CHAPTER II

POSITIVE ORDERED 0-SEMIFIELDS

In this chapter, we shall give some fundamental theorems of a theory of positive ordered semifields.

Definition 2.1. Let $\leq$ be a partial order on a semiring S. $\leq$ is said to be compatible iff it satisfies the following property for every $x, y, z \in S$, $x \leq y$ implies that 1) $x + z \leq y + z$ and 2) $xz \leq yz$ and $zx \leq zy$ if $z \geq 0$.

Definition 2.2. A system $(R, +, ., \leq)$ is said to be a partially ordered semiring iff $(R, +, .)$ is a semiring and $\leq$ is a compatible partial order on R. If $0 \leq x$ for all $x \in R$ then we say that R is a positive ordered semiring.

Examples 2.3. (1) Z_0^+ is a positive ordered semiring.

(2) $Z_0^+[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in Z_0^+ \}$ is a positive ordered. semiring where $a + b\sqrt{2} \leq c + d\sqrt{2}$ iff $a \leq c$ and $b \leq d$ for all $a, b, c, d \in Z_0^+$.

(3) From (2), $Z_0^+[\sqrt{2}]$ has a natural partial order as a subset of $\mathbb{R}_0^+$ is a positive ordered semiring.

Definition 2.4. Let $(R, +, ., \leq)$ be a positive ordered semiring. R is said to be a positive ordered semifield iff $(R^*, .)$ is a group.

Remark 2.5. Let K be a positive ordered semifield. Then the following statements hold :

(1) for every nonzero elements $x, y \in K$, $x \le y$ implies $y^{-1} \le x^{-1}$.

(2) for every $x, y, z \in K$, $xz \le yz$ implies that $z = 0$ or $x \le y$.

Examples 2.6. (1) Q_0^+, $\mathbb{R}_0^+$ are positive ordered semifields.

(2) Let K be a semifield such that $1 + 1 = 1$. Define a relation $\le$ on K by $x \le y$ if and only if $x + y = y$ for all $x, y \in K$. Then we have that $\le$ is a partial order on K. To show that $\le$ is a compatible, let $x, y \in K$ be such that $x \le y$. Then $x + y = y$. Let $z \in K$. Then $(x + z) + (y + z) = (x + y) + z = y + z$ and $xz + yz = (x + y)z = yz$ which imply that $(x + z) \le (y + z)$ and $xz \le yz$. Thus $\le$ is a compatible. Since $0 + x = x$ for all $x \in K$, $0 \le x$ for all $x \in K$.

Therefore K is a positive ordered semifield.

(3) Let K be a semifield which is additively cancellative. Define a relation $\le$ on K by $x \le y$ if and only if there exists $z \in K$ such that $x + z = y$. To show that $\le$ is a compatible partial order, clearly $\le$ is reflexive since $x + 0 = x$ for all $x \in K$. Let $x, y \in K$ be such that $x \le y$ and $y \le x$. Then there are $u, v \in K$ such that $x + u = y$ and $y + v = x$. Hence $y = x + u = (y + v) + u = y + (u + v)$. By A.C., $u + v = 0$ which implies that $u = v = 0$, so $x = y$. Let $x, y, z \in K$ be such that $x \le y$ and $y \le z$. Then there are $u, v \in K$ such that $x + u = y$ and $y + v = z$. Hence $x + (u + v) = (x + u) + v = y + v = z$, so $x \le z$. Next, let $x, y \in K$ be such that $x \le y$. Then there exists $u \in K$ such that $x + u = y$. Let $z \in K$. Thus $x + z + u = y + z$ and $xz + uz = (x + u)z = yz$, so $x + z \le y + z$ and $xz \le yz$. Therefore $\le$ is a compatible partial order on K. Obviously, $0 \le x$ for all $x \in K$. Hence K is a positive ordered semifield.

(4) Let K and L be positive ordered semifields. Define a relation $\le$ on $K^* \times L^* \cup \{(0, 0)\}$ by

$(x, y) \leq (z, w)$ if and only if $x \leq z$ and $y \leq w$ for all (x, y), $(z, w) \in K^* \times L^* \cup \{(0, 0)\}$. Then $K^* \times L^* \cup \{(0, 0)\}$ is a positive ordered semifield.

(5) Let K and L be positive ordered semifields such that K which is additively cancellative. Define a relation $\leq^*$ on $K^* \times L^* \cup \{(0, 0)\}$ by $(x, y) \leq^* (z, w)$ if and only if $x < z$ or $x = z$ and $y \leq w$ for all $(x, y), (z, w) \in K^* \times L^* \cup \{(0, 0)\}$. Then $K^* \times L^* \cup \{(0, 0)\}$ is a positive ordered semifield.

Note that the partial order $\leq$ defined in Example 2.6. (5), is called the lexicographic order.

Theorem 2.7. Let S be a positive ordered commutative semiring with multiplicative zero 0 having the M.C. property. If S satisfies that for every $x, y, z \in S$, $xz < yz$ implies that $x < y$ then S can be embeded into a positive ordered semifield.

Proof Using the construction of Theorem 1.28., we have that $K = \{S \times (S - \{0\})\}/_\sim$ is a semifield. Now define a relation $\leq$ on K as follows : let $\alpha, \beta \in K$ $\alpha \leq \beta$ iff there exists $(a, b) \in \alpha$ and $(c, d) \in \beta$ such that $ad \leq bc$. To show that $\leq$ is a partial order, clearly $\leq$ is reflexive. Let $\alpha, \beta \in K$ be such that $\alpha \leq \beta$ and $\beta \leq \alpha$. Then there are $(a, b), (c, d) \in \alpha$ and $(e, f), (g, h) \in \beta$ such that $af \leq be$, $dg \leq ch$, $ad = bc$ and $eh = fg$. Since $gd \leq ch$, $bcg = adg \leq ach$. Since $bcg \leq ach$, $bg \leq ah$. Since $eh = fg$, $bge \leq ahe = afg$. Then $be \leq af$. Since $af \leq be$ and $be \leq af$, $be = af$. Hence $\alpha = \beta$, so $\leq$ is anti-symetric.

