ปริพันธ์แบบเลอเบส์กบางรูปแบบที่ลู่เข้าอย่างไม่สัมบูรณ์

นางสาวสวรรยา ศกุนตะเสฐียร

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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SOME NONABSOLUTELY CONVERGENT LEBESGUE-TYPE INTEGRALS

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สวรรยา ศกุนตะเสฐียร : ปริพันธ์แบบเลอเบส์กบางรูปแบบที่ลู่เข้าอย่างไม่สัมบูรณ์ (SOME NONABSOLUTELY CONVERGENT LEBESGUE-TYPE INTEGRALS) อ.ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ สุวิมล ฮอลล์ อ.ที่ปรึกษาร่วม : ผู้ช่วยศาสตราจารย์ ดร. มาร์ค เอ็ดวิน ฮอลล์, 50 หน้า ISBN 974-03-0127-4

ปริพันธ์แบบเลอเบส์กเป็นเครื่องมือทั่วไปแบบหนึ่งที่มีประโยชน์มากสำหรับการหาปริพันธ์ บทนิยามของปริพันธ์แบบนี้มีความเป็นทั่วไปและปริพันธ์แบบนี้มีทฤษฎีที่พัฒนาอย่างดีแล้ว อย่างไรก็ตามปริพันธ์แบบนี้มีข้อจำกัดในการใช้บางประการ ข้อจำกัดที่สำคัญประการหนึ่งคือฟังก์ชันที่จะสามารถหาปริพันธ์แบบเลอเบส์กได้นั้นค่าปริพันธ์ของค่าสัมบูรณ์ของฟังก์ชันนั้นต้องมีค่าจำกัด แต่มีฟังก์ชันรูปแบบง่ายๆบางฟังก์ชันไม่มีคุณสมบัตินี้และมีแนวโน้มว่าฟังก์ชันเหล่านั้นน่าจะหาค่าปริพันธ์ได้ ปริพันธ์แบบรีมันน์เชิงทั่วไปสามารถแก้ปัญหานี้ได้ แต่เป็นที่น่าเสียดายว่าทฤษฎีที่มีประโยชน์สำหรับการหาปริพันธ์ของปริพันธ์แบบรีมันน์เชิงทั่วไปใช้ได้เฉพาะเมื่อหาปริพันธ์บนเซตย่อยของปริภูมิยูคลิเดียนที่มีมิติจำกัด

วิทยานิพนธ์ฉบับนี้นำเสนอปริพันธ์แบบใหม่เรียกว่าปริพันธ์แบบเลอเบส์กเชิงทั่วไปซึ่งนิยามบนปริภูมิเมเชอร์จำกัดซิกมาใดๆ และฟังก์ชันบางฟังก์ชันซึ่งปริพันธ์ของค่าสัมบูรณ์ของฟังก์ชันนั้นมีค่าไม่จำกัดสามารถหาปริพันธ์แบบนี้ได้ นิยามนี้รักษาสมบัติที่สำคัญของบทนิยามของปริพันธ์แบบเลอเบส์ก และนำเสนอแนวคิดใหม่สองแนวได้แก่ ลำดับกระจายซึ่งเป็นนัยทั่วไปของลำดับทางเดียว และการลู่เข้ากึ่งเอกรูปซึ่งเป็นรูปแบบอย่างอ่อนของการลู่เข้าเอกรูป นอกจากนี้เพื่อช่วยทำให้บทนิยามของการหาปริพันธ์แบบใหม่ง่ายต่อการเข้าใจและสะดวกต่อการนำไปใช้ จึงได้เสนอบทนิยามนามธรรมของปริพันธ์บนปริภูมิเมเชอร์ซึ่งจะเรียกว่าปริพันธ์มิวนามธรรม

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต...
สาขาวิชา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา......................................
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SAWANYA SAKUNTASATHIEN : SOME NONABSOLUTELY CONVERGENT LEBESGUE-TYPE INTEGRALS. THESIS ADVISOR : ASSISTANT PROFESSOR SUWIMON HALL . THESIS CO-ADVISOR : ASSISTANT PROFESSOR MARK E. HALL. 50pp. ISBN 974-03-0127-4.

As a general tool for integration, the Lebesgue integral is very useful. Its definition is quite general and it has a well-developed theory. However, it does have its limitations. An important limitation is that a function will be Lebesgue integrable only if its absolute value has finite integral, and there exist simple examples of functions that do not satisfy this property, yet intuition suggests they should be integrable. The generalized Riemann integral has helped to solve this problem. Unfortunately, it has a useful theory only for integration over subsets of finite-dimensional Euclidean space.

This thesis introduces a new integral, the generalized Lebesgue integral, which can be defined on any sigma-finite measure space, and allows the integration of some functions whose absolute values have infinite integrals. The definition retains some of the flavor of the definition of the Lebesgue integral, and introduces two new concepts: expanding sequences, which are a generalization of monotonic sequences, and semiuniform convergence, which is a weak form of uniform convergence. In addition, to help structure the presentation, an abstract definition of measure-based integrals, called the abstract mu-integral, is introduced.

Department **Mathematics**	Student’s signature……………………………
Field of Study **Mathematics**	Advisor’s signature…………………………...
Academic year **2001**	Co-advisor’s signature ……………………….

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CONTENT

CHAPTER I

INTRODUCTION

The history of the integral stretches back to the early days of mathematics. Over 2000 years ago Greek mathematicians used the basic idea of the Riemann integral, namely approximation by rectangles, to obtain formulas for the area of a circle and volume of a sphere. They called their technique the "method of exhaustion".

Many centuries later, Newton and Leibniz both came to the realization that this was in fact a very general idea with many applications, so they introduced a more abstract concept of integral. Their ideas were essentially what we now call the Riemann integral, but they lacked rigor.

The true theory of integration began with Riemann and Weirstrass. Riemann was the first person to give a completely rigorous definition in terms of limits, while Weierstrass introduced the key concept of uniform convergence. However, Riemann's definition has its limitations. For example, functions can only be integrated over finite intervals, and the functions themselves must be bounded. Furthermore, a function will be integrable only if its set of discontinuities has measure zero, and the convergence theorems require uniform convergence. Some of these limitations can be removed by the use of improper integrals, but this is not a complete solution, nor is it very elegant.

Since then many people have worked on this problem. Surely the biggest contribution was made by Lebesgue, whose definition allowed integration of unbounded, widely discontinuous functions over a wide variety of sets, including unbounded sets. Furthermore, the Lebesgue integral has very nice convergence properties, as illustrated by the Monotone and Dominated Convergence Theorems. The one real weakness of Lebesgue's definition is that a function taking on both positive and negative values is integrable only if the integral of its absolute value is finite. As a result, there are examples of differentiable functions f such that f' is not integrable on some intervals $[a, b]$, even though $f(b) - f(a)$ is clearly a reasonable value for $\int_{[a,b]} f' \, dm$. The generalized Riemann integral helps to solve this problem; however it has a standard definition and well-developed theory only for integration over intervals in $\mathbb{R}^n$.

In this thesis we introduce a new integral, which we call the generalized Lebesgue integral, that can be defined on any σ-finite measure space. The new integral retains some of the flavor of the Lebesgue integral, yet allows integration of some functions whose absolute values have infinite integrals.

The remainder of this thesis is organized as follows. In Chapter 2, we summarize some essential facts concerning the extended real numbers and the Lebesgue integral which will be used in the succeeding chapter.

The heart of our work is Chapter 3, which consists of 4 sections. The first section presents the definition and properties of abstract μ-integrals, which are an abstract formulation of the concept of measure-based integrals, and the definitions of absolutely and nonabsolutely convergent abstract μ-integrals. In sections 2 and 3 we introduce the definitions of expanding sequences and semiuniform convergence which are important tools to define the new integral. The last section concerns the new definition of the integral and the proof that it is an abstract μ-integral. We also give an example of a nonabsolutely convergent integral using the standard Lebesgue measure space on $\mathbb{R}$.

Chapter 4 summarizes the results of the previous chapters and discuss possible improvements and topics for further research.

CHAPTER II

PRELIMINARIES

In this chapter, we review the definition of the extended real numbers and a few essential facts concerning measure and the Lebesgue integral.

2.1 The Extended Real Number System

The extended real number system, consisting of the real numbers together with the two elements ∞ and $-\infty$, is denoted by $\bar{\mathbb{R}}$. We operate on the new elements by the following equations: for each $a \in \mathbb{R}$,

$$a \cdot \infty = \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$$

$$a \cdot (-\infty) = \begin{cases} 0 & \text{if } a = 0 \\ -\infty & \text{if } a > 0 \\ \infty & \text{if } a < 0 \end{cases}$$

$$a + \infty = \infty + a = \infty$$

$$a + (-\infty) = -\infty + a = -\infty$$

$$\infty + \infty = \infty$$

$$(-\infty) + (-\infty) = -\infty$$

$$\infty \cdot \infty = \infty$$

$$(-\infty)\cdot(-\infty)=\infty$$

$$\infty\cdot(-\infty)=-\infty$$

$$(-\infty)\cdot\infty=-\infty$$

$$\frac{a}{\pm\infty}=0.$$

The expressions $\infty-\infty$ and $\frac{\pm\infty}{\pm\infty}$ are not defined.

We extend the ordering on $\mathbb{R}$by defining $-\infty<a<\infty$ for all real numbers a. With this ordering, we can define the supremum and infimum of a subset E of the extended real numbers as follows.

1. if $E=\varnothing$, then $\sup E=-\infty$ and $\inf E=\infty$;
2. if $E\subseteq\mathbb{R}$ and E is bounded, then we define $\sup E$ and $\inf E$ as usual;
3. if $E\subseteq\mathbb{R}$ and E is not bounded above, then $\sup E=\infty$;
4. if $E\subseteq\mathbb{R}$ and E is not bounded below, then $\inf E=-\infty$;
5. if $\infty\in E$, then $\sup E=\infty$;
6. if $-\infty\in E$, then $\inf E=-\infty$;
7. if $\infty\notin E$ and $-\infty\in E$, then $\sup E=\sup\big(E\smallsetminus\{-\infty\}\big)$;
8. if $-\infty\notin E$ and $\infty\in E$, then $\inf E=\inf\big(E\smallsetminus\{\infty\}\big)$.

Consequently, every subset of the extended real numbers has a supremum and infimum. In particular, for monotonic sequences of elements of $\bar{\mathbb{R}}$:

(a) if $\{a_n\}$ is an increasing sequence, then $\lim_{n\to\infty}a_n=\sup_{n\in\mathbb{N}}a_n$; and

(b) if $\{a_n\}$ is an decreasing sequence, then $\lim_{n\to\infty}a_n=\inf_{n\in\mathbb{N}}a_n$.

2.2 Measure Theory and Integration

Let $(X, \mathscr{M}, \mu)$ be a measure space, where μ is complete (recall that every measure can be completed [2, p.29]). In this thesis we consider measurable functions on X which have range on $\bar{\mathbb{R}}$. The present section reviews some definitions and results from the theories of measure and the Lebesgue integral that will be used in the next chapter.

Definition 2.2.1. The set $E \in \mathscr{M}$ is said to be σ-finite if there exists a sequence $\{E_n\}$ of sets in $\mathscr{M}$ such that $\bigcup_{n=1}^{\infty} E_n = X$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. In particular, we say the measure space $(X, \mathscr{M}, \mu)$ is σ-finite whenever X is σ-finite.

Definition 2.2.2. Let f be a nonnegative measurable function. For any measurable set E, we define the integral of f on E with respect to μ by

$$\int_E f\,d\mu = \sup\{\textstyle\int_E s\,d\mu \mid s \text{ is a simple function with } 0 \leq s \leq f\}.$$

Theorem 2.2.3 (Lebesgue's Monotone Convergence Theorem). Let $\{f_n\}$ be a sequence of nonnegative measurable functions and $E \in \mathscr{M}$. If

(a) $f_1 \leq f_2 \leq f_3 \leq \dots$ on E, and

(b) $\lim_{n\to\infty} f_n = f$ on E,

then $\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$.

Measurable functions which have both positive and negative real values may or may not be integrable.

Definition 2.2.4. Let f be a measurable function and $E \in \mathscr{M}$. If $\int_E f^+\,d\mu < \infty$ or $\int_E f^-\,d\mu < \infty$, where f^+ and f^- are the positive and negative parts of f, respectively, then we define the integral of f on E by

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu.$$

We define $L^1(\mu)$ to be the collection of all measurable functions f on X for which $\int_X |f|\,d\mu < \infty$. The members of $L^1(\mu)$ are called Lebesgue integrable functions.

Note that since $|f| = f^+ + f^-$, $L^1(\mu)$ is the collection of all measurable functions f on X for which both $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$

Next, we consider the concept of almost everywhere. Let $P(x)$ be a property which which can be true or false for each point x in X. If $E \in \mathscr{M}$, the statement "$P(x)$ holds almost everywhere (a.e.) on E" means that there exists an $N \in \mathscr{M}$ such that $N \subseteq E$, $\mu(N) = 0$, and $P(x)$ holds at every point $x \in E \smallsetminus N$. This concept of a.e. depends on the given measure μ, and we normally write "a.e.[μ]".

In particular, if f and g are measurable functions and $\mu\Big(\{x \in X \mid f(x) \neq g(x)\}\Big) = 0$, we say that $f = g$ a.e.[μ] on X, which we may write as $f \sim g$. This is an equivalence relation. Note that if $f \sim g$, then

$$\int_E f\,d\mu = \int_E g\,d\mu.$$

for every $E \in \mathscr{M}$. Because of this, in integration theory it is not necessary to distinguish between functions that are equal almost everywhere. In particular, for some properties, if we have a measurable function f that satisfies such a property almost everywhere, then we can find another measurable function g such that $f \sim g$ and g satisfies the property everywhere. Because of this, in this thesis we will often simplify the statements of results and their proofs by stating that various properties hold everywhere rather than almost everywhere.

Theorem 2.2.5 (Lebesgue's Dominated Convergence Theorem). Let $\{f_n\}$ be a sequence of measurable functions and f a measurable function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. If there is a measurable function $g \in L^1(\mu)$ such that $|f_n| \le g$ for all $n \in \mathbb{N}$, then

(a) $f \in L^1(\mu)$,

(b) $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$, and

(c) $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

CHAPTER III

A NONABSOLUTELY CONVERGENT LEBESGUE-TYPE INTEGRAL

In this chapter, we define a new integral on an arbitrary σ-finite measure space, especially, for a function f that can be indicated its new appropriate integral and $\int_X |f|\, d\mu = \infty$. To provide a general framework for discussing measure-based integrals, we first define abstract μ-integrals, in which we are thinking of each integral as the set of integrable functions together with the integration operator. With this in hand we can defind absolutely and nonabsolutely convergent abstract μ-integrals. Next, we introduce expanding sequences and semiuniform convergence which are important tools in defining the new integral. The last step is to define the new integral and a nonabsolutely convergent abstract μ-integral.

Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, where μ is complete, and let $\mathscr{F}$ be the set of real-valued measurable functions on X. Note that $\mathscr{F}$ is an $\mathbb{R}$-vector space. Define a partial order $\leq$ on $\mathscr{F}$ by $f \leq g$ iff $f(x) \leq g(x)$ for almost all $x \in X$. Recall that if $\{f_n\}$ is a sequence of measurable functions such that $\sup_{n\in\mathbb{N}} f_n(x) < \infty$ for all $x \in X$, then $\sup_{n\in\mathbb{N}} f_n$ is also measurable.

3.1 Abstract μ-Integrals

To look for a nonabsolutely convergent integral, it is useful to have a clear idea of what an integral is. In this section, we introduce the concept of abstract μ-integral, an abstract definition of an integral based on measure.

Definition 3.1.1. An **abstract μ-integral** on X is a pair $(\mathscr{I}, \mathrm{I})$, where $\mathscr{I} \subseteq \mathscr{F}$ and $\mathrm{I} : \mathscr{I} \to \bar{\mathbb{R}}$, satisfying the following properties:

(a) For all $E \in \mathscr{M}$ we have $\chi_E \in \mathscr{I}$ and $\mathrm{I}(\chi_E) = \mu(E)$.

