CHAPTER III

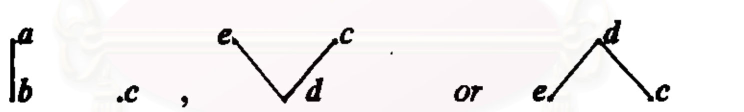
ORDER - PRESERVING TRANSFORMATION SEMIGROUPS ON PARTIALY ORDERED SETS WHICH ARE NOT CHAINS

In this chapter, we study regularity of order – preserving transformation semigroups $PT_{OP}(X)$, $T_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$, $V_{OP}(X)$ and $W_{OP}(X)$ where X is a partially ordered sets which is not a chain.

Let X be a partially ordered sets which is not a chain. There are two main results in this chapter. The first one is to prove that if S is one of $PT_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$, and $W_{OP}(X)$, then S is regular if and only if X is isolated. The second one is to give some necessary conditions and some sufficient conditions for X such that $T_{OP}(X)$ is regular.

The following lemmas are required.

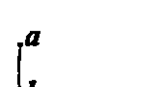
Lemma 3.1. *Let X be a partially ordered sets which is not a chain. If X is not isolated, then X contains a subposet of the forms*



Proof. Since X is not isolated, there exists $a \in X$ such that a is not an isolated point of X. Then there exists $b \in X$ such that $a < b$ or $b < a$. Without loss of generality, we assume that $a < b$. Let M be a maximal chain of X containing a and b. Since X is not a chain, $M \neq X$. Let $c \in X - M$. By maximality of M, the subposet $M \cup \{c\}$ is not a chain of X. Hence for all $x, y \in M$,

$$x < y \text{ implies that } x \not< c \text{ or } c \not< y \qquad (*)$$

Case 1: c is not comparable with any element of M. Then



is a subposet of X.

Case 2: $d < c$ for some $d \in M$. If for every $x \in M$, $d < x$ implies that $x < c$, then $M \cup \{c\}$ is a chain which contradicts the maximality of M. Thus there exists $e \in M$ such that $d < e$ and $e \nless c$. Since d, $e \in M$ and $d < c$, by (*), $c \nless e$. Hence e and c are not comparable. Therefore

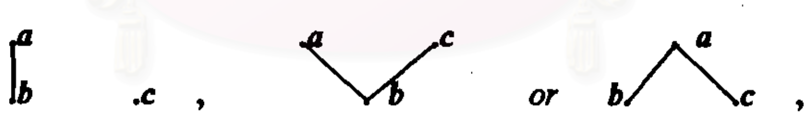
is a subposet of X.

Case 3: $c < d$ for some $d \in M$. If for $x \in M$, $x < d$ implies that $c < x$, then $M \cup \{c\}$ is a chain which is a contradiction since M is a maximal chain containing a and b. Consequently, there exists $e \in M$ such that $e < d$ and $c \nless e$. Since d, $e \in M$ and $c < d$, by (*), $e \nless c$. Therefore e and c are not comparable. Hence

is a subposet of X.

The lemma is completely proved. $\square$

Lemma 3.2. *If X is a partially ordered set containing a subposet of the forms*



then any of $PT_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$ and $W_{OP}(X)$ is not regular.

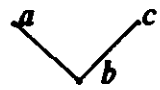
Proof. Let S be $PT_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$ or $W_{OP}(X)$.

Case 1: X contains

as a subposet. Let $\alpha \in PT(X)$ be such that $\Delta\alpha = \{b, c\}$, $\nabla\alpha = \{a, b\}$, $b\alpha = a$ and $c\alpha = b$. Then $\alpha \in S$. Suppose that $\alpha\beta\alpha = \alpha$ for some $\beta \in S$. Then $a = b\alpha = b\alpha\beta\alpha = (a\beta)\alpha$ and $b = c\alpha = c\alpha\beta\alpha = (b\beta)\alpha$ which implies that $a\beta = b$ and $b\beta = c$.

Since $b < a$ and $c \nless b$, β is not order – preserving, a contradiction. Hence α is not a regular element of S.