Let $\alpha, \beta, \gamma \in K$ be such that $\alpha \leq \beta$ and $\beta \leq \gamma$. Then there are $(a , b) \in \alpha$, $(c , d), (e , f) \in \beta$, $(g , h) \in \gamma$ such that $ad \leq bc$, $eh \leq fd$. Since $eh \leq fg$ and $cf = de$, $cfh = ehd \leq fgd$. Thus $ch \leq gd$, so $bch \leq bgd$. Since $ad \leq bc$ and $bch \leq bdg$, $adh \leq bch$. Therefore $ah \leq bg$. Hence $\alpha \leq \gamma$, so $\leq$ is transitive. Therefore $\leq$ is partial order. Let $\alpha, \beta \in K$ be such that $\alpha \leq \beta$. Then there are $(a , b) \in \alpha$ and $(c , d) \in \beta$ such that $ad \leq bc$. Let $\gamma \in K$.
Choose $(e , f) \in \gamma$. Since $ad \leq bc$, $adef \leq bcef$. Thus $\alpha\gamma = [(a , b)][(e , f)] = [(ae , bf)] \leq [(ce , df)] = [(c , d)][(e , f)] = \beta\gamma$. Since $ad \leq bc$, $adf \leq bcf$. Then $f(adf + bde) \leq f(bcf + bde)$, $df(af + be) \leq bf(cf + de)$. It follows that $\alpha + \gamma = [(a , b)] + [(e , f)] = [(af+be , bf)] \leq [(cf + de , df)] = [(c , d)] + [(e , f)] = \beta + \gamma$. Therefore $\leq$ is compatible. Clearly, $0 \leq \alpha$ for all $\alpha \in K$.

Therefore K is a positive ordered semifield. Fix $a \in S - \{0\}$. Define $f : S \to K$ by $f(x) = [(xa , a)]$ for all $x \in S$. Then f is a semiring homomorphism. To show that f is isotone, let $x, y \in S$ be such that $x \leq y$. Then $(xa)a \leq (ya)a$, so $f(x) = [(xa , a)] \leq [(ya , a)] = f(y)$. Hence f is isotone. #

Definition 2.8. Let C be a subset of a positive ordered semifield K. C is called a convex subset of K if is both an a-convex and o-convex subset of K.

Definiton 2.9. Let K be a positive ordered semifield.

The set $P = \{ x \in K \mid x \geq 1\}$ is called the positive cone of K.

Remark 2.10. Let P be the positive cone of a positive ordered semifield K. Then the following statements hold :

(1) If $P = \{1\}$ then $|K| = 2$.

(2) P is a multiplicative subsemigroup of K.

(3) $1 + x \in P$ for all $x \in K$, hence P is an additive ideal of K, that is $K + P \subseteq P$.

(4) P is a conic subset of K where $P^{-1} = \{x^{-1} \mid x \in P\}$.

(5) P is a convex subset of K.

(6) For every $x \in K^*$, $x = ab^{-1}$ for some $a, b \in P$.

(7) For every $x, y \in P$, $xy = 1$ implies that $x = y = 1$.

(8) If H is a subsemifield of K then $P_H = P \cap H$ where $P_H = \{ x \in H \mid x \geq 1\}$.

Proof (1) Assume that $P = \{1\}$. Let $x, y \in K^*$. Then $x \leq x + y$ and $y \leq x + y$. Thus $(x + y)x^{-1}$, $(x + y)y^{-1} \in P$. Since $P = \{1\}$, $(x + y)x^{-1} = (x + y)y^{-1} = 1$. Hence $x = y$.

(5) Let $x, y \in P$ and $a, b \in K$ be such that $a + b = 1$. Then $1 \leq x$ and $1 \leq y$, so $a \leq ax$ and $b \leq by$. Thus $1 = a + b \leq ax + by$. Therefore $ax + by \in P$. Hence P is an a-convex subset of K. Clearly P is an o-convex subset of K.

(6) Let $x \in K^*$. Then there $y \in K$ such that $1 \leq x$ and $x \leq y$. Thus $y, yx^{-1} \in P$. Hence $x = y(yx^{-1})^{-1}$ has indicated form. #

Theorem 2.11. Let K be a semifield and $P \subseteq K^*$. Suppose that P satisfies that

(1) P is a multiplicative subsemigroup of K^*,

(2) P is a conic subset of K,

(3) $1 + x \in P$ for all $x \in K$ and

(4) P is an a-convex subset of K.

Then there exists a unique positive compatible partial order on K induced by P such that P is the positive cone.

Proof Define $\leq_P$ on K as follows : for every $x, y \in K$,

$x \leq_P y$ if and only if $x = 0$ or $yx^{-1} \in P$.

Claim that $\leq_P$ is a partial order. Since $1 \in P$, $\leq_P$ is reflexive. Let $x, y \in K$ be such that $x \leq_P y$ and $y \leq_P x$. Then $x = 0$ or $yx^{-1} \in P$ and $y = 0$ or $xy^{-1} \in P$.

Case 1. $x = 0$ and $y = 0$. Then $x = y$.

Case 2. $x = 0$ and $xy^{-1} \in P$, a contradiction.

Case 3. $yx^{-1} \in P$ and $y = 0$. Similar to Case 2.

Case 4. yx^{-1}, $xy^{-1} \in P$. Then $xy^{-1} \in P \cap P^{-1}$. By (2), $xy^{-1} = 1$, so $x = y$.

Therefore $\leq_P$ is anti-symmetric. Let $x, y, z \in K$ be such that $x \leq_P y$ and $y \leq_P z$. Then $x = 0$ or $yx^{-1} \in P$ and $y = 0$ or $zy^{-1} \in P$. If $x = 0$ then done. Suppose that $x \neq 0$. Thus $yx^{-1} \in P$. If $y = 0$ then $0 = yx^{-1} \in P$, a contradiction. So $y \neq 0$, thus $zy^{-1} \in P$. By (1), $zx^{-1} = (zy^{-1})(yx^{-1}) \in P$. Hence $x \leq_P z$, so $\leq_P$ is transitive. So we have the claim.