(b) For all $f \in \mathscr{I}$ and all $r \in \mathbb{R}$ we have $rf \in \mathscr{I}$ and $\mathrm{I}(rf) = r\mathrm{I}(f)$.

(c) If $\{f_n\}$ is a monotonically increasing sequence of nonnegative members of $\mathscr{I}$ such that $\sup_{n\in\mathbb{N}} f_n(x) < \infty$ for all $x \in X$, then $\sup_{n\in\mathbb{N}} f_n \in \mathscr{I}$ and $\mathrm{I}(\sup_{n\in\mathbb{N}} f_n) = \sup_{n\in\mathbb{N}} \mathrm{I}(f_n)$.

(d) If we let $\mathscr{H} = \{f \in \mathscr{I} \mid |\mathrm{I}(f)| < \infty\}$, then $\mathscr{H}$ is closed under addition (so that $\mathscr{H}$ is a vector subspace of $\mathscr{F}$) and for all $f, g \in \mathscr{H}$ we have $\mathrm{I}(f+g) = \mathrm{I}(f) + \mathrm{I}(g)$. Thus $\mathrm{I}|_{\mathscr{H}}$ is a linear functional on $\mathscr{H}$.

Clearly, $\mathscr{I}$ is the set of integrable functions and I is the integration operator in the above definition. Our first task is to investigate the relationship between the Lebesgue integral, and its set of integrable functions, $L^1(\mu)$, and the abstract μ-integral, $(\mathscr{I}, \mathrm{I})$. The following definitions and lemmas will also be useful in our study of a nonabsolutely convergent integral.

Definition 3.1.2. A simple function s is said to be **μ-finite** iff it can be written as $\sum_{i=1}^{n} a_i \chi_{\mathrm{E}_i}$ with $\mu(\mathrm{E}_i) < \infty$ for all $i \in \{1, 2, 3, \ldots, n\}$.

Remark 3.1.3. Note every μ-finite simple function is in $L^1(\mu)$.

The phrase "a sequence $\{E_n\}$ in $\mathscr{M}$ is increasing" means that for each $n \in \mathbb{N}$, $E_n \subseteq E_{n+1}$. We will use this phrase often in the following pages.

Lemma 3.1.4. Let E be a σ-finite measurable set. For any $f \in \mathscr{F}$ with $f \geq 0$, there is an increasing sequence $\{s_n\}$ of μ-finite simple functions such that $\sup_{n\in\mathbb{N}} s_n = f$ on E.

Proof. Let $\{E_n\}$ be an increasing sequence of sets in $\mathscr{M}$ such that $\bigcup_{n=1}^{\infty} E_n = E$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $f \in \mathscr{F}$ be such that $f \geq 0$. Define a sequence $\{k_n\}$ of natural numbers by $k_n = n + \lceil \log_2(\mu(E_n) + 1) \rceil$. We have that $\{k_n\}$ is increasing. For each $n \in \mathbb{N}$, let $s_n : X \to [0, \infty)$ be defined by

$$s_n(x) = \begin{cases} \frac{i-1}{2^{k_n}} & \text{if } x \in E_n \text{ and } \frac{i-1}{2^{k_n}} \leq f(x) < \frac{i}{2^{k_n}} \text{ for some } i \in \{1, 2, 3, \ldots, n2^{k_n}\} \\ n & \text{if } x \in E_n \text{ and } f(x) \geq n \\ 0 & \text{if } x \notin E_n \end{cases}$$

for each $x \in X$. It is clear that for each $n \in \mathbb{N}$, $0 \leq s_n \leq f$ and $s_n = \sum_{i=1}^{n2^{k_n}+1} a_{n,i} \chi_{E_{n,i}}$, where $a_{n,i} = \frac{i-1}{2^{k_n}}$ and $E_{n,i} = E_n \cap f^{-1}([\frac{i-1}{2^{k_n}}, \frac{i}{2^{k_n}}))$ for all $i \in \{1, 2, 3, \ldots, n2^{k_n}\}$, $a_{n,n2^{k_n}+1} = n$ and $E_{n,n2^{k_n}+1} = E_n \cap f^{-1}([n, \infty))$. Since $f \in \mathscr{F}$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, $E_{n,i}$ is measurable and $\mu(E_{n,i}) < \infty$ for all $i \in \{1, \ldots, n2^{k_n} + 1\}$. Thus s_n is a μ-finite simple function for all $n \in \mathbb{N}$.

We will show that $\{s_n\}$ is increasing by proving that for all $n \in \mathbb{N}$ we have $s_n(x) \leq s_{n+1}(x)$ for all $x \in X$. Let $x \in X$ and $n \in \mathbb{N}$.

Case 1. *There exists $i \in \{1, \ldots, n2^{k_n}\}$ such that $x \in E_n$ and $\frac{i-1}{2^{k_n}} \leq f(x) < \frac{i}{2^{k_n}}$.* Then $s_n(x) = \frac{i-1}{2^{k_n}}$. Let $K = 2^{k_{n+1}-k_n}$. Since $f(x) \geq \frac{i-1}{2^{k_n}} = \frac{(K(i-1)+1)-1}{2^{k_{n+1}}}$ and $f(x) < \frac{i}{2^{k_n}} \leq n < n+1$, we have that $s_{n+1}(x) \geq \frac{(K(i-1)+1)-1}{2^{k_{n+1}}} = \frac{i-1}{2^{k_n}} = s_n(x)$.

Case 2. *$x \in E_n$ and $f(x) \geq n$.* We have that $f(x) \geq n$. Thus $s_{n+1}(x) \geq \frac{n2^{k_{n+1}}}{2^{k_{n+1}}} = n = s_n(x)$.

Case 3. *$x \notin E_n$.* Then $s_n(x) = 0 \leq s_{n+1}(x)$.

From the above cases, we can conclude that $\{s_n\}$ is increasing.

We will prove that $\sup_{n\in\mathbb{N}} s_n = f$ on E. Since $s_n \leq f$ for all $n \in \mathbb{N}$, $f(x)$ is an upper bound of $\{s_n(x) \mid n \in \mathbb{N}\}$ for all $x \in E$. Let $x \in E$ and $\epsilon > 0$. Then $x \in E_n$ for some $n \in \mathbb{N}$. Since f is real-valued and $\{E_n\}$ is increasing, there exists $n_x \in \mathbb{N}$ such that $\frac{1}{2^{n_x}} < \epsilon$, $x \in E_{n_x}$ and $0 \leq f(x) < n_x$. Then there exists $i \in \{1, 2, 3, \ldots, n_x 2^{k_{n_x}}\}$ such that $\frac{i-1}{2^{k_{n_x}}} \leq f(x) < \frac{i}{2^{k_{n_x}}}$. Since $n_x \leq k_{n_x}$, $f(x) - \epsilon < f(x) - \frac{1}{2^{n_x}} \leq f(x) - \frac{1}{2^{k_{n_x}}} < \frac{i-1}{2^{k_{n_x}}} = s_{n_x}(x)$. Therefore $\sup_{n\in\mathbb{N}} s_n = f$ on E. □

Remark 3.1.5. Since X is σ-finite, by Lemma 3.1.4, for any nonnegative $f \in \mathscr{F}$ there is an increasing sequence $\{s_n\}$ of μ-finite simple functions with $\sup_{n\in\mathbb{N}} s_n = f$ on X.

Let $(\mathscr{I}, \mathrm{I})$ be an arbitrary abstract μ-integral and let $\mathscr{H} = \big\{f \in \mathscr{I} \mid |\mathrm{I}(f)| < \infty\big\}$. The next propositions concern the relationship between $L^1(\mu)$ and $(\mathscr{I}, \mathrm{I})$.

Lemma 3.1.6. If s is a μ-finite simple function, then $s \in \mathscr{I}$ and $\mathrm{I}(s) = \int_X s\,d\mu$.

Proof. Assume that s is a μ-finite simple function. Then $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $\mu(\mathrm{E}_i) < \infty$ for all $i \in \{1, 2, 3, \ldots, n\}$. Since $\chi_{E_i} \in \mathscr{I}$ and $\mu(E_i) < \infty$ for all $i \in \{1, 2, 3, \ldots, n\}$, we have that $\chi_{E_i} \in \mathscr{H}$ for all $i \in \{1, 2, 3, \ldots, n\}$. Since $\mathscr{H}$ is a vector space, $s \in \mathscr{H}$. Thus $s \in \mathscr{I}$. Moreover, by the definition of abstract μ-integrals, we have that

$$\mathrm{I}(s) = \mathrm{I}\left(\sum_{i=1}^{n} a_i \chi_{E_i}\right) = \sum_{i=1}^{n} a_i \mathrm{I}(\chi_{E_i}) = \sum_{i=1}^{n} a_i \mu(E_i) = \int_X s\,d\mu. \qquad \square$$

Proposition 3.1.7. If $f \in \mathscr{F}$ and $f \geq 0$, then $f \in \mathscr{I}$, $\mathrm{I}(f) \geq 0$, and $\mathrm{I}(f) = \int_X f\,d\mu$.

Proof. Let $f \in \mathscr{F}$ be such that $f \geq 0$. Let $\{s_n\}$ be an increasing sequence of μ-finite simple functions such that $\sup_{n\in\mathbb{N}} s_n = f$ on X. By Lemma 3.1.6, we have that $s_n \in \mathscr{I}$ for all $n \in \mathbb{N}$, so by the definition of abstract μ-integral $f = \sup_{n\in\mathbb{N}} s_n \in \mathscr{I}$. The result $\mathrm{I}(f) \geq 0$ will follow from the result $\mathrm{I}(f) =$

$\int_X f\,d\mu$, which we will now prove. Indeed, by the definition of abstract μ-integral, Lemma 3.1.6, and the Monotone Convergence Theorem we have

$$\begin{aligned}\mathrm{I}(f) &= \mathrm{I}\left(\sup\nolimits_{n\in\mathbb{N}} s_n\right)\\ &= \sup_{n\in\mathbb{N}} \mathrm{I}(s_n)\\ &= \sup_{n\in\mathbb{N}} \int_X s_n\,d\mu\\ &= \int_X \sup_{n\in\mathbb{N}} s_n\,d\mu\\ &= \int_X f\,d\mu.\end{aligned}$$

□

Proposition 3.1.8. The set $L^1(\mu) \subseteq \mathscr{I}$ and $\mathrm{I}(f) = \int_X f\,d\mu$ for all $f \in L^1(\mu)$.

Proof. Let $f \in L^1(\mu)$. By Proposition 3.1.7, $f^+, f^- \in \mathscr{I}$, $0 \le \mathrm{I}(f^+) = \int_X f^+\,d\mu$ and $0 \le \mathrm{I}(f^-) = \int_X f^-\,d\mu$. Since $f \in L^1(\mu)$, $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$, and thus $f^+, f^- \in \mathscr{H}$. By the definition of abstract μ-integral, $f = f^+ - f^- \in \mathscr{H} \subseteq \mathscr{I}$, and

$$\mathrm{I}(f) = \mathrm{I}(f^+ - f^-) = \mathrm{I}(f^+) - \mathrm{I}(f^-) = \int_X f^+\,d\mu - \int_X f^-\,d\mu = \int_X f\,d\mu.$$

Thus $L^1(\mu) \subseteq \mathscr{I}$ and $\mathrm{I}(f) = \int_X f\,d\mu$ for all $f \in L^1(\mu)$. □

Proposition 3.1.9. $L^1(\mu) = \left\{f \in \mathscr{F} \;\middle|\; \mathrm{I}(|f|) < \infty\right\}$.

Proof. For any $f \in \mathscr{F}$, $|f| \in \mathscr{F}$ and $|f| \ge 0$, so by Proposition 3.1.7, $|f| \in \mathscr{I}$ and $\mathrm{I}(|f|) = \int_X |f|\,d\mu$. Thus $\mathrm{I}(|f|) < \infty$ iff $\int_X |f|\,d\mu < \infty$, and the result follows from the definition of $L^1(\mu)$. □

Definition 3.1.10. An abstract μ-integral $(\mathscr{I}, \mathrm{I})$ is **absolutely convergent** iff $|\mathrm{I}(f)| < \infty \Rightarrow \mathrm{I}(|f|) < \infty$ for all $f \in \mathscr{I}$. We say that $(\mathscr{I}, \mathrm{I})$ is **nonabsolutely convergent** iff there exists $f \in \mathscr{I}$ such that $|\mathrm{I}(f)| < \infty$ but $\mathrm{I}(|f|) = \infty$.

The following is an example of an absolutely convergent abstract μ-integral.

Example 3.1.11. Let

$$\mathscr{I}_{L^1} = \left\{ f \in \mathscr{F} \;\middle|\; \int_X f^+ \, d\mu < \infty \text{ or} \int_X f^- \, d\mu < \infty \right\},$$

and define

$$\mathrm{I}_{L^1}(f) = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

for all $f \in \mathscr{I}_{L^1}$. Then $L^1(\mu) \subseteq \mathscr{I}_{L^1}$, and in fact it is easy to check that $L^1(\mu) = \left\{ f \in \mathscr{I}_{L^1} \;\middle|\; |\mathrm{I}_{L^1}(f)| < \infty \right\}$. We will show that $(\mathscr{I}_{L^1}, \mathrm{I}_{L^1})$ is an abstract μ-integral. Since $(\chi_E)^- = 0$ for all measurable sets E, $\int_X (\chi_E)^- \, d\mu = 0$, so $\chi_E \in \mathscr{I}_{L^1}$ and $\mathrm{I}_{L^1}(\chi_E) = \int_X (\chi_E)^+ \, d\mu = \int_X \chi_E \, d\mu = \mu(E)$ for all measurable sets E. By the above characterization of $L^1(\mu)$ and the Monotone Convergence Theorem, we have that $(\mathscr{I}_{L^1}, \mathrm{I}_{L^1})$ is an abstract μ-integral. Note that if $|\mathrm{I}_{L^1}(f)| < \infty$, then $f \in L^1(\mu)$, so $\mathrm{I}_{L^1}(|f|) < \infty$. Hence $(\mathscr{I}_{L^1}, \mathrm{I}_{L^1})$ is absolutely convergent.

The rest of this thesis is devoted to finding a nonabsolutely convergent abstract μ-integral by extending some ideas from the Lebesgue integral.

3.2 Expanding Sequences

The first key to our nonabsolutely convergent integral is the concept of an expanding sequence of functions, which is a generalization of a monotonic sequence. Since measurability plays no role in this concept, all of the following definitions are phrased in terms of real-valued functions defined on an arbitrary nonempty set, which we will denote by A.

Definition 3.2.1. Let f and g be real-valued functions defined on A. We say that f **lies inside** g iff $f^+ \leq g^+$ and $f^- \leq g^-$. We will write $f \preceq g$ to denote that f lies inside g.

Definition 3.2.2. Let $\{f_n\}$ be a sequence of real-valued functions defined on A. The sequence $\{f_n\}$ is **expanding** iff $f_n \preceq f_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 3.2.3. Let $\{s_n\}$ be a sequence of real-valued functions defined on A and f a real-valued function defined on A. If $\{s_n\}$ converges pointwise to f on A, then $\{s_n^+\}$ and $\{s_n^-\}$ converge pointwise to f^+ and f^- on A, respectively.

Proof. Assume that $\{s_n\}$ converges pointwise to f and let $x \in A$.

Case 1. $f(x) = 0$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $|s_n(x)| < \epsilon$ for each $n \geq N$. Since $s_n^+(x) \leq |s_n(x)|$ and $s_n^-(x) \leq |s_n(x)|$ for all $n \in \mathbb{N}$, we have $|s_n^+(x)| = s_n^+(x) < \epsilon$ and $|s_n^-(x)| = s_n^-(x) < \epsilon$ for all $n \geq N$. Thus $\{s_n^+(x)\}$ and $\{s_n^-(x)\}$ converge to 0. Since $f(x) = 0$, $f^+(x) = 0 = f^-(x)$. Hence $s_n^+(x)$ converges to $f^+(x)$ and $s_n^-(x)$ converges to $f^-(x)$.

