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HEAT EQUATION ON A COMPACT LIE GROUP

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เราจะศึกษาส่วนกลางความร้อนบนกรุปลีแบบกระชับ ในงานนี้เราจะใช้โครงสร้างของทฤษฎีเซไมซิมเปิล เพื่อที่จะหารูปแบบที่สอดคล้องกับส่วนกลางความร้อนของลีกรุปแบบกระชับทั่ว ๆ ไป

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We study the heat kernel on a compact Lie group. In this work, we will use structure theory of semisimple Lie algebra to find equivalent forms of the heat kernel on a general compact Lie group.

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TABLE OF CONTENTS

CHAPTER 1
Introduction

Let K be a compact, connected Lie group, with Lie algebra $\mathfrak{k}$. Let $\langle\, , \rangle$ denote a fixed Ad-K-invariant inner product on $\mathfrak{k}$, the existence of which is guaranteed by the compactness of K. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{k}$, where we view the X_i's as left-invariant vector fields on K. Define the *Casimir operator* on K to be

$$\Delta = \sum_{i=1}^{n} X_i^2.$$

If π is a representation of K acting on some space V_π, and if π is irreducible, then

$$\pi(\Delta) = \sum \pi(X_i)^2 = -\lambda_\pi I.$$

Let ρ_t be the heat kernel for Δ, that is, the fundamental solution at the identity of the heat equation

$$\frac{du}{dt} = \Delta u.$$

Stein proved the existence of such a fundamental solution, and that ρ_t is a C^∞ function, which is real and strictly positive.

In addition, Using the Peter-Weyl theorem, we can derive a series expansion for ρ_t in terms of the character of K, namely

$$\rho_t(x) = \sum_{\pi \in \widehat{K}} \dim V_\pi e^{-\lambda_\pi t} \chi_\pi(x). \tag{1.1}$$

Here $\widehat{K}$ refers to the set of isomorphism classes of irreducible representation of K, and $\chi_\pi(x) = \operatorname{tr}(\pi(x))$ is the character of π.

There is another formula for the heat kernel in terms of Poisson's summation formula.

In this work, we illustrate how to find explicit formulas for the heat kernel on S^1 and $SU(2)$ and give another formula using Poisson's summation formula. Using structure theory of semisimple Lie algebra, we can give an analogous formulas for a general simply-connected compact semisimple Lie group. The main reference of this work is the paper [6].

CHAPTER 2
Lie group and Lie algebra

In this chapter, we give definitions of matrix Lie groups, Lie algebras, representations and their relevant concepts. Details can be found in [2].

Definition 2.1. The **general linear group** over the real numbers, denoted $\mathrm{GL}(n, \mathbb{R})$, is the group of all $n \times n$ invertible matrices with real entries. The general linear group over the complex numbers, denoted $\mathrm{GL}(n, \mathbb{C})$, is the group of all $n \times n$ invertible matrices with complex entries.

Definition 2.2. Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries.

Definition 2.3. Let $\{A_m\}$ be a sequence of complex matrices in $M_n(\mathbb{C})$. We say that $\{A_m\}$ **converges** to a matrix A if each entry of A_m converges (as $m \to \infty$) to the corresponding entry of A

Definition 2.4. A **matrix Lie group** is any subgroup G of $\mathrm{GL}(n, \mathbb{C})$ with the following property: If $\{A_m\}$ is any sequence of matrices in G, and $\{A_m\}$ converges to some matrix A then either $A \in G$, or A is not invertible.

Example 2.5. The following group are matrix Lie groups.

$$\begin{aligned}
\mathrm{GL}(n, \mathbb{C}) &= \{X \in M_n(\mathbb{C}) \mid \det X \neq 0\} \\
\mathrm{GL}(n, \mathbb{R}) &= \{X \in M_n(\mathbb{R}) \mid \det X \neq 0\} \\
\mathrm{SL}(n, \mathbb{C}) &= \{X \in M_n(\mathbb{C}) \mid \det X = 1\} \\
\mathrm{SL}(n, \mathbb{R}) &= \{X \in M_n(\mathbb{R}) \mid \det X = 1\} \\
\mathrm{O}(n) &= \{X \in M_n(\mathbb{R}) \mid X^t = X^{-1}\} \\
\mathrm{SO}(n) &= \{X \in M_n(\mathbb{R}) \mid X^t = X^{-1} \text{ and } \det X = 1\} \\
\mathrm{U}(n) &= \{X \in M_n(\mathbb{C}) \mid X^* = X^{-1}\} \quad ; X^* = \bar{X}^t \\
\mathrm{SU}(n) &= \{X \in M_n(\mathbb{C}) \mid X^* = X^{-1} \text{ and } \det X = 1\}
\end{aligned}$$

Definition 2.6. Let G and H be matrix Lie groups. A map Φ from G to H is called a **Lie group homomorphism** if Φ is a group homomorphism and Φ is continuous. If, in addition, Φ is one-to-one and onto and the inverse map Φ^{-1} is continuous, then Φ is called a **Lie group isomorphism**

Definition 2.7. A **finite-dimensional real or complex Lie algebra** is a finite-dimensional real or complex vector space $\mathfrak{g}$, together with a map $[\cdot,\cdot]$ from $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$, with the following properties:

1. $[\cdot,\cdot]$ is bilinear;
2. $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$;
3. $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$.

Condition 2 is called "skew symmetry". Condition 3 is called the **Jacobi's identity**. Note also that Condition 2 implies that [X,X] = 0 for all $X\in\mathfrak{g}$. We will deal only with finite-dimensional Lie algebras and will from now on interpret "Lie algebra" as "finite-dimensional Lie algebra".

Example 2.8. The space $M_n(\mathbb{R})$ of all $n\times n$ real matrices is a real Lie algebra if the bracket operation $[X,Y]$ is defined by

$$[X,Y]=XY-YX.$$

This Lie algebra is denoted by $\mathfrak{gl}(n,\mathbb{R})$.

Similarly, the space $M_n(\mathbb{C})$ of all $n\times n$ complex matrices is a complex Lie algebra with respect to the same bracket operation and denoted by $\mathfrak{gl}(n,\mathbb{C})$.

Let V be a finite-dimensional real or complex vector space, and let $\mathfrak{gl}(V)$ denote the space of linear maps of V into itself. The, $\mathfrak{gl}(V)$ becomes a real or complex Lie algebra with the bracket operation $[X,Y]=XY-YX$.

Definition 2.9. A **subalgebra** of a real or complex Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[H_1,H_2]\in\mathfrak{h}$ for all H_1 and $H_2\in\mathfrak{h}$. If $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{h}$ is a real subspace of $\mathfrak{g}$ which is closed under brackets, then $\mathfrak{h}$ is said to be a **real subalgebra** of $\mathfrak{g}$.

Theorem 2.10. *Let G be a matrix Lie group, then the set*

$$\mathfrak{g}=\{X\in\mathfrak{gl}(n,\mathbb{C})\,|\,e^{tX}\in G \text{ for all } t\in\mathbb{R}\}$$

is a Lie subalgebra of $M_n(\mathbb{C})$. We call $\mathfrak{g}$ the Lie algebra of a matrix Lie group G.

Example 2.11. By Theorem 2.10, we have the following sets are Lie algebras (of a Lie group in Example 2.5 respectively).

$$
\begin{aligned}
\mathfrak{gl}(n,\mathbb{C}) &= M_n(\mathbb{C}) \\
\mathfrak{gl}(n,\mathbb{R}) &= M_n(\mathbb{R}) \\
\mathfrak{sl}(n,\mathbb{C}) &= \{X \in M_n(\mathbb{C}) \mid \operatorname{tr}(X) = 0\} \\
\mathfrak{sl}(n,\mathbb{R}) &= \{X \in M_n(\mathbb{R}) \mid \operatorname{tr}(X) = 0\} \\
\mathfrak{o}(n) &= \{X \in M_n(\mathbb{R}) \mid X^t = -X\} \\
\mathfrak{so}(n) &= \{X \in M_n(\mathbb{R}) \mid X^t = -X\} \\
\mathfrak{u}(n) &= \{X \in M_n(\mathbb{C}) \mid X^* = -X\} \\
\mathfrak{su}(n) &= \{X \in M_n(\mathbb{C}) \mid X^* = -X \text{ and } \operatorname{tr}(X) = 0\}
\end{aligned}
$$

Definition 2.12. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. If, in addition, ϕ is one-to-one and onto, then ϕ is called a **Lie algebra isomorphism**. A Lie algebra isomorphism of a Lie algebra with itself is called a **Lie algebra automorphism**.

Definition 2.13. If V is a finite-dimensional real vector space, then **the complexification** of V, denoted $V_{\mathbb{C}}$, is the space of formal linear combinations

$$v_1 + iv_2,$$

with $v_1, v_2 \in V$. This becomes a real vector space in the obvious way and becomes a complex vector space if we define

$$i(v_1 + iv_2) = -v_2 + iv_1.$$

Proposition 2.14. *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification (as a real vector space). Then, the bracket operation on $\mathfrak{g}$ has a unique extension to $\mathfrak{g}_{\mathbb{C}}$ which makes $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra. The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is called the* ***complexification*** *of the real Lie algebra $\mathfrak{g}$.*

Proposition 2.15. *The Lie algebras $\mathfrak{gl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}), \mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are real Lie algebras and the complexifications are the following:*

$$
\begin{aligned}
\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}} &\cong \mathfrak{gl}(n,\mathbb{C}) \\
\mathfrak{sl}(n,\mathbb{R})_{\mathbb{C}} &\cong \mathfrak{sl}(n,\mathbb{C}) \\
\mathfrak{u}(n)_{\mathbb{C}} &\cong \mathfrak{gl}(n,\mathbb{C}) \\
\mathfrak{su}(n)_{\mathbb{C}} &\cong \mathfrak{sl}(n,\mathbb{C})
\end{aligned}
$$

Definition 2.16. Let G be a matrix Lie group. Then a **finite-dimensional complex representation** of G is a Lie group homomorphism

$$\Pi : G \to \mathrm{GL}(n, \mathbb{C})$$

($n \geq 1$) or more generally a Lie group homomorphism

$$\Pi : G \to \mathrm{Aut}(V)$$

where V is a finite-dimensional complex vector space (with $\dim V \geq 1$). A finite dimensional real representation of G is a Lie group homomorphism Π of G into $\mathrm{GL}(n, \mathbb{R})$ or into $\mathrm{GL}(V)$, where V is a finite-dimensional real vector space.