To show that $\leq_P$ is a compatible, let $x, y \in K$ be such that $x \leq_P y$. Then $x = 0$ or $yx^{-1} \in P$. Let $z \in K$.

Case 1. $x = 0$.

Subcase 1.1 $z = 0$. Then $0 = x + z \leq_P y + z$ and $0 = xz \leq_P yz$.

Subcase 1.2 $z \neq 0$. Then $0 = xz \leq_P yz$. By (3), $1 + yz^{-1} \in P$. So $(y + z)z^{-1} \in P$, hence $x + z = z \leq_P y + z$.

Case 2. $x \neq 0$. Then $yx^{-1} \in P$.

Subcase 2.1 $z = 0$. Then $0 = xz \leq_P yz$ and $x + z = x \leq_P y = y + z$.

Subcase 2.2 $z \neq 0$. Then $(yz)(xz)^{-1} = yx^{-1} \in P$, so $xz \leq_P yz$. By (4), $(y + z)(x + z)^{-1} = [x(x + z)^{-1}](yx^{-1}) + z(x + z)^{-1} \in P$. Hence $x + z \leq_P y + z$.

Therefore $\leq_P$ is a compatible. Clearly, $0 \leq_P x$ for all $x \in K$ and P is the positive cone of K.

Hence K is a positive ordered semifield having P as a positive cone. To prove the uniqueness, let $\leq^*$ be a compatible partial order of K such P is the positive cone. Let $x, y \in K$ be such that $x \leq^* y$. If $x = 0$ then done. Suppose that $x \neq 0$, so $1 \leq^* yx^{-1}$. Thus $yx^{-1} \in P$, so $x \leq_P y$. Hence $\leq^* \subseteq \leq_P$. Similarly, $\leq_P \subseteq \leq^*$. Therefore $\leq_P = \leq^*$, so $\leq_P$ is the unique compatible partial order on K having P as its positive cone. #

Corollary 2.12. Let K be a semifield. Let $\mathscr{A}$ be the set of all subsets of K^* which satisfy (1) – (4) in the Theorem 2.11. and $\mathscr{B}$ the set of all positive compatible partial orders on K. Then there exists an order isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.

Proof Define $\varphi : \mathscr{A} \to \mathscr{B}$ as follows : let $P \in \mathscr{A}$. Then Theorem 2.11. determines a unique positive compatible partial order $\leq_P$ induced by P on K, define $\varphi(P) = \leq_P$. Clearly φ is a bijection. To prove that φ is isotone, let $P, Q \in \mathscr{A}$ be such that $P \subseteq Q$. Then there exist compatible partial orders $\leq_P$ and $\leq_Q$ such that $P = \{ x \in K \mid 1 \leq_P x \}$ and $Q = \{ x \in K \mid 1 \leq_Q x \}$, respectively. Let $x, y \in K$ be such that $x \leq_P y$. If $x = 0$ then done. Suppose that $x \neq 0$. Then $yx^{-1} \in P$. Since $P \subseteq Q$, $yx^{-1} \in Q$. Then $x \leq_Q y$. This prove that $\leq_P \subseteq \leq_Q$, so $\varphi(P) \subseteq \varphi(Q)$. Hence φ is isotone. It remains to prove that φ^{-1} is isotone, let $\leq, \leq^* \in \mathscr{B}$ be such that $\leq \subseteq \leq^*$. Let $x \in \varphi^{-1}(\leq)$. Then $1 \leq x$ since $\varphi^{-1}(\leq) = \{ y \in K \mid 1 \leq y \}$. Since $\leq \subseteq \leq^*$, $1 \leq^* x$. Thus $x \in \varphi^{-1}(\leq^*)$ where $\varphi^{-1}(\leq^*) = \{ y \in K \mid 1 \leq^* y \}$. Therefore $\varphi^{-1}(\leq) \subseteq \varphi^{-1}(\leq^*)$, so φ^{-1} is isotone. Hence φ is an order isomorphism. #

Definition 2.13. Let C be a convex subset of a positive ordered semifield K. C is said to be a convex subgroup of K if C is a multiplicative subgroup of K^*.

Definition 2.14. Let K and M be positive ordered semifields. A function $f : K \rightarrow M$ is called an order homomorphism of K into M if f is an isotone homomorphism of semifields.

An order homomorphism $f : K \rightarrow M$ is called an order monomorphism iff f is injection and $f(P_K) = P_{f(K)}$, an order epimorphism if f is onto and $f(P_K) = P_M$ and an order isomorphism if f is bijection and f^{-1} is isotone. If there exists an order isomorphism of K onto M then we say that K and M are order isomorphic, denoted by $K \cong_o M$.

Remark 2.15. Let $f : K \rightarrow M$ be an order homomorphism of positive ordered semifields. Then the following statements hold :

(1) $f(P_K) \subseteq P_M$.

(2) ker f is a convex subgroup of K.

(3) If C' is a convex subgroup of M then $f^{-1}(C')$ is a convex subgroup of K containing ker f.

Proof (1) Obviously.

(2) By Remark 1.37 (2), ker f is an a-convex subgroup of K. Let $x, y \in \ker f$ and $z \in K$ be such that $x \leq z \leq y$. Since f is isotone, $1 = f(x) \leq f(z) \leq f(y) = 1$. Hence $f(z) = 1$, so $z \in \ker f$. Therefore ker f is a convex subgroup of K.

