dfrac{x^{1-r} - \tilde{\ell}^{1-r}}{1-r}\right)\left(\dfrac{L^{1-r} - \xi^{1-r}}{L^{1-r} - \tilde{\ell}^{1-r}}\right) & \text{for } \tilde{\ell} \le x < \xi, \\ \left(\dfrac{L^{1-r} - x^{1-r}}{1-r}\right)\left(\dfrac{\xi^{1-r} - \tilde{\ell}^{1-r}}{L^{1-r} - \tilde{\ell}^{1-r}}\right) & \text{for } \xi < x \le L. \end{cases}$$

We consider the time derivative of (3.4),

$$F_t(x,t) = \frac{\partial}{\partial t}\int_{\tilde{\ell}}^{L} \xi^r G(x;\xi)u(\xi,t)\,d\xi.$$

From (i) in Theorem 1.1 and the continuity of $G(x,\xi)$, we have, by the Leibniz integral rule ([16], p.422), that

$$\begin{aligned} F_t(x,t) &= \int_{\tilde{\ell}}^{L} \xi^r G(x;\xi)u_t(\xi,t)\,d\xi \\ &= \int_{\tilde{\ell}}^{L} G(x;\xi)(\xi^r u_\xi(\xi,t))_\xi\,d\xi + \int_{\tilde{\ell}}^{L} G(x;\xi)\xi^r f\left(u(\xi,t)\right)\,d\xi. \end{aligned} \tag{3.5}$$

Using Green's formula ([18], p.167) and the properties of the Green's function on the first term, we arrive at

$$F_t(x,t) = \tilde{\ell}^r u(\tilde{\ell},t)G_\xi(x;\tilde{\ell}) - u(x,t) + \int_{\tilde{\ell}}^{L} G(x;\xi)\xi^r f\left(u(\xi,t)\right)\,d\xi.$$

Since f is increasing, it follows from the Monotone Convergence Theorem ([17], p.87) and the continuity of f that

$$\lim_{t\to\infty} F_t(x,t) = \tilde{\ell}^r U(\tilde{\ell})G_\xi(x;\tilde{\ell}) - U(x) + \int_{\tilde{\ell}}^{L} G(x;\xi)\xi^r f\left(U(\xi)\right)\,d\xi.$$

According to (iv) in Theorem 1.1, we have from (3.5) that F is nondecreasing with respect to t. Furthermore, we have that

$$\lim_{t\to\infty} F_t(x,t) \ge 0.$$

Suppose that this limit were positive at some point x_0. $F(x_0,t)$ would be nondecreasing, and $\lim_{t\to\infty} F(x_0,t) = \infty$, which would contradict (ii) in Theorem 1.1. Therefore, the limit must be zero, and we have

$$\begin{aligned} U(x) &= \tilde{\ell}^r U(\tilde{\ell}) G_\xi(x;\tilde{\ell}) + \int_{\tilde{\ell}}^{L} G(x;\xi)\xi^r f\left(U(\xi)\right)\, d\xi \\ &= c\left(\frac{L^{1-r} - x^{1-r}}{L^{1-r} - \tilde{\ell}^{1-r}}\right) + \int_{\tilde{\ell}}^{L} G(x;\xi)\xi^r f\left(U(\xi)\right)\, d\xi. \end{aligned}$$

Differentiating with respect to x, we have that

$$U'(x) = \left(\frac{r-1}{L^{1-r} - \tilde{\ell}^{1-r}}\right) c x^{-r} + \int_{\tilde{\ell}}^{L} G_x(x;\xi)\xi^r f\left(U(\xi)\right)\, d\xi. \tag{3.6}$$

By multiplying x^r and differentiating both sides of (3.6), we have

$$\begin{aligned} -\left(x^r U'(x)\right)' &= \int_{\tilde{\ell}}^{L} -\left(x^r G_x(x;\xi)\right)_x \xi^r f\left(U(\xi)\right)\, d\xi \\ &= \int_{\tilde{\ell}}^{L} \delta(x-\xi)\xi^r f\left(U(\xi)\right)\, d\xi \\ &= x^r f\left(U(x)\right). \end{aligned}$$

Since $\tilde{\ell}$ is arbitrary, and U is continuous on $\bar{D}$, Lemma 3.2 is proven. □

Lemma 3.3. *For $x \in [\ell^*, L]$, $U(x) = c$.*

Proof. Suppose that there exists $x_0 \in [0, \ell^*]$ such that $U(x_0) < c$. By the continuity of U, there exists an interval (x_1, x_2) with $0 < x_1 < x_0 < x_2 < \ell^*$ such that $U(x_1) = c = U(x_2)$ and $U(x) < c$ for all $x \in (x_1, x_2)$.

The remaining part of the proof of Lemma 3.3 is identical to that of Lemma 2.3. □

Lemma 3.4. $U'(\ell_s^*) = 0$.

Proof. The proof of Lemma 3.4 is identical to that of Lemma 2.4. □

We modify the proof of Lemma 3.4 of Chan and Boonklurb [8], to prove the uniqueness of (3.3) resulting in Lemma 3.5.

Lemma 3.5. *(3.3) has a unique solution.*

Proof. We use the same idea presented in the proof of Lemma 2.5 with some changes in the limits of integrations. Let $W_1(x)$ and $W_2(x)$ be two distinct solutions of (3.3) and $Z(x) = W_1(x) - W_2(x)$. Consider

$$-\left(x^r Z'(x)\right)' = x^r \left(f\left(W_1(x)\right) - f\left(W_2(x)\right)\right).$$

By the Mean Value Theorem ([1], p.155),

$$-\left(x^r Z'(x)\right)' = x^r f'(\theta) Z(x)$$

for some θ between $W_1(x)$ and $W_2(x)$. Multiplying by $x^r Z'(x)$ and integrating both sides from ℓ_s^* to x, we have

$$\frac{\left(\left(\ell_s^*\right)^r Z'(\ell_s^*)\right)^2}{2} - \frac{\left(x^r Z'(x)\right)^2}{2} = \int_{\ell_s^*}^{x} \rho^{2r} f'(\theta) Z(\rho) Z'(\rho)\, d\rho.$$

Since $W_1'(\ell_s^*) = 0 = W_2'(\ell_s^*)$, $Z'(\ell_s^*) = 0$ and we have that

$$\frac{\left(x^r Z'(x)\right)^2}{2} = -\int_{\ell_s^*}^{x} \rho^{2r} f'(\theta) Z(\rho) Z'(\rho)\, d\rho. \tag{3.7}$$

Since $Z(\ell_s^*) = 0 = Z(L)$, it follows from Rolle's Theorem ([5], p.196) that there exists $x_0 = \min\{x \in (\ell_s^*, L) | Z'(x) = 0\}$. Without loss of generality, we assume $W_1(x) > W_2(x)$ for all $x \in (\ell_s^*, x_0)$. Then, $Z(x) > 0$ and $Z'(x) > 0$ for all $x \in (\ell_s^*, x_0)$. Thus, from (3.7),

$$\frac{\left(x_2^r Z'(x_2)\right)^2}{2} = -\int_{\ell_s^*}^{x_2} \rho^{2r} f'(\theta) Z(\rho) Z'(\rho)\, d\rho. \tag{3.8}$$

Since the left-hand side evaluates to zero, while the integrand is strictly positive, both sides of (3.8) are not equal. This contradiction proves Lemma 3.5. $\square$

From Lemma 3.2 to Lemma 3.4, we conclude the following

Theorem 3.1. *As $t \to \infty$, all weak solutions of (3.1) with the properties (i) and (iv) of Theorem 1.1 tend to the unique steady-state solution given by (3.2) and (3.3).*