Case 2. $f(x) > 0$. Then $f^+(x) = f(x) > 0$ and $f^-(x) = 0$. Since $s_n(x)$ converges to $f(x)$, there is an $N \in \mathbb{N}$ such that $s_n(x) > 0$ for each $n \geq N$. Then $s_n^+(x) = s_n(x)$ for all $n \geq N$ and $s_n^-(x) = 0$. This shows that $s_n^-(x)$ converges to $f^-(x)$. Since $s_n(x)$ converges to $f(x)$, $s_n^+(x) = s_n(x)$ converges to $f(x) = f^+(x)$.

Case 3. $f(x) < 0$. The proof of this case is similar to the proof of the case $f(x) > 0$, since $f^+(x) = 0$ and $f^-(x) = -f(x)$.

□

Definition 3.2.4. A sequence $\{f_n\}$ of real-valued functions defined on A **expands to** a real-valued function g defined on A iff $\{f_n\}$ is expanding and converges pointwise to g, which we denote by $f_n \prec g$.

Remark 3.2.5. Note that if $A \in \mathscr{M}$ then we could generalize the above definition and lemma by only requiring various properties to hold almost everywhere. However, as was pointed out at the end of the previous chapter, by redefining the functions involved on a set of measure zero, we could obtain equivalent functions such that the required properties hold everywhere, and thus the generalization has no practical effect. A more useful generalization is to restrict the various properties to a nonempty subset of A. For example, if B is a nonempty subset of A, then we can say that $f \preceq g$ on B iff $f|_B \preceq g|_B$. Definitions of a sequence expanding on B and a sequence expanding to a function on B are constructed similarly by restricting all functions involved to B.

Some properties of the last definition are described in the following lemmas.

Lemma 3.2.6. If a sequence $\{f_n\}$ of real-valued functions defined on A expands to a real-valued function g defined on A, then $f_n \preceq g$ for all $n \in \mathbb{N}$.

Proof. Let $\{f_n\}$ be a sequence of real-valued functions defined on A and g a real-valued function defined on A. Assume that $f_n \lessdot g$. For each $n \in \mathbb{N}$, since $f_n \preceq f_{n+1}$, by definition $f_n^+ \leq f_{n+1}^+$ and $f_n^- \leq f_{n+1}^-$. Thus the sequences $\{f_n^+\}$ and $\{f_n^-\}$ are nondecreasing. Since $\{f_n\}$ converges to g pointwise, by Lemma 3.2.3, $\{f_n^+\}$ converges to g^+ pointwise and f_n^- converges to g^- pointwise on A. This implies that $f_n^+ \leq g^+$ and $f_n^- \leq g^-$ for all $n \in \mathbb{N}$, which tells us that $f_n \preceq g$ for all $n \in \mathbb{N}$. □

Lemma 3.2.7. Let f be a real-valued function defined on A and $\{f_n\}$ a sequence of real-valued functions defined on A such that $f_n \lessdot f$. Let $x \in A$.

(a) If $f(x) > 0$, then $f_n(x) \geq 0$ for all $n \in \mathbb{N}$.

(b) If $f(x) = 0$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$.

(c) If $f(x) < 0$, then $f_n(x) \leq 0$ for all $n \in \mathbb{N}$.

Proof. All parts follow from Lemma 3.2.6 and the definition of the lies inside relation. □

Lemma 3.2.8. Let f be a real-valued function defined on A and $\{f_n\}$ a sequence of real-valued functions defined on A such that $f_n \lessdot f$. Then $-f_n \lessdot -f$.

Proof. Assume that $f_n \lessdot f$. Then $\{-f_n(x)\}$ converges to $-f(x)$ for all $x \in A$. It is easy to see that for each $n \in \mathbb{N}$, $(-f_n)^+ = f_n^-$ and $(-f_n)^- = f_n^+$ and thus $-f_1 \preceq -f_2 \preceq -f_3 \preceq \ldots$. It follows that $\{-f_n\}$ is expanding, so $-f_n \lessdot -f$. □

Lemma 3.2.9. Let $\{E_n\}$ be an increasing sequence in $\mathscr{M}$ such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. For all $f \in \mathscr{F}$ there exists a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \lessdot f$ on $\bigcup_{n=1}^{\infty} E_n$.

Proof. Let $f \in \mathscr{F}$ and let $E = \bigcup_{n=1}^{\infty} E_n$. By Lemma 3.1.4, there are increasing sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative μ-finite simple functions such that $\{s_n\}$ converges to f^+ pointwise on E and $\{t_n\}$ converges to f^- pointwise on E. Then $\{s_n - t_n\}$ is a sequence of μ-finite simple functions, i.e., $\{s_n - t_n\}$ is a sequence in $L^1(\mu)$. It remains to show that $s_n - t_n \lessdot f$ on E. Let $x \in E$. For each $n \in \mathbb{N}$, we have that

$$\begin{aligned}|(s_n - t_n)(x) - f(x)| &= |(s_n(x) - t_n(x)) - (f^+(x) - f^-(x))| \\ &\leq |s_n(x) - f^+(x)| + |t_n(x) - f^-(x)|.\end{aligned}$$

Since $s_n(x)$ converges to $f^+(x)$ and $t_n(x)$ converges to $f^-(x)$, this shows $|(s_n - t_n)(x) - f(x)|$ converges to 0. Thus $(s_n - t_n)(x)$ converges to $f(x)$. Finally, we have to prove that for all $n \in \mathbb{N}$, we have $(s_n - t_n)^+(x) \leq (s_{n+1} - t_{n+1})^+(x)$ and $(s_n - t_n)^-(x) \leq (s_{n+1} - t_{n+1})^-(x)$, which we will do by showing that $(s_n - t_n)^+(x) = s_n(x)$ and $(s_n - t_n)^-(x) = t_n(x)$. Let $n \in \mathbb{N}$.

Case 1. $f(x) = 0$. Then $f^+(x) = f^-(x) = 0$, so $s_n(x) = t_n(x) = 0$ as well. Hence $s_n(x) = 0 = (s_n - t_n)^+(x)$ and $t_n(x) = 0 = (s_n - t_n)^-(x)$.

Case 2. $f(x) > 0$. Then $f^+(x) = f(x)$ and $f^-(x) = 0$. This implies that $t_n(x) = 0$. Thus $(s_n - t_n)^+(x) = \max\{(s_n - t_n)(x), 0\} = s_n(x)$ and $(s_n - t_n)^-(x) = \max\{-(s_n - t_n)(x), 0\} = \max\{-s_n(x), 0\} = 0 = t_n(x)$.

Case 3. $f(x) < 0$. Then $f^+(x) = 0$ and $f^-(x) = -f(x)$. Thus $s_n(x) = 0$. As in the previous case, $(s_n - t_n)^+(x) = \max\{(s_n - t_n)(x), 0\} = \max\{-t_n(x), 0\} = 0 = s_n(x)$ and $(s_n - t_n)^-(x) = \max\{-(s_n - t_n)(x), 0\} = \max\{t_n(x), 0\} = t_n(x)$.

Thus for all $n \in \mathbb{N}$, $(s_n - t_n)^+ = s_n \leq s_{n+1} = (s_{n+1} - t_{n+1})^+$ and $(s_n - t_n)^- = t_n \leq t_{n+1} = (s_{n+1} - t_{n+1})^-$. Hence $s_n - t_n \lessdot f$ on E. $\square$

Remark 3.2.10. For any $f \in \mathscr{F}$, there exists a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \lessdot f$ by the same argument as in Remark 3.1.5.

Proposition 3.2.11. We have the following characterizations of $L^1(\mu)$ and $\mathscr{I}_{L^1}$.

$$L^1(\mu) = \Big\{ f \in \mathscr{F} \;\Big|\; \text{there exists } L \in \mathbb{R} \text{ such that } \lim_{n\to\infty} \int_X s_n \, d\mu = L \text{ for all sequences } \{s_n\} \text{ in } L^1(\mu) \text{ with the property that } s_n \lessdot f \Big\} \tag{3.1}$$

and

$$\mathscr{I}_{L^1} = \Big\{ f \in \mathscr{F} \;\Big|\; \text{there exists } L \in \bar{\mathbb{R}} \text{ such that } \lim_{n\to\infty} \int_X s_n \, d\mu = L \text{ for all sequences } \{s_n\} \text{ in } L^1(\mu) \text{ with the property that } s_n \lessdot f \Big\}. \tag{3.2}$$

Proof. Let A be the set on the right-hand side of equation 3.1 for $L^1(\mu)$ and B the set on the right-hand side of equation 3.2 for $\mathscr{I}_{L^1}$. We must show that $L^1(\mu) = A$ and $\mathscr{I}_{L^1} = B$.

First, we will show that $L^1(\mu) \subseteq A$. Let $f \in L^1(\mu)$ and let $L = \int_X f \, d\mu \in \mathbb{R}$. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ with the property that $s_n \lessdot f$. Then $\{s_n\}$ converges pointwise to f. By Lemma 3.2.6, $s_n \preceq f$ for all $n \in \mathbb{N}$, which tells us that $s_n^+ \leq f^+$ and $s_n^- \leq f^-$ for all $n \in \mathbb{N}$. Thus $|s_n| = s_n^+ + s_n^- \leq f^+ + f^- = |f|$. By the Dominated Convergence Theorem, $\lim_{n\to\infty} \int_X s_n \, d\mu = \int_X f \, d\mu = L$. This shows $f \in A$. Therefore $L^1(\mu) \subseteq A$.

Conversely, we have to show that $L^1(\mu) \supseteq A$. Let $f \in A$. Suppose that $f \notin L^1(\mu)$. Then $\int_X |f| \, d\mu = \infty$. Since $|f| = f^+ + f^-$, it must be the case that $\int_X f^+ \, d\mu = \infty$ or $\int_X f^- \, d\mu = \infty$. WLOG, assume that $\int_X f^+ \, d\mu = \infty$. By Lemma 3.1.4, there are increasing sequences $\{s_n\}$ and $\{t_n\}$ in $L^1(\mu)$ that $\{s_n\}$ converges pointwise to f^+ and $\lim_{n\to\infty} \int_X s_n \, d\mu = \int_X f^+ \, d\mu = \infty$, and $\{t_n\}$ converges pointwise to f^- and $\lim_{n\to\infty} \int_X t_n \, d\mu = \int_X f^- \, d\mu$. We can choose a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $\int_X s_{n_k} \, d\mu \geq 2 \int_X t_k \, d\mu$ for all $k \in \mathbb{N}$. Then $\{s_{n_k}\}$ is an increasing sequence of μ-finite simple functions that converges pointwise to f^+ and $\lim_{k\to\infty} \int_X s_{n_k} \, d\mu = \int_X f^+ \, d\mu = \infty$. For all $k \in \mathbb{N}$, let $u_k = s_{n_k} - t_k$. As in the proof of Lemma 3.2.9, $u_k \lessdot f$. We have that

$$\int_X u_k \, d\mu = \int_X s_{n_k} \, d\mu - \int_X t_k \, d\mu \geq \int_X s_{n_k} \, d\mu - \frac{1}{2} \int_X s_{n_k} \, d\mu = \frac{1}{2} \int_X s_{n_k} \, d\mu$$

for all $k \in \mathbb{N}$. Thus $\lim_{k\to\infty} \int_X u_k \, d\mu \geq \frac{1}{2} \lim_{k\to\infty} \int_X s_{n_k} \, d\mu = \infty$, which contradicts $f \in A$. Hence $f \in L^1(\mu)$. This proves $L^1(\mu) = A$.

We take a similar approach to proving $\mathscr{I}_{L^1} = B$, the first step being to show that $\mathscr{I}_{L^1} \subseteq B$. Let $f \in \mathscr{I}_{L^1}$. If $f \in L^1(\mu)$, then $f \in A$. By the above work, and the fact that A is clearly a subset of B, we conclude immediately that $f \in B$. Assume now that $f \in \mathscr{I}_{L^1} \smallsetminus L^1(\mu)$. WLOG, assume that $\int_X f^+ \, d\mu < \infty$. Since $f \notin L^1(\mu)$, we must have $\int_X f^- \, d\mu = \infty$. Let $L = -\infty$. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ that expands to f. Then $\{s_n^+\}$ converges pointwise to f^+ and $\{s_n^-\}$ converges pointwise to f^-. Since $s_n^+ \leq s_{n+1}^+$ and $s_n^- \leq s_{n+1}^-$ for all $n \in \mathbb{N}$, by the Monotone Convergence Theorem, $\lim_{n\to\infty} \int_X s_n^+ \, d\mu = \int_X f^+ \, d\mu < \infty$ and $\lim_{n\to\infty} \int_X s_n^- \, d\mu = \int_X f^- \, d\mu = \infty$. Then

$$\begin{aligned}
\lim_{n\to\infty} \int_X s_n \, d\mu &= \lim_{n\to\infty} \left(\int_X s_n^+ \, d\mu - \int_X s_n^- \, d\mu\right) \\
&= \lim_{n\to\infty} \int_X s_n^+ \, d\mu - \lim_{n\to\infty} \int_X s_n^- \, d\mu \\
&= \int_X f^+ \, d\mu - \int_X f^- \, d\mu \\
&= -\infty
\end{aligned}$$

$$= L.$$

Thus $f \in B$. Hence $\mathscr{I}_{L^1} \subseteq B$.

To finish the proof, we will prove that $\mathscr{I}_{L^1} \supseteq B$. Let $f \in B$ and suppose that $f \notin \mathscr{I}_{L^1}$. Then $\int_X f^+ \, d\mu = \infty$ and $\int_X f^- \, d\mu = \infty$. As usual there are increasing sequences $\{s_n\}$ and $\{t_n\}$ such that $\{s_n\}$ converges pointwise to f^+ and $\{t_n\}$ converges pointwise to f^-, and therefore $\lim_{n\to\infty} \int_X s_n \, d\mu = \int_X f^+ \, d\mu = \infty$ and $\lim_{n\to\infty} \int_X t_n \, d\mu = \int_X f^- \, d\mu = \infty$. We can choose subsequences $\{s_{n_k}\}$ and $\{t_{m_k}\}$ such that $\int_X s_{n_{2k-1}} \, d\mu \geq 2 \int_X t_{m_{2k-1}} \, d\mu$ and $\int_X t_{m_{2k}} \, d\mu \geq 2 \int_X s_{n_{2k}} \, d\mu$ for each $k \in \mathbb{N}$. Then $\{s_{n_k}\}$ and $\{t_{m_k}\}$ are increasing sequences, $\{s_{n_k}\}$ converges pointwise to f^+, $\{t_{m_k}\}$ converges pointwise to f^-, $\lim_{k\to\infty} \int_X s_{n_k} \, d\mu = \infty$, and $\lim_{k\to\infty} \int_X t_{m_k} \, d\mu = \infty$. For all $k \in \mathbb{N}$, let $u_k = s_{n_k} - t_{m_k}$. The usual argument shows that $\{u_k\}$ expands to f. However, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} \int_X u_{2k} \, d\mu &= \int_X s_{n_{2k}} \, d\mu - \int_X t_{m_{2k}} \, d\mu \\ &\leq \int_X s_{n_{2k}} \, d\mu - 2 \int_X s_{n_{2k}} \, d\mu \\ &= - \int_X s_{n_{2k}} \, d\mu \end{aligned}$$

and

$$\begin{aligned} \int_X u_{2k-1} \, d\mu &= \int_X s_{n_{2k-1}} \, d\mu - \int_X t_{m_{2k-1}} \, d\mu \\ &\geq 2 \int_X t_{m_{2k-1}} \, d\mu - \int_X t_{m_{2k-1}} \, d\mu \\ &= \int_X t_{m_{2k-1}} \, d\mu. \end{aligned}$$

Since $\lim_{k\to\infty} \int_X s_{n_{2k}} \, d\mu = \infty$, it follows that $\lim_{k\to\infty} \int_X u_{2k} \, d\mu = -\infty$. Similarly, $\lim_{k\to\infty} \int_X u_{2k-1} \, d\mu = \infty$. Thus $\{\int_X u_k \, d\mu\}$ is not convergent, which contradicts the fact that $f \in B$. Hence we must have $f \in \mathscr{I}_{L^1}$. This proves that $\mathscr{I}_{L^1} = B$. $\square$

Proposition 3.2.11 suggests we might be able to define a nonabsolutely convergent abstract μ-integral by modifying the set on the right-hand side of the equation for $\mathscr{I}_{L^1}$. This is indeed the approach we will take. However, as the following proposition shows, it will not be a trivial task.

