CHAPTER II

ORDER-PRESERVING TRANSFORMATION SEMIGROUPS ON CHAINS

The fact that $T_{OP}(X)$ is regular if X is a finite chain which we refer as Proposition 1.3 appears in [3] as an exercise. We generalize this fact by proving that if X is a chain which is order - isomorphic to a subset of $\mathbf{Z}$, then $T_{OP}(X)$ is regular.

Let A be a interval in $\mathbf{R}$. If $|A| > 1$, then A is not order - isomorphic to any subset of $\mathbf{Z}$. It is proved that $T_{OP}(A)$ is regular if and only if A is of the form $[a, b]$ for some a, $b \in \mathbf{R}$.

We refer from [1] as Proposition 1.4 that if X is a finite chain, then $I_{OP}(X)$ is a regular semigroup. The author of [1] quoted this fact without proof. In fact, it is not difficult to see this result. This result is generalized. We prove in this chapter that for any chain X, $I_{OP}(X)$, $W_{OP}(X)$, $PT_{OP}(X)$, $U_{OP}(X)$ and $V_{OP}(X)$ are regular.

Lemma 2.1. *Let X be a chain, $\alpha \in PT_{OP}(X)$, and a, $b \in \nabla\alpha$ such that $a < b$. Then $x < y$ for all $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$.*

Proof. Let $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$. Then $x\alpha = a$ and $y\alpha = b$. Since X is a chain, $x < y$ or $x \geq y$. If $x \geq y$, then $x\alpha \geq y\alpha$ since α is order - preserving which implies that $a \geq b$, a contradiction. Hence $x < y$. □

Theorem 2.2. *Let X be a chain. If X is order - isomorphic to a subset of $\mathbf{Z}$, then $T_{OP}(X)$ is a regular semigroup.*

Proof. First we note from the property of X that for any nonempty subset A of X, (1) if A has an upper bound in X, then $max(A)$ exists and (2) if A has a lower bound in X, then $min(X)$ exists. Then from this fact and Lemma 2.1, we have that for $\alpha \in PT_{OP}(X)$ and $a \in \nabla\alpha$, (i) if $a < b$ for some $b \in \nabla\alpha$, then $max(a\alpha^{-1})$ exists and (ii) if $b < a$ for some $b \in \nabla\alpha$, then $min(a\alpha^{-1})$ exists.

Let $\alpha \in T_{OP}(X)$. If $|\nabla\alpha| = 1$, then $\alpha^2 = \alpha$, so α is regular. Suppose that $|\nabla\alpha| > 1$. Since $\nabla\alpha \subseteq X$, by the property of X, there exists a set I such that $I = \{1, 2, 3, \ldots, n\}$ where $n > 1$, $I = \mathbf{N}$, $I = \mathbf{Z}^-$ or $I = \mathbf{Z}$ such that $\nabla\alpha = \{a_i \mid i \in I\}$ and $a_i < a_j$ if $i < j$ in I. Assume that $I = \{1, 2, 3, 4, \ldots, n\}$, $I = \mathbf{N}$ or $I = \mathbf{Z}$. Let $\beta : X \rightarrow X$ be defined as follows:

(1) If $I = \{1, 2, 3, \ldots, n\}$ where $n > 1$, define

$$x\beta = \begin{cases} \max(a_1\alpha^{-1}) & if \quad x \le a_1, \\ \min(a_{i+1}\alpha^{-1}) & if \quad a_i < x \le a_{i+1} \text{ for } i \in I - \{n\}, \\ \min(a_n\alpha^{-1}) & if \quad x \ge a_n. \end{cases}$$

(2) If $I = \mathbf{N}$, define

$$x\beta = \begin{cases} \max(a_1\alpha^{-1}) & if \quad x \le a_1, \\ \min(a_{i+1}\alpha^{-1}) & if \quad a_i < x \le a_{i+1} \; for \; i \in I. \end{cases}$$

and

(3) If $I = \mathbf{Z}$, define $x\beta = max(a_{i+1}\alpha^{-1})$ if $a_i < x \le a_{i+1}$ for all $i \in I$.

To show that $\alpha\beta\alpha = \alpha$, let $x \in X$. Then $x\alpha \in \nabla\alpha$, so there exists $k \in I$ such that $x\alpha = a_k$. By (1) - (3), $a_k\beta = min(a_k\alpha^{-1})$ or $max(a_k\alpha^{-1})$. Then $a_k\beta \in a_k\alpha^{-1}$ which implies that $a_k\beta\alpha = a_k$. Hence $x\alpha\beta\alpha = x\alpha$.

Next, to show that β is order - preserving, let $x, y \in X$ be such that $x < y$.

Case 1: I is in (1) or (2) and $x, y \le a_1$. Then $x\beta = y\beta = max(a_1\alpha^{-1})$.

Case 2: $a_k < x < y \le a_{k+1}$ for some $k \in I$ such that $k+1 \in I$. Then by (1) - (3), $x\beta = y\beta = min(a_{k+1}\alpha^{-1})$ or $max(a_{k+1}\alpha^{-1})$.

Case 3: $a_k < x \le a_{k+1} \le a_l < y \le a_{l+1}$ for some $k, l \in I$ such that $k+1, l+1 \in I$ and $k+1 \le l$. Then by (1) - (3), $x\beta = min(a_{k+1}\alpha^{-1})$ or $max(a_{k+1}\alpha^{-1})$ and $y\beta = min(a_{l+1}\alpha^{-1})$ or $max(a_{l+1}\alpha^{-1})$. Since $k+1 < l+1$, by Lemma 2.1, for all $u \in a_{k+1}\alpha^{-1}$ and $v \in a_{l+1}\alpha^{-1}$, $u < v$. But $x\beta \in a_{k+1}\alpha^{-1}$ and $y\beta \in a_{l+1}\alpha^{-1}$, so $x\beta < y\beta$.