If $\mathfrak{g}$ is a real or complex Lie algebra, then a **finite-dimensional complex representation** of $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(n, \mathbb{C})$ or into $\mathfrak{gl}(V)$, where V is a finite-dimensional complex vector space. If $\mathfrak{g}$ is a real Lie algebra, then a **finite-dimensional real representation** of $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(n, \mathbb{R})$ or into $\mathfrak{gl}(V)$.

Example 2.17. Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. For $A \in G$, define a linear map $\mathrm{Ad}_A : G \to \mathrm{GL}(\mathfrak{g})$ by the formula

$$\mathrm{Ad}_A(X) = AXA^{-1}.$$

Then Ad_A is a representation of G, acting on the space $\mathfrak{g}$.

Example 2.18. Let $\mathfrak{g}$ be a Lie algebra. For $X \in \mathfrak{g}$, define a linear map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by

$$\mathrm{ad}_X(Y) = [X, Y].$$

Note that $\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X, \mathrm{ad}_Y](Z)$. Thus $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism and hence ad is a representation of $\mathfrak{g}$, acting on the space $\mathfrak{g}$.

Definition 2.19. Let Π be a finite-dimensional real or complex representation of a matrix Lie group G, acting on a space V. A subspace W of V is called **invariant** if $\Pi(A)w \in W$ for all $w \in W$ and all $A \in G$. An invariant subspace W is called non-trivial if $W \neq \{0\}$ and $W \neq V$. A representation with no non-trivial invariant subspaces is called **irreducible**.

The terms **invariant, non-trivial**, and **irreducible** are defined analogously for a representation of a Lie algebra.

Definition 2.20. Let G be a matrix Lie group, let Π be a representation of G acting on the space V, and let Σ be a representation of G acting on the space W. A linear map $\phi : V \to W$ is called an **intertwining map** of representations if

$$\phi(\Pi(A)v) = \Sigma(A)\phi(v)$$

for all $A \in G$ and all $v \in V$. The analogous property defines intertwining maps of representations of a Lie algebra.

If ϕ is an intertwining map of representations and, in addition, ϕ is invertible, the ϕ is said to be an **equivalence** of representations. If there exists an isomorphism between V and W, then the representations are said to be **equivalent**.

Proposition 2.21. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let Π be a (finite-dimensional real or complex) representation of G, acting on the space V. Then, there is a unique representation π of $\mathfrak{g}$ acting on the same space such that*

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. The representation π can be computed as

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})|_{t=0}$$

and satisfies

$$\pi(AXA^{-1}) = \Pi(A)\pi(X)\Pi(A)^{-1}$$

for all $X \in \mathfrak{g}$ and all $A \in G$.

Proposition 2.22.

1. *Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$. Let Π be a representation of G and π the associated representation of $\mathfrak{g}$. Then, Π is irreducible if and only if π is irreducible.*

2. *Let G be a connected matrix Lie group. Let Π_1 and Π_2 be representations of G, and let π_1 and π_2 be the associated Lie algebra representations. Then π_1 and π_2 are equivalent if and only if Π_1 and Π_2 are equivalent.*

Proposition 2.23. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every finite-dimensional complex representation π of $\mathfrak{g}$ has a unique extension to a complex-linear representation of $\mathfrak{g}_{\mathbb{C}}$, also denoted π and given by*

$$\pi(X + iY) = \pi(X) + i\pi(Y)$$

for all $X, Y \in \mathfrak{g}$. Furthermore, π is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is irreducible as a representation of $\mathfrak{g}$.

Theorem 2.24 (**Schur's Lemma**).

1. *Let V and W be irreducible real or complex representations of a group or Lie algebra and let $\phi : V \to W$ be an intertwining map. Then either $\phi = 0$ or ϕ is an isomorphism.*

2. *Let V be an irreducible complex representation of a group or Lie algebra and let $\phi : V \to V$ be an intertwining map of V with itself. Then $\phi = \lambda I$, for some $\lambda \in \mathbb{C}$.*

3. *Let V and W be irreducible complex representations of a group or Lie algebra and let $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then $\phi_1 = \lambda\phi_2$, for some $\lambda \in \mathbb{C}$.*

Corollary 2.25. *Let Π be an irreducible complex representation of a matrix Lie group G. If A is in the center of G, then $\Pi(A) = \lambda I$. Similarly, if π is an irreducible complex representation of a Lie algebra $\mathfrak{g}$ and if X is in the center of $\mathfrak{g}$ (i.e., $[X, Y] = 0$ for all $Y \in \mathfrak{g}$), then $\pi(X) = \lambda I$.*

Corollary 2.26. *An irreducible complex representation of a commutative group or Lie algebra is one dimensional.*

Definition 2.27. Let $\mathfrak{g}$ be the Lie algebra over a field K. Denote T is the tensor algebra of $\mathfrak{g}$. Then T, as a vector space, is the direct sum

$$T = \bigoplus_{m=0}^{\infty} \mathfrak{g}^{\otimes m}$$

and the product in T is defined from the tensor product. Let I be the two-sided ideal of T generated by the set

$$\{X \otimes Y - Y \otimes X - [X, Y] \mid X, Y \in \mathfrak{g}\}.$$

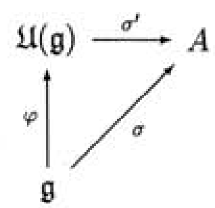
The associative algebra $\mathfrak{U}(\mathfrak{g}) = T/I$ is called the **universal enveloping algebra** of $\mathfrak{g}$.

Let ψ be the canonical projection of T onto $\mathfrak{U}(\mathfrak{g})$ and φ the restriction of ψ to $\mathfrak{g}$. Now φ is a linear map of $\mathfrak{g}$ into $\mathfrak{U}(\mathfrak{g})$ satisfying

$$\varphi([X, Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) = [\varphi(X), \varphi(Y)].$$

The map φ is called the **canonical map** of $\mathfrak{g}$ into $\mathfrak{U}(\mathfrak{g})$. Its universal mapping property is described in the following Proposition.

Proposition 2.28. *Let $\mathfrak{g}$ be a Lie algebra over a field K, A an associative algebra over K and σ a Lie algebra homomorphism of $\mathfrak{g}$ into A. Then there exists an algebra homomorphism σ' of $\mathfrak{U}(\mathfrak{g})$ into A such that $\sigma' \circ \varphi = \sigma$. That is, the following diagram commutes:*



Proof. The linear map σ of $\mathfrak{g}$ into A is extended to a homomorphism σ_0 of T into A by defining

$$\sigma_0(X_1 \otimes \cdots \otimes X_n) = \sigma(X_1) \cdots \sigma(X_n)$$

The extended homomorphism σ_0 sends the ideal I to 0 because

$$\sigma_0(X \otimes Y - Y \otimes X - [X,Y]) = \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) - \sigma([X,Y]) = 0.$$

Therefore, the image σ_0 of an element $t \in T$ depends only on the coset $t + I$ and hence there exists a homomorphism σ' of $\mathfrak{U}(\mathfrak{g})$ into A satisfying $\sigma' \circ \varphi = \sigma$. ☐

Definition 2.29. The bilinear form B on $\mathfrak{g} \times \mathfrak{g}$ defined by

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y)$$

is called the **Killing form** of $\mathfrak{g}$.

Proposition 2.30. *The Killing form has the following properties:*

1. $B(X,Y) = B(Y,X)$;
2. $B([X,Y],Z) = B(X,[Y,Z])$.

Proof. Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we have

$$\begin{aligned} B(X,Y) &= \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y) \\ &= \operatorname{tr}(\operatorname{ad}_Y \operatorname{ad}_X) \\ &= B(Y,X) \end{aligned}$$

and

$$\begin{aligned} B([X,Y],Z) &= \operatorname{tr}(\operatorname{ad}_{[X,Y]}, \operatorname{ad}_Z) \\ &= \operatorname{tr}([\operatorname{ad}_X, \operatorname{ad}_Y], \operatorname{ad}_Z) \\ &= \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y \operatorname{ad}_Z) - \operatorname{tr}(\operatorname{ad}_Y \operatorname{ad}_X \operatorname{ad}_Z) \\ &= \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y \operatorname{ad}_Z) - \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Z \operatorname{ad}_Y) \\ &= \operatorname{tr}(\operatorname{ad}_X [\operatorname{ad}_Y, \operatorname{ad}_Z]) \\ &= \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_{[Y,Z]}) \\ &= B(X,[Y,Z]) \end{aligned}$$

for any X, Y, $Z \in \mathfrak{g}$. ☐

Definition 2.31. A symmetric bilinear form $B : V \times V \to K$ is said to be **non-degenerate** if $B(v,w) = 0$ for all $w \in V$ implies $v = 0$.

Definition 2.32. A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is called **semisimple** if its Killing form is non-degenerate.

A connected matrix Lie group is called **semisimple** if its Lie algebra is semisimple.

Example 2.33. The Lie algebra $\mathfrak{su}(2)$ is semisimple.
Put

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

be a basis of $\mathfrak{su}(2)$. Thus

$$\mathrm{ad}_{X_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathrm{ad}_{X_3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we have

$$B(X_i, X_j) = -2\delta_{ij} \quad (1 \le i, j \le 3),$$

and $\det(B(X_i, X_j)) \neq 0$.

Definition 2.34. Let $(X_i)_{1 \le i \le n}$ be a basis of a semisimple Lie algebra. Write

$$g_{ij} = B(X_i, X_j)$$

and

$$(g^{ij}) = (g_{ij})^{-1}.$$

Then the element

$$\Omega = \sum_{i,j=1}^{n} g^{ij} X_i X_j$$

of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is called the **Casimir element**.

