INTRODUCTION

Mathematicians have studied commutative and noncommutative rings for more than 200 years. All mathematicians assumed that addition is commutative. The purpose of this research is to study objects which behave like rings except we do not assume addition is commutative which is called skewrings.

We want to see which theorems in ring theory can be generalize to skewring theory.

Our first discovery is that normal ideals, not ideals are the most important objects for us. Using normal ideals we can generalize many theorems in ring theory. For example, if R is a skewring and I is a normal ideal in R, then R/I has a natural multiplication and addition such that $\pi : R \to R/I$ is an epimorphism.

In Chapter I, we introduce some notations, give definitions, examples and fundamental theorems concerning skewrings.

In Chapter II, we give the definition of a quotient skewring and prove some theorems about quotient skewrings. Moreover, we generalize the four basic isomorphism theorems and the Jordan-Holder theorem for skewrings.

In Chapter III, we give some definitions and theorems of sums and products of skewrings. Moreover, we generalize the Krull-Schmidt theorem to skewrings.

In Chapter IV, we generalize theorems from ring theory to skewring theory, for example, the Levitzki theorem.

CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this research. Our notations are as follows:

Z is the set of all integers,

Z^+ is the set of all positive integers,

Id_A is the identity function on a set A,

$A \subset B$ means that A is a proper subset of B.

For any family of sets $\{I_\alpha / \alpha \in A\}$, every element of $\sum_{\alpha \in A} I_\alpha$ can be represented as the sum of a finite numbers of elements, each in some I_α.

Definition 1.1. *Let $(P,\leq)$ be a partially ordered set, P is called a **lower[upper] semilattice** if and only if $\inf\{x,y\}[\sup\{x,y\}]$ exists for all $x,y \in P$ and is denoted by $x \wedge y[x \vee y]$. A lattice P is said to be a **lattice** if and only if P is both a lower and upper semilattice. A lattice P is said to be a **modular lattice** if and only if for all $x,y,z \in P$, if $x \geq y$ then $x \wedge (y \vee z) = (x \wedge y) \vee z$.*

Definition 1.2. *Let $(P,\leq)$ and $(P',\leq')$ be partially ordered sets. A function $f: P \to P'$ is said to be **isotone** if and only if $x \leq y$ implies that $f(x) \leq' f(y)$ for all $x,y \in P$. f is said to be an **order-isomorphism** if and only if f is a bijection and is isotone and f^{-1} is isotone. In this case P is said to be **order-isomorphic to P'**.*

Remark 1.3. *Let $(P,\leq)$ and $(P',\leq')$ be partially ordered sets. Let $f: P \to P'$ be an order-isomorphism. If P is a lattice, then P' is a lattice.*

Definition 1.4. *A triple $(R,+,\cdot)$ is a **skewring** if and only if*

(I) $(R,+)$ is a group and 0 denotes its identity,

(2) $(R,\cdot)$ is a semigroup and

(3) for all $x,y,z\in R$, $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$.

It is clear that every ring is a skewring.

Remark 1.5. *Let R be a skewring. Then the following statements hold:*

(1) For every $x\in R$, $0\cdot x = x\cdot 0 = 0$.

(2) For all $x,y,w,z\in R$, $xy+wz = wz+xy$.

(3) If R has a left or a right multiplicative identity, then R is a ring.

Proof. Let $x,y,w,z\in R$.

(1) Same proof as for rings.

(2) Since $xz+xy+wz+wy = x(z+y)+w(z+y) = (x+w)(z+y) = (x+w)z+(x+w)y = xz+wz+xy+wy$, $xy+wz = wz+xy$.

(3) It follows from (2). #

Example 1.6. *Let $(R,+)$ be a group. Define a binary operation $\cdot$ on R as follows: for all $x,y\in R$, $xy = 0$.*

*Then $(R,+,\cdot)$ is a skewring. This skewring is called a **skewring with the trivial multiplication**.*

Example 1.7. *Let $(R,+,\cdot)$ be a noncommutative ring such that $R^3 = \{0\}$. Define a binary operation $\oplus$ on R by $x\oplus y = x+y+xy$ for all $x,y\in R$.*

Then $(R,\oplus,\cdot)$ is a skewring. (See Chapter II for an example of a noncommutative ring with the property that $R^3 = \{0\}$.)

Proof. Let $x,y,z\in R$. At first, we shall show that $(R,\oplus)$ is a group. Consider, $(x\oplus y)\oplus z = (x+y+xy)\oplus z = x+y+xy+z+(x+y+xy)z = x+y+xy+z+xz+yz+xyz$ and $x\oplus(y\oplus z) = x\oplus(y+z+yz) = x+y+z+yz+x(y+z+yz) = x+y+z+yz+xy+xz+xyz$. Since $(R,+)$ is commutative, $(x\oplus y)\oplus z = x\oplus(y\oplus z)$, so the associative law is true for $(R,\oplus)$. Since $x\oplus 0 = x+0+x\cdot 0 = x$, 0 is a right identity of $(R,\oplus)$. Note that

$x\oplus(-x+x^2) = x-x+x^2+x(-x+x^2) = x^2-x^2+x^3$. By assumption we have that $x^3 = 0$ which implies that $(-x+x^2)$ is a right inverse of x. Therefore $(R,\oplus)$ is a group. Since there exist $x,y\in R$ such $xy \neq yx$, $(R,\oplus)$ is not abelian.

Next, we shall show that the distributive law is true for $(R,\oplus,\cdot)$. Consider, $(x\oplus y)z = (x+y+xy)z = xz+yz+xyz = xz+yz+0 = xz+yz+xzyz = xz\oplus yz$ and $x(y\oplus z) = x(y+z+yz) = xy+xz+xyz = xy+xz+0 = xy+xz+xyxz = xy\oplus xz$. Hence $(R,\oplus,\cdot)$ is a skewring. #

Example 1.8. *Let $(G,+)$ be a nonabelain group, K an abelain subgroup of G and X be a set such that $X\cap G = \varnothing$ and $|X|>1$.*

Let $Map(G,X,K) = \{f:G\cup X\to G \,/\, f|_G:G\to K$ is a homomorphism.$\}$.

For all $f,g\in Map(G,X,K)$, define $(f+'g)(x) = f(x)+g(x)$ and $(f\cdot g)(x) = (f\circ g)(x)$ for all $x\in G\cup X$. Then $(Map(G,X,K),+',\cdot)$ is a skewring which is not always a ring.

Proof. Since the zero map 0 belongs to Map(G,X,K), Map(G,X,K) $\neq \varnothing$. Let $f,g,h\in$ Map(G,X,K) and $x,y\in G$. At first, shall show that (Map(G,X,K),+') is a group. Clearly, $f+'g$ and $f\cdot g$ are functions of $G\cup X$ to G. Since (K,+) is abelian, $(f+'g)(x+y) = f(x+y)+g(x+y) = f(x)+f(y)+g(x)+g(y) = f(x)+g(x)+f(y)+g(y) = (f+'g)(x)+(f+'g)(y)$. Moreover, $f(g(x+y)) = f(g(x)+g(y)) = f(g(x))+f(g(y))$. Then $f+'g$ and $f\cdot g$ are maps whose restrictions to G are homomorphisms. Therefore $f+'g$, $f\cdot g \in$ Map(G,X,K). Clearly, 0 is an additive identity and -f is an inverse of f. Since G is a group, the associative law is true for (Map(G,X,K),+'). Hence (Map(G,X,K),+') is a group.