This proves that $T_{OP}(X)$ is regular if I is finite, $I = \mathbf{Z}$ or $I = \mathbf{N}$.

If $I = \mathbf{N}$, by Proposition 1.5 (2) and the above proof, $T_{OP}((X, \leq_{opp}))$ is regular. But $(\mathbf{Z}^-, \leq)$ and $(\mathbf{N}, \leq_{opp})$ are order – isomorphic where $\leq$ is the natural partial order, so $T_{OP}(X)$ is regular if $I = \mathbf{Z}^-$. □

The converse of the Theorem 2.2 is not true. For $a, b \in \mathbf{R}$, $a < b$, we have that $[a, b)$ is not isomorphic to any subset of $\mathbf{Z}$. We show in the next theorem that $T_{OP}([a, b])$ is regular for all $a, b \in \mathbf{R}$ such that $a \leq b$. It is also shown in this theorem that if X is an interval in $\mathbf{R}$ which is not a closed and bounded interval, then $PT_{OP}(X)$ is not regular.

All nonempty intervals in $\mathbf{R}$ are of the forms

(1) $\mathbf{R}$,

(2) (a, ∞) where $a \in \mathbf{R}$,

(3) $[a, \infty)$ where $a \in \mathbf{R}$,

(4) $(-\infty, a)$ where $a \in \mathbf{R}$,

(5) $(-\infty, a]$ where $a \in \mathbf{R}$,

(6) (a, b) where $a, b \in \mathbf{R}$ such that $a < b$,

(7) $[a, b)$ where $a, b \in \mathbf{R}$ such that $a < b$,

(8) $(a, b]$ where $a, b \in \mathbf{R}$ such that $a < b$

and

(9) $[a, b]$ where $a, b \in \mathbf{R}$ such that $a \leq b$.

The theorem is proved by dividing it up into 7 lemmas. We know that the sets (1) - (8) have the same cardinality. The proofs of the lemmas also show that if X is one of the sets (1) - (8), then $|\{\alpha \in T_{op}(X) \mid \alpha \text{ is not regular}\}| \geq |\mathbf{R}|$. We have that for any partially ordered set X, all constant transformations of X are regular elements of $T_{OP}(X)$. These imply that if X is one of the set (1) - (8), then $|\{\alpha \in T_{op}(X) \mid \alpha \text{ is regular}\}| \geq |\mathbf{R}|$ and $|\{\alpha \in T_{op}(X) \mid \alpha \text{ is not regular}\}| \geq |\mathbf{R}|$.

Lemma 2.3. *$T_{OP}(\mathbf{R})$ is not a regular semigroup.*

Proof. Let $r \in (1, \infty)$ and $\alpha : \mathbf{R} \to \mathbf{R}$ be such that $x\alpha = r^x$ for all $x \in \mathbf{R}$. Then $\alpha \in T_{OP}(\mathbf{R})$, $\nabla\alpha = \mathbf{R}^+$ and α is 1-1. Suppose there exists $\beta \in T_{OP}(\mathbf{R})$ such that $\alpha = \alpha\beta\alpha$. Then for all $x \in \mathbf{R}$, $x\alpha = x\alpha\beta\alpha$. Since α is 1-1, $x = x\alpha\beta$ for every $x \in \mathbf{R}$, so $r^x\beta = x$ for all $x \in \mathbf{R}$. Thus $\mathbf{R}^+\beta = \mathbf{R}$. Since $0\beta \in \mathbf{R}$, there exists $a \in \mathbf{R}^+$ such that $0\beta = a\beta$. Let $b \in \mathbf{R}^+$ be such that $0 < b < a$. Since α and β is order - preserving, $0\beta\alpha \le b\beta\alpha \le a\beta\alpha$. Since $0\beta = a\beta$, $b\beta\alpha = a\beta\alpha$. Since $\nabla\alpha = \mathbf{R}^+$ and a, $b \in \mathbf{R}^+$, there exist $x, y \in \mathbf{R}$ such that $x\alpha = a$ and $y\alpha = b$. Consequently, $a = x\alpha = x\alpha\beta\alpha = a\beta\alpha = b\beta\alpha = y\alpha\beta\alpha = y\alpha = b$, a contradiction. Hence α is not a regular element of $T_{OP}(\mathbf{R})$, so $T_{OP}(\mathbf{R})$ is not a regular semigroup. □

Lemma 2.4. *For any $a \in \mathbf{R}$, $T_{OP}((a, \infty))$ is not a regular semigroup.*

Proof. Let $a \in \mathbf{R}$ and $l \in \mathbf{R}^+$. Define

$$x\alpha = x + l \text{ for all } x \in (a, \infty).$$

Then $\alpha \in T_{OP}((a, \infty))$, $\nabla\alpha = (a+l, \infty)$ and α is 1-1. Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in T_{OP}((a, \infty))$. Then $x\alpha = x\alpha\beta\alpha$ for all $x \in (a, \infty)$ which implies that $x = x\alpha\beta$ for all $x \in (a, \infty)$ since α is 1-1. Since $\nabla\alpha = (a + l, \infty)$, $(a + l, \infty)\beta = (a, \infty)$. Since $a + l > a$, $a + l \in (a, \infty)$, so there exists $b \in (a + l, \infty)$ such that $b\beta = (a + l)\beta$. Let $c \in (a + l, b)$. Then b, $c \in \nabla\alpha$ and $c < b$. Let x, $y \in (a, \infty)$ be such that $x\alpha = b$ and $y\alpha = c$. Since α and β are order-preserving and $a + l < c$, $(a + l)\beta\alpha \le c\beta\alpha$. Then $b = x\alpha = x\alpha\beta\alpha = b\beta\alpha = (a + l)\beta\alpha \le c\beta\alpha = y\alpha\beta\alpha = y\alpha = c$, a contradiction. Thus α is not a regular element of $T_{OP}((a, \infty))$. Hence $T_{OP}((a, \infty))$ is not a regular semigroup. □

Lemma 2.5. *For any $a \in \mathbf{R}$, $T_{OP}((-\infty, a))$ is not a regular semigroup.*

Proof. Let $\le$ be the natural partial order on $\mathbf{R}$ and $\le_{opp}$ its opposite partial order. Let $a \in \mathbf{R}$. Then $((-\infty, a), \le)$ and $((-a, \infty), \le_{opp})$ are order – isomorphic.