Example 2.35. The Casimir element Ω for $\mathfrak{su}(2)$ is given by

$$\Omega = -\frac{1}{2}(X_1^2 + X_2^2 + X_3^2).$$

Since $B(X_i, X_j) = \delta_{ij}$ for for $1 \le i, j \le 3$; , we have

$$(g_{ij}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Hence

$$(g^{ij}) = \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

Thus $g^{ij} = -\frac{1}{2}\delta_{ij}$ and $\Omega = \sum_{i,j=1}^{3} g^{ij} X_i X_j = -\frac{1}{2}\sum_{i=1}^{3} X_i^2$.

Proposition 2.36. *The Casimir element Ω of $\mathfrak{U}(\mathfrak{g})$ is independent of the choice of the basis (X_i). Moreover, Ω belongs to the center of $\mathfrak{U}(\mathfrak{g})$.*

Proof. Let $(Y_i)_{1\le i\le n}$ be another basis of $\mathfrak{g}$. Then Y_i can be written as

$$Y_i = \sum_j a_{ij} X_j.$$

Let

$$h_{ij} = B(Y_i, Y_j), (h^{ij}) = (h_{ij})^{-1} \quad \text{and} \quad (a^{ij}) = (a_{ij})^{-1}.$$

Then we have

$$X_k = \sum_i a^{ki} Y_i,$$

$$h_{ij} = \sum_{k,l} a_{ik} g_{kl} a_{jl} \quad \text{and} \quad h^{ij} = \sum_{k,l} a^{ki} g^{kl} a^{lj}.$$

So we have

$$\begin{aligned} \sum_{i,j} h^{ij} Y_i Y_j &= \sum_{i,j,k,l} a^{ki} g^{kl} a^{lj} Y_i Y_j \\ &= \sum_{k,l} g^{kl} X_k X_l. \end{aligned}$$

We have proved that Ω is independent of the choice of (X_i).
Let $X^i = \sum_j g^{ij} X_j$. Then we have

$$B(X^i, X_j) = \sum_k B(g^{ik} X_k, X_j) = \sum_k g^{ik} g_{kj} = \delta^i_j$$

and $\Omega = \sum_i X_i X^i$.

Let

$$[X, X_i] = \sum_j c_{ij} X_j \quad \text{and} \quad [X, X^i] = \sum_j d_{ij} X^j.$$

Then we have

$$d_{ij} = B([X, X^i], X_j) = -B(X^i, [X, X_j]) = -c_{ji}. \tag{2.1}$$

In any associative algebra A, the identity

$$[a, bc] = [a, b]c + b[a, c] \tag{2.2}$$

holds for any a, b and c in A. By (2.1) and (2.2), we have

$$\begin{aligned}
[X, \Omega] &= [X, \sum g^{ij} X_i X_j] \\
&= \sum g^{ij} [X, X_i X_j] \\
&= \sum_{i,j} g^{ij} [X, X_i] X_j + \sum_{i,j} g^{ij} X_i [X, X_j] \\
&= \sum_i [X, X_i] X^i + \sum_i X_i [X, X^i] \\
&= \sum_{i,j} c_{ij} X_j X^i + \sum_{i,j} d_{ij} X_i X^j = 0
\end{aligned}$$

for any X in $\mathfrak{g}$. Since $\mathfrak{g}$ and 1 generate the algebra $\mathfrak{U}(\mathfrak{g})$, we have proved that the Casimir element Ω belongs to the center of $\mathfrak{U}(\mathfrak{g})$. □

CHAPTER 3
Heat kernel on S^1 and $SU(2)$

In this chapter, we establish formulas for heat kernel on S^1 and $SU(2)$.

3.1 Heat kernel of the circle

Definition 3.1. The circle group is the set of all complex number z such that $|z| = 1$ and denoted by S^1.

Theorem 3.2. *Let χ be an irreducible representation of the group $\mathbb{R}$ of real numbers. Then χ is equivalent to $\chi_\theta(t) = e^{i\theta t}$ for some real number θ. Moreover, χ_θ is equivalent to χ_ϕ if and only if $\theta = \phi$.*

Proof. Since the group $\mathbb{R}$ is commutative, an irreducible representation χ is one-dimensional. By replacing χ with an equivalent one, we can assume that the representation space of χ is $\mathbb{C}$ and that χ is a continuous homomorphism of $\mathbb{R}$ into the multiplicative group $U(1)$ of complex numbers with absolute value 1. First we show that χ is a differentiable function. The homomorphic property of χ is expressed by

$$\chi(s+t) = \chi(s)\chi(t) \tag{3.1}$$

for all s and t in $\mathbb{R}$. Integrating (3.1) with respect to t from 0 to h, we obtain

$$\int_0^h \chi(s+t)\,dt = \chi(s)\int_0^h \chi(t)\,dt \tag{3.2}$$

Since $\chi(0) = 1$ and χ is continuous,

$$\int_0^h \chi(t)\,dt \neq 0 \tag{3.3}$$

for all $h \neq 0$ with sufficiently small absolute value. On the other hand

$$\int_0^h \chi(s+t)\,dt = \int_s^{s+h} \chi(u)\,du. \tag{3.4}$$

Since χ is continuous, the right-hand-side of the last equality (3.4) is a differentiable function of s. So by (3.2), (3.3) and (3.4), we have proved that χ is differentiable. Let $\chi'(0) = c$. Then (3.1) yields

$$\chi'(s) = \lim_{t\to 0}\frac{\chi(s+t)-\chi(s)}{t} = \chi(s)\lim_{t\to 0}\frac{\chi(t)-\chi(0)}{t} = c\chi(s). \tag{3.5}$$

The function χ also satisfies

$$\chi(0) = 1.$$

The function $\chi_c(t) = e^{ct}$ also satisfies the differential equation $\chi'(s) = c\chi(s)$ and the initial condition $\chi(0) = 1$. So χ coincides with χ_c, because the function $\varphi(s) = e^{-cs}\chi(s)$ has the derivative $\varphi'(s) = -ce^{cs}\chi(s) + ce^{-cs}\chi(s) = 0$ for all $s \in \mathbb{R}$ and is thus equal to the constant $\varphi(0) = 1$. Moreover since χ is a unitary representation, $\chi(s)\overline{\chi(s)} = 1$ for all $s \in \mathbb{R}$. Differentiating the last equation at $s = 0$, we get $c + \bar{c} = 0$. Thus c is a purely imaginary number. Thus there is a real number θ such that $\chi_\theta(t) = e^{i\theta t}$. Since $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$ is commutative, χ_θ is equivalent to χ_ϕ if and only if $\chi_\theta = \chi_\phi$. If $\chi_\theta = \chi_\phi$, then we get $\theta = \phi$ by differentiating $\chi_\theta = \chi_\phi$ at $t = 0$. □

Theorem 3.3. *Any irreducible representation Π of the circle is equivalent to $\Pi_n(t) = e^{int}$ for some integer n. Moreover $\Pi_n = \Pi_m$ if and only if $n = m$.*

Proof. Since the group S^1 is commutative, an irreducible representation Π is one-dimensional. Then $\Pi : S^1 \to \mathrm{GL}(1, \mathbb{C})$ is a homomorphism. Let $\phi : \mathbb{R} \to S^1$ be defined by $\phi(x) = e^{ix}$. Since ϕ is a homomorphism , $\Pi \circ \phi : \mathbb{R} \to \mathrm{GL}(1, \mathbb{C})$ is a homomorphism and hence $\Pi \circ \phi(2\pi) = 1$. By Theorem 3.2, we have $\Pi \circ \phi(x) = e^{i\theta x}$ for some real number θ. By substituting $x = 2\pi$, we have $\Pi \circ \phi(2\pi) = e^{2i\pi\theta}$. Thus $1 = e^{2i\pi\theta}$. Therefore θ is an integer. □

Theorem 3.4. *Heat kernel of the circle is given by the following formula*

$$\rho_t(x) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx}.$$

Proof. From Theorem 3.3, we can conclude that the set of isomorphism classes of irreducible representations of S^1 is $\{\Pi_n = e^{int} \mid n \in \mathbb{Z}\}$. Hence, we can identify $\widehat{S^1}$ with $\mathbb{Z}$, the set of integers. Since the group S^1 is commutative, the irreducible representation Π_n is one-dimensional. Thus

$$\dim(V_{\Pi_n}) = 1 \quad \text{for each } n \in \mathbb{Z}.$$

Next, we find the character of the representation. Since $\chi_{\Pi_n}(x) = \mathrm{tr}(\Pi_n(x)) = \mathrm{tr}(e^{inx}) = e^{inx}$ for all $x \in S^1$,

$$\chi_{\Pi_n}(x) = e^{inx}.$$

To determine λ_π, we find eigenvalues of Δ. Note that

$$\Delta(e^{inx}) = \frac{d^2}{dx^2}e^{inx} = \frac{d}{dx}(in)e^{inx} = -n^2 e^{inx}.$$

Thus

$$\lambda_{\Pi_n} = -n^2.$$

Since $\rho_t(x) = \sum_{\pi\in\widehat{K}} \dim V_\pi e^{-\lambda_\pi t/2}\chi_\Pi(x)$, we have

$$\rho_t(x) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx}.$$

□

Theorem 3.5. *The formulas for ρ_t on S^1 can be given by*

$$\rho_t(x) = (\frac{\pi}{t})^{1/2} \sum_{n\in\mathbb{Z}} e^{-(x+2\pi n)^2/4t}. \tag{3.6}$$