Case 2: X contains



as a subposet. Let $\alpha \in PT(X)$ be such that $\Delta\alpha = \{a, c\}$, $\nabla\alpha = \{a, b\}$, $a\alpha = a$ and $c\alpha = b$. Then $\alpha \in S$. Suppose that there exists $\beta \in S$ such that $\alpha\beta\alpha = \alpha$. Then $a = a\alpha = a\alpha\beta\alpha = (a\beta)\alpha$ and $b = c\alpha = c\alpha\beta\alpha = (b\beta)\alpha$, so $a\beta = a$ and $b\beta = c$. This implies that β is not order – preserving since $b < a$ but $a \nless c$. Therefore α is not a regular element of S.

Case 3: X contains

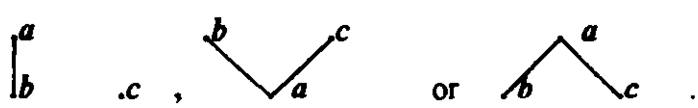
as a subposet. By Case 2, $T_{OP}((X, \leq_{opp}))$ is not regular. By Proposition 1.5 (2), $T_{OP}(X)$ is not regular.

Hence the lemma is proved. □

Theorem 3.3. *Let X be a partially ordered set which is not a chain and let S be $PT_{OP}(X)$, $I_{OP}(X)$, $U_{OP}(X)$ and $W_{OP}(X)$. Then S is regular if and only if X is isolated.*

Proof. If X is isolated, then $PT_{OP}(X) = PT(X)$, $I_{OP}(X) = I(X)$, $U_{OP}(X) = U(X)$ and $W_{OP}(X) = W(X)$, so S is a regular semigroup.

On the other hand, assume that X is not a chain and X is not isolated. By Lemma 3.1, X contains a poset of the forms



By Lemma 3.2, S is not regular. □

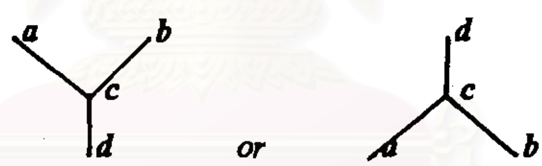
Theorem 3.4. *Let X be a partially ordered set containing disjoint components C_1 and C_2 with $|C_1| > 1$, then $T_{OP}(X)$ is not regular.*

Proof. Since $|C_1| > 1$, there exist a, $b \in C_1$ such that $a < b$.
Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} a & \text{if } x \in C_1, \\ b & \text{if } x \in C_2, \\ x & \text{if } x \in X - (C_1 \cup C_2). \end{cases}$$

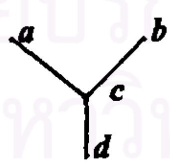
Then $\alpha \in T_{OP}(X)$. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in T_{OP}(X)$. Let $c \in C_2$. Then $a = a\alpha = a\alpha\beta\alpha = (a\beta)\alpha$ and $b = c\alpha = c\alpha\beta\alpha = (b\beta)\alpha$ which imply that $a\beta \in C_1$ and $b\beta \in C_2$. Since C_1 and C_2 are disjoint components, $a\beta$ and $b\beta$ are not comparable. It is a contradiction since $\beta \in T_{OP}(X)$ and $a < b$. Hence α is not regular in $T_{OP}(X)$. $\square$

Theorem 3.5. *If a partially ordered set X contains a subposet of the forms*

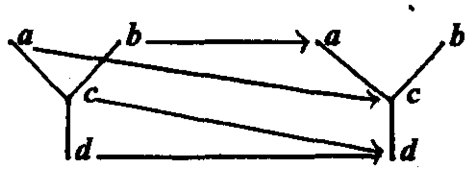


then $T_{OP}(X)$ is not regular.

Proof. First, let X have the subposet



Define $\alpha : X \to X$ by $a\alpha = c$, $b\alpha = a$, $c\alpha = d$ and $d\alpha = d$, i.e.,



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and for $x \in X - \{a, b, c, d\}$,

$$x\alpha = \begin{cases} x & \text{if } x > a \text{ and } x > b, \\ c & \text{if } x > a \text{ and } x \not> b, \\ a & \text{if } x \not> a \text{ and } x > b, \\ d & \text{if } x \not> a \text{ and } x \not> b. \end{cases} \qquad \text{.....(**)}$$

To show that α is order – preserving, let $y, z \in X$ be such that $y < z$. Because of that $x < y$, the following cases are all possible cases.