By Proposition 1.5 (2) and Lemma 2.4, $T_{OP}((-a, \infty), \leq_{opp})$ is not regular. Hence $T_{OP}((-\infty, a), \leq)$ is not a regular semigroup. □

Lemma 2.6. *For any $a \in \mathbf{R}$, $T_{OP}([a, \infty))$ is not a regular semigroup.*

Proof. Let $a \in \mathbf{R}$ and $l \in \mathbf{R}^+$. Define

$$x\alpha = a + \frac{x-a}{x-a+l} \text{ for all } x \in [a, \infty).$$

Then $\nabla\alpha = [a, a+l)$. Since the derivative of the function $\frac{x-a}{x-a+l}$ on (a, ∞) with respect to x is $\frac{l}{(x-a+l)^2}$ and $\frac{l}{(x-a+l)^2} > 0$ for all $x \in [a, \infty)$, it follows that α is increasing and 1-1. Therefore $\alpha \in T_{OP}([a, \infty))$. To show that α is not regular in $T_{OP}([a, \infty))$, suppose on the contrary that there exists $\beta \in T_{OP}([a, \infty))$ such that $\alpha = \alpha\beta\alpha$. Then $x\alpha = x\alpha\beta\alpha$ for all $x \in [a, \infty)$. Since α is 1-1, $x\alpha\beta = x$ for all $x \in [a, \infty)$. Thus $[a, a+l)\beta = [a, \infty)$ since $\nabla\alpha = [a, a+l)$. This implies that $(a+l)\beta = b\beta$ for some $b \in [a, a+l)$. Let $c \in (b, a+l)$. Then $b < c < a+l$, so $b\beta \leq c\beta \leq (a+l)\beta$ since β is order - preserving. But $(a+l)\beta = b\beta$, so $b\beta = c\beta$. Since $b, c \in [a, a+l) = \nabla\alpha$, there exist $x, y \in \Delta\alpha$ such that $x\alpha = b$ and $y\alpha = c$, Consequently, $x < y$ since $b < c$. Now we have $x\alpha = x\alpha\beta\alpha = b\beta\alpha = c\beta\alpha = y\alpha\beta\alpha = y\alpha$ which implies that $x = y$ since α is 1-1 which is a contradiction. Hence α is not a regular element in $T_{OP}([a, \infty))$. □

Lemma 2.7. *For any $a \in \mathbf{R}$, $T_{OP}((-\infty, a])$ is not a regular semigroup.*

Proof. Since for $a \in \mathbf{R}$, $((-\infty, a], \leq)$ and $([-a, \infty), \leq_{opp})$ are order - isomorphic where $\leq$ is the natural partial order on $\mathbf{R}$, by Proposition 1.5 (2) and Lemma 2.6, $T_{OP}((-\infty, a])$ is not a regular semigroup. □

Lemma 2.8. *If $a, b \in \mathbf{R}$ such that $a<b$, then $T_{OP}((a, b))$ is not regular semigroup.*

Proof. Let $l \in (0, b\text{-}a)$. Define

$$x\alpha = (1 - \frac{l}{b-a})x + \frac{lb}{b-a} \quad \text{for all } x \in (a, b).$$

Since $1 - \frac{l}{b-a} > 0$, α is increasing and 1-1. Then $(a, b)\alpha = (a+l, b) \subseteq (a, b)$, so $\nabla\alpha = (a+l, b)$. Suppose that α is regular in $T_{OP}(X)$. Then there exists $\beta \in T_{OP}(X)$ such that $\alpha = \alpha\beta\alpha$. Since α is 1-1, $x = x\alpha\beta$ for all $x \in (a, b)$. Then $(a, b) = (\nabla\alpha)\beta$. This implies that $\nabla\alpha\beta = (a+l, b)\beta = (a, b)$. Since $a+l \in (a, b)$, $(a+l)\beta \in (a, b)$. Then there exists $c \in (a, b)$ such that $a+l<c$ and $c\beta = (a+l)\beta$. Let $d \in (a+l, c)$. Then $d \in \nabla\alpha$. Since β is order - preserving and $a+l<d<c$, $(a+l)\beta \le d\beta \le c\beta$. But $(a+l)\beta = c\beta$, so $d\beta = c\beta$. Let $x, y \in X$ be such that $x\alpha = c$ and $y\alpha = d$. Consequently, $c = x\alpha = x\alpha\beta\alpha = c\beta\alpha = d\beta\alpha = y\alpha\beta\alpha = y\alpha = d$ which is a contradiction. Hence β is not a regular element in $T_{OP}(X)$. □

Lemma 2.9. *If $a, b \in \mathbf{R}$ are such that $a<b$, then $T_{OP}([a, b))$ is not a regular semigroup.*

Proof. Let $l \in (0, b\text{-}a)$. Define

$$x\alpha = \left(\frac{l}{b-a}\right)x + a - \frac{la}{b-a} \quad \text{for all } x \in [a, b).$$