Proof. Let $K_t(x) = \sum_{n\in\mathbb{Z}} e^{-(x+2\pi n)^2/4t}$. The m-th Fourier coefficients is determined by the integral for $m \in \mathbb{Z}$,

$$\begin{aligned}\int_0^{2\pi} K_t(x)e^{-imx}\,dx &= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} e^{-(x+2\pi n)^2/4t} e^{-imx}\,dx \\ &= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi(n+1)} e^{-y^2/4t} e^{-imy}\,dy,\end{aligned}$$

where we substitute $y = x + 2\pi n$ and use the fact that $e^{im2\pi n} = 1$. The last expression is

$$= \int_{-\infty}^{\infty} e^{-y^2/4t} e^{-imy}\,dy.$$

We make the substitution $y = (2t)^{1/2}u$, $dy = (2t)^{1/2}\,du$ and use a standard identity from calculus, namely

$$\int_{-\infty}^{\infty} e^{-u^2} e^{-iuv}\,du = (2\pi)^{1/2} e^{-v^2/2}.$$

We then obtain

$$\int_0^{2\pi} K_t(x)e^{-imx}dx = (\frac{t}{\pi})^{1/2} e^{-m^2 t}.$$

Hence

$$\sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx} = (\frac{\pi}{t})^{1/2} \sum_{n\in\mathbb{Z}} e^{-(x+2\pi n)^2/4t}.$$

□

3.2 Heat kernel of the group SU(2)

The group SU(2) consists of all 2×2 matrices X satisfying

$$X^* = X^{-1} \quad \text{and} \quad \det X = 1.$$

Consider the space V_m of homogeneous polynomials in two complex variables with total degree $m\,(m \geq 0)$; that is, V_m is the space of functions of the form

$$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \cdots + a_m z_2^m \tag{3.7}$$

with $z_1, z_2 \in \mathbb{C}$ and a_i's arbitrary complex constants. The space V_m is an $(m+1)$-dimensional complex vector space.

Now, by definition, an element U of SU(2) is a linear map on $\mathbb{C}^2$. Let z denote the pair $z = (z_1, z_2)$ in $\mathbb{C}^2$. Then, we may define a linear map $\Pi_m(U)$ on the space V_m by the formula

$$[\Pi_m(U)f](z) = f(U^{-1}z). \tag{3.8}$$

Explicitly, if f is as in (3.7), then

$$[\Pi_m(U)f](z_1, z_2) = \sum_{k=0}^{m} a_k (U_{11}^{-1} z_1 + U_{12}^{-1} z_2)^{m-k} (U_{21}^{-1} z_1 + U_{22}^{-1} z_2)^k.$$

By expanding out the right-hand side of this formula, we see that $\Pi_m(U)f$ is again a homogeneous polynomial of degree m. Thus, $\Pi_m(U)$ actually maps V_m into V_m.

Now compute

$$\begin{aligned}\Pi_m(U_1)[\Pi_m(U_2)f](z) &= [\Pi_m(U_2)f](U_1^{-1}z)\\ &= f(U_2^{-1}U_1^{-1}z)\\ &= \Pi_m(U_1U_2)f(z).\end{aligned}$$

Thus, Π_m is a (finite-dimensional complex) representation of SU(2).

Let us now compute the corresponding Lie algebra representation π_m. Then π_m can be computed as

$$\pi_m(X) = \frac{d}{dt}\Pi_m(e^{tX})|_{t=0}.$$

Hence,

$$(\pi_m(X)f)(z) = \frac{d}{dt} f(e^{-tX}z)|_{t=0}.$$

Now, let $z(t)$ be the curve in $\mathbb{C}^2$ defined as $z(t) = e^{-tX}z$, so that $z(0) = z$. Of course, $z(t)$ can be written as $z(t) = (z_1(t), z_2(t))$, with $z_i(t) \in \mathbb{C}$. By the chain rule,

$$\pi_m(X)f = \frac{\partial f}{\partial z_1}\frac{dz_1}{dt}|_{t=0} + \frac{\partial f}{\partial z_2}\frac{dz_2}{dt}|_{t=0}.$$

However, $dz/dt|_{t=0} = -Xz$, so we obtain the following formula for $\pi_m(X)$:

$$\pi_m(X)f = -\frac{\partial f}{\partial z_1}(X_{11}z_1 + X_{12}z_2) - \frac{\partial f}{\partial z_2}(X_{21}z_1 + X_{22}z_2). \tag{3.9}$$

Now, according to Proposition 2.23, every finite-dimensional complex representation of Lie algebra $\mathfrak{su}(2)$ extends uniquely to a complex-linear representation of the complexification of $\mathfrak{su}(2)$. However, the complexification of $\mathfrak{su}(2)$ is $\mathfrak{sl}(2;\mathbb{C})$. The representation π_m of $\mathfrak{su}(2)$ given by (3.9) thus extends to representation of $\mathfrak{sl}(2,\mathbb{C})$, which we will also call π_m and which is also given by (3.9).

So, for example, consider the element

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Applying formula (3.9) gives

$$(\pi_m(H)f)(z) = -\frac{\partial f}{\partial z_1}z_1 + \frac{\partial f}{\partial z_2}z_2.$$

Thus, we see that

$$\pi_m(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}. \tag{3.10}$$

Applying $\pi_m(H)$ to a basis element $z_1^k z_2^{m-k}$, we get

$$\begin{aligned}\pi_m(H)z_1^k z_2^{m-k} &= -kz_1^k z_2^{m-k} + (m-k)z_1^k z_2^{m-k} \\ &= (m-2k)z_1^k z_2^{m-k}.\end{aligned}$$

Thus, $z_1^k z_2^{m-k}$ is an eigenvector for $\pi_m(H)$ with eigenvalue $m-2k$. In particular, $\pi_m(H)$ is diagonalizable. Let X and Y be the elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in $\mathfrak{sl}(2,\mathbb{C})$. Then, (3.9) tells us that

$$\pi_m(X) = -z_2\frac{\partial}{\partial z_1}, \pi_m(Y) = -z_1\frac{\partial}{\partial z_2}$$

so that

$$\pi_m(X)z_1^k z_2^{m-k} = -kz_1^{k-1}z_2^{m-k+1} \tag{3.11}$$

$$\pi_m(Y)z_1^k z_2^{m-k} = (k-m)z_1^{k+1}z_2^{m-k+1}. \tag{3.12}$$

Theorem 3.6. *The representation π_m is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.*

Proof. It suffices to show that every nonzero invariant subspace of V_m is, in fact, equal to V_m. So, let W be such a space. Since W is assumed nonzero, there is at least one nonzero element w in W. Then, w can be written uniquely in the form

$$w = a_0 z_1^m + a_1 z_1^{m-1} z_2 + a_2 z_1^{m-2} z_2^2 + \cdots + a_m z_2^m$$

with at least one of the a_k's nonzero. Let k_0 be the smallest value of k for which $a_k \neq 0$ and consider

$$\pi_m(X)^{m-k_0} w.$$

Since each application of $\pi_m(X)$ lowers the power of z_1 by 1, $\pi_m(X)^{m-k_0}$ will kill all the terms in w except $a_{k_0} z_1^{m-k_0} z^{k_0}$. On the other hand, we compute easily that

$$\pi_m(X)^{m-k_0}(z_1^{m-k_0} z_2^{k_0}) = (-1)^{m-k_0}(m-k_0)! z_2^m.$$

We see, then, that $\pi_m(X)^{m-k_0} w$ is a nonzero multiple of z_2^m. Since W is assumed invariant, W must contain z_2^m. Furthermore, it follows from (3.12) that $\pi_m(Y)^k z_2^m$ is nonzero multiple of $z_1^k z_2^{m-k}$. Therefore, W must also contain $z_1^k z_2^{m-k}$ for all $0 \leq k \leq m$. Since these elements form a basis of V_m, we see that, in fact, $W = V_m$, as desired. □

Theorem 3.7. *For each integer $m \geq 0$, there is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $m+1$. Any two irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with the same dimension are equivalent. If π is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $m+1$, then π is equivalent to the representation π_m described in (3.9)*

Proof. See [2] pp. 102-106. □

Proposition 3.8. *Any element X in $SU(2)$ is conjugate to an element*

$$H(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \textit{for some } \theta \in [0, 2\pi].$$

Proof. A unitary matrix X is transformed to a diagonal matrix H by a unitary matrix $U : X = UHU^{-1}$. If X belongs to $SU(2)$ then H also belongs to $SU(2)$ and is equal to $H(\theta)$ for some $\theta \in [0, 2\pi]$. The unitary matrix X can be taken from $SU(2)$ because we can replace U by $\lambda U (\lambda \in U(1))$ □

Any function f on a group G satisfying $f(xyx^{-1}) = f(y)$ for all x, y in G is called a **class function**. The reason of this terminology is that the set of group elements of the form xyx^{-1}, with $y \in G$ fixed and x ranging over G, is called the **conjugacy class** of y. A class function is then a function that is constant on each conjugacy class.

A character of a representation Π of $SU(2)$ is a class function. To determine $\chi_\Pi(x)$ we can identify x in $SU(2)$ with $\theta \in [0, 2\pi]$ in Proposition 3.8 for each conjugacy class of θ.