Next, we shall show that the distributive law is true for (Map(G,X,K),+',·). Since $f((g+'h)(x)) = f(g(x)+h(x)) = f(g(x))+f(h(x))$ and $(f+'g)(h(x)) = f(h(x))+g(h(x))$, $f(g+'h) = fg+'fh$ and $(f+'g)h = fh+'gh$. Hence (Map(G,X,K),+',·) is a skewring.

Note that f(x) is arbitrary for each $x\in X$. This implies that $f+'g$ may not be equal to $g+'f$. #

Example 1.9. *Let $(H, +')$ be a group such that $0'$ is its identity and $(K,+,\cdot)$ a skewring. Let $f:K\to Aut(H)$ be a group homomorphism and for every $k\in K$, let $f_k = f(k)$. Let $R = H\times K$. For all $(h_1,k_1),(h_2,k_2)\in R$, we define $(h_1,k_1)\oplus(h_2,k_2) = (h_1+'f_{k_1}(h_2),k_1+k_2)$ and $(h_1,k_1)\otimes(h_2,k_2) = (0',k_1k_2)$. Then $(R,\oplus,\otimes)$ is a skewring.*

Proof. Clearly, $(R,\otimes)$ is a semigroup. Let $(h_1,k_1),(h_2,k_2),(h_3,k_3)\in R$.

First, we shall show that $(R,\oplus)$ is a group. Consider,

$$\begin{aligned}(h_1,k_1)\oplus[(h_2,k_2)\oplus(h_3,k_3)] &= (h_1,k_1)\oplus(h_2+'f_{k_2}(h_3),k_2+k_3)\\ &= (h_1+'f_{k_1}(h_2+'f_{k_2}(h_3)),k_1+(k_2+k_3))\\ &= (h_1+'f_{k_1}(h_2)+'f_{k_1}(f_{k_2}(h_3)),(k_1+k_2)+k_3)\\ &= (h_1+'f_{k_1}(h_2)+'f_{k_1+k_2}(h_3),(k_1+k_2)+k_3)\\ &= (h_1+'f_{k_1}(h_2),k_1+k_2)\oplus(h_3,k_3)\\ &= [(h_1,k_1)\oplus(h_2,k_2)]\oplus(h_3,k_3).\end{aligned}$$

Then the associative law is true for $(R,\oplus)$.

Since $(h_1,k_1)\oplus(0',0) = (h_1+'f_{k_1}(0'),k_1+0) = (h_1+'0',k_1) = (h_1,k_1)$, $(0',0)$ is a right additive identity. Since $(h_1,k_1)\oplus(f_{k_1}^{-1}(-h_1),-k_1) = (h_1+'f_{k_1}(f_{k_1}^{-1}(-h_1)),k_1-k_1) = (h_1+'Id_H(-h_1),0) = (h_1+'(-h_1),0) = (0',0)$, $(f_{k_1}^{-1}(-h_1),-k_1)$ is a right inverse of (h_1,k_1). Hence $(R,\oplus)$ is a group.

Next, we shall show that the distributive law is true for $(R,\oplus,\otimes)$. Consider, $[(h_1,k_1)\oplus(h_2,k_2)]\otimes(h_3,k_3) = (h_1+'f_{k_1}(h_2),k_1+k_2)\otimes(h_3,k_3) = (0',(k_1+k_2)k_3) = (0',(k_1k_3+k_2k_3)) = (0'+'0',k_1k_3+k_2k_3) = (0'+'f_{k_1k_3}(0'),k_1k_3+k_2k_3) = (0',k_1k_3)\oplus(0',k_2k_3) = [(h_1,k_1)\otimes(h_3,k_3)]\oplus[(h_2,k_2)\otimes(h_3,k_3)]$ and $(h_1,k_1)\otimes[(h_2,k_2)\oplus(h_3,k_3)] = (h_1,k_1)\otimes(h_2+'f_{k_2}(h_3),k_2+k_3) = (0',k_1(k_2+k_3)) = (0',(k_1k_2+k_1k_3)) = (0'+'0',k_1k_2+k_1k_3) = (0'+'f_{k_1k_2}(0'),k_1k_2+k_1k_3) = (0',k_1k_2)\oplus(0',k_1k_3) = [(h_1,k_1)\otimes(h_2,k_2)]\oplus[(h_1,k_1)\otimes(h_3,k_3)]$.

Hence $(R,\oplus,\otimes)$ is a skewring. #

Example 1.10. *Let $\{R_\alpha / \alpha \in A\}$ be a nonempty family of skewrings. Then the direct product of $\{R_\alpha / \alpha \in A\}$ is a skewring under addition and multiplication componentwise.*

Example 1.11. *Let R be a skewring. Then the set of all $n \times n$ metrices over R, M(n,R), under the usual addition and multiplication of metrices is a skewring.*

Proof. It can be proved in the same way as is done for rings. #

Example 1.12. *Let R be a skewring and R[x] be the set of all polynomials which are of the form $(a_0,a_1,\ldots)$ where $a_i \in R$ for all i. For all $p_1 = (a_0,a_1,\ldots)$ and $p_2 = (b_0,b_1,\ldots)$, we define $p_1+p_2 = (a_0+b_0,a_1+b_1,\ldots)$ and $p_1 \cdot p_2 = (c_0,c_1,\ldots)$ where $c_i = \sum_{0 \le j \le i} a_j b_{i-j}$. Then $(R[x],+,\cdot)$ is a skewring.*

Proof. It can be proved in the same way as is done for rings. #

Example 1.13. *Let R be a skewring. Define the binary operations on R^2 as followss: for all $(x_1,y_1),(x_2,y_2) \in R^2$, $(x_1,y_1)+(x_2,y_2) = (x_1+x_2\,,\,y_1+y_2)$ and $(x_1,y_1)(x_2,y_2) = (x_1x_2-y_1y_2\,,\,x_1y_2+y_1x_2)$. Then $(R^2,+,\cdot)$ is a skewring.*

Proof. Clearly, $(R^2,+)$ is a group. Let $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in R^2$. First, we shall show that $(R^2,\cdot)$ is a semigroup. Consider,

$$
\begin{aligned}
[(x_1,y_1)(x_2,y_2)](x_3,y_3) &= (x_1x_2-y_1y_2\,,\,x_1y_2+y_1x_2)(x_3,y_3) \\
&= ((x_1x_2-y_1y_2)x_3-(x_1y_2+y_1x_2)y_3\,,\,(x_1x_2-y_1y_2)y_3+(x_1y_2+y_1x_2)x_3) \\
&= (x_1x_2x_3-y_1y_2x_3-y_1x_2y_3-x_1y_2y_3\,,\,x_1x_2y_3-y_1y_2y_3+x_1y_2x_3+y_1x_2x_3) \\
&= (x_1x_2x_3-x_1y_2y_3-y_1y_2x_3-y_1x_2y_3\,,\,x_1x_2y_3+x_1y_2x_3+y_1x_2x_3-y_1y_2y_3) \\
&= (x_1(x_2x_3-y_2y_3)-y_1(x_2y_3+y_2x_3)\,,\,x_1(x_2y_3+y_2x_3)+y_1(x_2x_3-y_2y_3)) \\
&= (x_1,y_1)(x_2x_3-y_2y_3\,,\,x_2y_3+y_2x_3) \\
&= (x_1,y_1)[(x_2,y_2)(x_3,y_3)]
\end{aligned}
$$

Therefore $(R^2,\cdot)$ is a semigroup.