Then $[a, b)\alpha = [a, a+l)$. Since $\frac{l}{b-a} > 0$, α is increasing and 1-1. Therefore $\alpha \in T_{OP}([a, b))$. Suppose there exists $\beta \in T_{OP}([a, b))$ such that $\alpha = \alpha\beta\alpha$. Then $x\alpha = x\alpha\beta\alpha$ for all $x \in [a, b)$. Since α is 1-1, $x = x\alpha\beta$ for all $x \in [a, b)$. Thus $[a, a+l)\beta = [a, b)$. It follows that there exists $c \in [a, a+l)$ such that $c\beta = (a+l)\beta$. Let $d \in (c, a+l)$. Then $c<d<a+l$, so $c, d \in \nabla\alpha$. Since β is order - preserving, $c\beta \le d\beta \le (a+l)\beta$ which implies that $c\beta = d\beta$ since $c\beta =$

$(a+1)\beta$. Let $x, y \in [a, b)$ be such that $x\alpha = c$ and $y\alpha = d$. Hence we have $c = x\alpha = x\alpha\beta\alpha = c\beta\alpha = d\beta\alpha = y\alpha\beta\alpha = y\alpha = d$, a contradiction. Hence α is not a regular element of $T_{OP}([a, b))$. Hence $T_{OP}([a, b))$ is not regular. □

Lemma 2.10. *If $a, b \in R$ are such that $a < b$, then $T_{OP}((a, b])$ is not a regular semigroup.*

Proof. Let $a, b \in \mathbf{R}$ be such that $a < b$. Then $((a, b], \leq)$ and $([-b, -a), \leq_{opp})$ are order – isomorphic where $\leq$ is the natural partial order on **R**. By Proposition 1.5 (2) and Lemma 2.9, $T_{OP}((a, b])$ is not regular. □

Lemma 2.11. *Let $a, b \in R$ be such that $a < b$, $\alpha \in T_{OP}([a, b])$ and $x \in (a\alpha, b\alpha)$. If $A_x = [a, x]\alpha^{-1}$ and $B_x = (x, b]\alpha^{-1}$, then $A_x \neq \phi$ and $B_x \neq \phi$, $A_x \cup B_x = [a, b]$, $A_x \cap B_x = \phi$ and $c < d$ for all $c \in A_x$ and $d \in B_x$.*

Proof. Since $x \in (a\alpha, b\alpha)$, $a \leq a\alpha < x$ and $x < b\alpha \leq b$, so $a \in [a, x]\alpha^{-1}$ and $b \in (x, b]\alpha^{-1}$. Then $a \in A_x$ and $b \in B_x$. We have that $[a, b] = [a, b]\alpha^{-1} = ([a, x] \cup (x, b])\alpha^{-1} = [a, x]\alpha^{-1} \cup (x, b]\alpha^{-1} = A_x \cup B_x$. Since $[a, x] \cap (x, b] = \phi$, $[a, x]\alpha^{-1} \cap (x, b]\alpha^{-1} = \phi$, We get that $A_x \cap B_x = \phi$. Let $c \in A_x$ and $d \in B_x$. Then $c\alpha \in [a, x]$ and $d\alpha \in (x, b]$. Therefore $c\alpha < d\alpha$. Since $c \in (c\alpha)\alpha^{-1}$ and $d \in (d\alpha)\alpha^{-1}$, by Lemma 2.1, $c < d$. □

We know the following facts of real numbers.

(1) If A and B are nonempty subsets of **R** such that $A \cap B = \phi$ and $x < y$ for all $x \in A$ and $y \in B$, then $sup(A) \leq inf(B)$.

(2) If I is an interval in **R** and A and B are nonempty subsets of **R** such that $A \cup B = I$, $A \cap B = \phi$ and $x < y$ for all $x \in A$ and $y \in B$, then either $max(A)$ exists or $min(B)$ exists.

Lemma 2.12. *If a, $b \in \mathbf{R}$ are such that $a < b$, then $T_{OP}([a, b])$ is a regular semigroup.*

Proof. Let $\alpha \in T_{OP}([a, b])$. Since α is order - preserving, $a\alpha \le b\alpha$ and $\nabla\alpha \subseteq [a\alpha, b\alpha]$. We define d_x for each $x \in [a, b]$ as follows:

(i) If $x \in [a, a\alpha)$, let $d_x = a$.

(ii) If $x \in (b\alpha, b]$, let $d_x = b$.

(iii) If $x \in \nabla\alpha$, choose $d_x \in x\alpha^{-1}$.

Let $x \in (a\alpha, b\alpha) - \nabla\alpha$. Define $A_x = [a, x]\alpha^{-1}$ and $B_x = (x, b]\alpha^{-1}$. By Lemma 2.11, $A_x \neq \phi$, $B_x \neq \phi$, $A_x \cup B_x = [a, b]$, $A_x \cap B_x = \phi$ and $c < d$ for all $c \in A_x$ and $d \in B_x$. Therefore $sup(A_x) \le inf(B_x)$ and either $max(A_x)$ exists or $min(B_x)$ exists. Define

$$\text{(iv)} \quad d_x = \begin{cases} \max(A_x) & \textit{if} \quad \max(A_x) \quad \textit{exists}, \\ \min(B_x) & \textit{if} \quad \min(B_x) \quad \textit{exists}. \end{cases}$$

Next, define $x\beta = d_x$ for all $x \in [a, b]$. If $x \in [a, b]$, then $x\alpha \in \nabla\alpha$ which implies by (iii) that $d_{x\alpha}\alpha = x\alpha$ and hence $x\alpha\beta\alpha = ((x\alpha)\beta)\alpha = d_{x\alpha}\alpha = x\alpha$. This proves that $\alpha = \alpha\beta\alpha$ in $T([a, b])$.