Proposition 3.9. *The character of representation Π_n which is constructed in (3.8) is given by*

$$\chi_{\Pi_n}(H(\theta)) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Proof. For $0 \leq k \leq n$, let $v_k(z_1, z_2) = z_1^k z_2^{n-k}$

For $a \in U(1)$, let $H(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Then

$$\begin{aligned}
\Pi_n H(a) v_k(z_1, z_2) &= v_k(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) \\
&= v_k(a^{-1} z_1, a z_2) \\
&= (a^{-1} z_1)^k (a z_2)^{n-k} \\
&= a^{n-2k} v_k(z_1, z_2)
\end{aligned}$$

Thus

$$\begin{aligned}
\chi_{\Pi_n}(H(\theta)) &= \mathrm{tr}(\Pi_n(H(\theta))) \\
&= \sum_{k=0}^{n} a^{n-2k} \\
&= \frac{a^n(1 - a^{-2n-2})}{1 - a^{-2}} \\
&= \frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}}; a = e^{i\theta} \\
&= \frac{\sin(n+1)\theta}{\sin\theta}
\end{aligned}$$

□

Proposition 3.10. *Let Π_n be the irreducible representation of $SU(2)$ constructed in (3.8) and $(X_i)_{1 \leq i \leq n}$ the basis of $\mathfrak{su}(2)$ given in Example 2.33. Then the differential representation of Π_n is given by*

$$\begin{aligned}
\Pi_n(X_1) v_k &= \frac{-i}{2}\{k v_{k-1} + (n-k) v_{k+1}\}, \\
\Pi_n(X_2) v_k &= \frac{1}{2}\{k v_{k-1} - (n-k) v_{k+1}\}, \\
\Pi_n(X_3) v_k &= \frac{n-2k}{2} i v_k
\end{aligned}$$

where

$$v_k(z_1, z_2) = z_1^k z_2^{n-k}; (0 \le k \le n).$$

Proof. Use (3.9), we obtain

$$\begin{aligned}
\Pi_n(X_1)v_k &= -\frac{\partial v_k}{\partial z_1}(\frac{i}{2}z_2) - \frac{\partial v_k}{\partial z_2}(\frac{i}{2}z_1) \\
&= -\frac{i}{2}kz_1^{k-1}z_2^{n-k+1} - \frac{i}{2}(n-k)z_1^{k+1}z_2^{n-k-1} \\
&= -\frac{i}{2}[kv_{k-1} + (n-k)v_{k+1}]. \\
\Pi_n(X_2)v_k &= \frac{\partial v_k}{\partial z_1}(\frac{1}{2}z_2) - \frac{\partial v_k}{\partial z_2}(\frac{1}{2}z_1) \\
&= \frac{1}{2}kz_1^{k-1}z_2^{n-k+1} - \frac{1}{2}(n-k)z_1^{k+1}z_2^{n-k-1} \\
&= \frac{1}{2}[kv_{k-1} - (n-k)v_{k+1}] \\
\Pi_n(X_3)v_k &= -\frac{\partial v_k}{\partial z_1}(\frac{i}{2}z_1) + \frac{\partial v_k}{\partial z_2}(\frac{i}{2}z_2) \\
&= -\frac{i}{2}kz_1^k z_2^{n-k} + \frac{i}{2}(n-k)z_1^k z_2^{n-k} \\
&= \frac{n-2k}{2}iv_k.
\end{aligned}$$

□

Proposition 3.11. *In the representation* Π_n, $\Pi_n(\Omega)$ *is a scalar transformation given by*

$$\Pi_n(\Omega) = \frac{1}{8}n(n+2)I$$

Proof. The Casimir element Ω of $\mathfrak{su}(2)$ is given by Example 2.35. By Proposition 3.10, we have for any $k \in \{0, \ldots, n\}$.

$$\begin{aligned}
\Pi_n(\Omega)v_k &= \frac{-1}{2}\Big[\Pi_n(X_1)\Big\{\frac{-i}{2}(kv_{k-1} + (n-k)v_{k+1})\Big\} \\
&\quad + \Pi_n(X_2)\Big\{\frac{1}{2}(kv_{k-1} - (n-k)v_{k+1})\Big\} + \frac{n-2k}{2}i\Pi_n(X_3)v_k\Big] \\
&= \frac{1}{8}\Big\{2k(n-k+1) + 2(n-k)(k+1) + (n-2k)^2\Big\}v_k \\
&= \frac{1}{8}n(n+2)v_k.
\end{aligned}$$

□

Theorem 3.12. *The heat kernel of* $SU(2)$ *is given by*

$$\rho_t(x) = \frac{1}{2i\sin\theta}\sum_{n=-\infty}^{\infty} ne^{-\frac{(n^2-1)t}{8}}e^{in\theta}.$$

Proof. By Theorem 3.7, we have the set of isomorphism classes of irreducible representations of $SU(2)$ is $\{\Pi_n | n \geq 0\}$. Hence, we can identify $\widehat{SU(2)}$ with $\mathbb{N} \cup \{0\}$. Moreover $\dim V_n = n+1$. From Proposition 3.11, we have $\lambda_\pi = -\frac{n(n+2)}{8}$. For each $x \in SU(2)$, we can identify x with θ in Proposition 3.8. Hence $\chi_\pi(x) = \chi_\pi(H(\theta)) = \frac{\sin(n+1)\theta}{\sin\theta}$.

Now, the formula (1.1) for $SU(2)$ can be written as follows. For any $x \in SU(2)$. x can be conjugated to an element of the form $H(\theta)$, $\theta \in [0, 2\pi]$. Since ρ_t is a class function, we have

$$\begin{aligned}
\rho_t(x) = \rho_t(H(\theta)) &= \sum_{n=0}^{\infty}(n+1)e^{-\frac{n(n+2)t}{8}}\frac{\sin(n+1)\theta}{\sin\theta} \\
&= \sum_{n=1}^{\infty} ne^{-\frac{(n^2-1)t}{8}}\frac{\sin n\theta}{\sin\theta} \\
&= \frac{1}{2i\sin\theta}\sum_{n=1}^{\infty} ne^{-\frac{(n^2-1)t}{8}}(e^{in\theta} - e^{-in\theta}) \\
&= \frac{1}{2i\sin\theta}\Big(\sum_{n=1}^{\infty} ne^{-\frac{(n^2-1)t}{8}}e^{in\theta} + \sum_{n=1}^{\infty}(-n)e^{-\frac{(n^2-1)t}{16}}e^{-in\theta}\Big) \\
&= \frac{1}{2i\sin\theta}\sum_{n=-\infty}^{\infty} ne^{-\frac{(n^2-1)t}{8}}e^{in\theta}.
\end{aligned}$$

□

Theorem 3.13. *The heat kernel on $SU(2)$ can be given by the following formula*

$$\rho_t(x) = \frac{-4\sqrt{2}}{t^{3/2}\sin\theta}\sqrt{\pi}e^{t/8}\sum_{n=-\infty}^{\infty}(\theta + 2\pi n)e^{-2(\frac{\theta+2\pi n}{\sqrt{t}})^2}.$$

Proof. Recall the Poisson summation formula. Let $\phi(x)$ be a complex-valued smooth function with rapid decay on $\mathbb{R}$; if λ, θ are real parameter, then

$$\sum_{n=-\infty}^{\infty}\phi(\lambda n)e^{in\theta} = \frac{1}{|\lambda|}\sum_{n=-\infty}^{\infty}\hat{\phi}\Big(\frac{\theta + 2\pi n}{\lambda}\Big),$$

where

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\phi(x)dx$$

is the *Fourier transform* of $\phi(x)$.

Let $\phi(x) = xe^{-x^2+t}$. Then $\hat{\phi}(\omega) = -\frac{1}{2}ie^t\sqrt{\pi}\omega e^{-\frac{1}{4}\omega^2}$. Thus

$$\sum_{n=-\infty}^{\infty} \sqrt{t}ne^{-tn^2+t}e^{in\theta} = \frac{1}{\sqrt{t}}\sum_{-\infty}^{\infty}\hat{\phi}\Big(\frac{\theta+2\pi n}{\sqrt{t}}\Big)$$

$$\sum_{n=-\infty}^{\infty} ne^{-(n^2-1)t}e^{in\theta} = \frac{1}{t}\sum_{-\infty}^{\infty}\hat{\phi}\Big(\frac{\theta+2\pi n}{\sqrt{t}}\Big).$$

Therefore

$$\begin{aligned}\frac{1}{2i\sin\theta}\sum_{n=-\infty}^{\infty} ne^{-\frac{(n^2-1)t}{16}}e^{in\theta} &= \frac{1}{2it\sin\theta}\sum_{n=-\infty}^{\infty}\hat{\phi}\Big(\frac{\theta+2\pi n}{\sqrt{t}}\Big)\\ &= \frac{1}{2it\sin\theta}\sum_{n=-\infty}^{\infty} -\frac{1}{2}ie^t\sqrt{\pi}\Big(\frac{\theta+2\pi n}{\sqrt{t}}e^{-\frac{1}{4}(\frac{\theta+2\pi n}{\sqrt{t}})^2}\Big)\\ &= \frac{-1}{4t^{3/2}\sin\theta}\sqrt{\pi}e^t\sum_{n=-\infty}^{\infty}(\theta+2\pi n)e^{-\frac{1}{4}(\frac{\theta+2\pi n}{\sqrt{t}})^2}\end{aligned}$$

Replacing t with $t/8$, we have

$$\frac{1}{2i\sin\theta}\sum_{n=-\infty}^{\infty} ne^{-\frac{(n^2-1)t}{8}}e^{in\theta} = \frac{-4\sqrt{2}}{t^{3/2}\sin\theta}\sqrt{\pi}e^{t/8}\sum_{n=-\infty}^{\infty}(\theta+2\pi n)e^{-2(\frac{\theta+2\pi n}{\sqrt{t}})^2}.$$

□

CHAPTER 4
Structure Theory

In the next chapter, we will establish formulas for the heat kernel on a general compact Lie group. To do this, we need to know some structure theory of a semisimple Lie algebra. We recall relevant definitions and theorems that will be used in the next chapter.

4.1 Root space decomposition

Definition 4.1. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then a **Cartan subalgebra** of $\mathfrak{g}$ is a complex subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties:

1. for all H_1 and H_2 in $\mathfrak{h}$, $[H_1, H_2] = 0$;
2. for all X in $\mathfrak{g}$, if $[H, X] = 0$ for all H in $\mathfrak{h}$, then X is in $\mathfrak{h}$;
3. for all H in $\mathfrak{h}$, ad_H is diagonalizable.

Definition 4.2. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then a compact real form of $\mathfrak{g}$ is a real subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ with the property that every X in $\mathfrak{g}$ can be written uniquely as $X = X_1 + iX_2$ with X_1 and X_2 in $\mathfrak{k}$ and such that there is a compact simply-connected matrix Lie group K_1 such that the Lie algebra $\mathfrak{k}_1$ of K_1 is isomorphic to $\mathfrak{k}$.

It can be shown that a compact real form of a semisimple complex Lie algebra always exists.