3.2 Bound for ℓ_s^*

We use a similar technique in Section 2.2 to find an integral representation of ℓ_s^*. First, we multiply (3.3) by $x^r W'(x)$ and integrate from ℓ_s^* to x. Since $W'(\ell_s^*) = 0$, we have

$$\begin{aligned} \frac{1}{2}\left(x^r W'(x)\right)^2 &= -\int_{\ell_s^*}^{x} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho \\ &= \int_{x}^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho. \end{aligned}$$

From $W'(x) \le 0$ for all $x \in [\ell_s^*, L]$, we have

$$\frac{1}{x^r} = -\frac{1}{\sqrt{2}}\left(\int_{x}^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho\right)^{-\frac{1}{2}} W'(x).$$

Hence,

$$\int_{L}^{\ell_s^*} \frac{1}{x^r}\, dx = -\frac{1}{\sqrt{2}}\int_{L}^{\ell_s^*}\left(\int_{x}^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho\right)^{-\frac{1}{2}} W'(x)\, dx,$$

which gives

$$(\ell_s^*)^{1-r} = L^{1-r} + \frac{r-1}{\sqrt{2}}\int_{L}^{\ell_s^*}\left(\int_{x}^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho\right)^{-\frac{1}{2}} W'(x)\, dx.$$

Therefore,

$$\ell_s^* = \left(L^{1-r} + \frac{r-1}{\sqrt{2}}\int_{L}^{\ell_s^*}\left(\int_{x}^{\ell_s^*} \rho^{2r} f\left(W(\rho)\right) W'(\rho)\, d\rho\right)^{-\frac{1}{2}} W'(x)\, dx\right)^{\frac{1}{1-r}}. \tag{3.9}$$

We want to find an upper bound of ℓ_s^*. Since $\rho^{2r} \le x^{2r}$, where $\rho \in [\ell_s^*, x]$ and $-f(W(\rho))\,W'(\rho) > 0$, we have

$$\begin{aligned}\int_x^{\ell_s^*} \rho^{2r} f(W(\rho))\,W'(\rho)\,d\rho &\le x^{2r}\int_x^{\ell_s^*} f(W(\rho))\,W'(\rho)\,d\rho \\ &= x^{2r}\int_{W(x)}^{c} f(\eta)\,d\eta\end{aligned}$$

which gives

$$\left(\int_x^{\ell_s^*} \rho^{2r} f(W(\rho))\,W'(\rho)\,d\rho\right)^{-\frac{1}{2}} \ge x^{-r}\left(\int_{W(x)}^{c} f(\eta)\,d\eta\right)^{-\frac{1}{2}}.$$

For $x \in [\ell_s^*, L]$, we have that $x^{-r} \ge L^{-r}$, and

$$\left(\int_x^{\ell_s^*} \rho^{2r} f(W(\rho))\,W'(\rho)\,d\rho\right)^{-\frac{1}{2}} \ge L^{-r}\left(\int_{W(x)}^{c} f(\eta)\,d\eta\right)^{-\frac{1}{2}}.$$

Since $W'(x) < 0$, we obtain

$$\begin{aligned}&L^{1-r} + \frac{r-1}{\sqrt{2}L^r}\int_L^{\ell_s^*}\left(\int_x^{\ell_s^*} \rho^{2r} f(W(\rho))\,W'(\rho)\,d\rho\right)^{-\frac{1}{2}} W'(x)\,dx \\ &\le L^{1-r} + \frac{r-1}{\sqrt{2}L^r}\int_L^{\ell_s^*}\left(\int_{W(x)}^{c} f(\eta)\;d\eta\right)^{-\frac{1}{2}} W'(x)\,dx.\end{aligned}$$

From (3.9),

$$\begin{aligned}\ell_s^* &\le \left(L^{1-r} + \frac{r-1}{\sqrt{2}L^r}\int_L^{\ell_s^*}\left(\int_{W(x)}^{c} f(\eta)\,d\eta\right)^{-\frac{1}{2}} W'(x)\,dx\right)^{\frac{1}{1-r}} \\ &\le L\left(1 + \frac{r-1}{\sqrt{2}L}\int_0^{c}\left(\int_{\zeta}^{c} f(\eta)\,d\eta\right)^{-\frac{1}{2}} d\zeta\right)^{\frac{1}{1-r}}. \qquad (3.10)\end{aligned}$$

While the bound of ℓ_s^* can be obtained readily by evaluating (3.10), an exact value is needed to determine the beyond quenching steady-state solution profile. Therefore, we provide a numerical method for computing the exact value of ℓ_s^* for any given problem in Chapter IV. This numerical method is a crucial tool for accurately analyzing the long-term behavior of solutions to the semilinear parabolic partial differential equation under consideration.

CHAPTER IV

NUMERICAL METHOD FOR QUENCHING PROFILES

In Chapter II, we derived the steady-state problem of the Left IBVP. The results are restated as follows:

$$
\begin{aligned}
&W(x) = c,\ x \in [\ell_s^*, L], \\
&-(x^r W'(x))' = x^r f(W(x)),\ x \in (0, \ell_s^*), && (4.1) \\
&W(0) = 0, W(\ell_s^*) = c. && (4.2)
\end{aligned}
$$

Similarly, in Chapter III, we derived the steady-state problem of the Right IBVP restated as follows:

$$
\begin{aligned}
&W(x) = c,\ x \in [0, \ell_s^*], \\
&-(x^r W'(x))' = x^r f(W(x)),\ x \in (\ell_s^*, L), && (4.3) \\
&W(\ell_s^*) = c, W(L) = 0. && (4.4)
\end{aligned}
$$

In both problems, one must determine the value of the positive constant ℓ_s^* before beyond quenching profiles can be obtained. Therefore, in this chapter, we provide a method to numerically compute the value of ℓ_s^* for both problems.

To find an approximate value of ℓ_s^* for each problem, we use an iterative method to generate a sequence of ℓ's in the interval $[0, L]$ that converges to the true value. For each ℓ in the sequence, we solve the corresponding boundary-value problem using either (4.1) and (4.2) or (4.3) and (4.4). Depending on which problem, we observe that values of ℓ closer to one end of the domain can yield divergent results. Using this observation, we initialize ℓ to the value at the divergent end and use the bisection method to update its value so that the final ℓ is as close to the divergent end as possible while still producing a convergent solution.

4.1 Boundary value problem discretization

Consider (4.1) and (4.3) with their domains replaced by (x_L, x_R) where $0 \leq x_L < x_R \leq L$,

$$-(x^r W'(x))' = x^r f(W(x)),\ x \in (x_L, x_R). \tag{4.5}$$

We first expand the derivative terms and express (4.5) in a more compact form:

$$W'' + \frac{r}{x} W' + f(W) = 0. \tag{4.6}$$

Because the geometry of the domain is regular and by the continuity of u in Theorem 1.1 (i), we choose the finite-difference method (FDM) to discretize (4.6).

First, we partition $[x_L, x_R]$ uniformly into N subintervals of width Δ. Then, we have

$$x_i = x_L + i\Delta = x_L + i\left(\frac{x_R - x_L}{N}\right) \quad \text{for } i \in \{0, 1, 2, ..., N\}. \tag{4.7}$$

For $i \in \{1, 2, 3, ..., N-1\}$, we have from the Taylor's theorem [7] that

$$W'(x_i) = \frac{1}{2\Delta}\left(W(x_{i+1}) - W(x_{i-1})\right) - \frac{\Delta^2}{6} W'''(\eta_i) \tag{4.8}$$

and

$$W''(x_i) = \frac{1}{\Delta^2}\left(W(x_{i+1}) - 2W(x_i) + W(x_{i-1})\right) - \frac{\Delta^2}{12} W^{(4)}(\xi_i) \tag{4.9}$$

for some η_i and ξ_i in (x_{i-1}, x_{i+1}). Substituting (4.8) and (4.9) into (4.6), we have

$$\left(\frac{1}{\Delta^2} + \frac{r}{2x_i\Delta}\right) W(x_{i+1}) - \frac{2}{\Delta^2} W(x_i) + \left(\frac{1}{\Delta^2} - \frac{r}{2x_i\Delta}\right) W(x_{i-1}) + f(W(x_i)) + O(\Delta^2) = 0$$

Let w_i be an approximation to $W(x_i)$ and define

$$\gamma_i = \frac{1}{\Delta^2} + \frac{r}{2x_i\Delta} \quad \text{and} \quad \bar{\gamma}_i = \frac{1}{\Delta^2} - \frac{r}{2x_i\Delta}.$$

We arrive at the finite-difference approximation of (4.6) as follows

$$\gamma_i w_{i+1} - (\gamma_i + \bar{\gamma}_i)w_i + \bar{\gamma}_i w_{i-1} + f(w_i) = 0 \tag{4.10}$$

which we will use for numerically solving (4.1) and (4.2) or (4.3) and (4.4) in subsequent sections.

