Chapter II
Theoretical Background

Most coronal phenomena (including loop structures and solar flares) which astrophysicists can observe are generated by the magnetic field on the Sun and its dynamics. Thus, astrophysicists try to find and simulate many models and theories to explain these types of behavior. Because the Sun has very high temperatures, most of the matter is in the form of positive ions and negative electrons. Such a substance is called a plasma. The pressure of plasma in the magnetic loops is denser than that in its surroundings. The horizontal equilibrium of the loops is given by

$$P_e - P_i = \frac{B^2}{8\pi}, \tag{2.1}$$

where P_e and P_i are the external and internal plasma pressures and B is the magnetic field intensity, taken to be uniform across the flux tube (Foukal, 1990). The vertical force balance is maintained by the usual hydrostatic relation

$$\frac{dP}{dr} = -\rho g, \tag{2.2}$$

where ρ is the plasma density and g is the gravitational acceleration.

In general, the plasma and the magnetic field are tied to each other to some extent, depending on the bulk flow ram pressure (ρv^2) of the plasma. If the plasma ram pressure $\ll B^2/(8\pi)$, then the plasma will be held by the magnetic tension force and confined to flow parallel to the field. If the plasma ram pressure

$\gg B^2/(8\pi)$, then the magnetic field line will be pulled to follow the plasma motion.

In the inner corona, the magnetic field is very strong so the plasma is held in the flux tube, or possibly flowing along the magnetic field, allowing us see the loop in special filters or wavelengths. Magnetohydrodynamics (MHD) is the most general theory for explaining how the solar magnetic field and plasma flow interact.

In our model, we try to explain coronal magnetic loop structures on the solar corona which were observed during the total solar eclipse in Thailand on October 24^{th}, 1995. We are interested in the magnetic loops in a active region (AR7912) observed in a photograph (figure 1.2) during the total solar eclipse. MHD is used to explain the morphology of the plasma and loops in dynamic situations. In our work we used magnetohydrostatics only, in a simpler case of the "*force-free*" condition, which is a reasonable model for describing quiescent loops.

2.1 Magnetohydrodynamics (MHD)

MHD is the study of the interactions between a magnetic field and a plasma. It incorporates fluid dynamics and electromagnetism which explain the behavior of the fluid and electrically charged material moving in electromagnetic fields.

Therefore, the basic equations of MHD are the equations of slow electromagnetism and fluid mechanics. The Maxwell equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3}$$

$$\nabla \cdot \mathbf{D} = \rho_c, \tag{2.4}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2.5}$$

$$\nabla \cdot \mathbf{B} = 0 , \tag{2.6}$$

where $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \epsilon\mathbf{E}$ (using MKSA units; Jackson, 1975). Here $\mathbf{H}$ is the magnetic field, $\mathbf{B}$ is the magnetic induction, μ is the magnetic permeability, $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the electric displacement, ϵ is the electric permeability, ρ_c is the charge density, and $\mathbf{j}$ is the electric current density. Also Ohm's law is

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} , \tag{2.7}$$

where σ is the electrical conductivity.

The conservation equations of fluid mechanics include

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p , \tag{2.8}$$

$$\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{v} = 0 , \tag{2.9}$$

$$p = R\rho T , \tag{2.10}$$

and an energy equation, where ρ is the plasma density (the mass per unit volume), $\mathbf{v}$ is the plasma velocity, p is the plasma pressure, T is the temperature, and R is the gas constant per mean molecular weight, $\bar{\mu}$.

We can modify the previous equation by adding the Lorentz force and gravitational force on the right hand side, after which we get the equation of motion

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho\mathbf{g} , \tag{2.11}$$

where $\mathbf{g}$ is the gravitational acceleration and

$$\mathbf{j} = \nabla \times \frac{\mathbf{B}}{\mu}. \tag{2.12}$$

Next we will consider the magnetostatic case of the MHD model.