Proposition 4.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{k}$ a compact real form of $\mathfrak{g}$, and $\mathfrak{t}$ a maximal commutative subalgebra of $\mathfrak{k}$. Define $\mathfrak{h} \subset \mathfrak{g}$ to be $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Definition 4.4. A **root** of $\mathfrak{g}$ (relative to the Cartan subalgebra $\mathfrak{h}$) is a nonzero linear functional α on $\mathfrak{h}$ such that there exists a nonzero element X of $\mathfrak{g}$ with

$$[H, X] = \alpha(H)X$$

for all H in $\mathfrak{h}$. The set of all roots is denoted by R.

Theorem 4.5. *If α is a root, $\alpha(H)$ is imaginary for all H in* $\mathfrak{t}$.

Definition 4.6. If α is a root, then the **root space** $\mathfrak{g}_\alpha$ is the space of all X in $\mathfrak{g}$ for which $[H, X] = \alpha(H)X$ for all H in $\mathfrak{h}$. An element of $\mathfrak{g}_\alpha$ is called a **root vector** (for the root α).

More generally, if α is any element of $\mathfrak{h}^*$, we define $\mathfrak{g}_\alpha$ to be the space of all X in $\mathfrak{g}$ for which $[H, X] = \alpha(H)X$ for all H in $\mathfrak{h}$ (but we do not call $\mathfrak{g}_\alpha$ a root space unless α is actually a root).

Theorem 4.7. *The Lie algebra $\mathfrak{g}$ can be decomposed as a direct sum as follows:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

Theorem 4.8. *For any α and β in $\mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*

Theorem 4.9. *Let α be a root of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$.*

1. *If $\alpha \in \mathfrak{h}^*$ is a root, then so is $-\alpha$.*
2. *The roots span $\mathfrak{h}^*$.*

Theorem 4.10. *Let α be a root of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$.*

1. *If α is a root, then the only multiples of α that are roots are α and $-\alpha$.*
2. *If α is a root, then the root space $\mathfrak{g}_\alpha$ is one dimensional.*
3. *For each root α we can find nonzero elements X_α in $\mathfrak{g}_\alpha$, Y_α in $\mathfrak{g}_{-\alpha}$ and H_α in $\mathfrak{h}$ such that*

$$\begin{aligned}[H_\alpha, X_\alpha] &= 2X_\alpha \\ [H_\alpha, Y_\alpha] &= -2Y_\alpha \\ [X_\alpha, Y_\alpha] &= H_\alpha.\end{aligned}$$

The element H_α is unique and is called the co-roots.

Theorem 4.11. *Suppose α and β are roots and H_α is the co-root associated to root α. Then $\beta(H_\alpha)$ is an integer.*

Theorem 4.12. *Given any linear functional $\alpha \in \mathfrak{h}^*$ (not necessarily a root), there exists a unique element H^α in $\mathfrak{h}$ such that*

$$\alpha(H) = \langle H^\alpha, H \rangle$$

for all H in $\mathfrak{h}$.

From now on, we identify each root with the corresponding element of $\mathfrak{h}$ given by the previous theorem. Thus, we now regard a root α as a nonzero element of $\mathfrak{h}$ (not $\mathfrak{h}^*$).

Theorem 4.13. *Let α be a root and let H_α be the corresponding co-root. Then α and H_α are related by the formulas*

$$H_\alpha = 2\frac{\alpha}{\langle \alpha, \alpha \rangle}$$
$$\alpha = 2\frac{H_\alpha}{\langle H_\alpha, H_\alpha \rangle}.$$

Let $\mathfrak{g}$ be a complex semisimple Lie algebra given to us as a subalgebra of some $\mathfrak{gl}(n, \mathbb{C})$. We have chosen a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ and we let K be the compact subgroup of $GL(n, \mathbb{C})$ whose Lie algebra is $\mathfrak{k}$. We have chosen a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and we work with the associated Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. We have chosen an inner product on $\mathfrak{g}$ that is invariant under the adjoint action of K and that takes real values on $\mathfrak{k}$.
Consider the following two subgroups of K:

$$Z(\mathfrak{t}) = \{A \in K \mid \mathrm{Ad}_A(H) = H \text{ for all } H \text{ in } \mathfrak{t}\}$$
$$N(\mathfrak{t}) = \{A \in K \mid \mathrm{Ad}_A(H) \subset \mathfrak{t} \text{ for all } H \text{ in } \mathfrak{t}\}$$

Clearly, $Z(\mathfrak{t})$ is a subgroup of $N(\mathfrak{t})$, and it is easily seen that $Z(\mathfrak{t})$ is a normal subgroup of $N(\mathfrak{t})$.

Definition 4.14. The Weyl group for $\mathfrak{g}$ is the quotient group $W = N(\mathfrak{t})/Z(\mathfrak{t})$.

Theorem 4.15. *Under the above notation, we have*

1. *The inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$ is invariant under the action of W.*

2. *The set $R \subset \mathfrak{h}$ of roots is invariant under the action of W.*

3. *The set of co-roots is invariant under the action of W, and $w \cdot H_\alpha = H_{w \cdot \alpha}$ for all $w \in W$ and $\alpha \in R$.*

4. *The Weyl group is a finite group.*

Theorem 4.16. *For each root α, there exists an elements w_α of W such that*

$$w_\alpha \cdot \alpha = -\alpha$$

and such that

$$w_\alpha \cdot H = H$$

for all H in $\mathfrak{h}$ with $\langle \alpha, H \rangle = 0$.

Theorem 4.17. *The Weyl group W is generated by the elements w_α as α ranges over all roots.*

4.2 Root system

Theorem 4.18. *The roots form a finite set of nonzero elements of a real inner-product space $E = i\mathfrak{t} \subset \mathfrak{h}$ and have the following properties:*

1. *The roots span E.*

2. *If α is a root, then $-\alpha$ is a root and the only multiples of α that are roots are α and $-\alpha$.*

3. *If α is a root, let w_α denote the linear transformation of E given by*

$$w_\alpha \cdot \beta = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

   Then for all roots α and β, $w_\alpha \cdot \beta$ is also a root.

4. *If α and β are roots, the the quantity*

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$

   is an integer.

Any collection of vectors in a finite-dimensional real inner product space having these properties is called a **root system**.

Definition 4.19. Suppose that E is a finite-dimensional real inner product space and that $R \subset E$ is a root system. Then a **base** for R is a subset $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of R such that Δ forms a basis for E as a vector space and such that for each $\alpha \in R$, we have

$$\alpha = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r,$$

where the n_j's are integers and either all greater than or equal to zero or all less than or equal to zero.

Once a base Δ has been chosen, the α's for which $n_j \geq 0$ are called the **positive root** (with respect to the given choice of Δ) and the α's with $n_j \leq 0$ are called the **negative roots**. The set of all positive roots is denoted by R^+. The elements of Δ are called the **positive simple roots**.

Theorem 4.20. *For any root system, a base exists.*

Definition 4.21. For each $\alpha \in R$, let P_α denote the hyperplane perpendicular to α. The hyperplane P_α partition E into finitely many regions. An **open Weyl chamber** in E (relative to R) is a connected component of the set

$$E - \bigcup_{\alpha \in R} P_\alpha.$$

The set $P = \bigcup_{\alpha \in R} P_\alpha$ is called the **walls** of all Weyl chambers. An element in P is called **singular**, otherwise it is called **regular**.

Definition 4.22. The set of $\mu \in E$ such that $\langle \mu, \alpha \rangle \geq 0$ for all positive simple roots α is called the **closed fundamental Weyl chamber** relative to the given set of positive simple roots. The set of $\mu \in E$ such that $\langle \mu, \alpha \rangle > 0$ for all positive simple roots α is called the **open fundamental Weyl chamber** relative to the given set of positive simple roots.

4.3 Integral and dominant integral elements

Definition 4.23. An element μ of $\mathfrak{h}$ is called an **integral element** if $\langle \mu, H_\alpha \rangle$ is an integer for each root α.

Theorem 4.24. *The set of integral elements is invariant under the action of the Weyl group.*

Theorem 4.25. *If μ is an element of $\mathfrak{h}$ for which $\langle \mu, H_\alpha \rangle$ is an integer for all positive simple roots α, then $\langle \mu, H_\alpha \rangle$ is an integer for all roots α, and thus, μ is an integral element.*

Theorem 4.26. *An element μ of $\mathfrak{h}$ is integral if and only if*

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is an integer for each positive simple root α.

Corollary 4.27. *Every root is an integral element.*

Definition 4.28. An element μ of $\mathfrak{h}$ is called a **dominant integral element** if $\langle \mu, H_\alpha \rangle$ is a non-negative integer for each positive simple root α. Equivalently μ is a dominant integral element if

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is a non-negative integer for each positive simple root α.

Moreover, we say that μ is **strictly dominant integral element** if $\langle \mu, H_\alpha \rangle$ is a positive integer for each positive simple root α.

4.4 Theorem of the highest weight

Definition 4.29. Suppose π is a finite-dimensional representation of $\mathfrak{g}$ on a vector space V. Then $\mu \in \mathfrak{h}$ is called a **weight** for π if there exists a nonzero vector v in V such that

$$\pi(H)v = \langle \mu, H \rangle v \tag{4.1}$$

for all $H \in \mathfrak{h}$. A nonzero vector v satisfying (4.1) is called a **weight vector** for the weight μ, and the set of all vectors satisfying (4.1) (zero or nonzero) is called the **weight space** with weight μ. The dimension of the weight space is called the **multiplicity** of the weight.

Definition 4.30. Let μ_1 and μ_2 be two elements of $\mathfrak{h}$. Then μ_1 is **higher than** μ_2 (or, equivalently, μ_2 is **lower than** μ_1) if there exists non-negative real numbers $a_1, \ldots, a_r$ such that

$$\mu_1 - \mu_2 = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_r\alpha_r.$$

where $\{\alpha_1, \ldots, \alpha_r\} = \Delta$ is the set of positive simple roots. This relationship is written as $\mu_1 \succeq \mu_2$ or $\mu_2 \preceq \mu_1$.
If π is a representation of $\mathfrak{g}$, then a weight μ_0 for π is said to be a **highest weight** if for all weights μ for π, $\mu \preceq \mu_0$.