To show that β is order - preserving, let x, $y \in [a, b]$ be such that $x < y$. Then $x \in [a, y]$ and $y \in (x, b]$.

Case 1: $x < a\alpha$. Then $x\beta = d_x = a$ by (i), so $x\beta < y\beta$.

Case 2: $y > b\alpha$. By (ii), $y\beta = b$. Then $x\beta \le y\beta$.

Case 3: $x \in \nabla\alpha$ and $y \in \nabla\alpha$. By (iii) and Lemma 2.1, $d_x < d_y$. Then $x\beta < y\beta$.

Case 4: $x \in \nabla\alpha$ and $y \in (a\alpha, b\alpha) - \nabla\alpha$. Then $d_x \in x\alpha^{-1} \subseteq [a, y]\alpha^{-1} = A_y$. If $max(A_y)$ exists, then $d_x \le max(A_y)$ and by (iv), $d_y = max(A_y)$. If $min(B_y)$ exists, then $d_y = min(B_y)$ by (iv), so by Lemma 2.11, $d_x < d_y$. Thus $x\beta \le y\beta$.

Case 5: $x \in (a\alpha, b\alpha) - \nabla\alpha$ and $y \in \nabla\alpha$. Then $d_y \in y\alpha^{-1} \subseteq (x, b]\alpha^{-1} = B_x$. If $min(B_x)$ exists, then $min(B_x) \le d_y$ and by (iv), $d_x = min(B_x)$. If $max(A_x)$ exists, then by (iv) $d_x = max(A_x)$, so by Lemma 2.11, $d_x < d_y$. Thus $x\beta \le y\beta$.

Case 6: $x \in (a\alpha, b\alpha) - \nabla\alpha$ *and* $y \in (a\alpha, b\alpha) - \nabla\alpha$.

Case 6.1: $[x, y] \cap \nabla\alpha = \phi$. Then $[a, x]\alpha^{-1} = [a, y]\alpha^{-1}$ and $(x, b]\alpha^{-1} = (y, b]\alpha^{-1}$, so $A_x = A_y$ and $B_x = B_y$. By (iv), $d_x = d_y$. Hence $x\beta = y\beta$.

Case 6.2: $[x, y] \cap \nabla\alpha \neq \phi$. Then there exists $c \in \nabla\alpha$ such that $x < c < y$. Since $c \in \nabla\alpha$, there exists $p \in [a, b]$ such that $p\alpha = c$. Then $p \in [a, y]\alpha^{-1} \cap (x, b]\alpha^{-1}$. Thus $p \in B_x \cap A_y$. Therefore $sup(A_x) \leq inf(B_x) \leq p \leq sup(A_y) \leq inf(B_y)$. Hence by (iv), $d_x \leq d_y$, so we have $x\beta \leq y\beta$. □

From Lemma 2.2 – 2.10 and Lemma 2.12, the following theorem is obtained.

Theorem 2.13. *For any interval X of* **R**, *$T_{OP}(X)$ is regular if and only if X is a closed and bounded interval.*

Theorem 2.14. *If X is a chain, then $PT_{OP}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in PT_{OP}(X)$. For each $a \in \nabla\alpha$, choose $d_a \in a\alpha^{-1}$. Then $d_a\alpha = a$ for all $a \in \nabla\alpha$. Define $\beta \in PT_{OP}(X)$ by $a\beta = d_a$ for all $a \in \nabla\alpha$. Then $\Delta\beta = \nabla\alpha$, $\Delta\alpha\beta\alpha = \Delta\alpha$ and for every $x \in \Delta\alpha$, $x\alpha\beta\alpha = ((x\alpha)\beta)\alpha = (d_{x\alpha})\alpha = x\alpha$. Therefore $\alpha\beta\alpha = \alpha$. To show that β is order - preserving, let $a, b \in \Delta\beta$ be such that $a < b$. Then $a, b \in \nabla\alpha$ and $a < b$. By Lemma 2.1, $d_a < d_b$. Then $a\beta < b\beta$. Hence $\beta \in PT_{OP}(X)$. This proves that $PT_{OP}(X)$ is regular, as required. □

Theorem 2.15. *If X is a chain, then $U_{OP}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in U_{OP}(X)$. Then $s(\alpha)$ is finite. For each $a \in \nabla\alpha$, choose $d_a \in a\alpha^{-1}$. Then $d_a\alpha = a$ for all $a \in \nabla\alpha$. Define $\beta \in PT(X)$ by $a\beta = d_a$ for $a \in \nabla\alpha$. Then $\Delta\beta = \nabla\alpha$. By the proof of Theorem 2.14, $\alpha = \alpha\beta\alpha$ and β is order – preserving. By Proposition 1.2 (1), $a\alpha^{-1} = \{a\}$ for all $a \in \nabla\alpha - s(\alpha)\alpha$ Then $a = d_a = a\beta$ for all $a \in \nabla\alpha - s(\alpha)\alpha$ Then $s(\beta) \subseteq s(\alpha)\alpha$. Since $s(\alpha)$ is finite, $s(\alpha)\alpha$ is finite. Thus $s(\beta)$ is finite, so $\beta \in U_{OP}(X)$. Hence $U_{OP}(X)$ is a regular semigroup. □