(3) By Remark 1.37 (3), $f^{-1}(C')$ is an a-convex subgroup of K. Let $x \in f^{-1}(C')$ and $z \in K$ be such that $x \leq z \leq y$. Since f is isotone,

$f(x) \leq f(z) \leq f(y)$. By the o-convexity of C', $f(z) \in C'$. So $z \in f^{-1}(C')$, hence $f^{-1}(C')$ is a convex subgroup of K. #

Proposition 2.16. Let $f : K \to M$ be an order homomorphism of positive ordered semifields and f is a bijection. Then f^{-1} is isotone iff $f(P_K) = P_M$.

Proof Assume that f^{-1} is isotone. Clearly, $f(P_K) \subseteq P_M$. Let $y \in P_M$. Then $y \geq 1$. Since f is onto, $f(x) = y$ for some $x \in K$. Since f^{-1} is isotone, $1 = f^{-1}(1) \leq f^{-1}(y) = f^{-1}(f(x)) = x$. Thus $x \in P_K$, so $y \in f(P_K)$, it follows that $P_M \subseteq f(P_K)$. Therefore $f(P_K) = P_M$.

Conversely, assume that $f(P_K) = P_L$. Let $x, y \in L$ be such that $x \leq y$. If $x = 0$ then done. Suppose that $x \neq 0$. Then $yx^{-1} \in P_L$. By assumtion, there exists $p \in P_K$ such that $f(p) = yx^{-1}$. Since f is onto, there are $a, b \in K$ such that $f(a) = x$ and $f(b) = y$. Then $f(p) = yx^{-1} = f(b)f(a)^{-1} = f(ba^{-1})$. Since f is 1-1, $ba^{-1} \in P_K$. So $b \geq a$. Therefore $f^{-1}(x) = f^{-1}(f(a)) = a \leq b = f^{-1}(f(b)) = f^{-1}(y)$. Hence f^{-1} is an isotone. #

Let C be a convex subgroup of a positive ordered semifield K. Then $K/_C$ is a semifield. Define a relation $\leq$ on $K/_C$ as follows : for $\alpha, \beta \in K/_C$, define $\alpha \leq \beta$ if and only if there are $a \in \alpha$ and $b \in \beta$ such that $a \leq b$. To show that $\leq$ is a partial order, it is clear that $\leq$ is reflexive. Let $\alpha, \beta \in K/_C$ be such that $\alpha \leq \beta$ and $\beta \leq \alpha$. Then there are $a, d \in \alpha$ and $b, c \in \beta$ such that $a \leq b$ and $c \leq d$.

Case 1. $c = 0$. Then $b = 0$ since $b \in \beta = [0]$, so $a = 0$. Therefore $\alpha = aC = [0] = bC = \beta$.

Case 2. $d = 0$. Hence $c = 0$, so $\alpha = [0] = \beta$.

<u>Case 3</u>. $c \neq 0$ and $d \neq 0$. By definition of α and β, $bc^{-1}, ad^{-1} \in C$. Thus $ad^{-1} \leq bd^{-1} \leq bc^{-1}$. Since C is o-convex, $bd^{-1} \in C$. Hence $\beta = bC = dC = \alpha$. Therefore $\leq$ is anti-symmetric.

Let $\alpha, \beta, \gamma \in K/_C$ be such that $\alpha \leq \beta$ and $\beta \leq \gamma$. Then there exists $a \in \alpha$, $b, c \in \beta$ and $d \in \gamma$ such that $a \leq b$ and $c \leq d$. If $c = o$ then $b = 0$, so $a = 0$. Hence $\alpha = [0] \leq \gamma$. Suppose that $c \neq 0$. By definition of β, $bc^{-1} \in C$. Then $(bd)c^{-1} \in dC = \gamma$. Since $a \leq (bc)c^{-1} \leq (bd)c^{-1}$, $\alpha = aC \leq dC = \gamma$. Therefore $\leq$ is transitive, hence $\leq$ is a partial order.

Next, to show that $\leq$ is compatible, let $\alpha, \beta \in K/_C$ be such that $\alpha \leq \beta$. Then there exists $a \in \alpha$ and $b \in \beta$ such that $a \leq b$. Let $\gamma \in K/_C$. Choose $c \in \gamma$. Then we have that $a + c \leq b + c$ and $ac \leq bc$. So $\alpha + \gamma = aC + cC = (a + c)C \leq (b + c)C = bC + cC = \beta + \gamma$ and $\alpha\gamma = (aC)(cC) = (ac)C \leq (bc)C = (bC)(cC) = \beta\gamma$. Thus $\leq$ is a compatible on $K/_C$. Clearly, $[0] \leq \alpha$ for all $\alpha \in K/_C$. Therefore $K/_C$ is a positive ordered semifield.

From the above, we define $\leq^*$ on $K/_C$ as follows : let $\alpha, \beta \in K/_C$ define by $\alpha \leq \beta$ if and only if for every $a \in \alpha$, there exists $b \in \beta$ such that $a \leq b$. Then we get that $\leq^*$ is a positive compatible partial order on $K/_C$.

<u>Remark 2.17.</u> (1) The two definitions of compatible partial order of $K/_C$ as above are equivalent.

(2) Every element of $K/_C$ is a convex subset of K.

Proof (1) Obviously, $\leq^* \subseteq \leq$. Let $\alpha, \beta \in K/_C$ be such that $\alpha < \beta$. If $\alpha = 0$ then done. Suppose that $\alpha \neq 0$. Then there exist $a \in \alpha$ and $b \in \beta$ such that $a \leq b$. Since $\alpha \neq 0, a \neq 0$. Let $c \in \alpha$. By the definition of α, $ca^{-1} \in C$. Since $a \leq b$, $1 \leq ba^{-1}$. So $c \leq c(ba^{-1})$ and $c(ba^{-1}) = b(ca^{-1}) \in bC = \beta$, hence $\alpha \leq^* \beta$. Therefore $\leq^* \subseteq \leq$, so $\leq^* = \leq$.