Theorem 4.31 (Theorem of the Hightest Weight). *In a semisimple Lie algebra, the following statements hold.*

1. *Every irreducible representation has a highest weight.*
2. *Two irreducible representations with the same highest weight are equivalent.*
3. *The hightest weight of every irreducible representation is a dominant integral element.*
4. *Every dominant integral element occurs as the highest weight of an irreducible representation.*

4.5 Weyl character formula

Theorem 4.32. *If K is simply connected and λ is a (real) integral element, then there exists a function f on T satisfying*

$$f(e^H) = e^{i\lambda(H)} \tag{4.2}$$

for all H in $\mathfrak{t}$, where T is the connected Lie subgroup of K whose Lie algebra is $\mathfrak{t}$.

We note that T is commutative since $\mathfrak{t}$ is. Also, T is connected and compact. In fact, T is called a maximal torus of K. We have that the exponential map $\exp : \mathfrak{t} \to T$ is surjective. So every $t \in T$ can be written as $t = e^H$ for some $H \in \mathfrak{t}$. Now we define Γ to be the kernel of this map, that is,

$$\Gamma = \{H \in \mathfrak{t} \mid \exp H = I\}.$$

Let H_1 and H_2 in $\mathfrak{t}$ be such that $e^{H_1} = e^{H_2}$. Since $\mathfrak{t}$ is commutative, $e^{H_1 - H_2} = I$ and hence $H_1 - H_2 \in \Gamma$. This means every $t \in T$ can be written as $t = e^H$ where H is unique up to adding on an element of Γ. Now, back to equation (4.2), if we choose H_1 and H_2 so that $e^{H_1} = e^{H_2} = t$, then $f(t) = e^{i\lambda(H_1)}$ and concurrently $f(t) = e^{i\lambda(H_2)}$. We note that $H_2 = H_1 + \phi$ for some $\phi \in \Gamma$. So the function is well-defined if $e^{i\lambda(H_1)} = e^{i\lambda(H_2)} = e^{i\lambda(H_1)}e^{i\lambda(\phi)}$. This will hold precisely if $\lambda(\phi)$ is an integer multiple of 2π. So, the function in (4.2) is well-defined precisely if λ has the property that $\lambda(\phi)$ is an integer multiple of 2π for all $\phi \in \Gamma$. This leads to the following definition and theorem.

Definition 4.33. Let $\Lambda = \{H \in \mathfrak{t} \mid e^{2\pi H} = I\}$. An element μ of $\mathfrak{t}$ is called an **analytically integral element** if $\langle \mu, \lambda \rangle$ is an integer for all $\lambda \in \Lambda$ and denote by $\mathcal{I}$ the set of all analytically integral elements. That is

$$\mathcal{I} = \{\mu \in \mathfrak{t} \mid \langle \mu, \lambda \rangle \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

Meanwhile, we have another notion of integral element, namely that $\mu \in \mathfrak{h}$ is an integral element if $\langle \mu, H_\alpha \rangle$ is an integer for each co-root H_α. To distinguish this condition from the condition for an analytically integral element, we call μ an **algebraically integral element** if $\langle \mu, H_\alpha \rangle$ is an integer for each root α.

Theorem 4.34. *If K is simply connected, then the set of algebraically integral elements and the set of analytically integral elements are the same.*

Definition 4.35. Let Π be a finite dimensional representation of a group G. Then the function χ_Π on G defined by

$$\chi_\Pi(x) = \operatorname{tr}(\Pi(x))$$

is called the **character** of Π.

Definition 4.36. The **half sum** of positive roots is denoted by

$$\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

We summarize some properties that will be used later.

Theorem 4.37. *Let $\mathcal{D}$ be the set of dominant integral elements.*

(i) The set $\mathcal{D}+\delta$ is the set of strictly dominant integral elements.

(ii) $\bigcup_{s\in W} s\cdot\mathcal{D}=\mathcal{I}$.

(iii) $\bigcup_{s\in W} s\cdot(\mathcal{D}+\delta)=\mathcal{I}-P$, which is the set of regular integral elements.

(iv) For any $\gamma\in\mathcal{I}-P$, there exists a unique $s\in W$ and $\sigma\in\mathcal{D}$ such that $\gamma=s\cdot(\sigma+\delta)$. In other words, W acts freely on $\mathcal{I}-P$.

Theorem 4.38. *For any $H\in\mathfrak{t}$, there exists a unique μ in the closed fundamental Weyl chamber and (not necessarily unique) $s\in W$ such that $H=s\cdot\mu$, i.e., $s^{-1}\cdot H=\mu$.*

Theorem 4.39. *Let R be a root system in E, consider the function $\pi:E\to\mathbb{R}$ given by*

$$\pi(H)=\prod_{\alpha\in R^+}\langle\alpha,H\rangle.$$

Then $\pi(s\cdot H)=\det(s)\pi(H)$.

Theorem 4.40 (Weyl Character Formula). *If Σ is an irreducible representation of K with highest real weight μ, then we have*

$$\chi_\Sigma(e^H)=\frac{\sum_{w\in W}\det(w)e^{i\langle w\cdot(\mu+\delta),H\rangle}}{\sum_{w\in W}\det(w)e^{i\langle w\cdot\delta,H\rangle}} \tag{4.3}$$

for all H in $\mathfrak{t}$ for which the denominator of the right-hand side of (4.3) is nonzero. Here, δ denotes half the sum of the positive real roots.

The denominator of the above formula is called **Weyl denominator** and denoted by $j(H)$. That is

$$j(H)=\sum_{w\in W}\det(w)e^{i\langle w\cdot\alpha,H\rangle} \tag{4.4}$$

Theorem 4.41 (Weyl Dimension Formula). *Suppose that π is an irreducible representation of $\mathfrak{g}$ with highest weight μ. Then the dimension of π is given by*

$$\dim\pi=\frac{\prod_{\alpha\in R^+}\langle\alpha,\mu+\delta\rangle}{\prod_{\alpha\in R^+}\langle\alpha,\delta\rangle}=\frac{\pi(\mu+\delta)}{\pi(\delta)} \tag{4.5}$$

CHAPTER 5
Heat kernel on a compact Lie group

Let K be a compact simply-connected Lie group, with Lie algebra $\mathfrak{k}$. Fix a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and we work with the associated Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. Choose an inner product $\langle \, , \rangle$ on $\mathfrak{k}$ that is invariant under the adjoint action of K. Let R be a root system of $i\mathfrak{t}$ which has a basis $\{\alpha_1, \ldots, \alpha_r\}$ where $r = \dim \mathfrak{t}$ and let R^+ be the set of positive roots. Let $n = \dim \mathfrak{k}$, $l = \dim \mathfrak{h}$ and m the number of elements of R^+. Then $n = l + 2m$. Let W be the Weyl group and $\mathcal{D}$ the set of dominant integral elements. Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis of $\mathfrak{k}$, where we view the X_i's as left-invariant vector fields on K. Let $\Delta = \sum_{i=1}^{n} X_i^2$ be a Casimir operator on $\mathfrak{k}$.

Proposition 5.1. *Let π be a representation of K acting on some vector space V_π. Then $\pi(\Delta) = \langle \lambda, \lambda + 2\delta \rangle I$, where λ is a highest weight for π.*

Proof. Since the Casimir operator Δ belongs to the center of $\mathfrak{U}(\mathfrak{g})$, $\pi(\Delta)$ is a scalar operator $-\lambda_\pi I$ by Schur's lemma. The scalar c is determined as follows. We can choose a Weyl base E_α $(\alpha \in R)$, H_i $(1 \le i \le l)$ of $\mathfrak{g}_{\mathbb{C}}$ satisfying $\langle E_\alpha, E_{-\alpha} \rangle = 1$, $\langle H_i, H_j \rangle = \delta_{ij}$ and $E_\alpha + E_{-\alpha}, i(E_\alpha - E_{-\alpha}), H_i \in \mathfrak{g}$. Then we have

$$-\Delta = \sum_{\alpha \in R} E_{-\alpha} E_\alpha + \sum_{i=1}^{l} H_i^2 = \sum_{\alpha \in R^+} (2E_{-\alpha}E_\alpha + H_\alpha) + \sum_{i=1}^{l} H_i^2,$$

because $[E_\alpha, E_{-\alpha}] = H_\alpha$ where H_α is the element in the space $i\mathfrak{t}$ satisfying $\langle H, H_\alpha \rangle = \alpha(H)$ for every H in $\mathfrak{t}_{\mathbb{C}}$. Let $x \neq 0$ be the weight vector corresponding to the highest weight λ : $\pi(E_\alpha)x = 0$ for $\alpha \in P$, $\pi(H_\alpha)x = \lambda(H_\alpha)x = -\langle \lambda, \alpha \rangle x$. Then we have

$$\begin{aligned} \lambda_\pi x = \pi(\Delta)x &= (\sum_{\alpha \in R^+} \langle \lambda, \alpha \rangle - \sum_{i=1}^{l} \lambda(H_i)^2)x \\ &= (\langle \lambda, 2\delta \rangle + \langle \lambda, \lambda \rangle)x \\ &= \langle \lambda, \lambda + 2\delta \rangle x. \end{aligned}$$

□

Proposition 5.2. *The heat kernel of compact connected Lie group is given by*

$$\rho_t(e^H) = \frac{e^{|\delta|^2 t}}{j(H)\pi(\delta)} \sum_{\gamma \in \mathcal{I}} \pi(\gamma) e^{-|\gamma|^2 t} e^{i\langle \gamma, H \rangle}$$

for all $H \in \mathfrak{t}$.