Theorem 2.16. *If X is a chain, then $I_{OP}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in I_{OP}(X)$. For each $a \in \nabla\alpha$, choose $d_a \in a\alpha^{-1}$. Then $d_a\alpha = a$ for all $a \in \nabla\alpha$. Define $\beta \in PT(X)$ by $a\beta = d_a$ for all $a \in \nabla\alpha$. Then $\Delta\beta = \nabla\alpha$. By the proof of Theorem 2.14, $\alpha\beta\alpha = \alpha$ and $\beta \in PT_{OP}(X)$. Since for distinct $x, y \in \nabla\alpha$, $x\alpha^{-1}$ and $y\alpha^{-1}$ are disjoint, it follows that β is 1-1. Then $\beta \in I_{OP}(X)$. This proves that $I_{OP}(X)$ is regular. □

Theorem 2.17. *If X is a chain, then $W_{OP}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in W_{OP}(X)$. Then $\alpha \in I_{OP}(X)$ and $s(\alpha)$ is finite. Therefore $s(\alpha)\alpha$ is finite. By Proposition 1.2 (2), $s(\alpha^{-1})$ is finite. Then $\alpha^{-1} \in W_{OP}(X)$. Hence $W_{OP}(X)$ is a regular semigroup. □

Lemma 2.18. *Let X be a partially ordered set, $\alpha \in PT_{OP}(X)$ and $a \in \Delta\alpha$. Then $\{x \in \Delta\alpha \mid a\alpha < x < a\} \subseteq s(\alpha)$ and $\{x \in \Delta\alpha \mid a < x < a\alpha\} \subseteq s(\alpha)$.*

Proof. Let $x \in \Delta\alpha$ be such that $a\alpha < x < a$. Since α is order-preserving and $x < a$, $x\alpha \le a\alpha$. If $x\alpha = x$, then $x \le a\alpha$, a contradiction. Thus $x\alpha \ne x$ which implies that $x \in s(\alpha)$. Hence $\{x \in \Delta\alpha \mid a\alpha < x < a\} \subseteq s(\alpha)$. The fact that $\{x \in \Delta\alpha \mid a < x < a\alpha\} \subseteq s(\alpha)$ can be proved similarly. □

Lemma 2.19. *Let X be a partially ordered set, $\alpha \in PT_{OP}(X)$ and $A \subseteq \nabla\alpha$.*

(1) *If $max(A)$ and $max(A\alpha^{-1})$ exist, then $max(A) = max(A\alpha^{-1})\alpha$.*

(2) *If $min(A)$ and $min(A\alpha^{-1})$ exist, then $min(A) = min(A\alpha^{-1})\alpha$.*

Proof. (1) Since $max(A) \in A \subseteq \nabla\alpha$, there exists $x \in \Delta\alpha$ such that $max(A) = x\alpha$. Then $x \in A\alpha^{-1}$, so $x \le max(A\alpha^{-1})$. Since α is order-preserving, $x\alpha \le (max(A\alpha^{-1}))\alpha$. Then $max(A) \le (max(A\alpha^{-1}))\alpha$. Since $max(A\alpha^{-1}) \in A\alpha^{-1}$ and $A \subseteq \nabla\alpha$, $(max(A\alpha^{-1}))\alpha \in (A\alpha^{-1})\alpha = A$. This implies that $(max(A\alpha^{-1}))\alpha \le max(A)$. Hence

$max(A) = (max(A\alpha^{-1}))\alpha$.

(2) can be proved similarly. $\square$

Lemma 2.20. *Let X be a partially ordered set, $\alpha \in PT_{OP}(X)$ and $A, B \subseteq \nabla\alpha$ such that $max(A)$, $max(B)$, $max(A\alpha^{-1})$ and $max(B\alpha^{-1})$ exist.*

(1) If $max(A) = max(B)$, then $max(A\alpha^{-1}) = max(B\alpha^{-1})$.

(2) If X is a chain and $max(A) < max(B)$, then $max(A\alpha^{-1}) < max(B\alpha^{-1})$.

(3) If $min(A) = min(B)$, then $min(A\alpha^{-1}) = min(B\alpha^{-1})$.

(4) If X is a chain and $min(A) < min(B)$, then $min(A\alpha^{-1}) < min(B\alpha^{-1})$.

Proof. (1) By Lemma 2.19 (1), $max(A\alpha^{-1})\alpha = max(A)$ and $max(B\alpha^{-1})\alpha = max(B)$. Since $max(A) = max(B)$, $max(A) \in A$ and $max(B) \in B$, it follows that $max(A\alpha^{-1})\alpha \in B$ and $max(B\alpha^{-1})\alpha \in A$. These imply that $max(A\alpha^{-1}) \in B\alpha^{-1}$ and $max(B\alpha^{-1}) \in A\alpha^{-1}$. Thus $max(A\alpha^{-1}) \leq max(B\alpha^{-1})$ and $max(B\alpha^{-1}) \leq max(A\alpha^{-1})$. Hence $max(A\alpha^{-1}) = max(B\alpha^{-1})$.

(2) By Lemma 2.19 (1) and the assumption, we have

$$max(A\alpha^{-1})\alpha = max(A) < max(B) = max(B\alpha^{-1})\alpha \qquad (*)$$

Since X is a chain, $max(A\alpha^{-1}) < max(B\alpha^{-1})$ or $max(B\alpha^{-1}) \leq max(A\alpha^{-1})$. Since α is order - preserving, it follows that if $max(B\alpha^{-1}) \leq max(A\alpha^{-1})$, then $max(B\alpha^{-1})\alpha \leq max(A\alpha^{-1})\alpha$ which contradict (*). Hence $max(A\alpha^{-1}) < max(B\alpha^{-1})$.