Case 1: $y, z \in \{a, b, c, d\}$. By (*), $y\alpha \le z\alpha$.

Case 2: $y = a$ and $z \notin \{a, b, c, d\}$. Then $z > a > c$. By (*), $y\alpha = c$ and by (**),

$z\alpha \in \{z, c\}$. Then $y\alpha \le z\alpha$.

Case 3: $y = b$ and $z \notin \{a, b, c, d\}$. Then $z > b$. By (*), $y\alpha = a$. If $z > a$, then by (**), $z\alpha = z > a = y\alpha$. If $z \not> a$, then by (**), $z\alpha = a = y\alpha$..

Case 4: $y = c$ or d and $z \notin \{a, b, c, d\}$. Then $z > d$. By (*), $y\alpha = d$. By (**), $z\alpha \in \{z, a, c, d\}$, so $z\alpha \ge d = y\alpha$.

Case 5: $z = a$ and $y \notin \{a, b, c, d\}$. Then $y < a$ and $z\alpha = c$. Since $a \not> b$, $y \not> b$. Therefore $y \not> a$ and $y \not> b$. Then $y\alpha = d < c = z\alpha$.

Case 6: $z = b$ and $y \notin \{a, b, c, d\}$. Then $y < b$ and $z\alpha = a$. Since $y \not> b$, $y\alpha \in \{c, d\}$. Therefore $y\alpha \le z\alpha$.

Case 7: $z = c$ or d and $y \notin \{a, b, c, d\}$. Then $y < a$ and $z\alpha = d$. Since $a \not> b$, $y \not> b$. We have that $y \not> a$ and $y \not> b$. Then $y\alpha = d = z\alpha$.

Case 8: $y > a$ and $z \notin \{a, b, c, d\}$. Then $z > a$, so $z\alpha \in \{z, c\}$. If $y > b$, then $z > b$. so $y\alpha = y < z = z\alpha$. If $y \not> b$, then $y\alpha = c < a < z$, so $y\alpha \le z\alpha$.

Case 9: $y \not> a$ and $z \notin \{a, b, c, d\}$.

Subcase 9.1: $y > b$ and $z > a$. Then $z > b$. Thus $y\alpha = a < z = z\alpha$.

Subcase 9.2: $y > b$ and $z \not> a$. Then $z > b$. Thus $y\alpha = a = z\alpha$.

Subcase 9.3: $y \not> b$, $z > a$ and $z > b$. Thus $y\alpha = d < a < z = z\alpha$.

Subcase 9.4: $y \not> b$, $z > a$ and $z \not> b$. Thus $y\alpha = d < c = z\alpha$.

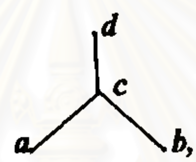
Subcase 9.5; $y \not> b$, $z \not> a$ and $z > b$. Thus $y\alpha = d < a = z\alpha$.

Subcase 9.6; $y \not> b$, $z \not> a$ and $z \not> b$. Thus $y\alpha = d = z\alpha$.

Next, to show that α is not regular, suppose it is.

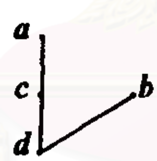
Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_{OP}(X)$. Since $c = a\alpha = a\alpha\beta\alpha = (c\beta)\alpha$ and $a = b\alpha = b\alpha\beta\alpha = (a\beta)\alpha$, by (*) and (**), $c\beta = a$ or ($c\beta > a$ and $c\beta \not> b$) and $a\beta = b$ or ($a\beta \not> a$ and $a\beta > b$). Then case that $c\beta = a$ and $a\beta = b$ can not occur because a and b are not comparable. If $c\beta = a$, $a\beta \not> a$ and $a\beta > b$, then $c\beta \not< a\beta$, so $a = c\beta = a\beta > b$, a contradiction. If $c\beta > a$, $c\beta \not> b$ and $a\beta = b$, then $a < c\beta \leq a\beta = b$, a contradiction. If $c\beta > a$, $c\beta \not> b$, $a\beta \not> a$ and $a\beta > b$, then $a < c\beta \leq a\beta$, a contradiction. Hence α is not regular, so $T_{OP}(X)$ is not a regular semigroup.