To prove (2), let $\alpha \in K/_C$. If $\alpha = [0]$ then done. Suppose that $\alpha \neq [0]$. Let $x, y \in \alpha$ and $z \in K$ be such that $x \leq z \leq y$. Since $\alpha \neq [0]$, $x \neq 0$. So $1 \leq zx^{-1} \leq yx^{-1}$. By the definition of α, $yx^{-1} \in C$. Since C is o-convex, $zx^{-1} \in C$. Thus $z = x(zx^{-1}) \in xC = \alpha$. Let $a, b \in K$ be such that $a + b = 1$. Since C is a-convex, $a + b(yx^{-1}) \in C$. So $ax + by = x[(ax + by)x^{-1}]$ $= x[a + b(yx^{-1})] \in xC = \alpha$. Therefore α is a convex subset of K. #

Proposition 2.18. Let K be a positive ordered semifield and C a convex subgroup of K. Then there exists a positive compatible partial order on $K/_C$ such that the projection map $\prod$ is an order epimorphism of K onto $K/_C$.

Proof Define $\prod : K \to K/_C$ by $\prod(x) = xC$ for all $x \in K$. Then $\prod$ is an onto homomorphism, let $x, y \in K$ be such that $x \leq y$. So $\prod(x) = xC \leq yC$ $= \prod(y)$. Hence $\prod$ is an isotone, so $\prod(P_K) \subseteq P_{K/_C}$.

Let $\alpha \in P_{K/_C}$. Then $\alpha \geq C$, so there are $c \in C$ and $a \in \alpha$ such that $c \leq a$. Thus $ac^{-1} \in P_K$. Hence $\prod(ac^{-1}) = (ac^{-1})C = aC = \alpha \in \prod(P_K)$, hence $P_{K/_C} \subseteq \prod(P_K)$. Therefore $\prod$ is an order epimorphism. #

Corollary 2.19. An a-convex subgroup C of a positive ordered semifield K. Then C is the kernel of some order homomorphism iff it is an o-convex subset of K.

Proof Assume that C is the kernel of some order homomorphism. By Remark 2.15. (2), C is an o-convex subset of K.

The converse follows from Proposition 2.18.. #

Theorem 2.20. (First Isomorphism Theorem)

Let $f : K \to M$ be an order epimorphism of positive ordered semifields. Then $K/_{\ker f} \cong_o M$.

Proof Let φ be the isomorphism defined in the proof of Theorem 1.45..

To show that φ is isotone, let $x, y \in K$ be such that $x\ker f \le y\ker f$. Then there are $a, b \in \ker f$ such that $xa \le yb$. If $y = 0$ then done. Suppose that $y \ne 0$. So $xy^{-1} \le ba^{-1}$. Since f is isotone and $ba^{-1} \in \ker f$, $f(xy^{-1}) \le f(ba^{-1}) = 1$. Thus $f(x) \le f(y)$, so φ is isotone.

Next, we shall prove that φ^{-1} is isotone. Since f is an order epimorphism, $f(P_K) = P_M$. Let $y \in P_M$. Then there exists $x \in P_K$ such that $f(x) = y$. So $x\ker f \in P_{K/_{\ker f}}$. Then $\varphi(x\ker f) = f(x) = y \in \varphi(P_{K/_{\ker f}})$. This show that $P_M \subseteq \varphi(P_{K/_{\ker f}})$. By Proposition 2.16., φ^{-1} is isotone. Therefore φ is an order isomorphism, so $K/_{\ker f} \cong_o M$. #

Lemma 2.21. Let K be a positive ordered semifield, H a subsemifield of K and C a convex subgroup of K. Then $H \cap C$ is a convex subgroup of H. And HC is a subsemifield of K.

Proof By Lemma 1.46., we shown that $H \cap C$ and HC are a-convex subgroup of H and a subsemifield of K, respectively. It remains to prove that $H \cap C$ is an o-convex of subgroup H, let $x, y \in H \cap C$ and $z \in H$ be such that $x \leq z \leq y$. By the o-convexity of C, $z \in C$. Then $H \cap C$ is a convex subgroup of H, as required. #

Theorem 2.22. (Second Isomorphism Theorem)

Let H be a subsemifield of a positive ordered semifield K. Let C be a convex subgroup of K such that $P_{HC} \subseteq P_H$. Then $H/_{H \cap C} \cong_o HC/_C$.

Proof Let φ be the epimorphism given in the proof of Theorem 1.47. To show that $\varphi(P_H) = P_{HC/_C}$, let $x \in P_H$. Then $x \in H$ and $x \geq 1$. So $\varphi(x) = xC \geq C$, hence $\varphi(P_H) \subseteq P_{HC/_C}$. Let $\alpha \in P_{HC/_C}$. Then there exists an $a \in P_{HC}$ such that $aC = \alpha$. Since $P_{HC} \subseteq P_H$, $a \in P_H$. Therefore $\varphi(a) = aC = \alpha \in \varphi(P_H)$. Hence $P_{HC/_C} \subseteq \varphi(P_H)$. Therefore $\varphi(P_H) = P_{HC/_C}$, so φ is an order epimorphism and we have that $\ker \varphi = H \cap C$. By Theorem 2.20., $H/_{H \cap C} \cong_o HC/_C$. #

Lemma 2.23. Let D and H be convex subgroups of a positive ordered semifield K such that $H \subseteq D$. Then $D/_H$ is a convex subgroup of $K/_H$.

Proof By Lemma 1.48. we proved that $D/_H$ is an a-convex subgroup of $K/_H$. Let $\alpha, \beta \in D/_H$ and $\gamma \in K/_H$ be such that $\alpha \leq \gamma \leq \beta$. Then there are $a \in \alpha$, $b, c \in \gamma$ and $d \in \beta$ such that $a \leq b$ and $c \leq d$. By the definition of γ, $cb^{-1} \in H$, since $H \subseteq D$, $cb^{-1} \in D$. Claim that $a, d \in D$. There is $x \in D$ such that $\alpha = xH$. Since $a \in \alpha$, there exists $h \in H$ such that $a = xh$. Since $H \subseteq D$, $a = xh \in D$. Similarly, $d \in D$. So we have the claim. Thus $da^{-1} \in D$. Since $a \leq b$ and $c \leq d$, $cb^{-1} \leq ca^{-1} \leq da^{-1}$. By the o-convexity of D,

$ca^{-1} \in D$. Since $a \in D$, $c \in D$, $\gamma = cH \in D/_H$, so $D/_H$ is a convex subgroup of $K/_H$. #

Theorem 2.24. (Third Isomorphism Theorem)

Let K be a positive ordered semifield, D and H are convex subgroups of K such that $H \subseteq D$. Then $(K/_H)/_{(D/_H)} \cong_o K/_D$.