Next, we shall show that the distributive law is true for $(R^2,+,\cdot)$. Consider,

$$\begin{aligned}(x_1,y_1)[(x_2,y_2)+(x_3,y_3)] &= (x_1,y_1)(x_2+x_3,y_2+y_3)\\ &= (x_1(x_2+x_3)-y_1(y_2+y_3)\ ,\ x_1(y_2+y_3)+y_1(x_2+x_3))\\ &= (x_1x_2+x_1x_3-y_1y_3-y_1y_2\ ,\ x_1y_2+x_1y_3+y_1x_2+y_1x_3)\\ &= (x_1x_2-y_1y_2+x_1x_3-y_1y_3\ ,\ x_1y_2+y_1x_2+x_1y_3+y_1x_3)\\ &= (x_1x_2-y_1y_2\ ,\ x_1y_2+y_1x_2)+(x_1x_3-y_1y_3\ ,\ x_1y_3+y_1x_3)\\ &= (x_1,y_1)(x_2,y_2)+(x_1,y_1)(x_3,y_3) \quad \text{and}\end{aligned}$$

$$\begin{aligned}[(x_1,y_1)+(x_2,y_2)](x_3,y_3) &= (x_1+x_2,y_1+y_2)(x_3,y_3)\\ &= ((x_1+x_2)x_3-(y_1+y_2)y_3)\ ,\ (x_1+x_2)y_3+(y_1+y_2)x_3)\\ &= (x_1x_3+x_2x_3-y_2y_3-y_1y_3\ ,\ x_1y_3+x_2y_3+y_1x_3+y_2x_3)\\ &= (x_1x_3-y_1y_3+x_2x_3-y_2y_3\ ,\ x_1y_3+y_1x_3+x_2y_3+y_2x_3)\\ &= (x_1x_3-y_1y_3\ ,\ x_1y_3+y_1x_3)+(x_2x_3-y_2y_3\ ,\ x_2y_3+y_2x_3)\\ &= (x_1,y_1)(x_3,y_3)+(x_2,y_2)(x_3,y_3).\end{aligned}$$

Hence $(R^2,+,\cdot)$ is a skewring. #

Example 1.14. *Let R be a skewring. Define the binary operations on R^4 as follows: for all $(x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)\in R^4$,*

$(x_0,x_1,x_2,x_3)+(y_0,y_1,y_2,y_3) = (x_0+y_0,x_1+y_1,x_2+y_2,x_3+y_3)$ *and*

$(x_0,x_1,x_2,x_3)(y_0,y_1,y_2,y_3) =$

$(x_0y_0-x_1y_1-x_2y_2-x_3y_3,x_0y_1+x_1y_0+x_2y_3-x_3y,x_0y_2+x_2y_0+x_3y_1-x_1y_3,x_0y_3+x_3y_0+x_1y_2-x_2y_1).$

Then $(R^4,+,\cdot)$ is a skewring.

Proof. Clearly, $(R^4,+)$ is a group. Let $(x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3),(z_0,z_1,z_2,z_3)\in R^4$

First, we shall show that $(R^4,\cdot)$ is a semigroup. Consider,

$$\begin{aligned}&[(x_0,x_1,x_2,x_3)(y_0,y_1,y_2,y_3)](z_0,z_1,z_2,z_3)\\ &= (x_0y_0-x_1y_1-x_2y_2-x_3y_3\ ,\ x_0y_1+x_1y_0+x_2y_3-x_3y_2\ ,\ x_0y_2+x_2y_0+x_3y_1-x_1y_3\ ,\\ &\quad x_0y_3+x_3y_0+x_1y_2-x_2y_1)(z_0,z_1,z_2,z_3)\end{aligned}$$

$= ((x_0y_0-x_1y_1-x_2y_2-x_3y_3)z_0-(x_0y_1+x_1y_0+x_2y_3-x_3y_2)z_1-(x_0y_2+x_2y_0+x_3y_1-x_1y_3)z_2-$

$(x_0y_3+x_3y_0+x_1y_2-x_2y_1)z_3 \ , \ (x_0y_0-x_1y_1-x_2y_2-x_3y_3)z_1+(x_0y_1+x_1y_0+x_2y_3-x_3y_2)z_0+$

$(x_0y_2+x_2y_0+x_3y_1-x_1y_3)z_3-(x_0y_3+x_3y_0+x_1y_2-x_2y_1)z_2 \ , \ (x_0y_0-x_1y_1-x_2y_2-x_3y_3)z_2+$

$(x_0y_2+x_2y_0+x_3y_1-x_1y_3)z_0+(x_0y_3+x_3y_0+x_1y_2-x_2y_1)z_1-(x_0y_1+x_1y_0+x_2y_3-x_3y_2)z_3 \ ,$

$(x_0y_0-x_1y_1-x_2y_2-x_3y_3)z_3+(x_0y_3+x_3y_0+x_1y_2-x_2y_1)z_0+(x_0y_1+x_1y_0+x_2y_3-x_3y_2)z_2-$

$(x_0y_2+x_2y_0+x_3y_1-x_1y_3)z_1)$

and $(x_0,x_1,x_2,x_3)[(y_0,y_1,y_2,y_3)(z_0,z_1,z_2,z_3)]$

$= (x_0,x_1,x_2,x_3)(y_0z_0-y_1z_1-y_2z_2-y_3z_3 \ , \ y_0z_1+y_1z_0+y_2z_3-y_3z_2 \ , \ y_0z_2+y_2z_0+y_3z_1-y_1z_3 \ ,$

$y_0z_3+y_3z_0+y_1z_2-y_2z_1)$

$= (x_0(y_0z_0-y_1z_1-y_2z_2-y_3z_3)-x_1(y_0z_1+y_1z_0+y_2z_3-y_3z_2)-x_2(y_0z_2+y_2z_0+y_3z_1-y_1z_3)-$

$x_3(y_0z_3+y_3z_0+y_1z_2-y_2z_1) \ , \ x_0(y_0z_1+y_1z_0+y_2z_3-y_3z_2)+x_1(y_0z_0-y_1z_1-y_2z_2-y_3z_3)+$

$x_2(y_0z_3+y_3z_0+y_1z_2-y_2z_1)-x_3(y_0z_2+y_2z_0+y_3z_1-y_1z_3) \ , \ x_0(y_0z_2+y_2z_0+y_3z_1-y_1z_3)+$

$x_2(y_0z_0-y_1z_1-y_2z_2-y_3z_3)+x_3(y_0z_1+y_1z_0+y_2z_3-y_3z_2)-x_1(y_0z_3+y_3z_0+y_1z_2-y_2z_1) \ ,$

$x_0(y_0z_3+y_3z_0+y_1z_2-y_2z_1)+x_3(y_0z_0-y_1z_1-y_2z_2-y_3z_3)+x_1(y_0z_2+y_2z_0+y_3z_1-y_1z_3)-$

$x_2(y_0z_1+y_1z_0+y_2z_3-y_3z_2))$.