If X contains a subposet of the form

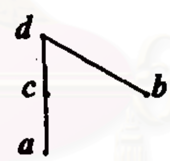


then by the above proof, $T_{OP}((X, \leq_{opp}))$ is not regular and hence by Proposition 1.5 (2), $T_{OP}(X)$ is not regular. $\square$

Lemma 3.6. *If a partially ordered set X contains a subposet of the forms*



or



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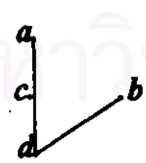
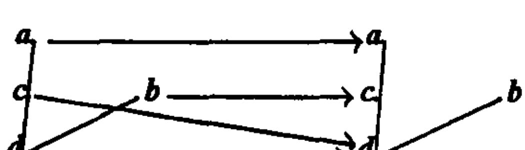
then $T_{OP}(X)$ is not regular.

Proof. First, let X have the subposet



Define $\alpha : X \to X$ as follows: $a\alpha = a$, $b\alpha = c$, $c\alpha = d\alpha = d$, i.e.,



.....(*)

and for $x \in X - \{a, b, c, d\}$,

$$x\alpha = \begin{cases} x & \text{if } x > a \text{ and } x > b, \\ a & \text{if } x > a \text{ and } x \not> b, \\ c & \text{if } x \not> a \text{ and } x > b, \\ d & \text{if } x \not> a \text{ and } x \not> b. \end{cases} \qquad .....(**)$$

To show that α is order – preserving, let $y,\ z \in X$ be such that $y < z$. Because of that $y < z$, the following cases are all possible cases.

Case 1: $y,\ z \in \{a,\ b,\ c,\ d\}$. By (*), $y\alpha \le z\alpha$.

Case 2: $y = a$ and $z \notin \{a,\ b,\ c,\ d\}$. Then $z > a$. By (*), $y\alpha = a$ and by (**), $z\alpha \in \{a,\ z\}$, so $y\alpha \le z\alpha$.

Case 3: $y = b$ and $z \notin \{a,\ b,\ c,\ d\}$. Then $z > b$. By (*), $y\alpha = c$. By (**), we have that $z > a$ implies $z\alpha = z > a > c > y\alpha$ and $z \not> a$ implies $z\alpha = c = y\alpha$..

Case 4: $y = c$ or d and $z \notin \{a,\ b,\ c,\ d\}$. Then $z > d$. By (*), $y\alpha = d$. By (**), $z\alpha \in \{z,\ a,\ c,\ d\}$, thus $z\alpha \ge y\alpha$.

Case 5: $z = a$ and $y \notin \{a,\ b,\ c,\ d\}$. Then $y < a$ and $z\alpha = a$. Since $y \not> a$, $y\alpha \in \{c,\ d\}$. Then $y\alpha < z\alpha$.

Case 6: $z = b$ and $y \notin \{a,\ b,\ c,\ d\}$. Then $y < b$ and $z\alpha = c$. Since $b \not> a$, $y \not> a$. Then $y \not> a$ and $y \not> b$. Therefore $y\alpha = d < c = z\alpha$.

Case 7: $z = c$ or d and $y \notin \{a,\ b,\ c,\ d\}$. Then $y < a$ and $z\alpha = d$. Since $a \not> b$, $y \not> b$. We have that $y \not> a$ and $y \not> b$. Then $y\alpha = d = z\alpha$.

Case 8: $y > a$ and $z \notin \{a,\ b,\ c,\ d\}$. Then $z > a$, so $z\alpha \in \{z,\ a\}$. If $y > b$, then $z > b$. so $y\alpha = y < z = z\alpha$. If $y \not> b$, then $y\alpha = a$ which implies that $y\alpha \le z\alpha$

Case 9: $y \not> a$ and $z \notin \{a,\ b,\ c,\ d\}$.

Subcase 9.1: $y > b$ and $z > a$. Then $z > b$. Thus $y\alpha = c < a < z = z\alpha$.

Subcase 9.2: $y > b$ and $z \not> a$. Then $z > b$. Thus $y\alpha = c = z\alpha$.

Subcase 9.3: $y \not> b$, $z > a$ and $z > b$. Thus $y\alpha = d < a < z = z\alpha$.

Subcase 9.4: $y \not> b$, $z > a$ and $z \not> b$. Thus $y\alpha = d < a = z\alpha$.