4.2 Numerical method for the Left IBVP

For a given value of ℓ, we create grid points by (4.7) with x_L replaced by 0 and x_R replaced by ℓ:

$$x_i = i\left(\frac{\ell}{N}\right) \quad \text{for } i \in \{0, 1, 2, ..., N\}.$$

We define $\mathbf{w}$ to be the vector of values of the solution at the internal grid points:

$$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 & ... & w_{N-1} \end{bmatrix}^T \tag{4.11}$$

with $w_0 = 0$ and $w_N = c$ from the boundary conditions (4.2).
From the left-hand side of (4.10), let

$$F_1(\mathbf{w}) = \gamma_1 w_2 - (\gamma_1 + \bar{\gamma}_1)w_1 + f(w_1), \tag{4.12}$$

$$F_{N-1}(\mathbf{w}) = c\gamma_{N-1} - (\gamma_{N-1} + \bar{\gamma}_i)w_{N-1} + \bar{\gamma}_{N-1}w_{N-2} + f(w_{N-1}) \tag{4.13}$$

and

$$F_i(\mathbf{w}) = \gamma_i w_{i+1} - (\gamma_i + \bar{\gamma}_i)w_i + \bar{\gamma}_i w_{i-1} + f(w_i) \tag{4.14}$$

for $i \in \{2, 3, 4, ..., N-2\}$. Furthermore, we let

$$\mathbf{F}(\mathbf{w}) = \begin{bmatrix} F_1(\mathbf{w}) & F_2(\mathbf{w}) & F_3(\mathbf{w}) & ... & F_{N-1}(\mathbf{w}) \end{bmatrix}^T .$$

Then, we must solve the system of algebraic equations

$$\mathbf{F}(\mathbf{w}) = \mathbf{0} \tag{4.15}$$

to obtain the solutions on the grid points. Since f contributes to non-linearity in (4.15), we use Newton's method to solve the root-finding problem.

Let $\mathbf{J}$ be the Jacobian matrix of $\mathbf{F}$. We have

$$\mathbf{J}(\mathbf{w}) = \begin{bmatrix} \frac{\partial \mathbf{F}(\mathbf{w})}{\partial w_1} & \frac{\partial \mathbf{F}(\mathbf{w})}{\partial w_2} & \frac{\partial \mathbf{F}(\mathbf{w})}{\partial w_3} & ... & \frac{\partial \mathbf{F}(\mathbf{w})}{\partial w_{N-1}} \end{bmatrix} . \tag{4.16}$$

By computing the derivatives of (4.12) – (4.14), we have each element of $\mathbf{J}$ as

$$\mathbf{J}_{ij} = \begin{cases} \dfrac{\partial F_i}{\partial w_{i-1}} = \bar{\gamma}_i & \text{if } j = i-1, \\ \dfrac{\partial F_i}{\partial w_i} = -\left(\gamma_i + \bar{\gamma}_i\right) + f'(w_i) & \text{if } j = i, \\ \dfrac{\partial F_i}{\partial w_{i+1}} = \gamma_i & \text{if } j = i+1, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j \in \{1, 2, 3, ..., N-1\}$. We want to generate a sequence of $\mathbf{w}$'s which converges to a root of (4.15).

Let $\mathbf{w}^{(k)}$ denote the k-th element in the sequence. Then, we construct the consecutive elements by

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \mathbf{J}^{-1}(\mathbf{w}^{(k)})\mathbf{F}(\mathbf{w}^{(k)}) \quad \text{for } k \in \{0, 1, 2, 3, ...\}, \tag{4.17}$$

where we choose $\mathbf{w}^{(0)}$ to be a linear function satisfying (4.2). Each component of $\mathbf{w}^{(0)}$ is given by

$$w_i^{(0)} = c\left(\frac{x_i}{\ell}\right) \quad \text{for } i \in \{1, 2, 3, ..., N-1\}.$$

For a specified tolerance ϵ, if $\|\mathbf{w}^{(K+1)} - \mathbf{w}^{(K)}\| < \epsilon$ for some K, then we terminate and check whether $\mathbf{w}^{(K+1)} \in \mathbb{R}^{N-1}$. If the condition is true, we have that the current value of ℓ yields a convergent solution.

Let $\ell^{(m)}$ denote the value of ℓ at the m-th iteration. Then, we update the value of ℓ by using the bisection method where

$$\ell^{(m+1)} = \begin{cases} \dfrac{\ell^{(m)} + L}{2} & \text{if } \ell^{(m)} \text{ gives a convergent solution,} \\ \dfrac{\ell^{(m)}}{2} & \text{otherwise} \end{cases}$$

for $m \in \{0, 1, 2, 3, ...\}$. From Lemma 2.1, we know that the value of ℓ_s^* is closer to 0 on D; therefore, we initialize ℓ to be L. Suppose that $\ell^{(M)}$ gives a convergent solution for some M. For a specified tolerance ω, if $|\ell^{(M+1)} - \ell^{(M)}| < \omega$, then we terminate. If $\ell^{(M+1)}$ gives a convergent result, we use $\ell^{(M+1)}$ as an approximation of ℓ_s^*; otherwise, use $\ell^{(M)}$ as the approximation. The algorithm for the Left IBVP is summarized as follows

Algorithm 1 Pseudocode for numerically compute ℓ_s^* (Left IBVP)

Input: $\{f, c, r, L\}$, $\{N, \epsilon, \omega\}$
Output: ℓ_s^*

$\ell \leftarrow L$
LEFT $\leftarrow$ 0, RIGHT $\leftarrow L$
$\delta \leftarrow$ RIGHT – LEFT
while $\delta \geq \omega$ **do**
  $\ell_{prev} \leftarrow \ell$
  $\mathbf{w} \leftarrow$ FDM(f, c, r, L, ℓ, N, ϵ)
  if $\mathbf{w}$ is divergent **then**
    RIGHT $\leftarrow \ell$
  else
    LEFT $\leftarrow \ell$
    $\ell_{conv} \leftarrow \ell$
  end if
  $\ell \leftarrow$ (LEFT+RIGHT)/2
  $\delta \leftarrow |\ell - \ell_{prev}|$
end while
return ℓ_{conv} **as** ℓ_s^*

4.3 Numerical method for the Right IBVP

The numerical method for the Right IBVP is largely the same as the previous section. We discuss the differences in their calculations.