Proposition 3.2.12. If $f \in \mathscr{F}$ and $\int_X f^+ \, d\mu = \int_X f^- \, d\mu = \infty$, then for every $L \in \bar{\mathbb{R}}$ there exists a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \lessdot f$ and $\lim_{n\to\infty} \int_X s_n \, d\mu = L$.

Proof. Let $f \in \mathscr{F}$, and assume that $\int_X f^+ \, d\mu = \int_X f^- \, d\mu = \infty$. Let $L \in \bar{\mathbb{R}}$.

Case 1. $L \geq 0$. Our first step will be to find sequences $\{t_n^*\}$ and $\{u_n^*\}$ of nonnegative functions in $L^1(\mu)$ such that $\{\sum_{n=1}^k t_n^*\}$ and $\{\sum_{n=1}^k u_n^*\}$ converge pointwise to f^+ and f^-, respectively, $\sum_{n=1}^\infty \int_X t_n^* \, d\mu = \sum_{n=1}^\infty \int_X u_n^* \, d\mu = \infty$, and $\lim_{n\to\infty} \int_X t_n^* \, d\mu = \lim_{n\to\infty} \int_X u_n^* \, d\mu = 0$. Since $f \in \mathscr{F}$, there exists a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \lessdot f$. Then $\{s_n^+\}$ converges pointwise to f^+ and $\{s_n^-\}$ converges pointwise to f^-. Let $t_1 = s_1^+$, $u_1 = s_1^-$, and for each $n \in \mathbb{N}$, let $t_{n+1} = s_{n+1}^+ - s_n^+$ and $u_{n+1} = s_{n+1}^- - s_n^-$. Then $\sum_{k=1}^n t_k = s_n^+$ and $\sum_{k=1}^n u_k = s_n^-$ for every $n \in \mathbb{N}$. In addition we have that $\int_X t_n \, d\mu \geq 0$ and $\int_X u_n \, d\mu \geq 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let p_n be a positive integer such that $p_n \geq n \int_X t_n \, d\mu$, so that $\frac{1}{p_n} \int_X t_n \, d\mu \leq \frac{1}{n}$. For each $n \in \mathbb{N}$, let $P_n = p_1 + \cdots + p_n$. For $i \in \{1, 2, \ldots, P_1\}$, let $t_i^* = \frac{1}{p_1} t_1$. Then $\sum_{i=1}^{P_1} t_i^* = t_1$. For all $n \in \mathbb{N}$ and $i \in \{\sum_{i=1}^{P_n} p_i + 1, \ldots, \sum_{i=1}^{P_{n+1}} p_i\}$, let $t_i^* = \frac{1}{p_{n+1}} t_{n+1}$, and observe that t_i^* is in $L^1(\mu)$ and $\int_X t_i^* \, d\mu = \frac{1}{p_{n+1}} \int_X t_{n+1} \, d\mu \leq \frac{1}{n+1}$. It follows from the above definitions that $\sum_{i=1}^{P_n} t_i^* = s_n^+$, and hence that $\sum_{i=1}^{P_n} \int_X t_i^* \, d\mu = \int_X s_n^+ \, d\mu$ for all $n \in \mathbb{N}$. This implies that $\{\sum_{n=1}^k t_n^*\}$ converges pointwise to f^+ and, since $\lim_{n\to\infty} \int_X s_n^+ \, d\mu = \infty$, that $\sum_{i=1}^\infty \int_X t_i^* \, d\mu = \infty$. Clearly $\lim_{n\to\infty} \int_X t_n^* \, d\mu = 0$. Similarly, we can define a sequence $\{u_n^*\}$ in $L^1(\mu)$ such that $\{\sum_{n=1}^k u_n^*\}$ converges pointwise to f^-, $\sum_{n=1}^\infty \int_X u_n^* \, d\mu = \infty$, and $\lim_{n\to\infty} \int_X u_n^* \, d\mu = 0$.

Choose an increasing sequence $\{L_n\}$ of real numbers such that $\lim_{n\to\infty} L_n = L$.

Since $\sum_{n=1}^{\infty}\int_X t_n^*\,d\mu=\infty$, we can find the smallest integer m_1 such that

$$\int_X t_1^*\,d\mu+\cdots+\int_X t_{m_1}^*\,d\mu>L_1.$$

Let $\widehat{t}_n=t_n^*$ and $\widehat{u}_n=0$ for $n=1,2,\ldots,m_1$. Since $\sum_{n=1}^{\infty}\int_X u_n^*\,d\mu=\infty$, we can find the smallest integer k_1 such that

$$\int_X t_1^*\,d\mu+\cdots+\int_X t_{m_1}^*\,d\mu-\int_X u_1^*\,d\mu-\cdots-\int_X u_{k_1}^*\,d\mu<L_1.$$

Let $\widehat{t}_n=0$ and $\widehat{u}_n=u_{n-m_1}^*$ for $n=m_1+1,\ldots,m_1+k_1$. Likewise, let m_2 be the smallest integer such that $m_2>m_1$ and

$$\int_X t_1^*\,d\mu+\cdots+\int_X t_{m_1}^*\,d\mu-\int_X u_1^*\,d\mu-\cdots-\int_X u_{k_1}^*\,d\mu$$
$$+\int_X t_{m_1+1}^*\,d\mu+\cdots+\int_X t_{m_2}^*\,d\mu>L_2.$$

Let $\widehat{t}_n=t_{n-k_1}^*$ and $\widehat{u}_n=0$ for all $n=(m_1+k_1+1),\ldots,(m_2+k_1)$. Continue this process to obtain infinite sequences $\{\widehat{t}_n\}$ and $\{\widehat{u}_n\}$. We have that $\{\widehat{t}_n\}$ is

$$(t_1^*,\ldots,t_{m_1}^*,0,\ldots,0,t_{m_1+1}^*,\ldots,t_{m_2}^*,\ldots)$$

and $\{\widehat{u}_n\}$ is

$$(0,\ldots,0,u_1^*,\ldots,u_{k_1}^*,0,\ldots,0,u_{k_1+1}^*,\ldots,u_{k_2},\ldots).$$

From the properties of the sequences $\{t_n^*\}$ and $\{u_n^*\}$, for each $n\in\mathbb{N}$ there exist $l_n,l_n'\in\mathbb{N}$ such that $s_n^+=\sum_{i=1}^{l_n}\widehat{t}_i$ and $s_n^-=\sum_{i=1}^{l_n'}\widehat{u}_i$. In particular, $\{\sum_{i=1}^{n}\widehat{t}_i\}$ converges pointwise to f^+ and $\{\sum_{i=1}^{n}\widehat{u}_i\}$ converges pointwise to f^-.

For each $n\in\mathbb{N}$, let $T_n=\sum_{k=1}^{n}\widehat{t}_k$ and $U_n=\sum_{k=1}^{n}\widehat{u}_k$. Since $\widehat{t}_k\geq 0$ and $\widehat{u}_k\geq 0$ for all $k\in\mathbb{N}$, we have that $\{T_n\}$ and $\{U_n\}$ are nonnegative increasing sequences in $L^1(\mu)$ such that $\{T_n\}$ converges pointwise to f^+ and $\{U_n\}$ converges pointwise to f^-. Let $v_n=T_n-U_n$ for all $n\in\mathbb{N}$. Then $v_n\lessdot f$, as in the proof of Lemma 3.2.9, and we have that

$$\int_X v_1\,d\mu=\int_X \widehat{t}_1\,d\mu$$

$$
\begin{aligned}
&= \int_X t_1^* \, d\mu \\
\int_X v_2 \, d\mu &= \int_X \widehat{t}_1 \, d\mu + \int_X \widehat{t}_2 \, d\mu \\
&= \int_X t_1^* \, d\mu + \int_X t_2^* \, d\mu \\
&\vdots \\
\int_X v_{m_1} \, d\mu &= \int_X \widehat{t}_1 \, d\mu + \cdots + \int_X \widehat{t}_{m_1} \, d\mu \\
&= \int_X t_1^* \, d\mu + \cdots + \int_X t_{m_1}^* \, d\mu \\
\int_X v_{m_1+1} \, d\mu &= \int_X \widehat{t}_1 \, d\mu + \cdots + \int_X \widehat{t}_{m_1} \, d\mu - \int_X \widehat{u}_{m_1+1} \, d\mu \\
&= \int_X t_1^* \, d\mu + \cdots + \int_X t_{m_1}^* \, d\mu - \int_X u_1^* \, d\mu \\
&\vdots \\
\int_X v_{m_1+k_1} \, d\mu &= \int_X \widehat{t}_1 \, d\mu + \cdots + \int_X \widehat{t}_{m_1} \, d\mu - \int_X \widehat{u}_{m_1+1} \, d\mu - \cdots - \int_X \widehat{u}_{m_1+k_1} \, d\mu \\
&= \int_X t_1^* \, d\mu + \cdots + \int_X t_{m_1}^* \, d\mu - \int_X u_1^* \, d\mu - \cdots - \int_X u_{k_1}^* \, d\mu \\
&\vdots
\end{aligned}
$$

That is, $\{\int_X v_n \, d\mu\}$ is the sequence of partial sums of the series

$$
\int_X t_1^* \, d\mu + \cdots + \int_X t_{m_1}^* \, d\mu - \int_X u_1^* \, d\mu - \cdots - \int_X u_{k_1}^* \, d\mu + \int_X t_{m_1+1}^* \, d\mu + \cdots + \int_X t_{m_2}^* \, d\mu - \cdots .
$$

Let us show that $\lim_{n\to\infty} \int_X v_n \, d\mu = L$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences of partial sums of the above series whose last terms are $\int_X t_{m_n}^* \, d\mu$ and $\int_X u_{k_n}^* \, d\mu$, respectively. By the properties of m_n and k_n, for each $n \in \mathbb{N}$ we have that $|x_n - L_n| \le \int_X t_{m_n}^* \, d\mu$ and $|y_n - L_n| \le \int_X u_{k_n}^* \, d\mu$. Since $\{\int_X t_n^* \, d\mu\}$ and $\{\int_X u_n^* \, d\mu\}$ converge to 0, and $\{L_n\}$ converges to L, we have that $\{x_n\}$ and $\{y_n\}$ converge to L. Furthermore, for each $n \in \mathbb{N}$ with $n \ge 2$, we have $y_n \le \int_X v_k \, d\mu \le x_n$ if $k \in \{m_n + k_{n-1}, \ldots, m_n + k_n\}$ and $y_n \le \int_X v_k \, d\mu \le x_{n+1}$ if $k \in \{m_n + k_n, \ldots, m_{n+1} + k_n\}$.

This shows that $\{\int_X v_n \, d\mu\}$ converges to L.

Case 2. $L < 0$. Then $-L > 0$. From Case 1, we have a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \lessdot -f$ and $\lim_{n\to\infty} \int_X s_n \, d\mu = -L$. Thus $\{-s_n\}$ expands to f and $\lim_{n\to\infty} \int_X (-s_n) \, d\mu = L$. □

3.3 Semiuniform Convergence

Let $\{E_n\}$ be an increasing sequence in $\mathscr{M}$ such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$.

Definition 3.3.1. A sequence $\{s_n\}$ of functions in $\mathscr{F}$ converges to a function f in $\mathscr{F}$ **semiuniformly with respect to $\{E_n\}$** iff there exists an increasing sequence $\{F_n\}$ in $\mathscr{M}$ such that $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_n) = 0$, with the property that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n) \cdot |s_n(x) - f(x)| < \epsilon$ holds for all $x \in F_n$.

Note that we will use the phrase "$s_n \lessdot f$ semiuniformly with respect to $\{E_n\}$" to mean that $s_n \lessdot f$ on $\bigcup_{n=1}^{\infty} E_n$ and $\{s_n\}$ converges to f semiuniformly with respect to $\{E_n\}$.

Lemma 3.3.2. Let f be a nonnegative function in $\mathscr{F}$. There is a sequence $\{t_n\}$ in $L^1(\mu)$ such that $t_n \lessdot f$ semiuniformly with respect to $\{E_n\}$.

Proof. For each $n, m \in \mathbb{N}$, let $E_{n,m} = \{x \in E_n \mid |f(x)| < m\}$. Then for each $n \in \mathbb{N}$, the sequence $\{E_{n,m}\}$ is increasing as a function of m and $E_n = \bigcup_{m=1}^{\infty} E_{m,n}$. We have that $\mu(E_n) = \mu\big(\bigcup_{m=1}^{\infty} E_{n,m}\big) = \lim_{m\to\infty} \mu(E_{n,m})$ for all $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there is an $m_n \in \mathbb{N}$ such that $\mu(E_n) - \mu(E_{n,k}) < \frac{1}{n}$ for all $k \geq m_n$. Let $l_1 = m_1$ and $l_n = \max\{m_n, l_{n-1}\} + 1$ for all $n \in \mathbb{N}$. Then $\{l_n\}$ is increasing and $\lim_{n\to\infty} l_n = \infty$. Let $\{s_n\}$ be the increasing sequence such that $\sup_{n\in\mathbb{N}} s_n = f$

defined in the proof of Lemma 3.1.4 and put $t_n = s_{l_n}$ and $F_n = E_{n,l_n}$. Then $t_n \mathrel{\substack{\nearrow \\ \searrow}} f$ on $\bigcup_{n=1}^{\infty} E_n$ and $F_n \subseteq E_n$ for all $n \in \mathbb{N}$.

Let us check that $\{F_n\}$ is increasing. Let $n \in \mathbb{N}$ and $x \in F_n$. Then $x \in E_{n,l_n}$, which means that $x \in E_n$ and $|f(x)| < l_n$. Since $\{E_n\}$ and $\{l_n\}$ are increasing, $x \in E_{n+1}$ and $|f(x)| < l_{n+1}$, which tell us $x \in E_{n+1,l_{n+1}} = F_{n+1}$. Thus $F_n \subseteq F_{n+1}$.

Next, let us check that $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$. It suffices to show that $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} F_n$. Let $x \in \bigcup_{n=1}^{\infty} E_n$. Then $x \in E_{n_\circ}$ for some $n_\circ \in \mathbb{N}$ and there is an $m \in \mathbb{N}$ such that $|f(x)| < m$. Since $\lim_{n\to\infty} l_n = \infty$, there is an $n \in \mathbb{N}$ such that $n_\circ \leq n$ and $m \leq l_n$. Then $|f(x)| < l_n$ and $x \in E_n$. Thus $x \in E_{n,l_n} = F_n \subseteq \bigcup_{n=1}^{\infty} F_n$. Hence $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$.

The third step is to show that $\lim_{n\to\infty} \mu(E_n \setminus F_n) = 0$. Since $\mu(E_n) - \mu(E_{n,l_n}) < \frac{1}{n}$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, we have that $\mu(E_n \setminus F_n) = \mu(E_n) - \mu(E_{n,ln}) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} \mu(E_n \setminus F_n) = 0$.