Then $[(x_0,x_1,x_2,x_3)(y_0,y_1,y_2,y_3)](z_0,z_1,z_2,z_3) = (x_0,x_1,x_2,x_3)[(y_0,y_1,y_2,y_3)(z_0,z_1,z_2,z_3)]$.

Therefore $(R^4,\cdot)$ is a semigroup.

Next, we shall show that the distributive law is true for $(R^4,+,\cdot)$.

Consider, $[(x_0,x_1,x_2,x_3)+(y_0,y_1,y_2,y_3)](z_0,z_1,z_2,z_3) = (x_0+y_0,x_1+y_1,x_2+y_2,x_3+y_3)(z_0,z_1,z_2,z_3)$

$= ((x_0+y_0)z_0-(x_1+y_1)z_1-(x_2+y_2)z_2-(x_3+y_3)z_3 \ , \ (x_0+y_0)z_1+(x_1+y_1)z_0+(x_2+y_2)z_3-(x_3+y_3)z_2 \ ,$

$(x_0+y_0)z_2+(x_2+y_2)z_0+(x_3+y_3)z_1-(x_1+y_1)z_3 \ , \ (x_0+y_0)z_3+(x_3+y_3)z_0+(x_1+y_1)z_2-(x_2+y_2)z_1)$

and $(x_0,x_1,x_2,x_3)(z_0,z_1,z_2,z_3)+(y_0,y_1,y_2,y_3)(z_0,z_1,z_2,z_3)$

$= (x_0z_0-x_1z_1-x_2z_2-x_3z_3 \ , \ x_0z_1+x_1z_0+x_2z_3-x_3z_2 \ , \ x_0z_2+x_2z_0+x_3z_1-x_1z_3 \ , x_0z_3+x_3z_0+x_1z_2-x_2y_1) +$

$(y_0z_0-y_1z_1-y_2z_2-y_3z_3 \ , \ y_0z_1+y_1z_0+y_2z_3-y_3z_2 \ , \ y_0z_2+y_2z_0+y_3z_1-y_1z_3, y_0z_3+y_3z_0+y_1z_2-y_2z_1)$.

Then $[(x_0,x_1,x_2,x_3)+(y_0,y_1,y_2,y_3)](z_0,z_1,z_2,z_3) = (x_0,x_1,x_2,x_3)(z_0,z_1,z_2,z_3)+$

$(y_0,y_1,y_2,y_3)(z_0,z_1,z_2,z_3)$.

Consider, $(x_0,x_1,x_2,x_3)[(y_0,y_1,y_2,y_3)+(z_0,z_1,z_2,z_3)] = (x_0,x_1,x_2,x_3)(y_0+z_0,y_1+z_1,y_2+z_2,y_3+z_3)$

$= (x_0(y_0+z_0)-x_1(y_1+z_1)-x_2(y_2+z_2)-x_3(y_3+z_3) \ , \ x_0(y_1+z_1)+x_1(y_0+z_0)+x_2(y_3+z_3)-x_3(y_2+z_2) \ ,$

$x_0(y_2+z_2)+x_2(y_0+z_0)+x_3(y_1+z_1)-x_1(y_3+z_3) \ , \ x_0(y_3+z_3)+x_3(y_0+z_0)+x_1(y_2+z_2)-x_2(y_1+z_1))$

and $(x_0,x_1,x_2,x_3)(y_0,y_1,y_2,y_3)+(x_0,x_1,x_2,x_3)(z_0,z_1,z_2,z_3)$

$=(x_0y_0-x_1y_1-x_2y_2-x_3y_3\,,\,x_0y_1+x_1y_0+x_2y_3-x_3y_2, x_0y_2+x_2y_0+x_3y_1-x_1y_3, x_0y_3+x_3y_0+x_1y_2-x_2y_1)$
$+\,(x_0z_0-x_1z_1-x_2z_2-x_3z_3\,,\,x_0z_1+x_1z_0+x_2z_3-x_3z_2\,,\,x_0z_2+x_2z_0+x_3z_1-x_1z_3\,, x_0z_3+x_3z_0+x_1z_2-x_2z_1)$.
Then $(x_0,x_1,x_2,x_3)[(y_0,y_1,y_2,y_3)+(z_0,z_1,z_2,z_3)] = (x_0,x_1,x_2,x_3)(y_0,y_1,y_2,y_3)+$
$(x_0,x_1,x_2,x_3)(z_0,z_1,z_2,z_3)$.
Hence $(R^4,+,\cdot)$ is a skewring. #

From [6] we get the following skewring with a commutative multiplication and a noncommutative addition.

Example 1.15. *Let S_n be the symmetric group of degree n where $n>1$. Define the binary operations $+, \cdot$ on S_n as follows: for all $f,g\in S_n$, $f+g = f\circ g$,*

$$fg = \begin{cases} Id & \text{if } f \text{ is even or } g \text{ is even,} \\ (1\ 2) & \text{if } f \text{ and } g \text{ are odd,} \end{cases}$$

where Id is an identity function on $\{1,2,\dots,n\}$. Then $(S_n,+,\cdot)$ is a skewring.

Proof. Clearly, $(S_n,+)$ is a nonabelian group and $(S_n,\cdot)$ is commutative semigroup. Let $f,g,h\in S_n$. First, we shall show that $(S_n,\cdot)$ is a semigroup. If f,g,h are odd, then $(fg)h = (1\ 2)h = (1\ 2)$ and $f(gh) = f(1\ 2) = (1\ 2)$. Otherwise, $(fg)h = Id = f(gh)$. Then $(S_n,\cdot)$ is a semigroup.

Next, we shall show that the distributive law is true for $(S_n,+,\cdot)$. If f is even, then $f(g+h) = Id$ and $fg+fh = Id+Id = Id\circ Id = Id$. If f is odd, then we consider 4 cases as follows:

Case1. g and h are odd. Then $g\circ h$ is even. Thus $f(g+h) = f(g\circ h) = Id$ and $fg+fh = (1\ 2)+(1\ 2) = (1\ 2)(1\ 2) = Id$.

Case2. g is even and h is odd. Then $g\circ h$ is odd. Thus $f(g+h) = f(g\circ h) = (1\ 2)$ and $fg+fh = Id+(1\ 2) = Id\circ(1\ 2) = (1\ 2)$.

Case3. g is odd and h is even. Then $g\circ h$ is odd. Thus $f(g+h) = f(g\circ h) = (1\ 2)$ and $fg+fh = (1\ 2)+Id = (1\ 2)\circ Id = (1\ 2)$.

Case4. g and h are even. Then $g\circ h$ is even. Thus $f(g+h) = f(g\circ h) = Id$ and $fg+fh = Id+Id = Id\circ Id = Id$.

Therefore $f(g+h) = fg+fh$ and $(f+g)h = h(f+g) = hf+hg = fh+gh$. Hence $(S_n,+,\cdot)$ is a skewring which has the commutative multiplication. #

Definition 1.16. *Let R be a skewring and I be a nonempty subset of R.*

(1) If I is a skewring under the operations of R, then I is a ***subskewring*** *of R, and it is denote by* $I \leq R$.