For a given value of ℓ, we create grid points by (4.7) with x_L replaced by ℓ and x_R replaced by L:

$$x_i = \ell + i\left(\frac{L-\ell}{N}\right) \quad \text{for } i \in \{0, 1, 2, ..., N\}.$$

The vector $\mathbf{w}$ is defined in the same manner as (4.11), but the values at the boundary points are switched: $w_0 = c$ and $w_N = 0$, according to the boundary conditions (4.4). Due to this fact, (4.12) and (4.13) need to be redefined as follows:

$$F_1(\mathbf{w}) = \gamma_1 w_2 - (\gamma_1 + \bar{\gamma}_1)w_1 + c\bar{\gamma}_1 + f(w_1)$$

and

$$F_{N-1}(\mathbf{w}) = -(\gamma_{N-1} + \bar{\gamma}_i)w_{N-1} + \bar{\gamma}_{N-1}w_{N-2} + f(w_{N-1}).$$

Then, we employ Newton's method to solve (4.15) by computing $\mathbf{J}$ as (4.16) and generate a sequence of vectors $\mathbf{w}$'s according to (4.17). However, we choose $\mathbf{w}^{(0)}$ to be a linear function satisfying (4.4). Each component of $\mathbf{w}^{(0)}$ is given by

$$w_i^{(0)} = c\left(1 - \frac{x_i - \ell}{L - \ell}\right) \quad \text{for } i \in \{1, 2, 3, ..., N-1\}.$$

For a specified tolerance ϵ, if $\|\mathbf{w}^{(K+1)} - \mathbf{w}^{(K)}\| < \epsilon$ for some K, then we terminate and check whether $\mathbf{w}^{(K+1)} \in \mathbb{R}^{N-1}$. If the condition is true, we have that the current value of ℓ yields a convergent solution.

Let $\ell^{(m)}$ denote the value of ℓ at the m-th iteration. Then, we update the value of

ℓ by using the bisection method where

$$\ell^{(m+1)} = \begin{cases} \dfrac{\ell^{(m)}}{2} & \text{if } \ell^{(m)} \text{ gives a convergent solution,} \\ \dfrac{\ell^{(m)} + L}{2} & \text{otherwise} \end{cases}$$

for $m \in \{0, 1, 2, 3, ...\}$. From Lemma 3.1, we know that the value of ℓ_s^* is closer to L on D; therefore, we initialize ℓ to be 0. Suppose that $\ell^{(M)}$ gives a convergent solution for some M. For a specified tolerance ω, if $|\ell^{(M+1)} - \ell^{(M)}| < \omega$, then we terminate. If $\ell^{(M+1)}$ gives a convergent result, we use $\ell^{(M+1)}$ as an approximation of ℓ_s^*; otherwise, use $\ell^{(M)}$ as the approximation. The algorithm for the Right IBVP is summarized as follows

Algorithm 2 Pseudocode for numerically compute ℓ_s^* (Right IBVP)

Input: $\{f, c, r, L\}$, $\{N, \epsilon, \omega\}$
Output: ℓ_s^*

$\ell \leftarrow 0$
LEFT $\leftarrow 0$, RIGHT $\leftarrow L$
$\delta \leftarrow$ RIGHT – LEFT
while $\delta \geq \omega$ **do**
  $\ell_{prev} \leftarrow \ell$
  $\mathbf{w} \leftarrow$ FDM$(f, c, r, L, \ell, N, \epsilon)$
  if $\mathbf{w}$ is divergent **then**
    LEFT $\leftarrow \ell$
  else
    RIGHT $\leftarrow \ell$
    $\ell_{conv} \leftarrow \ell$
  end if
  $\ell \leftarrow$ (LEFT+RIGHT)/2
  $\delta \leftarrow |\ell - \ell_{prev}|$
end while
return ℓ_{conv} **as** ℓ_s^*

4.4 Demonstrations

In this section, we illustrate the practical application of the numerical methods developed in the preceding sections through two examples. First, we consider the source function $f(u) = (1-u)^{-\beta}$ and simulate its behavior for various values of r, L, and β in Example 4.1. Then, in Example 4.2, we investigate the source function $f(u) = \alpha - \beta \ln(1 - u)$ using a similar approach as in the previous example.

We have developed a Python program (see Appendix B1 – B3) to compute ℓ_s^* for both examples. If necessary, one can modify some parts of the program to simulate other nonlinear functions in the future.

Example 4.1. We consider the source function $f(u) = (1-u)^{-\beta}$ where $0 < \beta < 1$. We have that $f(0) = 1$, $f'(u) = \beta(1-u)^{-\beta-1} > 0$, $f''(u) = \beta(\beta+1)(1-u)^{-\beta-2} > 0$, and $\lim_{u\to 1^-} f(u) = \infty$.

To compute a lower bound of ℓ_s^* in case of the Left IBVP, we refer to (2.10). The integral term can be computed as follows:

$$\int_0^1 \left(\int_\zeta^1 f(\eta)\, d\eta\right)^{-\frac{1}{2}} d\zeta = \sqrt{1-\beta}\int_0^1 (1-\zeta)^{\frac{\beta-1}{2}}\, d\zeta = \frac{2\sqrt{1-\beta}}{1+\beta}. \tag{4.18}$$

Then, we have

$$\ell_s^* \geq \left(\frac{(1-r)\sqrt{2(1-\beta)}}{(\ell_s^*)^r(1+\beta)}\right)^{\frac{1}{1-r}}. \tag{4.19}$$

Similarly, we refer to (3.10) for the Right IBVP. With the integral term in (4.18), we have that

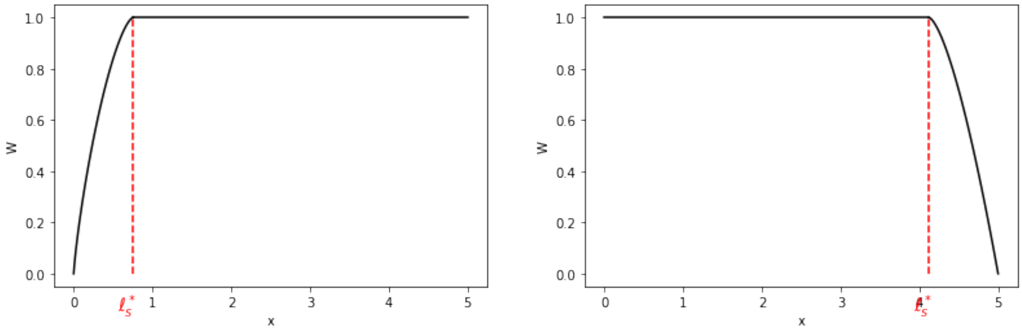
$$\ell_s^* \leq L\left(1+\frac{(r-1)\sqrt{2(1-\beta)}}{L(1+\beta)}\right)^{\frac{1}{1-r}}. \tag{4.20}$$

The following table summarizes the numerical approximations of ℓ_s^* for different values of r, β, and L for the Left and the Right IBVPs with $N = 100$ and $\epsilon = \omega = 1\times 10^{-6}$.

r	β	L	ℓ_s^* Left	ℓ_s^* Right
1/7	1/4	5	0.8983	4.0076
		7	0.8983	6.0098
		10	0.8983	9.0114
	1/3	5	0.7916	4.1207
		7	0.7916	6.1226
		10	0.7916	9.1239
	1/2	5	0.6107	4.3138
		7	0.6107	6.3152
		10	0.6107	9.3161
1/5	1/4	5	0.8622	4.0047
		7	0.8622	6.0078
		10	0.8622	9.0100
	1/3	5	0.7578	4.1182
		7	0.7578	6.1209
		10	0.7578	9.1227
	1/2	5	0.5816	4.3121
		7	0.5816	6.3140
		10	0.5816	9.3153

Table 4.1: Numerical approximations of ℓ_s^* for $f(u) = (1-u)^{-\beta}$

One can easily verify that all the values of ℓ_s^* in Table 4.1, with their corresponding values of r, β and L, satisfy (4.19) and (4.20) for the Left and the Right IBVPs, respectively.



(a) (b)

Figure 4.1: Steady-state solution profiles for (a) Left IBVP and (b) Right IBVP where $r = 1/5$, $L = 5$ and $f(u) = (1-u)^{-1/3}$.

Example 4.2. We consider the source function $f(u) = \alpha - \beta \ln(1-u)$ where $\alpha, \beta > 0$. We have that $f(0) = \alpha$, $f'(u) = \beta/(1-u) > 0$, $f''(u) = \beta/(1-u)^2 > 0$, and $\lim_{u \to 1^-} f(u) = \infty$.