Proof. By Theorem 4.31 of the highest weight, we can identify each isomorphism class of irreducible representations with a dominant integral element. Moreover, from Proposition 5.1, we have

$$\lambda_\pi = -\langle \lambda + 2\delta, \lambda \rangle = -[\langle \lambda + \delta, \lambda + \delta \rangle - \langle \delta, \delta \rangle].$$

Thus we can rewrite the heat kernel (1.1) by

$$\rho_t(H) = \sum_{\lambda \in \mathcal{D}} d_\lambda e^{-\langle \lambda + 2\delta, \lambda \rangle t} \chi_\lambda(H)$$

Next, using the Weyl dimension formula (4.5), the Weyl character formula (4.3) and the Weyl denominator formula (4.4), we have

$$\begin{aligned}
\rho_t(e^H) &= \sum_{\lambda \in \mathcal{D}} \left(\frac{\pi(\lambda + \delta)}{\pi(\delta)} e^{-\langle \lambda + \delta, \lambda + \delta \rangle t} e^{|\delta|^2 t} \frac{\sum_{s \in W} \det(s) e^{i\langle s(\lambda + \delta), H \rangle}}{j(H)} \right) \\
&= \frac{e^{|\delta|^2 t}}{j(H)\pi(\delta)} \sum_{\lambda \in \mathcal{D}} \sum_{s \in W} \det(s) \pi(\lambda + \delta) e^{-|\lambda + \delta|^2 t} e^{i\langle s(\lambda + \delta), H \rangle} \\
&= \frac{e^{|\delta|^2 t}}{j(H)\pi(\delta)} \sum_{\lambda \in \mathcal{D}} \sum_{s \in W} \det(s) \pi(\lambda + \delta) e^{-|s(\lambda + \delta)|^2 t} e^{i\langle s(\lambda + \delta), H \rangle} \\
&= \frac{e^{|\delta|^2 t}}{j(H)\pi(\delta)} \sum_{\lambda \in \mathcal{D}} \sum_{s \in W} \pi(s(\lambda + \delta)) e^{-|s(\lambda + \delta)|^2 t} e^{i\langle s(\lambda + \delta), H \rangle} \\
&= \frac{e^{|\delta|^2 t}}{j(H)\pi(\delta)} \sum_{\gamma \in \mathcal{I}} \pi(\gamma) e^{-|\gamma|^2 t} e^{i\langle \gamma, H \rangle},
\end{aligned}$$

where the last equality follows from Theorem 4.37 and the fact that $\pi(\gamma) = 0$ for any $\gamma \in P$. □

Now for every rapidly decreasing function f on t^*, we define a Fourier transform $\hat{f}$ as follows:

$$\hat{f}(\lambda) = \frac{1}{(2\pi)^{l/2}} \int_{t^*} f(\mu) e^{-i\langle \lambda, \mu \rangle} d\mu, \quad (\lambda \in t^*)$$

where the measure $d\mu = dx_1 \cdots dx_l \quad (\mu = \sum_{i=1}^{l} x_i \lambda_i)$. For $\alpha \in R$, we define a differential operator $\partial(\alpha)$ as follows:

$$\partial(\alpha) f(\lambda) = \left[\frac{d}{ds} f(\lambda + s\alpha) \right]_{s=0}$$

for every differentiable function f on t^* and put

$$\pi(\partial) = \prod_{\alpha \in R^+} \partial(\alpha).$$

Lemma 5.3. *For every rapidly decreasing function f on t^*, we have*

$$\widehat{\pi(\partial) f}(\lambda) = i^m \pi(\lambda) \hat{f}(\lambda)$$

and

$$\pi(\partial)\hat{f}(\lambda) = (-i)^m \widehat{\pi \cdot f}(\lambda)$$

for $\lambda \in t^*$.

Proof. For $\alpha \in R^+$, using integration by parts, we have

$$\int_{t^*} \partial(\alpha) f(\mu) e^{-i\langle\lambda,\mu\rangle} d\mu = i\langle\lambda,\alpha\rangle \int_{t^*} f(\mu) e^{-i\langle\lambda,\mu\rangle} d\mu.$$

Applying this repeatedly, we have the first claim. Differentiating $\hat{f}$ under the integral sign and noticing the fact that

$$\pi(\partial) e^{-i\langle\lambda,\mu\rangle} d\mu = (-i)^m \pi(\mu) e^{-i\langle\lambda,\mu\rangle},$$

the second assertion is obtained. □

Lemma 5.4.

$$\pi(\partial) e^{-\frac{a}{2}|\lambda|^2} = (-a)^m \pi(\lambda) e^{-\frac{a}{2}|\lambda|^2}$$

where a is an arbitrary constant

Proof. This is a straight forward calculation. □

Let $e_1, \ldots, e_l$ be an orthonormal basis for $\mathfrak{t}^*$ with respect to $\langle\, , \rangle$. We will indentify $\mathfrak{t}^*$ with $\mathbb{R}^l$. Hence $\mathfrak{t}^*$ has a translation-invariant measure defined on the Borel σ-algebra which will be nomalized to coincide with Lebesgue measure on $\mathbb{R}^l$.

Lemma 5.5. *For an arbitrary constant a, put $h_a(\lambda) = e^{-\frac{a}{2}|\lambda|^2}$. Then we have*

$$\widehat{h_a}(\lambda) = \frac{c}{a^{\frac{l}{2}}} e^{-\frac{|\lambda|^2}{2a}}.$$

Proof. For $\lambda = \sum_{j=1}^{l} x_i e_i \in t^*$, we have

$$\begin{aligned}
\widehat{h_a}(\lambda) &= \frac{1}{(2\pi)^{l/2} a^{l/2}} \int_{t^*} e^{-(|\mu^2|/2)} e^{-i\langle\lambda/\sqrt{a},\mu\rangle} d\mu \\
&= \frac{1}{a^{l/2}} \prod_{j=1}^{l} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 y_j^2} e^{-i(x_j/\sqrt{a})\cdot y_j} dy_j \\
&= \frac{1}{a^{l/2}} e^{-|\lambda|^2/2a}.
\end{aligned}$$

□

Lemma 5.6. *Under the assumption of Lemma 5.5, we have*

$$\widehat{\pi \cdot h_a}(\lambda) = (-i)^m a^{-\frac{n}{2}} \pi(\lambda) e^{-\frac{-|\lambda|^2}{2a}}.$$

Proof.

$$\begin{aligned}\widehat{\pi \cdot h_a}(\lambda) &= (-a)^{-m} \widehat{\pi(\partial) h_a}(\lambda) \\ &= (-a)^{-m} (i)^m \pi(\lambda) \widehat{h_a}(\lambda) \\ &= (-i)^m a^{-n/2} \pi(\lambda) e^{-|\lambda|^2/2a}.\end{aligned}$$

□

Now, we will find another form of $\rho_t(H)$. Let g be the inverse Fourier transform of $f_t(\gamma) = \pi(\gamma)e^{-|\gamma|^2}t$. That is,

$$g_t(H) = \frac{1}{(2\pi)^{l/2}} \int_{t^*} \pi(\gamma) e^{-|\gamma|^2 t} e^{i\langle\gamma, H\rangle}\, d\gamma.$$

We quote the following Poisson's summation formula. The standard proof on $\mathbb{R}^n$ works the same way if we change the sums to be over the lattice and its dual lattice of a Euclidean space.

Theorem 5.7 (Poisson's summation formula).

$$\sum_{\gamma \in \mathcal{I}} f_t(\gamma) e^{i\langle\gamma, H\rangle} = (2\pi)^{l/2} \sum_{\mu \in \Gamma} g_t(H + \mu).$$

Theorem 5.8. *The heat kernel of compact connected Lie group is given by*

$$\rho_t(e^H) = \frac{c}{j(H)} \sum_{\mu \in \Gamma} \pi(H + \mu) e^{-\frac{1}{4t}|H+\mu|^2}$$

where $c = \dfrac{e^{|\delta|^2 t}(2\pi)^{l/2} i^m}{(2t)^{n/2}\pi(\delta)}$.

Proof. We have

$$\begin{aligned}g_t(H) &= \frac{1}{(2\pi)^{l/2}} \int_{t^*} \pi(\gamma) e^{-|\lambda|^2 t} e^{i\langle\gamma, H\rangle}\, d\gamma \\ &= \frac{1}{(2\pi)^{l/2}} \int_{t^*} \pi(\gamma) h_{2t}(\gamma) e^{i\langle\gamma, H\rangle} d\,\gamma \\ &= \frac{1}{(2\pi)^{l/2}} \int_{t^*} (\pi \cdot h_{2t})(\gamma) e^{i\langle\gamma, H\rangle} d\,\gamma \\ &= \widehat{\pi \cdot h_{2t}}(-H) \\ &= (-i)^m (2t)^{-n/2} \pi(-H) e^{-\frac{|H|^2}{4t}} \\ &= i^m (2t)^{-n/2} \pi(H) e^{-\frac{|H|^2}{4t}}.\end{aligned}$$

By Theorem 5.7, we obtain

$$\begin{aligned}\sum_{\gamma\in\mathcal{I}}\pi(\gamma)e^{-|\lambda|^2t}e^{i\langle\gamma,H\rangle} &= \sum_{\mu\in\Gamma}(2\pi)^{l/2}i^m(2t)^{-n/2}\pi(H+\mu)e^{-\frac{|H+\mu|^2}{4t}}\\ &= \frac{(2\pi)^{l/2}i^m}{(2t)^{n/2}}\sum_{\mu\in\Gamma}\pi(H+\mu)e^{-\frac{1}{4t}|H+\mu|^2}.\end{aligned}$$

By Proposition 5.2, we have

$$\begin{aligned}\rho_t(e^H) &= \frac{e^{|\delta|^2t}(2\pi)^{l/2}i^m}{j(H)(2t)^{n/2}\pi(\delta)}\sum_{\mu\in\Gamma}\pi(H+\mu)e^{-\frac{1}{4t}|H+\mu|^2}\\ &= \frac{c}{j(H)}\sum_{\mu\in\Gamma}\pi(H+\mu)e^{-\frac{1}{4t}|H+\mu|^2}\end{aligned}$$

where $c = \dfrac{e^{|\delta|^2t}(2\pi)^{l/2}i^m}{(2t)^{n/2}\pi(\delta)}$.

□

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