(2) If I is a subskewring of R and $\{yx / x \in I, y \in R\} \subseteq I$ $[\{xy / x \in I, y \in R\} \subseteq I]$, *then I is a* ***left[right] ideal*** *of R.*

If I is both left ideal and right ideal, then I is a ***two-sided ideal*** *or* ***ideal*** *of R.*

(3) If I is a subskewring of R and $\{r+x-r / r \in R, x \in I\} \subseteq I$, *then I is a* ***normal subskewring*** *of R.*

(4) If I is a left[right] ideal of R and I is normal, then I is a ***left [right] normal ideal*** *of R, and it is denoted by* $I \lhd_{ln} R [I \lhd_{rn} R]$.

If I is both a left normal ideal and a right normal ideal, then I is a ***two-sided normal*** *ideal or* ***normal ideal****, and it is denoted by* $I \lhd_n R$.

Note. *An arbitrary intersection of subskewrings is a subskewring and an arbitrary intersection of left[right, two-sided] normal ideals is a left[right, two-sided] normal ideal.*

Definition 1.17. *A skewring R is* ***simple*** *if and only if {0} and R are the only normal ideal of R.*

Example 1.18. *(1) For any skewring R, {0} and R are normal ideals of R.*

(2) Let R be a skewring and $B = \{x \in R / x^n = 0 \text{ for some } n \in Z^+\}$. *If* $(R,\cdot)$ *is commutative, then B is a normal ideal of R.*

Proof. Clearly, $0\in B$. Let $x,y\in B$, $r\in R$. Then there exist $m,n\in \mathbf{Z}^+$ such that $x^n=y^m=0$. Then $(xy)^{m+n}=x^{m+n}y^{m+n}=0$ and $(x+y)^{m+n}=$

$x^{m+n}+\binom{m+n}{1}x^{m+n-1}y+\ldots+\binom{m+n}{m+n-1}xy^{m+n-1}+y^{m+n}=0$ which imply that

xy, $x+y\in B$. If n is even, $(-x)^n=x^n=0$ and otherwise, $(-x)^n=-x^n=0$,so that $-x\in B$. Therefore B is a subskewring of R. Since $(rx)^n = r^nx^n=0$, $rx\in B$. Similarly, $xr\in B$. Then B is an ideal of R.

Claim that for every $t\geq 2$, $(r+x-r)^t=x^t$. $(r+x-r)^2=(r+x-r)(r+x-r)=$ $r^2+rx-r^2+xr+x^2-xr-r^2-rx+r^2=x^2$. Assume that for some $k\geq 2$, $(r+x-r)^k=x^k$. Then $(r+x-r)^{k+1}=(r+x-r)^k(r+x-r)=x^k(r+x-r)=x^kr+x^{k+1}-x^kr=x^{k+1}$. By math induction, we have the claim. Then $(r+x-r)^n=x^n=0$,so that $r+x-r\in B$. Hence B is a normal ideal of R. #

Definition 1.19. *Let R be a skewring and* $A\subseteq R$.

The left[right, two-sided] normal ideal of R which is generated by A *is the intersection of all left[right, two-sided] normal ideals of R which contains A, and it is denoted by* $\langle A\rangle_{ln}[\langle A\rangle_{rn},\langle A\rangle_{n}]$, *hence* $\langle A\rangle_{ln}[\langle A\rangle_{rn},\langle A\rangle_{n}]$ *= the smallest left[right, two-sided] normal ideal of R which contains A.*

For $a_1,\ldots,a_m\in R$, *denote* $\langle\{a_1,\ldots,a_m\}\rangle_{ln}$ *by* $\langle a_1,\ldots,a_m\rangle_{ln}$. *For right and two-sided normal ideals are defined similarly.*

For each $A\subseteq R$, let $X=\{\sum_{i=1}^{m}(x_i+r_ia_is_i-x_i)/\, m\in\mathbf{Z}^+,\ x_i\in R,\ r_i,s_i\in R\cup\mathbf{Z},$ $a_i\in A$ for every $i\in\{1,\ldots,m\}\}$ where $na=an$ for all $a\in R, n\in\mathbf{Z}$. We shall show that $X=\langle A\rangle_n$. Clearly, $X\subseteq\langle A\rangle_n$ and $A\subseteq X$. Let $x\in X$. Then there exist $m\in\mathbf{Z}^+$, $x_i\in R$, $r_i,s_i\in R\cup\mathbf{Z}$, $a_i\in A$ for every $i\in\{1,\ldots,m\}$ such that $x=\sum_{i=1}^{m}(x_i+r_ia_is_i-x_i)$.

Let $r\in R$. Then $rx=r\sum_{i=1}^{m}(x_i+r_ia_is_i-x_i)=\sum_{i=1}^{m}(rx_i+rr_ia_is_i-rx_i) = \sum_{i=1}^{m}(rr_ia_is_i)$, by Remark 1.5 (2). Then $rx\in X$ which implies that $(X,\cdot)$ is a semigroup. Similarly, Remark 1.5 (2). Then $rx \in X$ which implies that $(X,\cdot)$ is a semigroup. Similarly, $xr \in X$. Therefore X is an ideal. Consider, $r+x-r = r+\sum_{i=1}^{m}(x_i+r_i a_i s_i - x_i) - r =$

$r + x_1 + r_1a_1s_1 - x_1 - r_1 + \sum_{i=2}^{m-1}(r+x_i+r_i a_i s_i - x_i - r) + r + x_m + r_m a_m s_m - x_m - r_1 =$

$\sum_{i=1}^{m}((r+x_i)+r_i a_i s_i - (r+x_i))$. Then $r+x-r \in X$ which implies that X is a normal ideal of R. Since $X \subseteq \langle A \rangle_n$ and $A \subseteq X$, $X = \langle A \rangle_n$.

Similarly, we can prove that left[right] normal ideal of R which is generated by A are equal to $\{\sum_{i=1}^{m}(x_i+r_i a_i - x_i) / m \in \mathbf{Z}^+$, $x_i \in R$, $r_i \in R \cup \mathbf{Z}$, $a_i \in A$ for every $i \in \{1,\ldots,m\}\}$ $[\langle A \rangle_m = \{\sum_{i=1}^{m}(x_i+a_i r_i - x_i) / m \in \mathbf{Z}^+$, $x_i \in R$, $r_i \in R \cup \mathbf{Z}$, $a_i \in A$ for every $i \in \{1,\ldots,m\}\}]$.#

Definition 1.20. *Let R be a skewring. A normal ideal I of R is* ***finitely generated*** *if and only if there exist $a_1,\ldots,a_m \in R$ such that $I = \langle a_1,\ldots,a_m \rangle_n$.*

For left[right] normal ideals, the definition is defined similarly.

Definition 1.21. *Let R be a skewring. For all left[right, two-sided] normal ideals I,J of R, we define $IJ = \langle\{xy/x \in I,\ y \in J\}\rangle_{ln}$ $[IJ = \langle\{xy/x \in I,\ y \in J\}\rangle_{rn}$, $IJ = \langle\{xy/x \in I,\ y \in J\}\rangle_n]$.*

We shall show that Definition 1.21 is well-defined. It is sufficient to show that for any normal ideals I,J of a skew ring R, $\langle\{xy/x \in I,\ y \in J\}\rangle_{ln} = \langle\{xy/x \in I,\ y \in J\}\rangle_{rn} = \langle\{xy/x \in I,\ y \in J\}\rangle_n$.