Subcase 9.5; $y \not> b$, $z \not> a$ and $z > b$. Thus $y\alpha = d < c = z\alpha$.

Subcase 9.6; $y \not> b$, $z \not> a$ and $z \not> b$. Thus $y\alpha = d = z\alpha$.

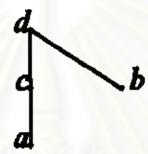
Next, to show that α is not regular, suppose it is.

Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T_{OP}(X)$. We have that $c\beta \le a\beta$.

Since $a = a\alpha = a\alpha\beta\alpha = (a\beta)\alpha$, $a\beta = a$ or ($a\beta > a$ and $a\beta \not> b$). Since $c = b\alpha = b\alpha\beta\alpha = (c\beta)\alpha$, $c\beta = b$ or ($c\beta \not> a$ and $c\beta > b$). The case that $a\beta = a$ and $c\beta = b$ can not occur because a and b are not comparable. If $a\beta = a$, $c\beta \not> a$ and $c\beta > b$, then $b < c\beta \le a\beta = a$, a contradiction. If $a\beta > a$, $a\beta \not> b$ and $c\beta = b$, then $a\beta \not> c\beta$, so $c\beta = a\beta$ and therefore $a < a\beta = c\beta = b$, a contradiction. If $a\beta > a$, $a\beta \not> b$, $c\beta \not> a$ and $c\beta > b$, then $b < c\beta \le a\beta$, a contradiction.

This proves that $T_{OP}(X)$ is not a regular semigroup.

If X contains a subposet of the form



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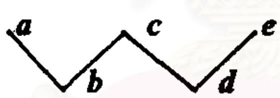
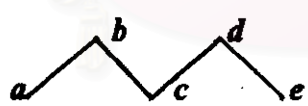
then by the above proof, $T_{OP}((X, \le_{opp}))$ is not regular and hence by Proposition 1.5 (2), $T_{OP}(X)$ is not regular. □

Theorem 3.7. *If a partially ordered set X contains a subposet of the forms*

(i) 

and {a, e} has no lower bound in X

or

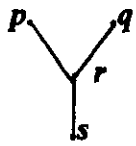
(ii) 

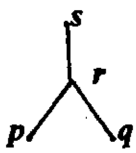
and {a, e} has no upper bound in X,

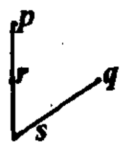
then $T_{OP}(X)$ is not regular.

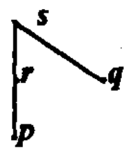
Proof. First, let X contain a subposet

where $\{a, e\}$ has no lower bound in X and let this subposet be denoted by Y. If X is not connected, by Lemma 3.4, $T_{OP}(X)$ is not regular. If X contains a subposet of the forms



, 

, 

or 

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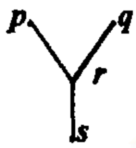
by Lemma 3.5 and Lemma 3.6, $T_{OP}(X)$ is not regular. Assume that

(1) X is connected

and

(2) X does not contain subposet of the forms



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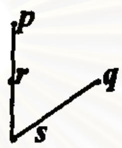
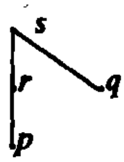
, 

and 

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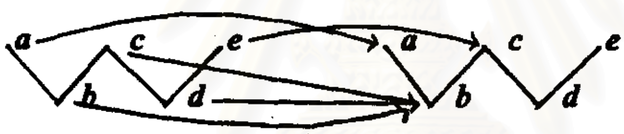
By (1) and (2),

(3) for $x \in X$, x is a maximal element or a minimal element of X.

Define $\alpha : X \to X$ by $a\alpha = a$, $b\alpha = d\alpha = c\alpha = b$, $e\alpha = c$, i.e.,



and $x\alpha = b$ for all $x \in X - Y$. Then

$$x\alpha = b \text{ if } x \text{ is a minimal element of } X$$

and ...(*)

$$x\alpha \in \{a, c, b\} \text{ if } x \text{ is a maximal element of } X.$$

If $y, z \in X$ are such that $y < z$, then by (3), y is a minimal element and z is a maximal element of X, so by (*), $y\alpha \leq z\alpha$.