Finally, we will show that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n) \cdot |t_n(x) - f(x)| < \epsilon$ holds for all $x \in F_n$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $\frac{1}{2^N} < \epsilon$. Fix $n \geq N$ and $x \in F_n$, and let $p = l_n$, to simplify some of the notation. Then $x \in E_n$ and $|f(x)| < l_n = p$. There is an $i \in \{1, 2, 3, \ldots, p2^{k_p}\}$ such that $\frac{i-1}{2^{k_p}} \leq f(x) < \frac{i}{2^{k_n}}$, so $t_n(x) = \frac{i-1}{2^{k_p}}$. By definition, $k_p \geq p + \log_2(\mu(E_p) + 1)$. Thus

$$2^{k_p} \geq 2^p \cdot 2^{\log_2(\mu(E_p)+1)} = 2^p \cdot (\mu(E_p) + 1) > 2^n \mu(E_n),$$

which implies $\frac{1}{2^n} \geq \frac{\mu(E_n)}{2^{k_n}} \geq \frac{\mu(F_n)}{2^{k_n}}$. Hence we have

$$\begin{aligned}
\mu(F_n) \cdot |t_n(x) - f(x)| &= \mu(F_n) \cdot \left| \frac{i-1}{2^{k_p}} - f(x) \right| \\
&\leq \mu(F_n) \cdot \left| \frac{i-1}{2^{k_p}} - \frac{i}{2^{k_p}} \right| \\
&= \frac{\mu(F_n)}{2^{k_p}} \\
&\leq \frac{1}{2^n}
\end{aligned}$$

$$\leq \frac{1}{2^N}$$

$$< \epsilon. \qquad \square$$

Lemma 3.3.3. Let f and g be functions in $\mathscr{F}$ and $\{s_n\}$ and $\{t_n\}$ sequences in $L^1(\mu)$. If $\{s_n\}$ and $\{t_n\}$ converge to f and g semiuniformly with respect to $\{E_n\}$, respectively, then $\{s_n + t_n\}$ converges to $f+g$ semiuniformly with respect to $\{E_n\}$.

Proof. Assume that $\{s_n\}$ and $\{t_n\}$ converge to f and g semiuniformly with respect to $\{E_n\}$, respectively. There are increasing sequences $\{F_{1,n}\}$ and $\{F_{2,n}\}$ in $\mathscr{M}$ such that $F_{1,n} \subseteq E_n$ and $F_{2,n} \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_{1,n} = \bigcup_{n=1}^{\infty} F_{2,n} = \bigcup_{n=1}^{\infty} E_n$ and $\lim_{n\to\infty} \mu(E_n \setminus F_{1,n}) = \lim_{n\to\infty} \mu(E_n \setminus F_{2,n}) = 0$, with the property that for every $\epsilon > 0$, there are $N_1, N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, the inequality $\mu(F_{1,n}) \cdot |s_n(x) - f(x)| < \epsilon$ holds for all $x \in F_{1,n}$ and for all $n \in \mathbb{N}$ with $n \geq N_2$, the inequality $\mu(F_{2,n}) \cdot |t_n(x) - g(x)| < \epsilon$ holds for all $x \in F_{2,n}$. For each $n \in \mathbb{N}$ let $F_n = F_{1,n} \cap F_{2,n}$. It follows from the corresponding properties of $\{F_{1,n}\}$ and $\{F_{2,n}\}$ that $\{F_n\}$ is increasing and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$.

Let us check that $\lim_{n\to\infty} \mu(E_n \setminus F_n) = 0$. Since for all $n \in \mathbb{N}$, $E_n \setminus F_n = (E_n \setminus F_{1,n}) \cup (E_n \setminus F_{2,n})$, we have that $\mu(E_n \setminus F_n) \leq \mu(E_n \setminus F_{1,n}) + \mu(E_n \setminus F_{2,n})$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \mu(E_n \setminus F_{1,n}) = 0$ and $\lim_{n\to\infty} \mu(E_n \setminus F_{2,n}) = 0$, it follows that $\lim_{n\to\infty} \mu(E_n \setminus F_n) = 0$.

Let $\epsilon > 0$. Let N_1 and N_2 be as above for the case $\frac{\epsilon}{2}$, and choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ be such that $n \geq N$ and let $x \in F_n$. Then $x \in F_{1,n} \cap F_{2,n}$. Thus

$$\begin{aligned}
\mu(F_n) \cdot |(s_n + t_n)(x) - (f+g)(x)| &\leq \mu(F_n) \cdot \big[|s_n(x) - f(x)| + |t_n(x) - g(x)|\big] \\
&= \mu(F_n) \cdot |s_n(x) - f(x)| \\
&\quad + \mu(F_n) \cdot |t_n(x) - g(x)| \\
&\leq \mu(F_{1,n}) \cdot |s_n(x) - f(x)|
\end{aligned}$$

$$
\begin{aligned}
&+ \mu(F_{2,n}) \cdot |t_n(x) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Therefore $\{s_n + t_n\}$ converges to $f + g$ semiuniformly with respect to $\{E_n\}$. $\square$

Lemma 3.3.4. Let f be a function in $\mathscr{F}$. There is a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \prec f$ semiuniformly with respect to $\{E_n\}$.

Proof. By Lemma 3.3.2, there is a sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \prec f^+$ semiuniformly with respect to $\{E_n\}$ and a sequence $\{t_n\}$ in $L^1(\mu)$ such that $t_n \prec f^-$ semiuniformly with respect to $\{E_n\}$. By the proof of Lemma 3.1.4, $s_n - t_n \prec f$ on $\bigcup_{n=1}^{\infty} E_n$. By Lemma 3.3.3, $\{s_n - t_n\}$ converges to $f^+ - f^- = f$ semiuniformly with respect to $\{E_n\}$. $\square$

Lemma 3.3.5. Let $\{f_n\}$ be a sequence in $\mathscr{F}$ and $f \in \mathscr{F}$. If $f_n \prec f$ semiuniformly with respect to $\{E_n\}$, then $-f_n \prec -f$ semiuniformly with respect to $\{E_n\}$.

Proof. Assume $f_n \prec f$ semiuniformly with respect to $\{E_n\}$. By Lemma 3.2.8, $-f_n \prec -f$ on $\bigcup_{n=1}^{\infty} E_n$. We will show that $\{-f_n\}$ converges to $-f$ semiuniformly with respect to $\{E_n\}$.

Since $\{f_n\}$ converges to f semiuniformly with respect to $\{E_n\}$, there is an increasing sequence $\{F_n\}$ in $\mathscr{M}$ such that $F_n \subseteq E_n$for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \setminus F_n) = 0$, with the property that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n) \cdot |f_n(x) - f(x)| < \epsilon$ holds for all $x \in F_n$. Let $\epsilon > 0$ and let N be as in the previous sentence. Let $n \in \mathbb{N}$ be such that $n \geq N$, and let $x \in F_n$. Then

$$\mu(F_n) \cdot | - f_n(x) - (-f)(x)| = \mu(F_n) \cdot |f_n(x) - f(x)| < \epsilon.$$

Hence $\{-f_n\}$ converges to $-f$ semiuniformly with respect to $\{E_n\}$. $\square$

Lemma 3.3.6. Let $A \in \mathscr{M}$, $\{f_n\}$ be a sequence in $\mathscr{F}$, and $f \in \mathscr{F}$. If $\{f_n\}$ converges to f semiuniformly with respect to $\{E_n\}$, then $\{f_n \cdot \chi_A\}$ converges to $f \cdot \chi_A$ semiuniformly with respect to $\{E_n\}$.

Proof. Assume that $\{f_n\}$ converges to f semiuniformly with respect to $\{E_n\}$. Then there exists an increasing sequence $\{F_n\}$ in $\mathscr{M}$ such that $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_n) = 0$, with the property that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n) \cdot |f_n(x) - f(x)| < \epsilon$ holds for all $x \in F_n$. Let $\epsilon > 0$ and let N be as in the previous sentence. Let $n \in \mathbb{N}$ be such that $n \geq N$ and let $x \in F_n$. If $x \notin A$, then

$$\mu(F_n) \cdot |f_n \cdot \chi_A(x) - f \cdot \chi_A(x)| = \mu(F_n) \cdot 0 = 0 < \epsilon,$$

while if $x \in A$, then

$$\mu(F_n) \cdot |f_n \cdot \chi_A(x) - f \cdot \chi_A(x)| = \mu(F_n) \cdot |f_n(x) - f(x)| < \epsilon.$$

Hence $\{f_n \cdot \chi_A\}$ converges to $f \cdot \chi_A$ semiuniformly with respect to $\{E_n\}$ □

Lemma 3.3.7. Let $f, g \in \mathscr{F}$. If a sequence $\{s_n\}$ in $L^1(\mu)$ expands to $f + g$ semiuniformly with respect to $\{E_n\}$, then there exist sequences $\{f_n\}$ and $\{g_n\}$ in $L^1(\mu)$ such that $f_n \lessdot f$ and $g_n \lessdot g$ semiuniformly with respect to $\{E_n\}$, and for each $n \in \mathbb{N}$ we have $f_n + g_n = s_n$ on $\bigcup_{n=1}^{\infty} E_n$.

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$ and define the following subsets of E:

$$A = \{x \in E \mid f(x) \cdot g(x) \geq 0\},$$

$$B = \{x \in E \mid f(x) \cdot g(x) < 0\},$$

$$B_\circ = \{x \in B \mid f(x) \cdot (f+g)(x) = 0\},$$

$$B^- = \{x \in B \mid f(x) \cdot (f+g)(x) < 0\}, \quad \text{and}$$

$$B^+ = \{x \in B \mid f(x) \cdot (f+g)(x) > 0\}.$$

Let $\{p_n\}$ and $\{q_n\}$ be sequences in $L^1(\mu)$ such that $p_n \lessdot f$ and $q_n \lessdot g$ semi-uniformly with respect to $\{E_n\}$. For each $x \in X$, we define $\{f_n\}$ and $\{g_n\}$ as follows:

Case 1. $x \in X \setminus E$. Let $f_n(x) = 0$ and $g_n(x) = 0$ for all $n \in \mathbb{N}$.

Case 2. $x \in A$. If $(f+g)(x) = 0$, then let $f_n(x) = 0$ and $g_n(x) = 0$ for all $n \in \mathbb{N}$. If $(f+g)(x) \neq 0$, then let $f_n(x) = \frac{f(x)}{(f+g)(x)} \cdot s_n(x)$ and $g_n(x) = \frac{g(x)}{(f+g)(x)} \cdot s_n(x)$ for all $n \in \mathbb{N}$.

Case 3. $x \in B$. If $x \in B_\circ$, then let $f_n(x) = p_n(x)$ and $g_n(x) = -f_n(x)$ for all $n \in \mathbb{N}$. If $x \in B^-$, then let $f_n(x) = p_n(x)$ and $g_n(x) = (s_n - f_n)(x)$ for all $n \in \mathbb{N}$. If $x \in B^+$, then let $g_n(x) = q_n(x)$ and $f_n(x) = (s_n - g_n)(x)$ for all $n \in \mathbb{N}$.

Let us check that f_n and g_n are in $L^1(\mu)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and consider $\int_A |f_n|\, d\mu$. Let $A_\circ = \{x \in A \,|\, (f+g)(x) = 0\}$. Then $\int_{A_\circ} |f_n|\, d\mu = 0$. We claim that $|f_n(x)| \leq |s_n(x)|$ for all $x \in A \setminus A_\circ$. Let $x \in A \setminus A_\circ$. Note that since $f(x)$ and $g(x)$ have the same sign, $0 \leq \frac{f(x)}{(f+g)(x)} \leq 1$. Thus

$$|f_n(x)| = \left| \frac{g(x)}{(f+g)(x)} \cdot s_n(x) \right| = \frac{g(x)}{(f+g)(x)} \cdot |s_n(x)| \leq |s_n(x)|.$$

This shows that $|f_n(x)| \leq |s_n(x)|$ for all $x \in A \setminus A_\circ$. Since $s_n \in L^1(\mu)$, we have

$$\int_A |f_n|\, d\mu = \int_{A_\circ} |f_n|\, d\mu + \int_{A \setminus A_\circ} |f_n|\, d\mu = \int_{A \setminus A_\circ} |s_n|\, d\mu < \infty.$$

Similarly, $\int_A |g_n|\, d\mu < \infty$. Now consider $\int_B |f_n|\, d\mu$ and $\int_B |g_n|\, d\mu$. We have that

$$\begin{aligned} \int_B |f_n|\, d\mu &= \int_{B_\circ \cup B^-} |f_n|\, d\mu + \int_{B^+} |f_n|\, d\mu \\ &= \int_{B_\circ \cup B^-} |p_n|\, d\mu + \int_{B^+} |s_n - g_n|\, d\mu \end{aligned}$$

$$
\begin{aligned}
&\le \int_{B_\circ \cup B^-} |p_n|\, d\mu + \int_{B^+} |s_n|\, d\mu + \int_{B^+} |g_n|\, d\mu \\
&\le \int_{B_\circ \cup B^-} |p_n|\, d\mu + \int_{B^+} |s_n|\, d\mu + \int_{B^+} |q_n|\, d\mu \\
&< \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\int_B |g_n|\, d\mu &= \int_{B_\circ} |g_n|\, d\mu + \int_{B^+} |g_n|\, d\mu + \int_{B^-} |g_n|\, d\mu \\
&= \int_{B_\circ} |f_n|\, d\mu + \int_{B^+} |q_n|\, d\mu + \int_{B^-} |s_n - p_n|\, d\mu \\
&\le \int_{B_\circ} |p_n|\, d\mu + \int_{B^+} |q_n|\, d\mu + \int_{B^-} |s_n|\, d\mu + \int_{B^-} |p_n|\, d\mu \\
&< \infty.
\end{aligned}
$$

Combining these we obtain

$$
\begin{aligned}
\int_X |f_n|\, d\mu &= \int_{X \setminus E} |f_n|\, d\mu + \int_E |f_n|\, d\mu \\
&= 0 + \int_A |f_n|\, d\mu + \int_B |f_n|\, d\mu \\
&< \infty.
\end{aligned}
$$

Similarly, $\int_X |g_n|\, d\mu < \infty$. Thus f_n and g_n are in $L^1(\mu)$ for all $n \in \mathbb{N}$.

The next step is to show that $f_n \lessdot f$ and $g_n \lessdot g$ on E. Let $x \in E$.

Case 1. $x \in A$. If $(f+g)(x) = 0$, then $f(x) = 0 = g(x)$, and we have $f_n(x) = 0 = g_n(x)$ for all $n \in \mathbb{N}$. Thus for all $n \in \mathbb{N}$, $f_n^+(x) = 0 \le f_{n+1}^+(x)$, $f_n^-(x) = 0 \le f_{n+1}^-(x)$, and $\{f_n(x)\}$ converges to $f(x)$. The same argument shows that $\{g_n(x)\}$ has the corresponding properties.

If $(f+g)(x) \neq 0$, then note as before that $0 \le \frac{f(x)}{(f+g)(x)} \le 1$. We have two subcases to consider.

Subcase 1.1. $f(x) \geq 0$ *and* $g(x) \geq 0$. This implies that $s_n(x) \geq 0$ and hence that $f_n(x) \geq 0$ as well, for all $n \in \mathbb{N}$, so

$$f_n^+(x) = f_n(x) = \frac{f(x)}{(f+g)(x)} \cdot s_n(x) \leq \frac{f(x)}{(f+g)(x)} \cdot s_{n+1}(x) = f_{n+1}(x) = f_{n+1}^+(x)$$

and $f_n^-(x) = 0 \leq f_{n+1}^-(x)$ for all $n \in \mathbb{N}$. A similar argument shows that $\{g_n(x)\}$ has the corresponding properties.

Subcase 1.2. $f(x) \leq 0$ *and* $g(x) \leq 0$. Then $s_n(x) \leq 0$ for all $n \in \mathbb{N}$. Thus $f_n(x) = \frac{f(x)}{(f+g)(x)} \cdot s_n(x) \leq 0$ for all $n \in \mathbb{N}$. It follows that $f_n^+ = 0$ and

$$f_n^-(x) = -f_n(x) = \frac{f(x)}{(f+g)(x)} \cdot (-s_n(x)) = \frac{f(x)}{(f+g)(x)} \cdot s_n^-(x)$$

for all $n \in \mathbb{N}$. We have that $f_n^+(x) = 0 \leq f_{n+1}^+(x)$ and

$$f_n^-(x) = \frac{f(x)}{(f+g)(x)} \cdot s_n^-(x) \leq \frac{f(x)}{(f+g)(x)} \cdot s_{n+1}^-(x) = f_{n+1}^-(x).$$

Similarly, $\{g_n(x)\}$ has the corresponding properties.