Let $z \in \langle\{xy/x \in I,\ y \in J\}\rangle_n$. Then $z = \sum_{i=1}^{m}(x_i + r_i y_i z_i s_i - x_i)$ for some $m \in \mathbf{Z}^+$, $x_i \in R$, $r_i, s_i \in R \cup \mathbf{Z}$, $y_i \in I$, $z_i \in J$ for every $i \in \{1,\ldots,m\}$. Since J is an ideal, $z_i s_i \in J$ for every $i \in \{1,\ldots,m\}$ which implies that $z \in \langle\{xy/x \in I,\ y \in J\}\rangle_{ln}$ and $\langle\{xy/x \in I,\ y \in J\}\rangle_n$

$\subseteq \{xy/x\in I,\ y\in J\}\rangle_{ln}$. Clearly, $\langle\{xy/x\in I,\ y\in J\}\rangle_{ln} \subseteq \langle\{xy/x\in I,\ y\in J\}\rangle_n$ hence they are equal. Similarly, $\langle\{xy/x\in I,\ y\in J\}\rangle_m = \langle\{xy/x\in I,\ y\in J\}\rangle_n$.

Proposition 1.22. *For any left[right, two-sided] normal ideals I,J and K of a skewring R, (IJ)K = I(JK).*

Proof. Let I,J,K be normal ideals. By definition 1.21., IJ =

$\{\sum_{i=1}^{m}(x_i + r_i y_i z_i s_i - x_i) / m\in \mathbf{Z}^+, x_i\in R, r_i,s_i\in R\cup \mathbf{Z}, y_i\in I, z_i\in J$ for every $i\in\{1,\ldots,m\}\}$.

Claim that $\{xk/ x\in IJ, k\in K\}\subseteq I(JK)$. Let $x\in IJ$. Then there exist $m\in \mathbf{Z}^+$, $x_i\in R$, $r_i,s_i\in R\cup \mathbf{Z}$, $y_i\in I$, $z_i\in J$ for every $i\in\{1,\ldots,m\}$ such that $x = \sum_{i=1}^{m}(x_i + r_i y_i z_i s_i - x_i)$.

Let $k\in K$. Then $xk = \sum_{i=1}^{m}(x_i + r_i y_i z_i s_i - x_i)k = \sum_{i=1}^{m}(x_i k + r_i y_i z_i s_i k - x_i k) =$

$\sum_{i=1}^{m}(r_i y_i z_i s_i k)$, by Remark 1.5 (2). Then $xk\in I(JK)$ and hence we have the claim. Since (IJ)K is a normal ideal which is generated by $\{xk/ x\in IJ, k\in K\}$, $(IJ)K\subseteq I(JK)$. The converse is proved similarly. For left[right] normal ideals, we can prove similarly. #

Proposition 1.23. *For any normal ideal I of a skewring R and for any $m\in \mathbf{Z}^+$, $I^m = \langle\{x_1 \ldots x_m/ x_i\in I$ for every $i\in\{1, \ldots, m\}\}\rangle_n$.*

This proposition is similar for left[right] normal ideals.

Proof. We will prove by math induction on m.

If $m = 1$, obvious. Suppose that this proposition is true for $m\geq 1$.

By Proposition 1.22, $I^{m+1} = I^m I = \langle\{x_1\ldots x_m/ x_i\in I$ for every $i\in\{1,\ldots,m\}\}\rangle_n I$. Claim that $\langle\{x_1\ldots x_m/ x_i\in I$ for every $i\in\{1,\ldots,m\}\}\rangle_n I = \langle\{x_1\ldots x_{m+1}/ x_i\in I$ for every $i\in\{1,\ldots,m+1\}\}\rangle_n$. Let $y\in\langle\{x_1\ldots x_m/ x_i\in I$ for every $i\in\{1,\ldots,m\}\}\rangle_n$. Then there exist $k\in \mathbf{Z}^+$, $r_i\in R$, $s_i,t_i\in R\cup \mathbf{Z}$, $x_{1i},\ldots,x_{mi}\in I$ for every $i\in\{1,\ldots,k\}$ such that

$y = \sum_{i=1}^{k}(r_i + s_i x_{1i}\cdots x_{mi} t_i - r_i)$. Let $z\in I$. Then $yz = \sum_{i=1}^{k}(r_i z + s_i x_{1i}\cdots x_{mi} t_i z - r_i z) =$

$\sum_{i=1}^{k}(s_i x_{1i} \cdots x_{mi} t_i z)$, by Remark 1.5(2). Then $yz \in \langle\{x_1 \ldots x_{m+1} / x_i \in I$ for every $i \in \{1,\ldots,m+1\}\}\rangle_n$ which implies that $\langle\{x_1 \ldots x_m / x_i \in I$ for every $i \in \{1,\ldots,m\}\rangle_n I \subseteq \langle\{x_1 \ldots x_{m+1} / x_i \in I$ for every $i \in \{1,\ldots,m+1\}\}\rangle_n$. The converse is obvious. Therefore we have the claim. Hence this proposition is true. For left[right] normal ideals, we can prove it similarly. #

Proposition 1.24. *For any left[right, two-sided] normal ideals I,J and K of a skewring R, I(J+K) = IJ+IK.*

Proof. Let I,J and K be normal ideals. Since $\{i(j+k) / i \in I, j \in J, k \in K\} \subseteq IJ+IK$, $I(J+K) \subseteq IJ+IK$. Since $J,K \subseteq J+K$, $IJ, IK \subseteq I(J+K)$ which implies that $IJ+IK \subseteq I(J+K)$. Hence we have the proposition. #

Definition 1.25. *A proper left[right, two-sided] normal ideal M of a skewring R is called a **maximal left[right, two-sided] normal ideal** of R if and only if every left[right, two-sided] normal ideal I of R such that $M \subseteq I \subseteq R$ implies that $I = M$ or $I = R$.*

Remark 1.26. *If R is a finitely generated nonzero skewring (as normal ideal), then every left[right, two-sided] normal ideal $I \neq R$ is contained in a maximal left[right, two-sided] normal ideal.*

Proof. Let I be a proper normal ideal of R and L = {J/ J is a proper normal ideal of R which contains I}. Since $I \in L$, L is not empty. Let $\{J_\alpha\}_{\alpha\in\Lambda}$ be a nonempty chain in L. Clearly, $\bigcup_{\alpha\in\Lambda} J_\alpha$ is a normal ideal and. Claim that $\bigcup_{\alpha\in\Lambda} J_\alpha \neq R$. Suppose not. Since R is finitely generated as a normal ideal, there exist $x_1,\ldots,x_m \in R$ such that $R = \langle x_1,\ldots,x_m \rangle_n$. Then there exists $\alpha_0 \in \Lambda$ such that $x_1,\ldots,x_m \in J_{\alpha_0}$. Then $J_{\alpha_0} = R$, which contradicts $J_{\alpha_0} \in L$. Thus $\bigcup_{\alpha\in\Lambda} J_\alpha \neq R$ which implies that $\bigcup_{\alpha\in\Lambda} J_\alpha$ is an upper bound of $\{J_\alpha\}_{\alpha\in\Lambda}$. By Zorn's Lemma, L has a maximal element. Hence this remark is true. Similarly, we can prove for left [right] normal ideal. #

Moreover, every finite skewring has a maximal left[right, two-sided] normal ideal.