Proof Let φ be the epimorphism given in the proof of Theorem 1.49. show that $\varphi(P_{K/_H}) = P_{K/_D}$, let $\alpha \in K/_H$ be such that $H \leq \alpha$. Then there are $a \in \alpha$ and $h \in H$ be such that $h \leq a$. Since $H \subseteq D$, $h \in D$. Thus $\varphi(\alpha) = \varphi(aH) = aD$. Since $h \leq a$ and $h \in D$, $\varphi(\alpha) = aD \geq D$, so $\varphi(\alpha) \in P_{K/_D}$. Let $\alpha \in P_{K/_D}$. Then $\alpha \geq D$. Then there exist $x \in D$ and $a \in \alpha$ such that $a \geq x$. So $xH \leq aH$. Thus $(ax^{-1})H \in P_{K/_H}$ and $\varphi((ax^{-1})H) = (ax^{-1})D = aD = \alpha \in \varphi(P_{K/_H})$. Hence $\varphi(P_{K/_H}) = P_{K/_D}$. Therefore φ is an order epimorphism and $\ker \varphi = D/_H$, by Theorem 2.20., $(K/_H)/_{(D/_H)} \cong_o K/_D$. #

Proposition 2.25 Let $f : K \to M$ be an epimorphism of positive ordered semifields. If C' is a convex subgroup of M then $K/_{f^{-1}(C')} \cong_o M/_{C'}$.

Proof By Remark 2.15. (3), $f^{-1}(C')$ is a convex subgroup of K. Let φ be an epimorphism as the proof of Theorem 1.50.. To show that φ is isotone, let $x, y \in K$ be such that $x \leq y$. Since f is isotone, $f(x) \leq f(y)$. So $\varphi(x) = f(x)C' \leq f(y)C' = \varphi(y)$. Hence φ is isotone.

Let $\alpha \in P_{M/_{C'}}$. Then $C' \leq \alpha$, so there are $a \in \alpha$ and $c \in C'$ such that $c \leq a$. Hence $ac^{-1} \in P_M$. Since $f(P_K) = P_M$, $f(p) = ac^{-1}$ for some $p \in P_K$. Then

$\varphi(p) = f(p)C' = (ac^{-1})C' = aC' = \alpha \in \varphi(P_K)$. So $P_{M/C'} \subseteq \varphi(P_K)$. By Proposition 2.16., φ^{-1} is isotone. Therefore φ is order isomorphism and we have that $\ker \varphi = f^{-1}(C')$. By Theorem 2.20., $K/_{f^{-1}(C')} \cong_o M/_{C'}$. #

Proposition 2.26. Let K be a positively ordered semifield, C a convex subgroup of K, and P the positive cone of K. Let $\prod : K \to K/_C$ be the projection map. Then $\prod(P)$ is the positive cone of $K/_C$. Furthermore, if $P - C \neq \varnothing$ then $P - C$ is a multiplicative subsemigroup of C.

Proof : Clear that $\prod(P)$ is the positive cone of $K/_C$.

Let $x, y \in P - C$. Then $xy \in P$. Suppose that $xy \in C$. Then $xyC = C$. Since $x,y \in P$, xC and $yC \in \prod(P)$. So $(xC)(yC) = xyC = C$. Since $\prod(P)$ is the positive cone of $K/_C$, $xC = yC = C$. Thus $x, y \in C$, a contradiction. Hence $xy \notin C$, so $xy \in P - C$. #

Theorem 2.27. Let P be a commutative semiring with 1. Then there exists a positive ordered semifield K having its the positive cone isomorphic to P iff P satisfies the following statements :

(1) P is M.C. with zero,

(2) for every $x, y \in P$, $xy = 1$ implies $x = y = 1$,

(3) for every $x, y, a, b \in P$ there is $d \in P$ such that $ax + by = da + db$.

Proof Let K be the semifield as in Theorem 1.27..

Define a relation on K as follows : for $\alpha, \beta \in K$.

$\alpha \le \beta$ iff there exists $(a, b) \in \alpha$, $(c, d) \in \beta$ and $p \in P$ such that $pad = bc$ and $0 \le \alpha$ for all $\alpha \in K$.

To show that $\le$ is compatible on K, it is clear that $\le$ is reflexive. Let $\alpha, \beta \in K$ be such that $\alpha \le \beta$ and $\beta \le \alpha$. There exist (a, b), $(x, y) \in \alpha$, $(c, d), (z, w) \in \beta$ and $p, q \in P$ such that $pad = bc$ and $qyz = xw$. Since $ay = bx$, $(ay)w = (bx)w = b(xw) = b(qyz)$. By (1), $aw = qbz$. Then $daw = dqbz$. Since $cw = zd$, $daw = cwbq$. By (1), $da = cbq$. Since $bc = pad$, $ad = padq$. By (1), we have $1 = pq$. By (2), $p = q = 1$. Thus $ad = bc$, hence $\alpha = \beta$. Thus $\le$ is anti-symmetric. Let $\alpha, \beta, \gamma \in K$ be such that $\alpha \le \beta$ and $\beta \le \gamma$. Then there are $(a, b) \in \alpha$, $(c, d), (e, f) \in \beta$, $(g, h) \in \gamma$ and $p, q \in P$ such that $pad = bc$ and $qeh = fg$. Since $qeh = fg$, $qehc = fgc$. Then $gehc = fgc = deg$. By (1), $qhc = dg$. Then $paqhc = padg$. Since $pad = bc$, $pahqc = bcg$. By (1), $ahpq = bg$. Hence $\alpha \le \gamma$, so $\le$ is transitive. Let $\alpha, \beta \in K$ be such that $\alpha \le \beta$. Then there exist $(a, b) \in \alpha$, $(c, d) \in \beta$ and $p \in P$ such that $pad = bc$. Let $\gamma \in K$. Choose $(e, f) \in \gamma$. Then $padef = bcef$. Thus $\alpha\gamma \le \beta\gamma$. And by (3), there exists $q \in P$ such that $adfp + bde = qadf + qbde$. Then $qd(af + be) = bcf + bde = b(cf + de)$, so $qdf(af + be) = bf(cf + de)$. Hence $\alpha + \gamma \le \beta + \gamma$. Therefore $\le$ is a compatible partial order.