In addition, for each $n \in \mathbb{N}$,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{f(x)}{(f+g)(x)} \cdot s_n(x) - f(x) \right| \\ &= \left| \frac{f(x)}{(f+g)(x)} \right| \cdot |s_n(x) - (f+g)(x)|. \end{aligned}$$

Since $\{s_n(x)\}$ converges to $(f+g)(x)$, this shows $\{f_n(x)\}$ converges to $f(x)$. Similarly, $\{g_n(x)\}$ converges to $g(x)$.

Case 2. $x \in B$. It is easy to check that $\{f_n(x)\}$ and $\{g_n(x)\}$ converge to $f(x)$ and $g(x)$, respectively.

If $x \in B_\circ$, then $f_n^+(x) \leq f_n^+(x)$ and $f_n^-(x) \leq f_n^-(x)$ for all $n \in \mathbb{N}$, by the definition of $\{f_n\}$ and Lemma 3.2.7. The corresponding properties hold for $\{g_n(x)\}$ by the same argument.

If $x \in B^-$, then two subcases must be considered.

Subcase 2.1. $f(x) > 0$ *and* $(f+g)(x) < 0$. Then $g(x) < 0, s_n(x) \leq 0$, and $f_n(x) = p_n(x) \geq 0$ for all $n \in \mathbb{N}$. It follows that $(s_n - f_n)(x) \leq 0$ for all $n \in \mathbb{N}$, and hence we have that $g_n^+(x) = (s_n - f_n)^+(x) = 0 \leq g_{n+1}^+(x)$ and

$$\begin{aligned} g_n^-(x) &= (s_n - f_n)^-(x) = -(s_n - f_n)(x) = -s_n(x) + f_n(x) \\ &= s_n^-(x) + f_n^+(x) \ \leq s_{n+1}^-(x) + f_{n+1}^+(x) = -s_{n+1}(x) + f_{n+1}(x) \\ &= -(s_{n+1} - f_{n+1})(x) = (s_{n+1} - f_{n+1})^-(x) = g_{n+1}^-(x). \end{aligned}$$

By the definition of $f_n(x)$, we have $f_n^+(x) \leq f_{n+1}^+(x)$ and $f_n^-(x) \leq f_{n+1}^-(x)$ for all $n \in \mathbb{N}$.

Subcase 2.2. $f(x) < 0$ *and* $(f+g)(x) > 0$. Then $g(x) > 0$, $s_n(x) \geq 0$, and $f_n(x) \leq 0$ for all $n \in \mathbb{N}$. It follows that $(s_n - f_n)(x) \geq 0$ for all $n \in \mathbb{N}$ and hence we have that, $g_n^-(x) = (s_n - f_n)^-(x) = 0 \leq g_{n+1}^-(x)$ and

$$\begin{aligned} g_n^+(x) &= (s_n - f_n)^+(x) = (s_n - f_n)(x) = s_n(x) - f_n(x) \\ &= s_n^+(x) + f_n^-(x) \leq s_{n+1}^+(x) + f_{n+1}^-(x) = s_{n+1}(x) - f_{n+1}(x) \\ &= (s_{n+1} - f_{n+1})(x) = (s_{n+1} - f_{n+1})^+(x) = g_{n+1}^+(x). \end{aligned}$$

By the definition of $f_n(x)$, we have $f_n^+(x) \leq f_{n+1}^+(x)$ and $f_n^-(x) \leq f_{n+1}^-(x)$ for all $n \in \mathbb{N}$.

If $x \in B^+$, then note that $g(x) \cdot (f+g)(x) < 0$, so the proof in this case is the same as the proof in the case $x \in B^-$, with the roles of f and g interchanged.

We now show that $\{f_n\}$ converges to f semiuniformly with respect to $\{E_n\}$, by considering each of the sets $A, B_\circ, B^+, B^-$. The proof that $\{g_n\}$ converges to g semiuniformly with respect to $\{E_n\}$ is similar, and is therefore omitted.

For the set A, let $A_\circ = \{x \in A \mid (f+g)(x) = 0\}$. Let $a \in \mathscr{F}$ be such that $a(x) = 0$ for all $x \in X$ and let $a_n(x) = 0$ for all $x \in X$ and all $n \in \mathbb{N}$. Then $\{a_n\}$

converges to a semiuniformly with respect to $\{E_n\}$. By Lemma 3.3.6, $\{a_n \cdot \chi_{A_\circ}\}$ converges to $a \cdot \chi_{A_\circ}$ semiuniformly with respect to $\{E_n\}$. Since $a_n \cdot \chi_{A_\circ} = f_n \cdot \chi_{A_\circ}$ for all $n \in \mathbb{N}$ and $a \cdot \chi_{A_\circ} = f \cdot \chi_{A_\circ}$, we have that $\{f_n \cdot \chi_{A_\circ}\}$ converges to $f \cdot \chi_{A_\circ}$ semiuniformly with respect to $\{E_n\}$.

Now consider $A \setminus A_\circ$. We claim that $\{f_n \cdot \chi_{A \setminus A_\circ}\}$ converges to $f \cdot \chi_{A_\circ}$ semiuniformly with respect to $\{E_n\}$. Since $\{s_n\}$ converges to $f+g$ semiuniformly with respect to $\{E_n\}$, there is an increasing sequence $\{F_n\}$ in $\mathscr{M}$ such that $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \setminus F_n) = 0$, with the property that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n) \cdot |s_n(x) - (f+g)(x)| < \epsilon$ holds for all $x \in F_n$. Let $\epsilon > 0$ and let N be as in the previous sentence. Let $n \in \mathbb{N}$ be such that $n \geq N$ and let $x \in F_n$. If $x \notin A \setminus A_\circ$, then

$$\mu(F_n) \cdot |f_n \cdot \chi_{A \setminus A_\circ}(x) - f \cdot \chi_{A \setminus A_\circ}(x)| = \mu(F_n) \cdot 0 = 0 < \epsilon.$$

If $x \in A \setminus A_\circ$, then, since $\left|\frac{f(x)}{(f+g)(x)}\right| \leq 1$, we have that

$$\begin{aligned}
\mu(F_n) \cdot |f_n \cdot \chi_{A \setminus A_\circ}(x) - f \cdot \chi_{A \setminus A_\circ}(x)| &= \mu(F_n) \cdot |f_n(x) - (f+g)(x)| \\
&= \mu(F_n) \cdot \left|\frac{f(x)}{(f+g)(x)} \cdot s_n(x) - (f+g)(x)\right| \\
&= \mu(F_n) \cdot \left|\frac{f(x)}{(f+g)(x)}\right| \\
&\quad \cdot |s_n(x) - (f+g)(x)| \\
&\leq \mu(F_n) \cdot |s_n(x) - (f+g)(x)| \\
&< \epsilon.
\end{aligned}$$

Thus, we have the claim.

By Lemma 3.3.3, $\{f_n \cdot \chi_{A_\circ} + f_n \cdot \chi_{A \setminus A_\circ}\}$ converges to $f \cdot \chi_{A_\circ} + f \cdot \chi_{A \setminus A_\circ}$ semiuniformly with respect to $\{E_n\}$. Since $f_n \cdot \chi_A = f_n \cdot \chi_{A_\circ} + f_n \cdot \chi_{A \setminus A_\circ}$ and $f \cdot \chi_A = f \cdot \chi_{A_\circ} + f \cdot \chi_{A \setminus A_\circ}$ for all $n \in \mathbb{N}$, $\{f_n \cdot \chi_A\}$ converges to $f \cdot \chi_A$ semiuniformly with respect to $\{E_n\}$.

For $B_\circ \cup B^-$, by Lemma 3.3.6 we have that $\{p_n \cdot \chi_{B_\circ \cup B^-}\}$ converges to $f \cdot \chi_{B_\circ \cup B^-}$ semiuniformly with respect to $\{E_n\}$. Since $p_n \cdot \chi_{B_\circ \cup B^-} = f_n \cdot \chi_{B_\circ \cup B^-}$ for all $n \in \mathbb{N}$, this tells us that $\{f_n \cdot \chi_{B_\circ \cup B^-}\}$ converges to $f \cdot \chi_{B_\circ \cup B^-}$ semiuniformly with respect to $\{E_n\}$.

Now consider B^+. Since $\{q_n\}$ converges to g semiuniformly with respect to $\{E_n\}$, by Lemma 3.3.3, $\{s_n - g_n\}$ converges to $(f+g)-g = f$ semiuniformly with respect to $\{E_n\}$. Because $f_n \cdot \chi_{B^+} = (s_n - g_n) \cdot \chi_{B^+}$ for all $n \in \mathbb{N}$, we have that $\{f_n \cdot \chi_{B^+}\}$ converges to $f \cdot \chi_{B^+}$ semiuniformly with respect to $\{E_n\}$. As in the above paragraph, we may combine the cases $B_\circ \cup B^-$ and B^+ to conclude that $\{f_n \cdot \chi_B\}$ converges to $f \cdot \chi_B$ semiuniformly with respect to $\{E_n\}$.

Putting all of the above cases together, we conclude that $\{f_n \cdot \chi_{A \cup B}\}$ converges to $f \cdot \chi_{A \cup B}$ semiuniformly with respect to $\{E_n\}$. Since $A \cup B = E$, this tells that $\{f_n \cdot \chi_E\}$ converges to $f \cdot \chi_E$ semiuniformly with respect to $\{E_n\}$. By the definition of semiuniform convergence, $\{f_n\}$ converges to f semiuniformly with respect to $\{E_n\}$.

Finally, observe that $f_n + g_n = s_n$ on E for all $n \in \mathbb{N}$, by the definitions of f_n and g_n Therefore the lemma holds. $\square$

Lemma 3.3.8. Let $f, g \in \mathscr{F}$ be such that $f \leq g$ on $\bigcup_{n=1}^{\infty} E_n$, and $\{s_n\}$ and $\{t_n\}$ be sequences in $L^1(\mu)$ that expand to f and g semiuniformly with respect to $\{E_n\}$, respectively.

(a) If $u_n = \max\{s_n, t_n\}$ for all $n \in \mathbb{N}$, then $u_n \lessdot g$ semiuniformly with respect to $\{E_n\}$.

(b) If $u_n = \min\{s_n, t_n\}$ for all $n \in \mathbb{N}$, then $u_n \lessdot f$ semiuniformly with respect to $\{E_n\}$.

Proof. **(a)** Let $u_n = \max\{s_n, t_n\}$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ is a sequence in $L^1(\mu)$. We will show that $u_n \lessdot g$ on $E = \bigcup_{n=1}^{\infty} E_n$. Let $x \in E$.

Case 1. $g(x) \geq 0$ *and* $f(x) \geq 0$. Then $\{u_n(x)\}$ converges to $g(x)$ because $t_n(x) \leq u_n(x) \leq g(x)$ for all $n \in \mathbb{N}$ and $\{t_n(x)\}$ converges to $g(x)$. Since $g(x) \geq 0$ and $f(x) \geq 0$, we have that $s_n(x) \geq 0$ and $t_n(x) \geq 0$ for all $n \in \mathbb{N}$. By the definition of $\{u_n\}$, $u_n(x) \geq 0$ for all $n \in \mathbb{N}$. We have that $u_n^-(x) = 0 \leq u_{n+1}^-(x)$ and $u_n^+(x) = u_n(x)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. If $s_n(x) \geq t_n(x)$, then $u_n(x) = s_n(x)$. Thus

$$u_n^+(x) = s_n(x) = s_n^+(x) \leq s_{n+1}^+(x) = s_{n+1}(x) \leq u_{n+1}(x) = u_{n+1}^+(x).$$

If $t_n(x) \geq s_n(x)$, then $u_n(x) = t_n(x)$. Thus

$$u_n^+(x) = t_n(x) = t_n^+(x) \leq t_{n+1}^+(x) = t_{n+1}(x) \leq u_{n+1}(x) = u_{n+1}^+(x).$$

Case 2. $g(x) \geq 0$ *and* $f(x) \leq 0$. Then $s_n(x) \leq 0 \leq t_n(x)$ for all $n \in \mathbb{N}$, which implies $u_n(x) = t_n(x)$ for all $n \in \mathbb{N}$. This shows that $u_n^+(x) \leq u_{n+1}^+(x)$ and $u_n^-(x) \leq u_{n+1}^-(x)$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} u_n(x) = g(x)$.

Case 3. $g(x) \leq 0$ *and* $f(x) \leq 0$. Then $s_n(x) \leq 0$ and $t_n(x) \leq 0$ for all $n \in \mathbb{N}$, and hence $u_n^+(x) = 0 \leq u_{n+1}^+(x)$ and $u_n^-(x) = -u_n(x)$ for all $n \in \mathbb{N}$.

Subcase 3.1. $s_n(x) \geq t_n(x)$. Then $u_n(x) = s_n(x)$. If $u_{n+1}(x) = s_{n+1}(x)$, then

$$u_n^-(x) = -s_n(x) = s_n^-(x) \leq s_{n+1}^-(x) = -s_{n+1}(x) = -u_{n+1}(x) = u_{n+1}^-(x).$$

If $u_{n+1}(x) = t_{n+1}(x)$, then

$$\begin{aligned} u_n^-(x) &= -s_n(x) \leq -t_n(x) = t_n^-(x) \\ &\leq t_{n+1}^-(x) = -t_{n+1}(x) = -u_{n+1}(x) = u_{n+1}^-(x). \end{aligned}$$

Subcase 3.2. $t_n(x) > s_n(x)$. Then $u_n(x) = t_n(x)$. If $u_{n+1}(x) = s_{n+1}(x)$, then

$$u_n^-(x) = -t_n(x) \le -s_n(x) = s_n^-(x) \le s_{n+1}^-(x) = -s_{n+1}(x) = -u_{n+1}(x) = u_{n+1}^-(x).$$

If $u_{n+1}(x) = t_{n+1}(x)$, then

$$u_n^-(x) = -t_n(x) = t_n^-(x) \le t_{n+1}^-(x) = -t_{n+1}(x) = -u_{n+1}(x) = u_{n+1}^-(x).$$

In both subcases we have that $u_n(x) \le u_{n+1}(x)$.

Let us show that $\lim_{n\to\infty} u_n(x) = g(x)$. If $f(x) = g(x)$, then we have two subcases as follows:

Subcase 3.1. *For all $N \in \mathbb{N}$ there exists n_N in $\mathbb{N}$ with $n_N \ge N$ such that $u_{n_N}(x) = s_{n_N}(x)$.* Then we can choose a subsequence $\{u_{n_N}\}$ of $\{u_n\}$ such that $u_{n_N} = s_{n_N}$ for all $N \in \mathbb{N}$. Hence $\{u_{n_N}\}$ converges to $f(x)$. Since $\{u_n\}$ is decreasing and $u_n(x) \ge g(x) = f(x)$ for all $n \in \mathbb{N}$, $u_n(x)$ is convergent. Hence $\lim_{n\to\infty} u_n(x) = f(x) = g(x)$.

Subcase 3.2. *There exists $N \in \mathbb{N}$ such that $u_n(x) = t_n(x)$ for all $n \in \mathbb{N}$ with $n \ge N$.* Then clearly $\lim_{n\to\infty} u_n(x) = g(x)$.

If $f(x) \ne g(x)$, then $f(x) < g(x)$, and hence $g(x) - f(x) > 0$. Since $\{s_n(x)\}$ is decreasing and $\lim_{n\to\infty} s_n(x) = f(x)$, there is an $N \in \mathbb{N}$ such that $s_n(x) - f(x) < g(x) - f(x)$ for all $n \in \mathbb{N}$ with $n \ge N$. Then $s_n(x) < g(x)$ for all $n \ge N$. Thus $u_n(x) = t_n(x)$ for all $n \ge N$. Hence $\lim_{n\to\infty} u_n(x) = g(x)$.