Similar to the previous example, we compute a lower bound of ℓ_s^* in case of the Left IBVP by refering to (2.10). The integral term can be computed as follows:

$$\int_0^1 \left(\int_\zeta^1 f(\eta)\, d\eta \right)^{-\frac{1}{2}} d\zeta = \int_0^1 \left(\sqrt{(1-\zeta)(\alpha + \beta - \beta \ln(1-\zeta))} \right)^{-1} d\zeta. \tag{4.21}$$

Then, we have

$$\ell_s^* \geq \left(\frac{1-r}{(\ell_s^*)^r} \int_0^1 \left(\sqrt{(1-\zeta)(\alpha + \beta - \beta \ln(1-\zeta))} \right)^{-1} d\zeta \right)^{\frac{1}{1-r}}.$$

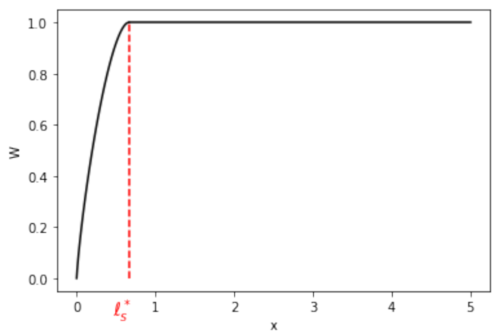
Similarly, we refer to (3.10) for the Right IBVP. With the integral term in (4.21), we have that

$$\ell_s^* \leq L \left(1 + \frac{r-1}{L} \int_0^1 \left(\sqrt{(1-\zeta)(\alpha + \beta - \beta \ln(1-\zeta))} \right)^{-1} d\zeta \right)^{\frac{1}{1-r}}.$$

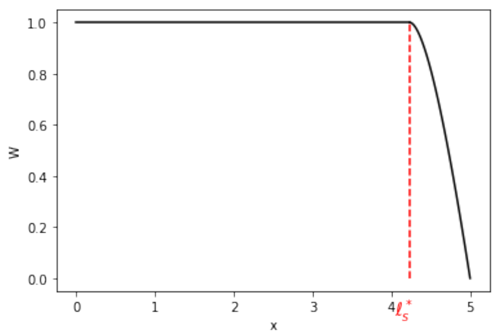
The following table summarizes the numerical approximations of ℓ_s^* for different values of r, α, β, and L for Left and the Right IBVPs with $N = 100$ and $\epsilon = \omega = 1 \times 10^{-6}$.

r	α	β	L	ℓ_s^*	
				Left	Right
1/7	1	1/4	5	1.0088	3.8932
			7	1.0088	5.8957
			10	1.0088	8.8974
		1/2	5	0.8578	4.0549
			7	0.8578	6.0568
			10	0.8578	9.0581
		1	5	0.6915	4.2351
			7	0.6915	6.2364
			10	0.6915	9.2373
	2	1/4	5	0.7971	4.1305
			7	0.7971	6.1319
			10	0.7971	9.1329
		1/2	5	0.7133	4.2192
			7	0.7133	6.2203
			10	0.7133	9.2211
		1	5	0.6066	4.3331
			7	0.6066	6.3340
			10	0.6066	9.3347
1/5	1	1/4	5	0.9713	3.8901
			7	0.9713	5.8935
			10	0.9713	8.8960
		1/2	5	0.8242	4.0525
			7	0.8242	6.0552
			10	0.8242	9.0570
		1	5	0.6630	4.2334
			7	0.6630	6.2352
			10	0.6630	9.2365
	2	1/4	5	0.7688	4.1288
			7	0.7688	6.1307
			10	0.7688	9.1321
		1/2	5	0.6868	4.2177
			7	0.6868	6.2193
			10	0.6868	9.2204
		1	5	0.5828	4.3320
			7	0.5828	6.3332
			10	0.5828	9.3341

Table 4.2: Numerical approximations of ℓ_s^* for $f(u) = \alpha - \beta \ln(1-u)$



(a)



(b)

Figure 4.2: Steady-state solution profiles for (a) Left IBVP and (b) Right IBVP where $r = 1/5$, $L = 5$ and $f(u) = 1 - \ln(1-u)$.

CHAPTER V

CONCLUSION

This thesis investigated the singular convection-diffusion problems with mixed boundary conditions. Specifically, we focused on studying the beyond quenching steady-state solution profiles for the following problems:

$$
\begin{aligned}
&u_t - u_{xx} - \frac{r}{x} u_x = f(u)\chi_{\{u<c\}}(u) \text{ in } \Omega, \\
&u(x,0) = 0 \text{ on } \bar{D}
\end{aligned}
$$

subject to the mixed boundary conditions:

$$
\begin{aligned}
&u(0,t) = 0 = u_x(L,t) \quad \text{for } 0 < t < T \quad \text{(Left IBVP)} \\
\text{or} \qquad &u_x(0,t) = 0 = u(L,t) \quad \text{for } 0 < t < T \quad \text{(Right IBVP)}
\end{aligned}
$$

where $0 < r < 1$, $L > 0$, $T \leq \infty$, $D = (0, L)$, $\Omega = D \times (0, T]$, $\chi_{\mathbb{S}}$ is the characteristic function of the set $\mathbb{S}$ and f is a twice continuously differentiable function on $[0, c)$, for some constant c, with $f(0) > 0$, $f' > 0$, $f'' \geq 0$ and $\lim_{u \to c^-} f(u) = \infty$.

With some conditions on f, we assumed that classical solutions to both problems exist before quenching, and weak solutions exist after quenching.

We proved that all weak solutions of the Left IBVP tend to the unique of solution of the steady-state problem:

$$
\begin{aligned}
&W(x) = c, \ x \in [\ell_s^*, L], \\
&-(x^r W'(x))' = x^r f(W(x)), \ x \in (0, \ell_s^*), \ W(0) = 0, W(\ell_s^*) = c.
\end{aligned}
$$

Using a similar approach, we found that all weak solutions of the the Right IBVP tend to the unique of solution of the steady-state problem:

$$W(x) = c,\ x \in [0, \ell_s^*],$$
$$-(x^r W'(x))' = x^r f(W(x)),\ x \in (\ell_s^*, L),\ W(\ell_s^*) = c, W(L) = 0.$$

The key factor in determining the beyond quenching profiles is the parameter ℓ_s^*. We established integral representations of ℓ_s^* and provided a lower and an upper bounds for the Left and the Right IBVPs, respectively.

Although explicit calculations of ℓ_s^* are not feasible due to the dependence on the function f, we developed a numerical method to approximate its value for any f that satisfies the aforementioned conditions. Two examples were used to illustrate the use of these numerical methods, and we demonstrated how the elongation of L affects the value of ℓ_s^* in the Left and the Right IBVPs.

In the Left IBVP, we observed that if L is less than ℓ_s^*, the solution does not quench. On the other hand, elongation of L beyond ℓ_s^* only elongates the quenching region but does not affect the value of ℓ_s^* itself. In contrast, for the Right IBVP, we found that the elongation of L increases the value of ℓ_s^* by approximately the same amount.

Overall, our findings contribute to the understanding of the solution behavior of singular convection-diffusion problems beyond quenching. For future research, there are several avenues for future research in this area.

One possible direction for future research is to develop a more efficient numerical method for solving for ℓ_s^* based on (2.9) and (3.9). The method presented in Chapter IV works well; however, there may be other approaches that can further improve the computational efficiency and accuracy of the solution.

Another suggestion for future research is to investigate the behavior of the parabolic equation with nonhomogeneous boundary conditions. While this study focused on homogeneous boundary conditions, many practical applications involve nonhomogeneous boundary conditions, and it would be interesting to explore how the steady-state solution profile changes in these cases.