2.2 Magnetohydrostatics

The equilibrium of coronal gas within each magnetic flux tube is hydrostatic. The pressure from flux tube to flux tube will vary, the difference being compensated by slight deformation of the magnetic field. We have force balance if the first term in eq. (2.11) is much smaller than the third. Then eq. (2.11) reduces to the equation for magnetohydrostatic force balance

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}, \tag{2.13}$$

where $\mathbf{j} = \nabla \times (\mathbf{B}/\mu)$, $\nabla \cdot \mathbf{B} = 0$, and $\rho = P/(RT)$.

In this equation, when gravity is negligible we have magnetostatic balance

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B}. \tag{2.14}$$

If the fourth term in eq. (2.11) is much smaller than third, then eq. (2.14) reduces to the equation

$$\mathbf{j} \times \mathbf{B} = 0. \tag{2.15}$$

This equation describes a "*force-free*" condition as will be explained below.

2.2.1 Force-free field condition

Under the force-free condition, the magnetic field is in equilibrium under a balance between the magnetic and kinetic pressure and magnetic tension force, i.e., is at a minimum energy configuration of the field.

If pressure gradients and gravity are negligible, and $\rho\dfrac{dv}{dt} \approx 0$, we can use eq. (2.15). Since $\mathbf{j} = \nabla \times \mathbf{B}$, and in such a tenuous plasma $\mu \approx \mu_0$ to a good approximation, we must have $\nabla \times \mathbf{B} \parallel \mathbf{B}$ (the corona can only carry an electric current parallel to the magnetic field), and can write (Priest, 1994)

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \tag{2.16}$$

where α is a scalar function of position. Such a magnetic field is called *force-free.*

For simplicity, we set α to a constant since this is a case where $\mathbf{B}(\mathbf{r})$ can be computed numerically from observational data of the line-of-sight component of $\mathbf{B}$ at the photosphere. Eq. (2.16) is a linear equation so we can solve this equation by a Fourier transform (FT) technique. This computational technique was explained by Alissandrakis (1981), and similar methods were used by Nakagawa et al. (1972) and Seehafer (1978). They solved the constant-α force-free field by using a Fourier transform and found the input data from the vertical component B_z. In another method, Semel (1988) used a Green's function method to analytically generalize the existing solutions for the constant-α force free field to the oblique case. Cuperman et al. (1989) numerically integrated for the constant-α force-free magnetic fields using boundary conditions in three dimensions. Antiochos (1987), Wu et al. (1990), and Cuperman et al. (1990) considered the case of a nonlinear force-free field.

2.2.2 Fourier transform technique to solve this problem

We will solve the boundary value problem for a constant α, force-free magnetic field by using the Fourier transform in 2 dimensions, which is

$$\hat{\mathbf{B}}(u,v,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{B}(x,y,z)\, e^{-i2\pi(ux+vy)} dx dy. \tag{2.17}$$

Thus the Fourier transforms of the components of the field with respect to the variables x and y at a height z in 2 dimensions are

$$B_x(x,y,z) \rightarrow \hat{B}_x(u,v,z)$$
$$B_y(x,y,z) \rightarrow \hat{B}_y(u,v,z)$$
$$B_z(x,y,z) \rightarrow \hat{B}_z(u,v,z).$$

Let the z-axis be perpendicular to the solar surface and set the boundary condition at the $z = 0$ plane (photosphere). Then we are only interested in the solution in the upper half-space ($z > 0$). Applying the boundary condition that $\mathbf{B} \to 0$ as $z \to \infty$, we only consider solutions with the FT decreasing exponentially with z:

$$\hat{\mathbf{B}}(u,v,z) = \hat{\mathbf{B}}(u,v,0)\ e^{-kz}, \tag{2.18}$$

so

$$\begin{aligned}
\hat{B}_x(u,v,z) &= \hat{B}_x(u,v,0)\ e^{-kz} \\
\hat{B}_y(u,v,z) &= \hat{B}_y(u,v,0)\ e^{-kz} \\
\hat{B}_z(u,v,z) &= \hat{B}_z(u,v,0)\ e^{-kz}.
\end{aligned}$$

From the force-free magnetic field in eq. (2.16),

$$\nabla \times \mathbf{B} = \alpha \mathbf{B},$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \alpha \mathbf{B},$$

then we have:

x-axis:

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \alpha B_x, \tag{2.19}$$

y-axis:

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \alpha B_y, \tag{2.20}$$

z-axis:

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \alpha B_z. \tag{2.21}$$

The Fourier transform equations are

$$\begin{aligned}
\hat{B}_x(u,v,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_x(x,y,z)\ e^{-i2\pi(ux+vy)}dxdy, \\
\hat{B}_y(u,v,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_y(x,y,z)\ e^{-i2\pi(ux+vy)}dxdy, \\
\hat{B}_z(u,v,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_z(x,y,z)\ e^{-i2\pi(ux+vy)}dxdy.
\end{aligned}$$

Taking the Fourier transform of both sides of eqs. (2.19)-(2.21) and we get

$$\alpha\hat{B}_x(u,v,0) - k\hat{B}_y(u,v,0) - i2\pi v\hat{B}_z = 0, \quad (2.22)$$

$$k\hat{B}_x(u,v,0) + \alpha\hat{B}_y(u,v,0) + i2\pi u\hat{B}_z = 0, \quad (2.23)$$

$$i2\pi v\hat{B}_x(u,v,0) - i2\pi u\hat{B}_y(u,v,0) + \alpha\hat{B}_z = 0. \quad (2.24)$$

For a non-trivial solution, we require the a determinant of the coefficient matrix to be zero

$$\begin{vmatrix} \alpha & -k & -i2\pi v \\ k & \alpha & i2\pi u \\ i2\pi v & -i2\pi u & \alpha \end{vmatrix} = 0$$

$$\alpha^3 + 4\pi^2 kuv - 4\pi^2 kuv - 4\pi^2 v^2\alpha - 4\pi^2 u^2\alpha + k^2\alpha = 0$$

$$\alpha^3 - 4\pi^2 v^2\alpha - 4\pi^2 u^2\alpha + k^2\alpha = 0$$

$$\alpha(\alpha^2 - 4\pi^2(u^2+v^2) + k^2) = 0,$$

but $\alpha \neq 0$ so

$$\alpha^2 - 4\pi^2(u^2+v^2) + k^2 = 0$$

$$k^2 = 4\pi^2(u^2+v^2) - \alpha^2$$

$$k = \pm(4\pi^2(u^2+v^2) - \alpha^2)^{\frac{1}{2}}$$

$$k = \pm(4\pi^2 q^2 - \alpha^2)^{\frac{1}{2}} \quad (2.25)$$

where $q^2 = u^2 + v^2$. We choose $k = +(4\pi^2 q^2 - \alpha^2)^{\frac{1}{2}}$ since:

If $q < \frac{|\alpha|}{2\pi}$ then k is imaginary for a small scale.

If $q \geq \frac{|\alpha|}{2\pi}$ then k is real for a large scale.

In the case in which k is real, we can get the well-known solution (Nakagawa and Raadu, 1972, cited by Alissandrakis, 1981) from eq. (2.18) and eqs.

(2.22)-(2.25). If we can express $\hat{B}_x$, $\hat{B}_y$ and $\hat{B}_z$ in terms of the FT of the vertical component of the field at the boundary, we get

$$\hat{\mathbf{B}}(u,v,z) = \hat{B}_z(u,v,0)\hat{\mathbf{G}}(u,v,z). \tag{2.26}$$

Comparing with eq. (2.18),

x-axis:

$$\hat{B}_x(u,v,0)e^{-kz} = \hat{G}_x(u,v,z)\hat{B}_z(u,v,0), \tag{2.27}$$

y-axis:

$$\hat{B}_y(u,v,0)e^{-kz} = \hat{G}_y(u,v,z)\hat{B}_z(u,v,0), \tag{2.28}$$

z-axis:

$$e^{-kz} = \hat{G}_z(u,v,z). \tag{2.29}$$

Substituting these equations in eqs. (2.22)-(2.24), we will get

$$\begin{aligned}
\alpha\hat{G}_x(u,v,z)\hat{B}_z(u,v,0)\ e^{kz} - k\hat{G}_y(u,v,z)\hat{B}_z(u,v,0)\ e^{kz} & \\
-i2\pi v\hat{B}_z(u,v,0) &= 0, \\
k\hat{G}_x(u,v,z)\hat{B}_z(u,v,0)\ e^{kz} + \alpha\hat{G}_y(u,v,z)\hat{B}_z(u,v,0)\ e^{kz} & \\
+i2\pi u\hat{B}_z(u,v,0) &= 0, \\
i2\pi v\hat{G}_x(u,v,z)\hat{B}_z(u,v,0)\ e^{kz} - i2\pi u\hat{G}_y(u,v,z)\hat{B}_z(u,v,0)\ e^{kz} & \\
+i2\pi u\hat{B}_z(u,v,0) &= 0.
\end{aligned}$$

When $\hat{B}_z(u,v,0) \neq 0$,

$$\alpha\hat{G}_x\ e^{kz} - k\hat{G}_y\ e^{kz} - i2\pi v = 0, \tag{2.30}$$

$$k\hat{G}_x\ e^{kz} + \alpha\hat{G}_y\ e^{kz} + i2\pi u = 0, \tag{2.31}$$

$$i2\pi v\hat{G}_x\ e^{kz} - i2\pi u\hat{G}_y\ e^{kz} + i2\pi u = 0. \tag{2.32}$$

Eq. (2.30) $\times\alpha$:

$$\alpha^2\ \hat{G}_x\ e^{kz} - k\alpha\hat{G}_y\ e^{kz} + i2\pi v\alpha = 0, \tag{2.33}$$

Eq. (2.31) $\times k$;

$$k^2\hat{G}_x\, e^{kz} + \alpha k\hat{G}_y\, e^{kz} + i2\pi uk = 0, \tag{2.34}$$

Eq. (2.33) + Eq. (2.34) yields

$$(\alpha^2 + k^2)\ \hat{G}_x\, e^{kz} + i2\pi\ (uk - v\alpha) = 0$$

$$\begin{aligned}\hat{G}_x &= \frac{-i2\pi(uk - v\alpha)\ e^{-kz}}{\alpha^2 + k^2}\\ &= \frac{-i2\pi(uk - v\alpha)\ e^{-kz}}{4\pi^2 q^2}\\ &= \frac{-i(uk - v\alpha)\ e^{-kz}}{2\pi q^2}.\end{aligned}$$

where $k^2 = 4\pi^2(u^2 + v^2) - \alpha^2$. Substituting $\hat{G}_x$ into eq. (2.30),

$$\begin{aligned}\frac{-i\alpha(uk - v\alpha)}{2\pi q^2}\, e^{-kz}\, e^{kz} - k\hat{G}_y\, e^{kz} - i2\pi v &= 0\\ \frac{-i\alpha uk + iv\alpha^2}{2\pi q^2} - i2\pi v &= k\hat{G}_y\, e^{kz}\\ \frac{-i\alpha uk + iv\alpha^2 - i4\pi^2 q^2 v}{2\pi q^2} &= k\hat{G}_y\, e^{kz}\\ \frac{-i\alpha uk + iv\alpha^2 - iv(k^2 + \alpha^2)}{2\pi q^2} &= k\hat{G}_y\, e^{kz}\\ \frac{-ik(u\alpha + vk)}{2\pi q^2} &= k\hat{G}_y\, e^{kz}\\ \hat{G}_y(u, v, z) &= \frac{-i(u\alpha + vk)}{2\pi q^2}\, e^{-kz}.\end{aligned}$$

Thus we get

$$\begin{aligned}\hat{G}_x(u, v, z) &= \frac{-i(uk - v\alpha)}{2\pi q^2}\, e^{-kz},\\ \hat{G}_y(u, v, z) &= \frac{-i(u\alpha + vk)}{2\pi q^2}\, e^{-kz},\\ \hat{G}_z(u, v, z) &= e^{-kz}.\end{aligned}$$