(3) and (4) can be proved similarly. $\square$

Theorem 2.21. *If X is a chain, then $V_{OP}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in V_{OP}(X)$. Since $X - \nabla\alpha \subseteq s(\alpha)$ by Proposition 1.1 (1) and $s(\alpha)$ is finite, we have that $X - \nabla\alpha$ is finite. Since $s(\alpha)$ is finite, for every $a \in \nabla\alpha$, $a\alpha^{-1}$ is finite by Proposition 1.1 (2). Consequently, $max(a\alpha^{-1})$ exists for every $a \in \nabla\alpha$ since X is a chain. For each $x \in X$, define $d_x \in X$ as follows: Let $x \in X$.

Case I : $x \in \nabla\alpha$. Define

$$d_x = max(x\alpha^{-1}). \qquad (*)$$

Case II : $x \in X - \nabla\alpha$. Then $x \in s(\alpha)$ since $X - \nabla\alpha \subsetneq s(\alpha)$. Therefore $x\alpha \neq x$ which implies that $x\alpha < x$ or $x < x\alpha$ since X is a chain. By Lemma 2.18, both $\{y \in X \mid x\alpha < y < x\}$ and $\{y \in X \mid x < y < x\alpha\}$ are subsets of $s(\alpha)$. It follows that each of $\{y \in X \mid x\alpha < y < x\}$ and $\{y \in X \mid x < y < x\alpha\}$ is finite. Thus each of $\{y \in \nabla\alpha \mid x\alpha \leq y < x\}$ and $\{y \in \nabla\alpha \mid x < y \leq x\alpha\}$ is finite. Since for every $a \in \nabla\alpha$, $a\alpha^{-1}$ is finite, we have that both $\{y \in \nabla\alpha \mid x\alpha \leq y < x\}\alpha^{-1}$ and $\{y \in \nabla\alpha \mid x < y \leq x\alpha\}\alpha^{-1}$ are finite. If $x\alpha < x$, then $\{y \in \nabla\alpha \mid x\alpha \leq y < x\} \neq \phi$, so $max(\{y \in \nabla\alpha \mid x\alpha \leq y < x\}\alpha^{-1})$ exists. If $x < x\alpha$, then $\{y \in \nabla\alpha \mid x < y \leq x\alpha\} \neq \phi$, so $min(\{y \in \nabla\alpha \mid x < y \leq x\alpha\}\alpha^{-1})$ exists. Define

$$d_x = \begin{cases} \max(\{y \in \nabla\alpha \ / \ x\alpha \leq y < x\}\alpha^{-1}) & if \ x\alpha < x, \qquad (**) \\ \min(\{y \in \nabla\alpha \ / \ x < y \leq x\alpha\}\alpha^{-1}) & if \ x < x\alpha. \qquad (***) \end{cases}$$

From defining d_x for all $x \in X$, we have from Case I that $d_x\alpha = x$ for all $x \in \nabla\alpha$. Next define $\beta : X \to X$ by $x\beta = d_x$ for all $x \in X$. If $x \in X$, then $x\alpha \in \nabla\alpha$, so $x\alpha\beta\alpha = (x\alpha)\beta\alpha = d_{x\alpha}\alpha = x\alpha$. This proves that $\alpha = \alpha\beta\alpha$. To show $s(\beta)$ is finite, it suffices to show that $\{x \in \nabla\alpha \mid x\beta \neq x\}$ is finite since $s(\beta) \subsetneq (X - \nabla\alpha) \cup \{x \in \nabla\alpha \mid x\beta \neq x\}$ and $X - \nabla\alpha$ is finite. By the definition of β, we have $\{x \in \nabla\alpha \mid x\beta \neq x\} = \{x \in \nabla\alpha \mid max(x\alpha^{-1}) \neq x\} \subsetneq \{x \in \nabla\alpha \mid x\alpha^{-1} \neq \{x\}\}$. But $\{x \in \nabla\alpha \mid x\alpha^{-1} \neq \{x\}\}$ is finite by Proposition 1.1 (2), so $\{x \in \nabla\alpha \mid x\beta \neq x\}$ is finite. Hence $s(\beta)$ is finite.

Finally, we shall show β is order - preserving. Let $a, b \in X$ be such that $a < b$. Then $a\alpha \leq b\alpha$.

Case 1: $a, b \in \nabla\alpha$. From (*), $d_a \in a\alpha^{-1}$ and $d_b \in b\alpha^{-1}$. Since $a < b$, by Lemma 2.1, $d_a < d_b$.

Case 2: $a \in \nabla\alpha$ and $b \notin \nabla\alpha$. Since $b \notin \nabla\alpha$, $b\alpha \neq b$, so $b\alpha < b$ or $b < b\alpha$.

Case 2.1: $b\alpha < b$. If $a \in \{y \in \nabla\alpha \mid b\alpha \leq y < b\}$, then $a\alpha^{-1} \subsetneq \{y \in \nabla\alpha \mid b\alpha \leq y < b\}\alpha^{-1}$ which implies that $max(a\alpha^{-1}) \leq max(\{ y \in \nabla\alpha \mid b\alpha \leq y < b \}\alpha^{-1})$. By (*) and (**), $d_a \leq d_b$. Next assume that $a \notin \{y \in \nabla\alpha \mid b\alpha \leq y < b\}$. Since $a \in \nabla\alpha$ and $a < y$ for all $y \in \nabla\alpha$ such that $b\alpha \leq y < b$. By Lemma 2.1, we have that $u \in a\alpha^{-1}$ and $v \in \{y \in \nabla\alpha \mid b\alpha \leq y < b\}\alpha^{-1}$ imply $u < v$. Hence $max(a\alpha^{-1}) <$

$max(\{ y \in \nabla\alpha \mid b\alpha \leq y < b \}\alpha^{-1})$. By (*) and (**), we have that $d_a < d_b$.