Last but not least, it would be valuable to investigate quenching phenomena in other types of problems, such as those that arise in hyperbolic equations. For example, quenching waves have been observed in some hyperbolic systems, and it would be interesting to explore how these phenomena are affected by different parameter regimes and boundary conditions.

Overall, the results presented in this thesis provide a solid foundation for further research in the area of parabolic partial differential equations beyond quenching, and we hope that the suggestions outlined here will inspire and guide future investigations in this field.

REFERENCES

[1] S. Abbot. *Understanding Analysis*, volume 2. Springer New York, 2016.

[2] A. Acker and W. Walter. The quenching problem for nonlinear parabolic differential equations. In *Ordinary and Partial Differential Equations: Proceedings of the Fourth Conference Held at Dundee, Scotland March 30–April 2, 1976*, pages 1–12. Springer, 2006.

[3] V. Alexiades. Generalized axially symmetric heat potentials and singular parabolic initial boundary value problems. *Archive for Rational Mechanics and Analysis*, 79(4):325–350, 1982.

[4] V. Alexiades and C. Chan. A singular fourier problem with nonlinear radiation in a noncylindrical domain. *Nonlinear Analysis: Theory, Methods & Applications*, 5(8):835–844, 1981.

[5] R. G. Bartle. *The elements of real analysis*, volume 2. Wiley New York, 1964.

[6] R. Boonklurb, K. Kaewrak, and T. Treeyaprasert. Solution profiles beyond quenching for singular semilinear parabolic problems. *Dynamic Systems and Applications*, 27(3):673–689, 2018.

[7] R. L. Burden and J. D. Faires. *Numerical analysis*. Cengage Learning, Boston, MA, USA, 9th ed. edition, 2010.

[8] C. Chan and R. Boonklurb. Solution profiles beyond quenching for a radially symmetric multi-dimensional parabolic problem. *Nonlinear Analysis: Theory, Methods & Applications*, 76:68–79, 2013.

[9] C. Chan and C. Chen. A numerical method for semilinear singular parabolic quenching problems. *Quarterly of Applied Mathematics*, 47(1):45–57, 1989.

[10] C. Chan and L. Ke. Beyond quenching for singular reaction-diffusion problems. *Mathematical methods in the applied sciences*, 17(1):1–9, 1994.

[11] C. Chan and P. Kong. Quenching for degenerate semilinear parabolic equations. *Applicable Analysis*, 54(1-2):17–25, 1994.

[12] C. Chan and P. Kong. Solution profiles beyond quenching for degenerate reaction-diffusion problems. *Nonlinear Analysis: Theory, Methods & Applications*, 24(12):1755–1763, 1995.

[13] C. Hirota and K. Ozawa. Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations—an application to the blow-up problems of partial differential equations. *Journal of computational and applied mathematics*, 193(2):614–637, 2006.

[14] K. Kaewrak. Steady-state solution after quenching for heat equation with singularity. Master's thesis, Chulalongkorn University, 2016.

[15] H. Kawarada. On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1-u)$. *Publications of the Research Institute for Mathematical Sciences*, 10(3):729–736, 1974.

[16] M. H. Protter and C. B. Morrey. *Intermediate Calculus.* Springer New York, 1985.

[17] H. Royden. *Real analysis.* Macmillan Publishing, New York, 1988.

[18] I. Stakgold and M. Holst. *Green's functions and boundary value problems.* John Wiley & Sons, Hoboken, NJ, USA, 3rd ed. edition, 2011.

[19] K. Stromberg. *An introduction to classical real analysis.* Wadsworth International Group, Belmont, 1981.

APPENDICES

APPENDIX A1 : Green's function derivation (Lemma 2.2)

We would like to find a Green's function G of (2.3). The Green's function is the solution to the following boundary-value problem [18]:

$$-(x^r G_x(x;\xi))_x = \delta(x-\xi),\ 0 < x, \xi < \tilde{\ell}, \tag{A1.1}$$

$$G(0;\xi) = 0, \tag{A1.2}$$

$$G(\tilde{\ell};\xi) = 0. \tag{A1.3}$$

We divide the interval $(0,\tilde{\ell})$ into two parts at ξ. Consider the region to the left of ξ where $0 < x < \xi$. Let G^- denote the Green's function on this region. From (A1.1), G^- must satisfy

$$-(x^r G^-_x(x;\xi))_x = 0,\ 0 < x < \xi. \tag{A1.4}$$

On the other hand, we let G^+ be the Green's function on the region to the right of ξ where $\xi < x < \tilde{\ell}$. From (A1.1), G^+ must satisfy

$$-(x^r G^+_x(x;\xi))_x = 0,\ \xi < x < \tilde{\ell}. \tag{A1.5}$$

We propose that the Green's functions be

$$G^-(x;\xi) = \left(\frac{A^-}{1-r}\right) x^{1-r} + B^-$$

and

$$G^+(x;\xi) = \left(\frac{A^+}{1-r}\right) x^{1-r} + B^+$$

so that G^- and G^+ satisfy (A1.4) and (A1.5), respectively.

We need four equations to solve for A^+, A^-, B^+ and B^-. First, we use the boundary conditions (A1.2) and (A1.3) to create two equations:

$$G^-(0;\xi) = B^- = 0 \tag{A1.6}$$

and

$$G^+(\tilde{\ell};\xi) = \left(\frac{\tilde{\ell}^{1-r}}{1-r}\right) A^+ + B^+ = 0. \tag{A1.7}$$

Consider

$$G^+(\xi+\epsilon;\xi) - G^-(\xi-\epsilon;\xi) = \left(\frac{(\xi+\epsilon)^{1-r}}{1-r}\right) A^+ + B^+ - \left(\frac{(\xi-\epsilon)^{1-r}}{1-r}\right) A^- - B^-$$

for some $\epsilon > 0$. We have, by the continuity of the Green's function G, that $\lim_{\epsilon\to 0}(G^+(\xi+\epsilon;\xi) - G^-(\xi-\epsilon;\xi)) = 0$. Therefore, we have the third equation

$$\left(\frac{\xi^{1-r}}{1-r}\right)(A^+ - A^-) + B^+ - B^- = 0. \tag{A1.8}$$

To obtain the last equation, we integrate (A1.1) with respect to x from $\xi - \epsilon$ to $\xi + \epsilon$

$$\begin{aligned}
\int_{\xi-\epsilon}^{\xi+\epsilon} -(x^r G_x(x;\xi))_x \, dx &= \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx \\
-\left(x^r G_x(x;\xi)\right)\Big|_{\xi-\epsilon}^{\xi+\epsilon} &= 1 \\
(\xi-\epsilon)^{1-r} G_x^-(\xi-\epsilon;\xi) - (\xi+\epsilon)^{1-r} G_x^+(\xi+\epsilon;\xi) &= 1.
\end{aligned}$$

Let $\epsilon \to 0$, we have that

$$\begin{aligned}
\xi^{1-r}\left(G_x^-(\xi;\xi) - G_x^+(\xi;\xi)\right) &= 1 \\
A^- - A^+ &= 1. \qquad \text{(A1.9)}
\end{aligned}$$

Solving (A1.6) – (A1.9) simultaneously, we get the Green's function G as follows:

$$G(x;\xi) = \begin{cases} \dfrac{x^{1-r}}{1-r}\left(1-\left(\dfrac{\xi}{\tilde{\ell}}\right)^{1-r}\right) & \text{for } 0 \leq x < \xi, \\ \dfrac{\xi^{1-r}}{1-r}\left(1-\left(\dfrac{x}{\tilde{\ell}}\right)^{1-r}\right) & \text{for } \xi < x \leq \tilde{\ell}. \end{cases}$$

Furthermore, the solution to (2.3) can be expressed using the Green's formula ([18], p.167) as follows:

$$W(x) = \int_0^{\tilde{\ell}} \xi^r G(x;\xi) f(W(\xi))\, d\xi + c\left(\frac{x}{\tilde{\ell}}\right)^{1-r}, \; x \in [0, \tilde{\ell}].$$