Definition 1.27. *A proper left[right, two-sided] normal ideal P of a skewring R is called a* ***prime left****[right, two-sided]* ***normal ideal*** *of R if and only if for any left[right, two-sided] normal ideals I,J of R, such that $IJ\subseteq P$ implies that $I\subseteq P$ or $J\subseteq P$.*

Definition 1.28. *Let R be a skewring and I be a normal ideal of R.*

(1) For each $x\in R$, x is called a ***nilpotent element*** *if and only if there exists $n\in \mathbb{Z}^+$ such that $x^n = 0$,*

(2) I is called a ***nilpotent normal ideal*** *if and only if there exists $n\in \mathbb{Z}^+$ such that $I^n = \{0\}$ and*

(3) I is called a ***normal nilideal*** *if and only if every element in I is a nilpotent.*

Left[right] nilpotent normal ideals and left[right] normal nilideals are defined similarly.

Definition 1.29. *Let R,S be skewrings and $f:R\to S$.*

(1) f is called a ***homomorphism*** *if and only if for all $x,y\in R$, $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$.*

(2) f is called an ***epimorphism*** *if and only if f is a surjective homomorphism.*

(3) f is called an ***isomorphism*** *if and only if f is an injective epimorphism. In this case R and S are said to be* ***isomorphic****, and is denoted*

by $R \cong S$.

(4) f is called an ***endomorphism*** *if and only if $R = S$ and f is a homomorphism.*

(5) f is called an ***automorphism*** *if and only if f is a bijective endomorphism.*

If $f:R \to S$ and $g:S \to T$ are homomorphisms of skewrings, it is easy to see that $g \circ f:R \to T$ is also a homomorphism. Likewise, the composition of monomorphisms is a monomorphism ; similarly for epimorphisms and isomorphisms. Moreover, $f(0) = 0$ and $f(-a) = -f(a)$ for every $a \in R$.

Definition 1.30. *Let R,S be skewrings. Then R is called a* ***quotient skewring of S*** *if and only if there exists an epimorphism $f:R \to S$. Denoted by (R,f).*

Let R be a skewring and I be a normal ideal of R. Let $R/I = \{x+I \,/\, x \in R\}$ and define the binary operations $+, \cdot$ on R/I as follows:

for all $x+I, y+I \in R/I$, $(x+I)+(y+I) = x+y+I$ and $(x+I)(y+I) = xy+I$.

Then $(R/I, +)$ is a group. We shall show that $\cdot$ is well-defined. Let $x+I, x'+I, y+I, y'+I \in R/I$ be such that $x+I = x'+I$ and $y+I = y'+I$. Since $xy \in (x'+I)(y'+I) \subseteq x'y'+x'I+y'I+I = x'y'+I$, $xy+I = x'y'+I$. Then $\cdot$ is well-defined and $(R/I, \cdot)$ is a semigroup. Let $x+I, y+I, z+I \in R/I$. Then $(x+I+y+I)(z+I) = (x+y+I)(z+I) = (x+y)z+I = xz+yz+I = xz+I+yz+I = (x+I)(z+I)+(y+I)(z+I)$ and $(x+I)(y+I+z+I) = (x+I)(y+z+I) = x(y+z)+I = xy+xz+I = (x+I)(y+I)+(x+I)(z+I)$. Hence the distributive law is true for $(R/I, +, \cdot)$ and it is a skewring.#

For any normal ideal I of a skewring R, define $\pi: R \to R/I$ *by* $\pi(x) = x+I$ *for every* $x \in R$. *Clearly,* π *is an epimorphism.* π *is called the* ***canonical epimorphism****. And we have* R/I *is a quotient skewing of R.*

Remark 1.31. *For any normal ideals I,J of a skewring R and for any* $n \in Z^+$, $(I/J)^n = I^n/J$.

Proof. It follows from Proposition 1.22 and Proposition 1.23. #

Definition 1.32. *Let R,S be skewrings and* $f: R \to S$ *be a homomorphism. The* ***kernel of f*** *is* $\{a \in R \,/\, f(a) = 0 \in S\}$ *and it is denoted by Ker(f).*

Remark 1.33. *Let* $f: R \to S$ *be a homomorphism of skewrings. Then*

(1) f is a monomorphism if and only if $Ker(f) = \{0\}$ *and*

(2) f is an isomorphism if and only if there is a homomorphism $g: S \to R$ *such that* $f \circ g = Id_S$ *and* $g \circ f = Id_R$.

Remark 1.34. *Let* $f: R \to S$ *be a hommomorphism of skewrings. Then Ker(f) is a normal ideal of R.*

Proof. It is well-known that Ker(f) is a normal subgroup of (R,+). Let $x,y \in Ker(f)$ and $r \in R$. Since $f(xy) = f(x)f(y) = 0$ and $f(rx) = f(r)f(x) = f(r) \cdot 0 = 0$, $xy, rx \in Ker(f)$. Similarly, $xr \in Ker(f)$. Hence Ker(f) is a normal ideal of R. #

Proposotion 1.35. *An ideal I of a skewring R is a normal ideal if and only if it is the kernel of a homomorphism.*

Proposition 1.36. *Let $f:R\to S$ be a homomorphism of skewrings. Then the following statements hold:*

(1) If I is a subskewring of R, then $f[I]$ is a subskewring of S.

(2) If f is surjective and I is a normal ideal of R, Then $f[I]$ is a normal ideal of S.

(3) If I is a subskewring of R which contains $Ker(f)$, then $f^{-1}[f[I]] = I$.

(4) If I' is a subskewring of S, then $f^{-1}[I']$ is a subskewring of R which contains $Ker(f)$.

(5) If I' is a normal ideal of S, then $f^{-1}[I']$ is a normal ideal of R which contains $Ker(f)$.

Furthermore, this proposition is true for left[right, two-sided] ideals and left[right] normal ideals.

Proof. (1) Let I be a subskewring of R. It is well-known that f[I] is a subgroup of (S,+). Let $x,y\in I$. Then $f(x)f(y) = f(xy)\in f[I]$. Hence f[I] is a subskewring of S.

(2) Suppose that f is surjective and let I be a normal ideal of R. It is well-known that f[I] is a normal subgroup of (S,+). By(1), f[I] is a subskewring of S. Let $s\in S$ and $x\in I$. Since f is surjective, there exists an $r\in R$ such that $f(r) = s$. Then $sf(x) = f(r)f(x) = f(rx)\in f[I]$. Similarly, $f(x)s\in f[I]$. Therefore f[I] is a normal ideal of R.

(3) Let I be a subskewring of R which contains Ker(f). Since I is a subgroup of (R,+), $f^{-1}[f[I]] = I$.

(4) Let I′ be a subskewring of S. It is well-known that $f^{-1}[I']$ is a subgroup of (R,+). Let $x,y\in f^{-1}[I']$. Then $f(x),f(y)\in I'$. Thus $f(xy) = f(x)f(y)\in I'$ which implies that $xy\in f^{-1}[I']$. Hence $f^{-1}[I']$ is a subskewring of R.