Case 2.2: $b < b\alpha$. Then $a < b < b\alpha$, so $a < y$ for all $y \in \nabla\alpha$ such that $b < y \leq b\alpha$. By Lemma 2.1, $max(a\alpha^{-1}) < min(\{ y \in \nabla\alpha \mid b < y \leq b\alpha\}\alpha^{-1})$. Hence $d_a < d_b$ by (*) and (***).

Case 3: $a \notin \nabla\alpha$ and $b \in \nabla\alpha$. Then $a\alpha \neq a$, so $a\alpha < a$ or $a < a\alpha$.

Case 3.1: $a\alpha < a$. Since $a < b$, it follows that for $y \in \nabla\alpha$, $a\alpha \leq y < a$ implies $y < b$. By Lemma 2.1, $max(\{y \in \nabla\alpha \mid a\alpha \leq y < a\}\alpha^{-1}) < max(b\alpha^{-1})$. By (**) and (*), $d_a < d_b$

Case 3.2: $a < a\alpha$. Since $a < b$, $a < b \leq a\alpha$ or $a\alpha < b$. If $a < b \leq a\alpha$, then $b\alpha^{-1} \subsetneq \{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}$ which implies that $min(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}) \leq max(b\alpha^{-1})$. Then $d_a \leq d_b$ by (***) and (*). If $a\alpha < b$, then $y < b$ for all $y \in \nabla\alpha$ such that $a < y \leq a\alpha$. By Lemma 2.1, we have that $u \in \{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}$ and $v \in b\alpha^{-1}$ imply $u < v$. Hence $min(\{ y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}) < max(b\alpha^{-1})$. By (***) and (*), we have that $d_a < d_b$.

Case 4: $a \notin \nabla\alpha$ and $b \notin \nabla\alpha$. Since $a \notin \nabla\alpha$, $a\alpha < a$ or $a < a\alpha$. We also have $b\alpha < b$ or $b < b\alpha$ since $b \notin \nabla\alpha$.

Case 4.1: $a\alpha < a$ and $b\alpha < b$. Since X is a chain and $a\alpha \leq b\alpha$, $max(\{y \in \nabla\alpha \mid a\alpha \leq y < b\}) = max(\{y \in \nabla\alpha \mid b\alpha \leq y < b\})$. From the fact that $a < b$, we have $\{y \in \nabla\alpha \mid a\alpha \leq y < a\}) \subsetneq \{y \in \nabla\alpha \mid a\alpha \leq y < b\}$. Consequently, $max(\{y \in \nabla\alpha \mid a\alpha \leq y < a\}) \leq max(\{y \in \nabla\alpha \mid a\alpha \leq y < b\}) = max(\{y \in \nabla\alpha \mid b\alpha \leq y < b\})$. By Lemma 2.20 ((1) and (2)), $max(\{y \in \nabla\alpha \mid a\alpha \leq y < a\}\alpha^{-1}) \leq max(\{y \in \nabla\alpha \mid b\alpha \leq y < b\}\alpha^{-1})$, so $d_a \leq d_b$ by (**).

Case 4.2: $a\alpha < a$ and $b < b\alpha$. Then $a\alpha < a < b < b\alpha$, so for all $u \in \{y \in \nabla\alpha \mid a\alpha \leq y < a\}$ and $u' \in \{y \in \nabla\alpha \mid b < y \leq b\alpha\}$, $u < u'$. Thus $v \in \{y \in \nabla\alpha \mid a\alpha \leq y < a\}\alpha^{-1}$ and $v' \in \{y \in \nabla\alpha \mid b < y \leq b\alpha\}\alpha^{-1}$, $v < v'$. Then $max(\{y \in \nabla\alpha \mid a\alpha \leq y < a\}\alpha^{-1}) < min(\{y \in \nabla\alpha \mid b < y \leq b\alpha\}\alpha^{-1})$. Hence $d_a < d_b$ by (**) and (***).

Case 4.3: $a < a\alpha$ and $b\alpha < b$. Then $a < a\alpha \leq b\alpha < b$. Thus for all $u \in \{y \in \nabla\alpha \mid a < y \leq a\alpha\}$ and $u' \in \{y \in \nabla\alpha \mid b\alpha \leq y < b\}$, $u \leq u'$. Then

$max(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}) \leq max(\{y \in \nabla\alpha \mid b\alpha \leq y < b\})$. By Lemma 2.20,((1) and (2)), $max(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}) \leq max(\{y \in \nabla\alpha \mid b\alpha \leq y < b\}\alpha^{-1})$ which implies that $min(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}) \leq max(\{y \in \nabla\alpha \mid b\alpha \leq y < b\}\alpha^{-1})$. Hence $d_a \leq d_b$ by (***) and (**).

Case 4.4: $a < a\alpha$ and $b < b\alpha$. Since X is a chain and $a\alpha \leq b\alpha$. $min(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}) = min(\{y \in \nabla\alpha \mid a < y \leq b\alpha\})$. By Lemma 2.20 (3), $min(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}) = min(\{y \in \nabla\alpha \mid a < y \leq b\alpha\}\alpha^{-1})$. But $\{y \in \nabla\alpha \mid b < y \leq b\alpha\} \subsetneq \{y \in \nabla\alpha \mid a < y \leq b\alpha\}$ since $a < b$, so $min(\{y \in \nabla\alpha \mid a < y \leq b\alpha\}) \leq min(\{y \in \nabla\alpha \mid b < y \leq b\alpha\})$. Therefore $min(\{y \in \nabla\alpha \mid a < y \leq a\alpha\}\alpha^{-1}) \leq min(\{y \in \nabla\alpha \mid b < y \leq b\alpha\}\alpha^{-1})$. Hence by (***), $d_a \leq d_b$.

Therefore we have $d_a \leq d_b$ for all possible cases, so $a\beta \leq b\beta$.

Hence the theorem is completely proved. $\square$