To finish the proof we will show that $\{u_n\}$ converges to g semiuniformly with respect to $\{E_n\}$. Since $\{s_n\}$ converges to f semiuniformly with respect to $\{E_n\}$, there exists an increasing sequence $\{F_{1,n}\}$ in $\mathscr{M}$ such that $F_{1,n} \subseteq E_n$ for all

$n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_{1,n} = \bigcup_{n=1}^{\infty} E_n$ and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_{1,n}) = 0$, with the property that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_{1,n}) \cdot |s_n(x) - f(x)| < \epsilon$ holds for all $x \in F_{1,n}$. Similarly, there exists an increasing sequence $\{F_{2,n}\}$ in $\mathscr{M}$ such that $F_{2,n} \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_{2,n} = \bigcup_{n=1}^{\infty} E_n$ and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_{2,n}) = 0$, with the property that for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the equality $\mu(F_{2,n}) \cdot |t_n(x) - g(x)| < \epsilon$ holds for all $x \in F_{2,n}$. For each $n \in \mathbb{N}$, let $F_n = F_{1,n} \cap F_{2,n}$. As in the proof of Lemma 3.3.3, we have that $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_n) = 0$. Let $\epsilon > 0$. There exist $N_1, N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, the inequality $\mu(F_{1,n}) \cdot |s_n(x) - f(x)| < \epsilon$ holds for all $x \in F_{1,n}$; and for all $n \in \mathbb{N}$ with $n \geq N_2$, the inequality $\mu(F_{2,n}) \cdot |t_n(x) - g(x)| < \epsilon$ holds for all $x \in F_{2,n}$. Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ be such that $n \geq N$ and let $x \in F_n$. Then $x \in F_{1,n} \cap F_{2,n}$.

If $g(x) \geq 0$, then $g(x) \geq u_n(x) \geq t_n(x) \geq 0$. Thus

$$\begin{aligned}
\mu(F_n) \cdot |g(x) - u_n(x)| &\leq \mu(F_n) \cdot |g(x) - t_n(x)| \\
&\leq \mu(F_{2,n}) \cdot |g(x) - t_n(x)| \\
&< \epsilon.
\end{aligned}$$

Now suppose that $g(x) < 0$, so that $f(x) \leq g(x) < 0$. If $s_n(x) \geq t_n(x)$, then $u_n(x) = s_n(x)$ and $f(x) \leq g(x) \leq t_n(x) \leq s_n(x)$. Thus

$$\begin{aligned}
\mu(F_n) \cdot |g(x) - u_n(x)| &= \mu(F_n) \cdot |g(x) - s_n(x)| \\
&\leq \mu(F_n) \cdot |f(x) - s_n(x)| \\
&\leq \mu(F_{1,n}) \cdot |f(x) - s_n(x)| \\
&< \epsilon.
\end{aligned}$$

If $s_n(x) < t_n(x)$, then $u_n(x) = t_n(x)$, and thus

$$\mu(F_n) \cdot |g(x) - u_n(x)| = \mu(F_n) \cdot |g(x) - t_n(x)|$$

$$\leq \mu(F_{2,n}) \cdot |g(x) - t_n(x)|$$

$$< \epsilon.$$

Therefore (a) holds.

(b) For each $n \in \mathbb{N}$, let $u_n = \min\{s_n, t_n\}$, and let $v_n = -u_n$, for all $n \in \mathbb{N}$. Then $v_n = -\min\{s_n, t_n\} = \max\{-s_n, -t_n\}$ for all $n \in \mathbb{N}$. Since $-s_n \lessdot -f$ and $-t_n \lessdot -g$ semiuniformly with respect to $\{E_n\}$ and $-g \leq -f$, by part (a) $v_n \lessdot -f$ semiuniformly with respect to $\{E_n\}$. By Lemma 3.3.5, $-v_n \lessdot f$ semiuniformly with respect to $\{E_n\}$. Since $-v_n = u_n$ for all $n \in \mathbb{N}$, this says that $u_n \lessdot f$ semiuniformly with respect to $\{E_n\}$. □

3.4 A Nonabsolutely Convergent Abstract μ-integral

In this section we will show how the concepts of expanding sequences and semi-uniform convergence can be combined to yield a family of abstract μ-integrals, at least some of which are nonabsolutely convergent.

Throughout this section $\{E_n\}$ will denote an increasing sequence in $\mathscr{M}$ such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} E_n = X$.

Definition 3.4.1. A function f in $\mathscr{F}$ is said to be **generalized Lebesgue integrable with respect to $\{E_n\}$** iff there exists an $L \in \bar{\mathbb{R}}$ with the property that $\lim_{n\to\infty} \int_{E_n} s_n \, d\mu = L$ for every sequence $\{s_n\}$ in $L^1(\mu)$ such that $s_n \lessdot f$ semiuniformly with respect to $\{E_n\}$, in which case the generalized Lebesgue integral of f is L. We denote the generalized Lebesgue integral of a function f with respect to $\{E_n\}$ by $\{E_n\}$-$\int_X f \, d\mu$.

Let $\mathscr{I} = \big\{f \in \mathscr{F} \;\big|\; f \text{ is generalised Lebesgue integrable with respect to} \{E_n\}\big\}$ and $\mathrm{I}(f) = \{E_n\}$-$\int_X f \, d\mu$ for every $f \in \mathscr{I}$. We will show that $(\mathscr{I}, \mathrm{I})$ is an abstract μ-integral and $\mathscr{I}_{L^1} \subseteq \mathscr{I}$.

Proposition 3.4.2. Let $f \in \mathscr{F}$. If f is in $\mathscr{I}_{L^1}$, then f is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X f\,d\mu = \int_X f\,d\mu.$$

Proof. This follows directly from Proposition 3.2.11. □

Corollary 3.4.3. Let $f \in \mathscr{F}$. If $f \geq 0$, then f is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X f\,d\mu = \int_X f\,d\mu.$$

Proposition 3.4.4. For all E in $\mathscr{M}$, χ_E is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X \chi_E\,d\mu = \mu(E).$$

Proof. Let $E \in \mathscr{M}$. Since $(\chi_E)^- = 0$, $\int_X (\chi_E)^-\,d\mu = 0$. This shows that $f \in \mathscr{I}_{L^1}$. By Lemma 3.4.2, χ_E is generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}\text{-}\int_X \chi_E\,d\mu = \int_X \chi_E\,d\mu = \mu(E)$. □

Proposition 3.4.5. For every $f \in \mathscr{F}$ and $r \in \mathbb{R}$, if f is generalized Lebesgue integrable with respect to $\{E_n\}$, then rf is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X rf\,d\mu = r\big(\{E_n\}\text{-}\int_X f\,d\mu\big).$$

Proof. Let $f \in \mathscr{F}$ and $r \in \mathbb{R}$. Assume that f is generalized Lebesgue integrable with respect to $\{E_n\}$. There exists $L \in [-\infty, \infty]$ such that $\lim_{n\to\infty} \int_{E_n} s_n\,d\mu = L$ for all sequences $\{s_n\}$ in $L^1(\mu)$ with the property that $s_n \lessdot f$ semiuniformly with respect to $\{E_n\}$.

If $r = 0$, then $rf = 0$. Clearly, rf is generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}\text{-}\int_X rf\,d\mu = 0 = r\big(\{E_n\}\text{-}\int_X f\,d\mu\big)$.

Assume that $r \neq 0$, and consider rL. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ such that s_n expands to rf semiuniformly with respect to $\{E_n\}$. Then $\{\frac{s_n}{r}\}$ is a sequence in $L^1(\mu)$.

We will show that $\frac{s_n}{r} \lessdot f$ on $\bigcup_{n=1}^{\infty} E_n = X$. Since $\{s_n(x)\}$ converges to $rf(x)$ for all $x \in X$, $\{\frac{s_n}{r}(x)\}$ converges to $f(x)$ for all $x \in X$.

Case 1. $r > 0$. Then $(\frac{s_n}{r})^+ = \frac{(s_n)^+}{r}$ and $(\frac{s_n}{r})^- = \frac{(s_n)^-}{r}$ for all $n \in \mathbb{N}$. Since $s_n \lessdot rf$ on X, we have that $(\frac{s_n}{r})^+ = \frac{s_n^+}{r} \leq \frac{s_{n+1}^+}{r} = (\frac{s_{n+1}}{r})^+$ and $(\frac{s_n}{r})^- = \frac{s_n^-}{r} \leq \frac{s_{n+1}^-}{r} = (\frac{s_{n+1}}{r})^-$ for all $n \in \mathbb{N}$.

Case 2. $r < 0$. Then $(\frac{s_n}{r})^+ = -\frac{s_n^-}{r}$ and $(\frac{s_n}{r})^- = -\frac{s_n^+}{r}$ for all $n \in \mathbb{N}$. Since $s_n \lessdot rf$ on X, we have that $(\frac{s_n}{r})^+ = -\frac{s_n^-}{r} \leq -\frac{s_{n+1}^-}{r} = (\frac{s_{n+1}}{r})^+$ and $(\frac{s_n}{r})^- = -\frac{s_n^+}{r} \leq -\frac{s_{n+1}^+}{r} = (\frac{s_{n+1}}{r})^-$.

Thus $\frac{s_n}{r} \lessdot f$ on X.

We will show that $\{\frac{s_n}{r}\}$ converges to f semiuniformly with respect to $\{E_n\}$. Since $\{s_n\}$ converges to rf semiuniformly with respect to $\{E_n\}$, there exists an increasing sequence $\{F_n\}$ in $\mathscr{M}$ such that $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_n) = 0$, with the property that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n) \cdot |s_n(x) - rf(x)| < \epsilon$ holds for all $x \in F_n$. Let $\epsilon > 0$. Then $\epsilon|r| > 0$. There is an $N_{\epsilon|r|}$ in $\mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq N_{\epsilon|r|}$, the inequality $\mu(F_n) \cdot |s_n(x) - rf(x)| < \epsilon|r|$ holds for all $x \in F_n$. Then for each $n \in \mathbb{N}$ such that $n \geq N_{\epsilon|r|}$ we have that

$$\begin{aligned} \mu(F_n) \cdot \left|\frac{s_n}{r}(x) - f(x)\right| &= \mu(F_n) \cdot \left|\frac{1}{r}\right| |s_n(x) - rf(x)| \\ &< \frac{1}{|r|}\,\epsilon\,|r| \\ &= \epsilon. \end{aligned}$$

We have that $\{\frac{s_n}{r}\}$ is a sequence in $L^1(\mu)$ such that $\frac{s_n}{r} \lessdot f$ semiuniformly with respect to $\{E_n\}$. Thus $\lim_{n\to\infty} \int_{E_n} \frac{s_n}{r}\, d\mu = L$. This implies that $\lim_{n\to\infty} \int_{E_n} s_n\, d\mu =$

rL. Thus rf is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X rf\,d\mu = rL = r\left(\{E_n\}\text{-}\int_X f\,d\mu\right). \qquad \square$$

Proposition 3.4.6. Let $f, g \in \mathscr{F}$ be such that f and g are generalized Lebesgue integrable with respect to $\{E_n\}$, $\left|\{E_n\}\text{-}\int_X f\,d\mu\right| < \infty$ and $\left|\{E_n\}\text{-}\int_X g\,d\mu\right| < \infty$. Then $f+g$ is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X (f+g)\,d\mu = \{E_n\}\text{-}\int_X f\,d\mu + \{E_n\}\text{-}\int_X g\,d\mu.$$

Proof. Let $L_1 = \{E_n\}\text{-}\int_X f\,d\mu$ and $L_2 = \{E_n\}\text{-}\int_X g\,d\mu$ and let $L = L_1 + L_2$. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ such that s_n expands to $f+g$ semiuniformly with respect to $\{E_n\}$. By Lemma 3.3.7, there exist sequences $\{f_n\}$ and $\{g_n\}$ in $L^1(\mu)$ such that $f_n \lessdot f$ and $g_n \lessdot g$ semiuniformly with respect to $\{E_n\}$ and $f_n+g_n = s_n$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty}\int_{E_n} f_n\,d\mu = L_1$, $\lim_{n\to\infty}\int_{E_n} g_n\,d\mu = L_2$, and $\int_{E_n} f_n\,d\mu + \int_{E_n} g_n\,d\mu = \int_{E_n} s_n\,d\mu$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\left|\int_{E_n} s_n\,d\mu - L\right| &= \left|\int_{E_n} f_n\,d\mu + \int_{E_n} g_n\,d\mu - (L_1+L_2)\right| \\ &\le \left|\int_{E_n} f_n\,d\mu - L_1\right| + \left|\int_{E_n} g_n\,d\mu - L_2\right|.\end{aligned}$$

Thus $\lim_{n\to\infty}\int_{E_n} s_n\,d\mu = L$, which proves that $f+g$ is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X (f+g)\,d\mu = \{E_n\}\text{-}\int_X f\,d\mu + \{E_n\}\text{-}\int_X g\,d\mu. \qquad \square$$

Lemma 3.4.7. Let $f, g \in \mathscr{F}$ be such that f and g are generalized Lebesgue integrable with respect to $\{E_n\}$ and $f \le g$.

(a) If $\{E_n\}\text{-}\int_X f\,d\mu = \infty$ then $\{E_n\}\text{-}\int_X g\,d\mu = \infty$.

(b) If $\{E_n\}\text{-}\int_X g\,d\mu = -\infty$ then $\{E_n\}\text{-}\int_X f\,d\mu = -\infty$.

Proof. **(a)** Assume that $\{E_n\}$-$\int_X f\,d\mu = \infty$. Let $\{t_n\}$ be a sequence in $L^1(\mu)$ such that $t_n \mathrel{\substack{\nearrow \\ \searrow}} g$ semiuniformly with respect to $\{E_n\}$. We must show that $\lim_{n\to\infty} \int_{E_n} t_n\,d\mu = \infty$. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ such that $s_n \mathrel{\substack{\nearrow \\ \searrow}} f$ semiuniformly with respect to $\{E_n\}$. Let $u_n = \min\{s_n, t_n\}$ for all $n \in \mathbb{N}$. By Lemma 3.3.8, $u_n \mathrel{\substack{\nearrow \\ \searrow}} f$ semiuniformly with respect to $\{E_n\}$. Thus $\lim_{n\to\infty} \int_{E_n} u_n\,d\mu = \infty$. Since $u_n \le t_n$ for all $n \in \mathbb{N}$, $\int_{E_n} u_n\,d\mu \le \int_{E_n} t_n\,d\mu$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \int_{E_n} t_n\,d\mu = \infty$. Therefore $\{E_n\}$-$\int_X g\,d\mu = \infty$.

(b). Since $f \le g$, $-f \le -g$. Because f and g are generalized Lebesgue integrable with respect to $\{E_n\}$, $-f$ and $-g$ are generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}$-$\int_X (-g)\,d\mu = -\{E_n\}$-$\int_X g\,d\mu = \infty$. By part (a) $\{E_n\}$-$\int_X (-f)\,d\mu = \infty$, and hence by Proposition 3.4.5, we have

$$\{E_n\}\text{-}\int_X f\,d\mu = -\left(\{E_n\}\text{-}\int_X (-f)\,d\mu\right) = -\infty. \qquad \square$$

Proposition 3.4.8. Let $f, g \in \mathscr{F}$ be such that $f \le g$. If f and g are generalized Lebesgue integrable with respect to $\{E_n\}$, then $\{E_n\}$-$\int_X f\,d\mu \le \{E_n\}$-$\int_X g\,d\mu$.

Proof. Assume that f and g are generalized Lebesgue integrable with respect to $\{E_n\}$. If $\{E_n\}$-$\int_X g\,d\mu = \infty$, then we are finished. If $\{E_n\}$-$\int_X g\,d\mu = -\infty$, then by Lemma 3.4.7, $\{E_n\}$-$\int_X f\,d\mu = -\infty$ and again we are finished.