Define $\varphi : P \to K$ by $\varphi(x) = [(x, 1)]$ for all $x \in P$. We have that φ is a monomorphism. To show that $\varphi(P) = \{\alpha \in K \mid \alpha \ge [(1, 1)]\}$, let $x \in P$. Then $\varphi(x) = [(x, 1)] \ge [(1, 1)]$. Thus $\varphi(x) \in \{\alpha \in K \mid \alpha \ge [(1, 1)]\}$. Let $\beta \in \{\alpha \in K \mid \alpha \ge [(1, 1)]\}$. Then there exists $(a, b) \in \beta$ and $p \in P$ such that $pb = a$. Thus $\varphi(p) = [(p, 1)] = [(bp, b)] = [(a, b)] = \beta \in \varphi(P)$. Hence $\varphi(P) = \{\alpha \in K \mid \alpha \ge [(1, 1)]\}$, $P \cong \varphi(P)$. This proves that $\varphi(P)$ is the positive cone of K.

Conversely, let P be a positive cone of some positive ordered semifield K. Then (1) and (2) clearly hold. Let $x, y, a, b \in P$. By the a-convexity of P, $(ax + by)(a + b)^{-1} = [a(a + b)^{-1}]x + [b(a + b)^{-1}]y \in P$. Then $(ax + by)(a + b)^{-1} = p$ for some $p \in P$. Thus $ax + by = pa + pb$, (3) holds. #

Definition 2.28. Let K be a semifield and C an a-convex subgroup of K. A compatible partial order on C is a partial order $\leq$ on C such that for every $x, y, z \in C$, $x \leq y$ implies $xz \leq yz$.

Proposition 2.29. Let C be an a-convex subgroup of semifield K. Let $\leq$ be a compatible partial order on C and $\leq^*$ a compatible partial order on semifield $K/_C$. Suppose that

(1) for every $x, y \in P_C$ and $a, b \in K$ are such that $a + b = 1$, $ax + by \in P_C$ where $P_C = \{ x \in C \mid x \geq 1 \}$

(2) for every $x \in K$, $1 + x \in C$ implies $1 + x \in P_C$.

(3) for every $x \in C$, $y \in \bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha$ and $a, b \in K$ such that $a + b = 1$ and $ax + by \in K - C$.

Then there exists a compatible partial order $\leq$ on K such that $\leq$ is the restriction of the partial order on C and the projection map Π is an order epimorphism.

Proof Let $P = P_C \cup \bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha$. We shall show that P satisfies (1) - (4) of Theorem 2.11.

(1) Let $x, y \in P$

Case 1. $x, y \in P_C$. Then $xy \in P_C \subseteq P$.

Case 2. $x \in P_C$ and $y \in \beta$ for some $\beta \in P_{K/C} - \{C\}$. Thus $xy \in \bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha$.

Case 3. $x \in \alpha$ and $y \in \beta$ for some $\alpha, \beta \in P_{K/C} - \{C\}$. Then $(xy)C = (xC)(yC) = \alpha\beta \in P_{K/C}$. If $xy \in C$ then $C = (xy)C = \alpha\beta$. Thus $\alpha = \beta = C$, a contradiction. Thus $xy \notin C$, so $xy \in P_{K/C} - \{C\}$. Hence $P^2 \subseteq P$.

(2) Let $x \in P \cap P^{-1}$. Then $x, x^{-1} \in P$.

Case 1. $x, x^{-1} \in P_C$. Thus $x, x^{-1} \geq 1$, so $x = 1$.

Case 2. $x \in P_C$ and $x^{-1} \in \beta$ for some $\beta \in P_{K/C} - \{C\}$. Then $C = (xx^{-1})C = (xC)(x^{-1}C) = C\beta = \beta$, a contradiction.

Case 3. $x \in \alpha$ and $y \in \beta$ for some $\alpha, \beta \in P_{K/C} - \{C\}$. So $C = (xx^{-1})C = (xC)(x^{-1}C) = \alpha\beta$. Thus $\alpha = \beta = C$, a contradiction. Therefore $P \cap P^{-1} = \{1\}$.

(3) Let $x \in K$.

Case 1. $1 + x \notin C$. Then $(1 + x)C = C + xC \in P_{K/C}$. Thus $1 + x \in \bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha$ since $1 + x \notin C$

Case 2. $1 + x \in C$. Then $1 + x \in P_C$.

Hence $1 + x \in P$ for all $x \in K$.

(4) Let $x, y \in P$ and $a, b \in K$ be such that $a + b = 1$. So we have that $aC + bC = C$.

Case 1. $x, y \in P_C$, so done.

Case 2. $x, y \in P_C$ and $y \in \beta$ for some $\beta \in P_{K/C} - \{C\}$. Thus $y \in \bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha$, so by assumption $ax + by \in K - C$. Since $aC + bC = C$ and $\alpha, \beta \in P_{K/C}$, $(ax + by)C = (aC)\alpha + (bC)\beta \in P_{K/C}$. Therefore $(ax + by)C \in P_{K/C} - \{C\}$, so $ax + by \in \bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha$.