(5) Let I′ be a normal ideal of S. By (4), $f^{-1}[I']$ is a subskewring of R which contains Ker(f). It is well-known that $f^{-1}[I']$ is normal subgroup of (R,+). Let $r\in R$ and $x\in f^{-1}[I']$. Then $f(r)\in S$ and $f(x)\in I'$. Then $f(rx) = f(r)f(x)\in I'$. Similarly, $f(xr)\in I'$. Therefore $rx, xr\in f^{-1}[I']$ and hence $f^{-1}[I']$ is a normal ideal

of R which contains Ker(f). #

It is well-known that every group is isomorphic to a subgroup of S_X for some set X and every ring is isomorphic to a subring of End(A) for some abelian group A.

We shall show that Map(G,X,K) is a universal skewring in the following theorem .

Theorem 1.37. *Let R be a skewring. Then R isomorphic to a subskewring of Map(G,X,K) for some group G, some abelain subgroup K of G and some nonempty set X such that $X \cap G = \varnothing$.*

Proof. Let $(G,+) = (R,+)$ and $(K,+) = (R^2,+)$ where $R^2 = \{\sum_{i=1}^{n} x_i y_i / n \in \mathbf{Z}^+, x_i, y_i \in R$ for every $i \in \{1,\ldots,n\}\}$. By Remark 1.5 (2), (K,+) is abelian. Let X be a nonempty set such that $G \cap X = \varnothing$. For each $r \in R$, define $l_r: G \cup X \to G$ by

$$l_r(x) = \begin{cases} r & \text{if } x \in X, \\ rx & \text{if } x \in G. \end{cases}$$

Then l_r is well-defined for every $r \in R$, since $G \cap X = \varnothing$.

We shall show that $l_r|_G \in \mathrm{Hom}(G,K)$. By definition of l_r, $\mathrm{Im}(l_r|_G) \subseteq K$. Let $x,y \in G$. Then $l_r|_G(x+y) = r(x+y) = rx+ry = l_r|_G(x)+lr|_G(y)$. Therefore $l_r|_G \in \mathrm{Hom}(G,K)$.

Define $\Phi: R \to \mathrm{map}(G,X,K)$ by $\Phi(r) = l_r$ for every $r \in R$. We shall show that Φ is a monomorphism. Let $r_1, r_2 \in R$ and $x \in G \cup X$. If $x \in X$, then $l_{r_1+r_2}(x) = r_1+r_2 = l_{r_1}(x) + l_{r_2}(x)$ and $l_{r_1r_2}(x) = r_1r_2 = l_{r_1}(r_2) = l_{r_1}(l_{r_2}(x))$. If $x \in G$, then $l_{r_1+r_2}(x) = (r_1+r_2)x = r_1x+r_2x = l_{r_1}(x) + l_{r_2}(x)$ and $l_{r_1r_2}(x) = r_1r_2x = r_1 l_{r_2}(x) = l_{r_1}(l_{r_2}(x))$. Then $l_{r_1+r_2} = l_{r_1} + l_{r_2}$ and $l_{r_1r_2} = l_{r_1} \circ l_{r_2}$, so Φ is a homomorphism. Let $r \in \mathrm{Ker}(\Phi)$. Then $l_r = \Phi(r) = 0$. If $x \in X$, then $r = l_r(x) = 0$ which implies that $\mathrm{Ker}(\Phi) = \{0\}$. By Remark 1.31 (1), Φ is a monomorphism. #

Definition1.38. *A triple $(S,+,\cdot)$ is a skewsemifield if and only if*

(1) $(S,\cdot)$ is a group with 0,

(2) $(S,+)$ is a commutative semigroup and

(3) for all $x,y,z\in R$, $x(y+z)=xy+xz$ and $(y+z)x=yx+zx$.

Let S be a skewsemifield, $X\subseteq S$ and C be a normal subgroup of S^* where $S^*=S\setminus\{0\}$. Define $Co(X)=\{y\in S^*\,/\,yx+y'x'\in X$ for every $y'\in S$ such that $y+y'=1$ for all $x,x'\in X\}$ and $N_S(C)=\{y\in S^*\,/\,yCy^{-1}=C\}$.

Next, we shall generalize the following theorem from skewsemifield theory to the case of skewring.

Theorem 1.39. *Let H be a subskewsemifield of a skewsemifield S, C be a normal subgroup of S^*. Suppose that $(HC)^*\subseteq Co(C)$ and $H^*\subseteq N_S(C)$. Then $H/_{H\cap C}\cong HC/_C$.*

Proof. Claim1. HC is a subskewsemifield of S.

Let $h_1,h_2\in H$ and $c_1,c_2\in C$. If $h_1=0$ or $h_2=0$, then $(h_1c_1)(h_2c_2)=0\in HC$. If $h_1\neq 0$ and $h_2\neq 0$, then $(h_1c_1)(h_2c_2)=h_1h_2(h_2^{-1}c_1h_2c_2)\in HC$ since C is a normal subgroup and $(h_1c_1)(c_1^{-1}h_1^{-1})=1=(c_1^{-1}h_1^{-1})(h_1c_1)$ where $c_1^{-1}h_1^{-1}=h_1^{-1}(h_1c_1^{-1}h_1^{-1})\in HC$ since C is a normal subgroup. Therefore $((HC)^*,\cdot)$ is a group. If $h_1+h_2=0$, by Proposition 3.12 in [8], then $h_1=h_2=0$. So $h_1c_1+h_2c_2=0\in HC$. If $h_1+h_2\neq 0$, then $(h_1c_1+h_2c_2)=(h_1+h_2)((h_1+h_2)^{-1}h_1c_1+(h_1+h_2)^{-1}h_2c_2)\in HC$, since $(h_1+h_2)^{-1}h_1+(h_1+h_2)^{-1}h_2=1\in HC$ and $(HC)^*\subseteq Co(C)$. So that $(HC,+)$ is a semigroup. Therefore HC is a subskewsemifield of S and we have Claim1.

Claim2. C is a normal convex subgroup of $(HC)^*$.

Let $h_1,h_2\in H$ and $c,c_1,c_2\in C$ be such that $h_1c_1+h_2c_2=1$. Since $(HC)^*\subseteq Co(C)$, $h_1c_1c+h_2c_2=h_1c_1c+h_2c_2\cdot 1\in C$. Hence C is a normal convex subgroup of $(HC)^*$ and hence we have Claim2.

Define $f:H\to HC/_C$ by $f(h)=hC$ for every $h\in H$. Let $h_1,h_2\in H$. Then

$f(h_1+h_2) = (h_1+h_2)C = h_1C+h_2C = f(h_1)+f(h_2)$ and $f(h_1h_2) = (h_1h_2)C = h_1Ch_2C = f(h_1)f(h_2)$, so f is a homomorphism. Let $h \in H$ and $c \in C$. Then $f(h) = hC = hcC$, so f is an epimorphism. By First Isomorphism Theorem in [9],

$$H/_{Ker(f)} \cong HC/_{C}.$$

Claim3. $Ker(f) = H \cap C$.

Let $x \in Ker(f)$. Then $x \in H$ and $xC = f(x) = C$, so $x \in C$ and $x \in H \cap C$. Thus $Ker(f) \subseteq H \cap C$. Let $x \in H \cap C$. Then $f(x) = xC = C$, so $x \in Ker(f)$ and $H \cap C \subseteq Ker(f)$. Hence we have Claim3.

Therefore this theorem is true. #