APPENDIX A2 : Green's function derivation (Lemma 2.3)

We would like to find a Green's function G of the following boundary-value problem:

$$\left.\begin{aligned} &-(x^r W'(x))' = x^r f(W(x)),\ x_1 < x < x_2, \\ &W(x_1) = c = W(x_2). \end{aligned}\right\} \tag{A2.1}$$

The Green's function is the solution to the following boundary-value problem [18]:

$$-(x^r G_x(x;\xi))_x = \delta(x-\xi),\ x_1 < x, \xi < x_2, \tag{A2.2}$$

$$G(x_1;\xi) = 0, \tag{A2.3}$$

$$G(x_2;\xi) = 0. \tag{A2.4}$$

We divide the interval (x_1, x_2) into two parts at ξ. Consider the region to the left of ξ where $x_1 < x < \xi$. Let G^- denote the Green's function on this region. From (A2.2), G^- must satisfy

$$-(x^r G_x^-(x;\xi))_x = 0,\ x_1 < x < \xi. \tag{A2.5}$$

On the other hand, we let G^+ be the Green's function on the region to the right of ξ where $\xi < x < x_2$. From (A2.2), G^+ must satisfy

$$-(x^r G_x^+(x;\xi))_x = 0,\ \xi < x < x_2. \tag{A2.6}$$

We propose that the Green's functions be

$$G^-(x;\xi) = \left(\frac{A^-}{1-r}\right) x^{1-r} + B^-$$

and

$$G^+(x;\xi) = \left(\frac{A^+}{1-r}\right) x^{1-r} + B^+$$

so that G^- and G^+ satisfy (A2.5) and (A2.6), respectively.

We need four equations to solve for A^+, A^-, B^+ and B^-. First, we use the boundary conditions (A2.3) and (A2.4) to create two equations:

$$G^-(x_1;\xi) = \left(\frac{x_1^{1-r}}{1-r}\right) A^- + B^- = 0 \tag{A2.7}$$

and

$$G^+(x_2;\xi) = \left(\frac{x_2^{1-r}}{1-r}\right) A^+ + B^+ = 0. \tag{A2.8}$$

Consider

$$G^+(\xi+\epsilon;\xi) - G^-(\xi-\epsilon;\xi) = \left(\frac{(\xi+\epsilon)^{1-r}}{1-r}\right) A^+ + B^+ - \left(\frac{(\xi-\epsilon)^{1-r}}{1-r}\right) A^- - B^-$$

for some $\epsilon > 0$. We have, by the continuity of the Green's function G, that $\lim_{\epsilon\to 0}(G^+(\xi+\epsilon;\xi) - G^-(\xi-\epsilon;\xi)) = 0$. Therefore, we have the third equation

$$\left(\frac{\xi^{1-r}}{1-r}\right)(A^+ - A^-) + B^+ - B^- = 0. \tag{A2.9}$$

To obtain the last equation, we integrate (A2.2) with respect to x from $\xi-\epsilon$ to $\xi+\epsilon$

$$\begin{aligned} \int_{\xi-\epsilon}^{\xi+\epsilon} -(x^r G_x(x;\xi))_x \, dx &= \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\, dx \\ -\left(x^r G_x(x;\xi)\right)\Big|_{\xi-\epsilon}^{\xi+\epsilon} &= 1 \\ (\xi-\epsilon)^{1-r} G_x^-(\xi-\epsilon;\xi) - (\xi+\epsilon)^{1-r} G_x^+(\xi+\epsilon;\xi) &= 1. \end{aligned}$$

Let $\epsilon \to 0$, we have that

$$\begin{aligned} \xi^{1-r}\left(G_x^-(\xi;\xi) - G_x^+(\xi;\xi)\right) &= 1 \\ A^- - A^+ &= 1. \end{aligned} \tag{A2.10}$$

Solving (A2.7) – (A2.10) simultaneously, we get the Green's function G as follows:

$$G(x;\xi) = \begin{cases} \left(\dfrac{x^{1-r} - x_1^{1-r}}{1-r}\right)\left(\dfrac{x_2^{1-r} - \xi^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) & \text{for } x_1 \leq x < \xi, \\ \left(\dfrac{x_2^{1-r} - x^{1-r}}{1-r}\right)\left(\dfrac{\xi^{1-r} - x_1^{1-r}}{x_2^{1-r} - x_1^{1-r}}\right) & \text{for } \xi < x \leq x_2 \end{cases}$$

Furthermore, the solution to (A2.1) can be expressed using the Green's formula ([18], p.167) as follows:

$$W(x) = \int_0^{\tilde{\ell}} \xi^r G(x;\xi) f(W(\xi))\, d\xi + c,\ x \in [x_1, x_2].$$

APPENDIX A3 : Green's function derivation (Lemma 3.2)

We would like to find a Green's function G of (3.3). The Green's function is the solution to the following boundary-value problem [18]:

$$-(x^r G_x(x;\xi))_x = \delta(x-\xi),\ \tilde{\ell} < x, \xi < L, \tag{A3.1}$$

$$G(\tilde{\ell};\xi) = 0, \tag{A3.2}$$

$$G(L;\xi) = 0. \tag{A3.3}$$

We divide the interval $(\tilde{\ell}, L)$ into two parts at ξ. Consider the region to the left of ξ where $\tilde{\ell} < x < \xi$. Let G^- denote the Green's function on this region. From (A3.1), G^- must satisfy

$$-(x^r G_x^-(x;\xi))_x = 0,\ \tilde{\ell} < x < \xi. \tag{A3.4}$$

On the other hand, we let G^+ be the Green's function on the region to the right of ξ where $\xi < x < L$. From (A3.1), G^+ must satisfy

$$-(x^r G_x^+(x;\xi))_x = 0,\ \xi < x < L. \tag{A3.5}$$

We propose that the Green's functions be

$$G^-(x;\xi) = \left(\frac{A^-}{1-r}\right) x^{1-r} + B^-$$

and

$$G^+(x;\xi) = \left(\frac{A^+}{1-r}\right) x^{1-r} + B^+$$

so that G^- and G^+ satisfy (A3.4) and (A3.5), respectively.

We need four equations to solve for A^+, A^-, B^+ and B^-. First, we use the boundary conditions (A3.2) and (A3.3) to create two equations:

$$G^-(\tilde{\ell};\xi) = \left(\frac{\tilde{\ell}^{1-r}}{1-r}\right)A^- + B^- = 0 \tag{A3.6}$$

and

$$G^+(L;\xi) = \left(\frac{L^{1-r}}{1-r}\right)A^+ + B^+ = 0. \tag{A3.7}$$

Consider

$$G^+(\xi+\epsilon;\xi) - G^-(\xi-\epsilon;\xi) = \left(\frac{(\xi+\epsilon)^{1-r}}{1-r}\right)A^+ + B^+ - \left(\frac{(\xi-\epsilon)^{1-r}}{1-r}\right)A^- - B^-$$

for some $\epsilon > 0$. We have, by the continuity of the Green's function G, that $\lim_{\epsilon\to 0}(G^+(\xi+\epsilon;\xi) - G^-(\xi-\epsilon;\xi)) = 0$. Therefore, we have the third equation

$$\left(\frac{\xi^{1-r}}{1-r}\right)(A^+ - A^-) + B^+ - B^- = 0. \tag{A3.8}$$

To obtain the last equation, we integrate (A3.1) with respect to x from $\xi-\epsilon$ to $\xi+\epsilon$

$$\begin{aligned} \int_{\xi-\epsilon}^{\xi+\epsilon} -(x^r G_x(x;\xi))_x\,dx &= \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\,dx \\ -(x^r G_x(x;\xi))\,\Big|_{\xi-\epsilon}^{\xi+\epsilon} &= 1 \\ (\xi-\epsilon)^{1-r}G_x^-(\xi-\epsilon;\xi) - (\xi+\epsilon)^{1-r}G_x^+(\xi+\epsilon;\xi) &= 1. \end{aligned}$$