Suppose their exists $\beta \in T_{OP}(X)$ such that $\alpha = \alpha\beta\alpha$. Then $a = a\alpha = a\alpha\beta\alpha = (a\beta)\alpha$ and $c = e\alpha = e\alpha\beta\alpha = (c\beta)\alpha$ which imply that $a\beta = a$ and $c\beta = e$. Since $b < a$, $b\beta \leq a\beta$, so $b\beta \leq a$. Since $b < c$, $b\beta \leq c\beta$, so $b\beta \leq e$. Then $b\beta$ is lower bound of $\{a, e\}$, a contradiction. Hence α is not regular, so $T_{OP}(X)$ is not regular.

If X contains a subposet of the form (ii), by the above proof, $T_{OP}((X, \leq_{opp}))$ is not regular. By Proposition 1.5 (2), $T_{OP}(X)$is not regular. □

Theorem 3.8. *Let X be a partially ordered set and $M(X)$ and $m(X)$ denote the set of all maximal elements of X and minimal elements of X, respectively. If (i) $X = M(X) \cup m(X)$ and (ii) for $x \in m(X)$ and $y \in M(X)$, $x < y$, then $T_{OP}(X)$ is regular.*

Proof. Let $\alpha \in T_{OP}(X)$. Since α is order – preserving and for $x \in \mathrm{m}(X)$ and $y \in \mathrm{M}(X)$, $x < y$, it follows that

(i) if $x \in \nabla\alpha \cap \mathrm{M}(X)$, then $x\alpha^{-1} \cap \mathrm{M}(X) \neq \varnothing$ and

(ii) if $x \in \nabla\alpha \cap \mathrm{m}(X)$, then $x\alpha^{-1} \cap \mathrm{m}(X) \neq \varnothing$.

For $x \in \nabla\alpha$, choose $d_x \in x\alpha^{-1}$ which satisfies the following properties

(1) If $x \in \mathrm{M}(X)$, choose $d_x \in \mathrm{M}(X)$.

(2) If $x \in \mathrm{m}(X)$, choose $d_x \in \mathrm{m}(X)$.

(1) and (2) can be obtained because of (i) and (ii). Define $\beta : X \to X$ by

$$x\beta = \begin{cases} d_x & \text{if } x \in \nabla\alpha, \\ x & \text{if } x \notin \nabla\alpha. \end{cases}$$

Then for $x \in X$, $x\alpha\beta\alpha = d_{x\alpha}\alpha = x\alpha$. thus $\alpha = \alpha\beta\alpha$. To show that β is order – preserving, let $x, y \in X$ be such that $x < y$. Then $x \in \mathrm{m}(X)$ and $y \in \mathrm{M}(X)$.

Case 1: $x \notin \nabla\alpha$ and $y \notin \nabla\alpha$. Then $x\alpha = x < y = y\alpha$.

Case 2: $x \in \nabla\alpha$ and $y \notin \nabla\alpha$. Then $y\beta = y \in \mathrm{M}(X)$. By (2), $d_x \in \mathrm{m}(X)$. Then $x\beta = d_x < y = y\beta$.

Case 3: $x \notin \nabla\alpha$ and $y \in \nabla\alpha$. Then $x\beta = x \in \mathrm{m}(X)$, By (1), $d_y \in \mathrm{M}(X)$. Then $x\beta = x < d_y = y\beta$.

Case 4: $x \in \nabla\alpha$ and $y \in \nabla\alpha$. Then $d_x \in \mathrm{m}(X)$ and $d_y \in \mathrm{M}(X)$. Thus $x\beta = d_x < d_y = y\beta$. □

Theorem 3.9. *Let X be a partially ordered set. If X has a maximum element a and a minimum element b such that for all distinct $x, y \in X - \{a, b\}$, x and y are not comparable, then $T_{OP}(X)$ is regular.*

Proof. Let $\alpha \in T_{OP}(X)$. Since α is order – preserving, we have that

(i) if $a \in \nabla\alpha$, then $a \in a\alpha^{-1}$ and

(ii) if $b \in \nabla\alpha$, then $b \in b\alpha^{-1}$.

For $x \in \nabla\alpha$, choose $d_x \in x\alpha^{-1}$ which satisfies the following properties

(1) If $x = a$, then $d_x = a$.

(2) If $x = b$, then $d_x = b$.

(1) and (2) can be obtained because of (i) and (ii), respectively.