Case 3. $x \in \alpha$ and $y \in \beta$ for some $\alpha, \beta \in P_{K/C}-\{C\}$. Then $x, y \in \bigcup_{\alpha \in P_{K/C}-\{C\}} \alpha$, so $(ax + by)(a + by)^{-1} = [a(a + b)^{-1}]x + [b(a + b)^{-1}]y \in K - C$ and $a + by = [b(a + b)^{-1}]y + a(a + b)^{-1} \in K - C$. If $ax + by \in C$ then $C = (ax + by)C = [(ax + by)(a + by)^{-1}]C(a + by)C$. Since $[(ax + by)(a + by)^{-1}]C$ and $(a + by)C \in P_{K/C}$, $[(ax + by)(a+by)^{-1}]C = (a + by)C = C$. So $a + by \in C$, a contradiction. Thus $ax + by \notin C$. Hence $ax + by \in \bigcup_{\alpha \in P_{K/C}-\{C\}} \alpha$, so P is an a-convex subset of K. By Theorem 2.11., P is the positive cone of K. Let $\leq'$ be a positive compatible partial order induced by P.

Next, to show that $\leq$ is the restriction of $\leq'$ on C, let $x, y \in C$ be such that $x \leq' y$ in C. Then $yx^{-1} \in P$. Since $yx^{-1} \in C$, $yx^{-1} \in P_C$. Thus $y \geq x$. Hence $\leq$ is the restriction of $\leq'$ on C.

Finally, to prove that $\Pi(P) = P_{K/C}$, let $x \in P$.

Case 1. $x \in P_C$. Then $x \in C$, $\Pi(x) = xC = C \in P_{K/C}$.

Case 2. $x \in \beta$ for some $\beta \in P_{K/C}-\{C\}$. So $\Pi(x) = xC = \beta \in P_{K/C}-\{C\} \subseteq P_{K/C}$.

Therefore $\Pi(P) \subseteq P_{K/C}$. Let $\beta \in P_{K/C}$.

Case 1. $\beta = C$. Then $\Pi(1) = C = \beta \in \Pi(P_C) \subseteq \Pi(P)$.

Case 2. $\beta \neq C$. Then $\beta \in \bigcup_{\alpha \in P_{K/C}-\{C\}} \alpha$. Choose $x \in \beta$. Then $\Pi(x) = xC = \beta \in \Pi(\bigcup_{\alpha \in P_{K/C}-\{C\}} \alpha) \subseteq \Pi(P)$. Hence $P_{K/C} \subseteq \Pi(P)$, so $\Pi(P) = P_{K/C}$. #

Definition 2.30. Let $\{K_i \mid i \in I\}$ be a family of positive ordered semifields. The direct product of a family $\{K_i \mid i \in I\}$, denoted by $\prod_{i \in I} K_i$, defined as a direct product semifield with natural partial order $\leq$, that is for every $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} K_i$,

$$(x_i)_{i \in I} \leq (y_i)_{i \in I} \text{ iff } x_i \leq y_i \text{ for all } i \in I.$$

Remark 2.31. Let $\{K_i \mid i \in I\}$ be a family of positive ordered semifields. Then $P_{\prod_{i\in I} K_i} = \prod_{i\in I} P_i$ where $P_i = \{x \in K_i \mid x \geq 1_i\}$ for all $i \in I$.

Proposition 2.32. Let $\{K_i \mid i \in I\}$ be a family of positive ordered semifields and C_i a convex subgroup of K_i for all $i \in I$. Then $\prod_{i\in I} C_i$ is a convex subgroup of $\prod_{i\in I} K_i$ and $\prod_{i\in I} K_i / \prod_{i\in I} C_i \cong_o \prod_{i\in I} (K_i/C_i)$.

Proof Let φ be an epimorphism as the proof of Proposition 1.53., To show thta φ is isotone, let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} K_i$ be such that $(x_i)_{i\in I} \leq (y_i)_{i\in I}$. Then $x_i \leq y_i$ for all $i \in I$, $x_iC_i \leq y_iC_i$ for all $i \in I$. Hence $\varphi((x_i)_{i\in I}) = (x_iC_i)_{i\in I} \leq (y_iC_i)_{i\in I} = \varphi((y_i)_{i\in I})$. Hence φ is isotone, so $\varphi(P_{\prod_{i\in I} K_i}) \subseteq P_{\prod_{i\in I}(K_i/C_i)}$. Next, let $(x_iC_i)_{i\in I} \in P_{\prod_{i\in I}(K_i/C_i)}$. Then $(x_iC_i)_{i\in I} \geq (C_i)_{i\in I}$, so $x_iC_i \geq C_i$ for all $i \in I$. Thus there exist $c_i, d_i \in C_i$ such that $x_ic_i \geq d_i$ for all $i \in I$. Thus $(x_ic_i)d_i^{-1} \in P_{K_i}$ for all I, so $((x_ic_i)d_i^{-1})_{i\in I} \in P_{\prod_{i\in I} K_i}$. Then $\varphi(((x_ic_i)d_i^{-1})_{i\in I}) = ([(x_ic_i)d_i^{-1}]C_i)_{i\in I} = (x_iC_i)_{i\in I} \in \varphi(P_{\prod_{i\in I} K_i})$, hence $P_{\prod_{i\in I}(K_i/C_i)} \subseteq \varphi(P_{\prod_{i\in I} K_i})$. Therefore φ is an order epimorphism.

Clearly $\ker\varphi = \prod_{i\in I} C_i$, by Theorem 2.20., $\prod_{i\in I} K_i / \prod_{i\in I} C_i \cong_o \prod_{i\in I} (K_i/C_i)$. #

Proposition 2.33. Let D be a partially ordered semiring. If (D , .) is a group then D can be embedded into a positive ordered semifield iff for every $x, y \in D$, $x \leq x + y$.

Proof Obviously. #