Let $\epsilon \to 0$, we have that

$$\begin{aligned} \xi^{1-r}\left(G_x^-(\xi;\xi) - G_x^+(\xi;\xi)\right) &= 1 \\ A^- - A^+ &= 1. \end{aligned} \tag{A3.9}$$

Solving (A3.6) – (A3.9) simultaneously, we get the Green's function G as follows:

$$G(x;\xi) = \begin{cases} \left(\dfrac{x^{1-r} - \tilde{\ell}^{1-r}}{1-r}\right)\left(\dfrac{L^{1-r} - \xi^{1-r}}{L^{1-r} - \tilde{\ell}^{1-r}}\right) & \text{for } \tilde{\ell} \leq x < \xi, \\ \left(\dfrac{L^{1-r} - x^{1-r}}{1-r}\right)\left(\dfrac{\xi^{1-r} - \tilde{\ell}^{1-r}}{L^{1-r} - \tilde{\ell}^{1-r}}\right) & \text{for } \xi < x \leq L. \end{cases}$$

Furthermore, the solution to (3.3) can be expressed using the Green's formula ([18], p.167) as follows:

$$W(x) = \int_0^{\tilde{\ell}} \xi^r G(x;\xi) f(W(\xi))\, d\xi + c\left(\frac{L^{1-r} - x^{1-r}}{L^{1-r} - \tilde{\ell}^{1-r}}\right)^{1-r},\ x \in [\tilde{\ell}, L].$$

APPENDIX B1 : Numerical method implementation in Python

```
## Numerical Method

import autograd.numpy as np
from autograd import grad

def computeL(f,c,r,L,tol_bisect,tol_newton,N,type_bvp):
  if type_bvp == 1:
    l = L
  elif type_bvp == 2:
    l = 0
  else:
    print("Invalid Problem Type")
    l_conv,x,w_conv = np.inf,np.inf,np.inf
    return l_conv,x,w_conv

  def gradf(f,w):
    g = grad(f)
    n = len(w)
    f_prime = np.zeros(n)
    for i in range(n):
      f_prime[i] = g(w[i])
    return f_prime

  left,right = 0,L

  del_bisect = tol+1
  while del_bisect > tol_bisect:
    if type_bvp == 1:
      h = l/(N+1)
      x = np.linspace(0,l,N+2)
      w = c*(x/l)
    else:
      h = (L-l)/(N+1)
      x = np.linspace(l,L,N+2)
      w = c-(x-l)/(L-l)

    gamma = 1/h**2+r/(2*h*x[1:N+1])
    gamma_bar = 1/h**2-r/(2*h*x[1:N+1])
    Jh = np.diag(-(gamma+gamma_bar))+np.diag(gamma_bar[1:N],-1)+np.diag(gamma[0:N-1],1)
    F = np.zeros(N)
    j,flag_div = 0,0

    del_newton = tol_newton+1
    while flag_div == 0 and del_newton > tol_newton:
      l_prev = l
      j = j+1
      for i in range(N):
        F[i] = gamma[i]*w[i+2]+Jh[i,i]*w[i+1]+gamma_bar[i]*w[i]+f(w[i+1])
      J = Jh+np.diag(gradf(f,w[1:N+1]))
      w_prev = w.copy()
      w[1:N+1] = w[1:N+1]-np.linalg.solve(J,F)
      if sum(w[1:N+1] >= c) > 0:
        flag_div = 1
      del_newton = np.linalg.norm(w-w_prev,np.inf)

    if flag_div == 1:
      if type_bvp == 1:
        right = l
      else:
        left = l
    else:
      if type_bvp == 1:
        left = l
      else:
        right = l
      l_conv = l
      w_conv = w.copy()

    l = (left+right)/2
    del_bisect = abs(l-l_prev)

  return l_conv,x,w_conv
```

APPENDIX B2 : Example 4.1 and Example 4.2

```
## Example 4.1.

import pandas as pd

table = pd.DataFrame(columns=['r','b','L','l1','l2'])
eps,tol = 1e-6,1e-6
N = 100
c = 1
rr = [1/7,1/5]
bb = [1/4,1/3,1/2]
LL = [5,7,10]

for i in rr:
  for j in bb:
    for k in LL:
      l1 = computeL(lambda u:1/(c-u)**j,c,i,k,tol,eps,N,1)[0]
      l2 = computeL(lambda u:1/(c-u)**j,c,i,k,tol,eps,N,2)[0]
      table.loc[len(table)] = [i,j,k,l1,l2]

print(table)
```

```
## Example 4.2.

import pandas as pd

table = pd.DataFrame(columns=['r','a','b','L','l1','l2'])
eps,tol = 1e-6,1e-6
N = 100
c = 1
rr = [1/7,1/5]
aa = [1,2]
bb = [1/4,1/2,1]
LL = [5,7,10]

for i in rr:
  for a in aa:
    for j in bb:
      for k in LL:
        l1 = computeL(lambda u:a-j*np.log(c-u),c,i,k,tol,eps,N,1)[0]
        l2 = computeL(lambda u:a-j*np.log(c-u),c,i,k,tol,eps,N,2)[0]
        table.loc[len(table)] = [i,a,j,k,l1,l2]

print(table)
```

APPENDIX B3 : Solution profiles plotting

```
## Plotting

import matplotlib.pyplot as plt

def plotSol1(l_conv,x,w_conv,c,L):
  plt.plot([l_conv,l_conv], [0,c],'r--')
  plt.plot([l_conv,L], [c,c],'k')
  plt.plot(x, w_conv, 'k')
  plt.xlabel("x")
  plt.ylabel("W")
  plt.text(l_conv-0.2,-0.15,"$\ell^*_s$",fontsize=14,color='r')
  plt.show()

def plotSol2(l_conv,x,w_conv,c,L):
  plt.plot([l_conv,l_conv], [0,c],'r--')
  plt.plot([0,l_conv], [c,c],'k')
  plt.plot(x, w_conv, 'k')
  plt.xlabel("x")
  plt.ylabel("W")
  plt.text(l_conv-0.2,-0.15,"$\ell^*_s$",fontsize=14,color='r')
  plt.show()



# Example 4.1.
eps,tol = 1e-6,1e-6
N = 100
c,r,b,L = 1,1/5,1/3,5

l1,x1,w1 = computeL(lambda u:1/(c-u)**b,c,r,L,tol,eps,N,1)
l2,x2,w2 = computeL(lambda u:1/(c-u)**b,c,r,L,tol,eps,N,2)

plotSol1(l1,x1,w1,c,L)
plotSol2(l2,x2,w2,c,L)


# Example 4.2.
eps,tol = 1e-6,1e-6
N = 100
c,r,L = 1,1/5,5

l1,x1,w1 = computeL(lambda u:-np.log(c-u)+1,c,r,L,tol,eps,N,1)
l2,x2,w2 = computeL(lambda u:-np.log(c-u)+1,c,r,L,tol,eps,N,2)

plotSol1(l1,x1,w1,c,L)
plotSol2(l2,x2,w2,c,L)
```

BIOGRAPHY

Name	Benjamin Thaitavorn
Date of Birth	28 July 1996
Place of Birth	Chiang Mai, Thailand
Educations	B.Eng. in Aerospace Engineering (First-class Honours), Chulalongkorn University, 2020
Awards Received	Vidyabundit Scholarship for Graduate Studies in Science (2021)

Publications

- B. Thaitavorn and R. Boonklurb. Beyond quenching profile for a singular semilinear parabolic problem with mixed boundary conditions. In *Proceedings of the 26th Annual Meeting in Mathematics*, pages 62–69. Suranaree University of Technology, May 2022.