Define $\beta : X \to X$ by

$$x\beta = \begin{cases} d_x & if \quad x \in \nabla\alpha, \\ x & if \quad x \notin \nabla\alpha. \end{cases} \qquad \text{....(*)}$$

For $x \in X$, $x\alpha\beta\alpha = (d_{x\alpha})\alpha = x\alpha$, so $\alpha = \alpha\beta\alpha$.

To show that β is order – preserving, let $y, z \in X$ be such that $y < z$. Then $y = b$ or $z = a$.

Case 1: $z = a$. If $z \notin \nabla\alpha$, then by (*), $z\beta = z = a$. If $z \in \nabla\alpha$, then by (1) and (*), $z\beta = d_z = a$. Since a is the maximum element of X, $y\beta \le z\beta$.

Case 2: $z \ne a$. Then $y = b$. If $y \notin \nabla\alpha$, then by (*), $y\beta = y = b$. If $y \in \nabla\alpha$, then by (2) and (*), $y\beta = d_y = b$. Since b is the minimum element of X, $y\beta \le z\beta$.

Hence β is order – preserving.

This proves that $T_{OP}(X)$ is regular. □

Example. For each $n \in \mathbf{N}$, let $X_n = \{1, 2, 3, \ldots, n\}$. Under the natural partial order, X_n is a finite chain for every $n \in \mathbf{N}$, so $T_{OP}(X_n)$ is regular for all $n \in \mathbf{N}$. Define the partial order $\le_d$ on $\mathbf{N}$ by

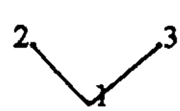
$$a \le_d b \text{ if and only if } a \mid b.$$

The pictures of $(X_1, \le_d)$, $(X_2, \le_d)$, $(X_3, \le_d)$ and $(X_4, \le_d)$ are as follows:

$(X_1, \le_d)$: .1

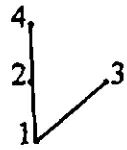
$(X_2, \le_d)$:

$(X_3, \le_d)$:



and

$(X_4, \leq_d)$:



Since $(X_1, \leq_d)$ and $(X_2, \leq_d)$ are finite chains, $T_{OP}((X_1, \leq_d))$ and $T_{OP}((X_2, \leq_d))$ are regular. By Theorem 3.6, $T_{OP}((X_4, \leq_d))$ is not regular. By Theorem 3.8, $T_{OP}((X_3, \leq_d))$ is regular. Hence we have that for $n \in \mathbf{N}$, $T_{OP}((X_n, \leq_d))$ is regular if and only if $n \leq 3$.

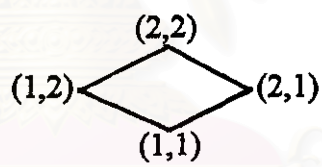
Example. For each $n \in \mathbf{N}$, let $Y_n = \{(x, y) \mid x, y \in \{1, 2, 3, \ldots, n\}\}$.
Define the partial order $\leq_c$ on $\mathbf{N} \times \mathbf{N}$ by

$$(a, b) \leq_c (c, d) \text{ if and only if } a \leq c \text{ and } b \leq d.$$

The pictures of $(Y_1, \leq_c)$, $(Y_2, \leq_c)$ and $(Y_3, \leq_c)$ are as follows:

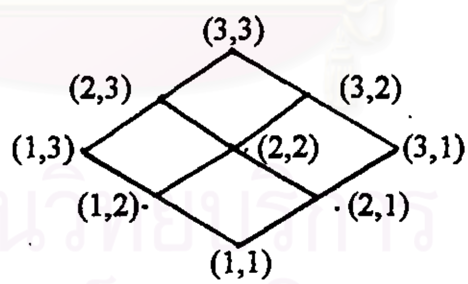
$(Y_1, \leq_c)$: $\cdot(1,1)$

$(Y_2, \leq_c)$:



and

$(Y_3, \leq_c)$:



Then $T_{OP}((Y_1, \leq_c))$ is regular and by Theorem 3.9, $T_{OP}((Y_2, \leq_c))$ is regular. By Theorem 3.6, $T_{OP}((Y_3, \leq_c))$ is not regular. Hence we have that for $n \in \mathbf{N}$, $T_{OP}((Y_n, \leq_c))$ is regular if and only if $n \leq 2$.