Then the solutions are

$$\hat{B}_x(u, v, z) = \frac{-i(uk - v\alpha)}{2\pi q^2}\, e^{-kz}\hat{B}_z(u, v, 0), \tag{2.35}$$

$$\hat{B}_y(u,v,z) = \frac{-i(vk+u\alpha)}{2\pi q^2}\, e^{-kz}\hat{B}_z(u,v,0), \tag{2.36}$$

$$\hat{B}_z(u,v,z) = e^{-kz}\hat{B}_z(u,v,0). \tag{2.37}$$

So far, we have followed the derivation of Alissandrakis (1981), which assumes that the vertical component of the field at the boundary is available from magnetogram observations. However, standard magnetograms, based on the Zeeman effect, actually give the line-of-sight component of the magnetic field. Therefore, Alissandrakis's derivation is only applicable for magnetogram observations of active regions near the disk center.

For our thesis, we modified the boundary condition of Alissandrakis (1981) to instead specify the line-of-sight component of the photospheric field:

$$\sin\varphi\ B_x(x,y,0) + \cos\varphi\ B_z(x,y,0) = B_l(x,y,0), \tag{2.38}$$

where $B_l(x,y,0)$ is the magnetic field in the line of sight direction, and φ is the angle between the line-of-sight and the z-direction, which is perpendicular to the solar surface.

To solve this problem, we used the previous solutions as follows. Taking Fourier transforms on the both sides, we get

$$\hat{B}_l(u,v,0) = \cos\varphi\hat{B}_z(u,v,0) + \sin\varphi\hat{B}_x(u,v,0). \tag{2.39}$$

Substituting $\hat{B}_x(u,v,z)$ from eq. (2.35) into eq. (2.39) to find $\hat{B}_z(u,v,0)$, then we get

$$\hat{B}_z(u,v,0) = \frac{\hat{B}_l(u,v,0)}{\left(\cos\varphi - i\sin\varphi\dfrac{(uk-v\alpha)}{2\pi q^2}\right)}. \tag{2.40}$$

Then our solutions are

$$\hat{B}_x(u,v,z) = \frac{-i(uk-v\alpha)\hat{B}_l(u,v,0)}{2\pi q^2\ \left(\cos\varphi\ e^{-kz} - i\sin\varphi\dfrac{(uk-v\alpha)}{2\pi q^2}\right)}$$

$$= \frac{(uk - v\alpha)^2 \ e^{-kz} \sin\varphi \hat{B}_l(u,v,0)}{\left(4\pi^2 q^4 \cos^2\varphi + \sin^2\varphi (uk - v\alpha)^2\right)} - \frac{i2\pi q^2 \cos\varphi (uk - v\alpha) \ e^{-kz} \ \hat{B}_l(u,v,0)}{\left(4\pi^2 q^4 \cos^2\varphi + \sin^2\varphi (uk - v\alpha)^2\right)}, \tag{2.41}$$

$$\begin{aligned}\hat{B}_y(u,v,z) &= \frac{-i(vk + u\alpha) \ e^{-kz} \ \hat{B}_l(u,v,0)}{2\pi q^2 \ \left(\cos\varphi - i\sin\varphi \frac{(uk - v\alpha)}{2\pi q^2}\right)} \\ &= \frac{(uk - v\alpha)(vk + u\alpha) \ e^{-kz} \ \sin\varphi \hat{B}_l(u,v,0)}{\left(4\pi^2 q^4 \cos^2\varphi + \sin^2\varphi (uk - v\alpha)^2\right)} \\ &\quad - \frac{i2\pi q^2 \cos\varphi (vk + u\alpha) \ e^{-kz} \ \hat{B}_l(u,v,0))}{\left(4\pi^2 q^4 \cos^2\varphi + \sin^2\varphi (uk - v\alpha)^2\right)},\end{aligned} \tag{2.42}$$

$$\hat{B}_z(u,v,z) = \frac{\hat{B}_l(u,v,0) \ e^{-kz}}{\left(\cos\varphi - i\sin\varphi \frac{(uk - v\alpha)}{2\pi q^2}\right)}. \tag{2.43}$$

Finally, we use inverse Fourier transforms (see chapter IV) to transform these solutions back into (x, y, z) coordinates.