Thus, assume that $-\infty < \{E_n\}$-$\int_X g\,d\mu < \infty$. By Lemma 3.4.7, we have that $\{E_n\}$-$\int_X f\,d\mu < \infty$ also. If $\{E_n\}$-$\int_X f\,d\mu = -\infty$, then again we are finished, so we may assume that $-\infty < \{E_n\}$-$\int_X f\,d\mu < \infty$. By Corollary 3.4.3 and Proposition 3.4.6 and the fact that $g - f \ge 0$, we have

$$\{E_n\}\text{-}\int_X g\,d\mu - \{E_n\}\text{-}\int_X f\,d\mu = \{E_n\}\text{-}\int_X (g-f)\,d\mu \ge 0.$$

Thus $\{E_n\}$-$\int_X g\,d\mu \ge \{E_n\}$-$\int_X f\,d\mu$. $\square$

Theorem 3.4.9. If $\{f_n\}$ is a monotonically increasing sequence of generalized Lebesgue integrable functions with respect to $\{E_n\}$, $f_1 \le f_2 \le f_3 \le \ldots$, such that

$\sup_{n\in\mathbb{N}} f_n(x) < \infty$ for all $x \in X$ and there exists $N \in \mathbb{N}$ with $\{E_n\}$-$\int_X f_N \, d\mu > -\infty$, then $\sup_{n\in\mathbb{N}} f_n$ is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X \sup_{n\in\mathbb{N}} f_n \, d\mu = \sup_{n\in\mathbb{N}} \left(\{E_n\}\text{-}\int_X f_n \, d\mu \right).$$

Proof. For each $n \in \mathbb{N}$ let $L_n = \{E_n\}$-$\int_X f_n \, d\mu$ and let $L = \sup_{n\in\mathbb{N}} L_n$. By hypothesis, we have that $L_n > -\infty$ for all $n \in \mathbb{N}$ such that $n \geq N$. By dropping a finite number of terms at the beginning of the sequence $\{f_n\}$, we may assume $N = 1$. Let $f = \sup_{n\in\mathbb{N}} f_n$. We must prove that f is generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}$-$\int_X f \, d\mu = L$.

Case 1. $L = \infty$. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ such that $s_n \lessdot f$ semiuniformly with respect to $\{E_n\}$. We must show that $\lim_{n\to\infty} \int_{E_n} s_n \, d\mu = \infty$. It suffices to show $\liminf_{n\to\infty} \int_{E_n} s_n \, d\mu \geq L_k$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$, and let $\{t_n\}$ be a sequence in $L^1(\mu)$ such that $t_n \lessdot f_k$ semiuniformly with respect to $\{E_n\}$. Let $u_n = \min\{s_n, t_n\}$ for all $n \in \mathbb{N}$. Since $f_k \leq f$, by Lemma 3.3.8 $\{u_n\}$ is a sequence in $L^1(\mu)$ such that $u_n \lessdot f_k$ semiuniformly with respect to $\{E_n\}$. Thus $\lim_{n\to\infty} \int_{E_n} u_n \, d\mu = L_k$. Since $u_n \leq s_n$ for all $n \in \mathbb{N}$, it follows that

$$\liminf_{n\to\infty} \int_{E_n} s_n \, d\mu \geq \liminf_{n\to\infty} \int_{E_n} u_n \, d\mu = L_k,$$

which is the inequality we need to finish this case.

Case 2. $L < \infty$. Let $g_n = f_n - f_1$ for each $n \in \mathbb{N}$. Then $\{g_n\}$ is an increasing sequence of nonnegative functions, which by Corollary 3.4.3 are all generalized Lebesgue integrable with respect to $\{E_n\}$. Furthermore, $\{E_n\}$-$\int_X g_n \, d\mu = \{E_n\}$-$\int_X f_n \, d\mu - \{E_n\}$-$\int_X f_1 \, d\mu$ and $\{E_n\}$-$\int_X g_n \, d\mu = \int_X g_n \, d\mu$ for all $n \in \mathbb{N}$. Let $g = \sup_{n\in\mathbb{N}} g_n$ and observe that

$$g = \sup_{n\in\mathbb{N}} (f_n - f_1) = \left(\sup_{n\in\mathbb{N}} f_n \right) - f_1 = f - f_1.$$

In particular, g is a nonnegative, measurable, real-valued function, and thus by Corollary 3.4.3 again, g is generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}\text{-}\int_X g\,d\mu = \int_X g\,d\mu$. By the Monotone Convergence theorem,

$$\begin{aligned}
\{E_n\}\text{-}\int_X g\,d\mu &= \int_X g\,d\mu \\
&= \int_X \sup_{n\in\mathbb{N}} g_n\,d\mu \\
&= \sup_{n\in\mathbb{N}} \int_X g_n\,d\mu \\
&= \sup_{n\in\mathbb{N}} \left[\{E_n\}\text{-}\int_X f_n\,d\mu - \{E_n\}\text{-}\int_X f_1\,d\mu\right] \\
&= \sup_{n\in\mathbb{N}} \left[\{E_n\}\text{-}\int_X f_n\,d\mu\right] - \{E_n\}\text{-}\int_X f_1\,d\mu \\
&= \sup_{n\in\mathbb{N}} L_n - L_1 \\
&= L - L_1.
\end{aligned}$$

This tells us that $\{E_n\}\text{-}\int_X g\,d\mu$ is finite. Since $\{E_n\}\text{-}\int_X f_1\,d\mu = L_1$ is also finite, by Proposition 3.4.6 $f = g + f_1$ is generalized Lebesgue integrable with respect to $\{E_n\}$ and

$$\{E_n\}\text{-}\int_X f\,d\mu = \{E_n\}\text{-}\int_X g\,d\mu + \{E_n\}\text{-}\int_X f_1\,d\mu = (L - L_1) + L_1 = L. \qquad \square$$

By Propositions 3.4.4, 3.4.5, and 3.4.6, and Theorem 3.4.9, we have that $(\mathscr{I}, \mathrm{I})$ is an abstract μ-integral. Also, we have $\mathscr{I}_{L^1} \subseteq \mathscr{I}$ by Proposition 3.4.2.

The following is an example of a choice of measure space $(X, \mathscr{M}, \mu)$ and sequence of sets $\{E_n\}$ such that the corresponding generalized Lebesgue integral is nonabsolutely convergent.

Example 3.4.10. Let $(X, \mathscr{M}, \mu)$ be the standard Lebesgue measure space on $\mathbb{R}$ and let $E_n = [-n, n]$ for all $n \in \mathbb{N}$. Then $(X, \mathscr{M}, \mu)$ is a σ-finite measure space and the corresponding generalized Lebesgue integral is an abstract μ-integral with

$\mathscr{I}_{L^1} \subseteq \mathscr{I}$. We will show that $(\mathscr{I}, \mathrm{I})$ is nonabsolutely convergent by exhibiting a function $f \in \mathscr{I}$ with $|\mathrm{I}(f)| < \infty$ but $\mathrm{I}\big(|f|\big) = \infty$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

for all $x \in X$.

Since $|f|(x) = 1$ for all $x \in X$, we have $|f| = \chi_X$, and thus $\mathrm{I}\big(|f|\big) = \mu(X) = \infty$. Hence we will be finished if we can show that f is generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}$-$\int_X f\,d\mu = 0$. Let $\{s_n\}$ be a sequence in $L^1(\mu)$ such that $s_n \lessdot f$ semiuniformly with respect to $\{E_n\}$. We must prove that $\lim_{n\to\infty} \int_{E_n} s_n\,d\mu = 0$.

By definition there exists an increasing sequence $\{F_n\}$ in $\mathscr{M}$ such that $F_n \subseteq E_n$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$, and $\lim_{n\to\infty} \mu(E_n \smallsetminus F_n) = 0$, with the property that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, the inequality $\mu(F_n)\cdot|s_n(x) - f(x)| < \epsilon$ holds for all $x \in F_n$. Since $\lim_{n\to\infty} \mu(E_n \smallsetminus F_n) = 0$ and $\lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} 2n = \infty$, we have that there exists an $N_1 \in \mathbb{N}$ such that $\mu(F_n) \neq 0$ for all $n \geq N_1$. Let $\epsilon > 0$. There is an $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$, the inequality $\mu(F_n) \cdot |s_n(x) - f(x)| < \frac{\epsilon}{2}$ holds for all $x \in F_n$. There exists an $N_3 \in \mathbb{N}$ such that $\mu(E_n \smallsetminus F_n) < \frac{\epsilon}{2}$ for all $n \geq N_3$. Choose $N = \max\{N_1, N_2, N_3\}$. Let $n \in \mathbb{N}$ be such that $n \geq N$. Then for each $x \in F_n$, we have $\mu(F_n) \cdot |s_n(x) - f(x)| < \frac{\epsilon}{2}$, i.e., $|s_n(x) - f(x)| < \frac{\epsilon}{2\mu(F_n)}$.

Consider $\int_{F_n \cap [0,n]} s_n\,d\mu$. Since $s_n \lessdot f$, we have $s_n(x) \leq f(x)$ for all $x \in [0, n]$. Thus

$$1 - \frac{\epsilon}{2\mu(F_n)} = f(x) - \frac{\epsilon}{2\mu(F_n)} < s_n(x) \leq f(x) = 1.$$

Hence

$$\int_{F_n \cap [0,n]} 1 - \frac{\epsilon}{2\mu(F_n)}\,d\mu < \int_{F_n \cap [0,n]} s_n\,d\mu$$

$$\leq \int_{F_n \cap [0,n]} 1 \, d\mu$$
$$= \mu\big(F_n \cap [0,n]\big).$$

Next, consider $\int_{F_n \cap [-n,0]} s_n \, d\mu$. Since $s_n \prec f$, we have $s_n(x) \geq f(x)$ for all $x \in [-n, 0]$. Thus

$$-1 = f(x) \leq s_n(x) < f(x) + \frac{\epsilon}{2\mu(F_n)} = -1 + \frac{\epsilon}{2\mu(F_n)}.$$

Hence

$$\begin{aligned}(-1) \cdot \mu\big(F_n \cap [-n,0]\big) &\leq \int_{F_n \cap [-n,0]} s_n \, d\mu \\ &< \int_{F_n \cap [-n,0]} -1 + \frac{\epsilon}{2\mu(F_n)} \, d\mu.\end{aligned}$$

Now, consider $\int_{(E_n \smallsetminus F_n) \cap [0,n]} s_n \, d\mu$. Since $s_n \prec f$, we have $0 \leq s_n(x) \leq f(x) = 1$ for all $x \in [0, n]$. Thus

$$0 \leq \int_{(E_n \smallsetminus F_n) \cap [0,n]} s_n \, d\mu \leq \int_{(E_n \smallsetminus F_n) \cap [0,n]} f \, d\mu = \mu\big((E_n \smallsetminus F_n) \cap [0,n]\big).$$

Finally, consider $\int_{(E_n \smallsetminus F_n) \cap [-n,0]} s_n \, d\mu$. Since $s_n \prec f$, we have $0 \geq s_n(x) \geq f(x) = -1$ for all $x \in [-n, 0]$. Thus

$$(-1)\mu\big((E_n \smallsetminus F_n) \cap [-n,0]\big) = \int_{(E_n \smallsetminus F_n) \cap [-n,0]} f \, d\mu \leq \int_{(E_n \smallsetminus F_n) \cap [-n,0]} s_n \, d\mu \leq 0.$$

Combining all of these we obtain

$$\begin{aligned}\int_{E_n} s_n \, d\mu &= \int_{F_n \cap [0,n]} s_n \, d\mu + \int_{F_n \cap [-n,0]} s_n \, d\mu + \int_{(E_n \smallsetminus F_n) \cap [0,n]} s_n \, d\mu \\ &\quad + \int_{(E_n \smallsetminus F_n) \cap [-n,0]} s_n \, d\mu \\ &> \int_{F_n \cap [0,n]} 1 - \frac{\epsilon}{2\mu(F_n)} \, d\mu + (-1) \cdot \mu\big(F_n \cap [-n,0]\big) + 0 \\ &\quad + (-1) \cdot \mu\big((E_n \smallsetminus F_n) \cap [-n,0]\big) \\ &= \left(1 - \frac{\epsilon}{2\mu(F_n)}\right) \mu\big(F_n \cap [0,n]\big) - \mu\big(F_n \cap [-n,0]\big)\end{aligned}$$

$$
\begin{aligned}
&\quad - \mu\big((E_n \smallsetminus F_n) \cap [-n,0]\big) \\
&= \mu\big(F_n \cap [0,n]\big) - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} - \mu\big(F_n \cap [-n,0]\big) \\
&\quad - \mu\big([-n,0] \smallsetminus (F_n \cap [-n,0])\big) \\
&= \mu\big(F_n \cap [0,n]\big) - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} - \mu\big(F_n \cap [-n,0]\big) \\
&\quad - \mu\big([-n,0]\big) + \mu\big(F_n \cap [-n,0]\big) \\
&= -\big[\mu([-n,0]) - \mu(F_n \cap [0,n])\big] - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} \\
&= -\big[\mu([0,n]) - \mu(F_n \cap [0,n])\big] - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} \\
&= -\mu\big([0,n] \smallsetminus (F_n \cap [0,n])\big) - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} \\
&= -\mu\big((E_n \smallsetminus F_n) \cap [0,n]\big) - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} \\
&\geq -\mu(E_n \smallsetminus F_n) - \frac{\epsilon\mu\big(F_n \cap [0,n]\big)}{2\mu(F_n)} \\
&> -\frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&= -\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{E_n} s_n \, d\mu &= \int_{F_n \cap [0,n]} s_n \, d\mu + \int_{F_n \cap [-n,0]} s_n \, d\mu + \int_{(E_n \smallsetminus F_n) \cap [0,n]} s_n \, d\mu \\
&\quad + \int_{(E_n \smallsetminus F_n) \cap [-n,0]} s_n \, d\mu \\
&< \mu\big(F_n \cap [0,n]\big) + \int_{F_n \cap [-n,0]} -1 + \frac{\epsilon}{2\mu(F_n)} \, d\mu + \mu\big((E_n \smallsetminus F_n) \cap [0,n]\big) \\
&\quad + 0 \\
&= -\mu\big(F_n \cap [-n,0]\big) + \frac{\epsilon\mu\big(F_n \cap [-n,0]\big)}{2\mu(F_n)} + \mu\big([0,n]\big) \\
&= -\mu\big(F_n \cap [-n,0]\big) + \frac{\epsilon\mu\big(F_n \cap [-n,0]\big)}{2\mu(F_n)} + \mu\big([-n,0]\big) \\
&= \mu\big([-n,0] \smallsetminus (F_n \cap [-n,0])\big) + \frac{\epsilon\mu\big(F_n \cap [-n,0]\big)}{2\mu(F_n)}
\end{aligned}
$$

$$\leq \mu(E_n \setminus F_n) + \frac{\epsilon}{2}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon.$$

Together, the two strings of inequalities above tell us that $\left|\int_{E_n} s_n \, d\mu\right| < \epsilon$. In other words, given $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that $\left|\int_{E_n} s_n \, d\mu\right| < \epsilon$ for all $n \geq \mathbb{N}$.Thus, we have that $\lim_{n\to\infty} \int_{E_n} s_n \, d\mu = 0$. Hence f is generalized Lebesgue integrable with respect to $\{E_n\}$ and $\{E_n\}$-$\int_X f \, d\mu = 0$. This completes our proof that $(\mathscr{I}, \mathrm{I})$ is a nonabsolutely convergent abstract μ-integral.

CHAPTER IV

CONCLUSIONS

In this thesis, we have defined the generalized Lebesgue integral on an arbitrary σ-finite measure space by using the concepts of expanding sequences and semi-uniform convergence. We have shown that the generalized Lebesgue integral is always an abstract μ-integral, and given an example of a generalized Lebesgue integral which is nonabsolutely convergent using the standard Lebesgue measure space on $\mathbb{R}$.

The definition of generalized Lebesgue integral we have given may not be the best possible definition. It can be observed that it depends on a designated sequence of measurable sets. Thus, a single function may have many different integrals when we choose different sequences of measurable sets. It would be better if the definition could be improved so that the integral of a given function is unique. The key to an improved definition is probably a better concept of semiuniform convergence, or perhaps even an alternate type of convergence. Also, the relationship between the generalized Riemann integral on $\mathbb{R}$ and the generalized Lebesgue integral using the standard Lebesgue measure on $\mathbb{R}$ should be considered.

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