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จุฬาลงกรณ์มหาวิทยาลัย

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SUPER EDGE-MAGIC LABELINGS OF SOME GRAPHS

Miss Wannaporn Sanprasert

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

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การกำกับกลบนเส้นเชื่อมอย่างยวดยิ่งของกราฟ G ที่มี $V(G)$ เป็นเซตของจุดยอด และ $E(G)$ เป็นเซตของด้าน คือฟังก์ชัน f ที่เป็นฟังก์ชันหนึ่งต่อหนึ่งจาก $V(G) \cup E(G)$ ไปทั่วถึง $\{1,2,\ldots,p+q\}$ เมื่อ $p = |V(G)|$ และ $q = |E(G)|$ ที่มีสมบัติว่า สำหรับด้าน ab ใดๆ จะได้ว่า $f(a) + f(ab) + f(b) = k$ เมื่อ k เป็นค่าคงตัวบางค่า และ $f(V(G)) = \{1,2,\ldots,p\}$

วิทยานิพนธ์นี้ได้ศึกษาและรวบรวมกราฟที่มีการกำกับกลบนเส้นเชื่อมอย่างยวดยิ่ง นอกจากนี้เรายังพิสูจน์ว่ากราฟต่อไปนี้มีการกำกับกลบนเส้นเชื่อมอย่างยวดยิ่ง กราฟว่าว n เหลี่ยมหางยาว 1 เมื่อ n เป็นจำนวนคี่ กราฟสับปะรด n เหลี่ยมจุกมี m ใบ เมื่อ n เป็นจำนวนคี่ กราฟที่ประกอบด้วยผลผนวกที่แยกออกจากกันของกราฟว่าวขนาด n หางยาว 1 จำนวน n ชุด และกราฟที่ประกอบด้วยผลผนวกที่แยกออกจากกันของกราฟดาวขนาด n และ $n+1$ จำนวนอย่างละ 1 ชุด

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A super edge-magic labeling on a graph G with the vertex-set $V(G)$ and the edge-set $E(G)$ is a one-to-one function f from $V(G) \cup E(G)$ onto the set $\{1,2,\ldots,p+q\}$ where $p = |V(G)|$ and $q = |E(G)|$ with the property that, for any edge ab,
$f(a) + f(ab) + f(b) = k$ for some k and $f(V(G)) = \{1,2,\ldots,p\}$.

This thesis surveys and collects many classes of graphs that can admit a super edge-magic labeling. Moreover, we prove that the following graphs are super edge-magic: the $(n,1)$-kite when n is odd, the (n,m)-pineapple when n is odd, the disjoint union of n copies of $(n,1)$-kite, when n is odd and the disjoint union of $K_{1,n}$ and $K_{1,n+1}$.

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CONTENTS

CHAPTER I

INTRODUCTION

A super edge-magic labeling of a graph is motivated by the concept of edge-magic labeling. In 1998 H. Enomoto, A.S. Llado, T. Nakamigawa and G. Ringel [2] defined a super edge-magic labeling of a graph G as a bijection from $V(G) \cup E(G)$ to the set of integers from 1 to $|V(G)|+|E(G)|$ such that the sum of labels on an edge and its two endpoints is the same for all edges and the set of vertex labels is $\{1,2,\ldots,|V(G)|\}$. A graph G is called super edge-magic if it admits a super edge-magic labeling. Enomoto et al. proved that the cycle C_n is super edge-magic when n is odd, the star $K_{1,n}$ is super edge-magic and if a graph G with p vertices and q edges is super edge-magic then $q \le 2p-3$. G. Ringel and A.S. Llado [6] proved that a caterpillar is super edge-magic. In 2001 R.M. Figueroa - Centeno, R.Ichishima and F.A. Muntaner - Batle [3] enumerated a necessary and sufficient condition for a graph to be super edge-magic, which is most of the time more useful than the definition itself and proved that the fan F_n is super edge-magic when $n \le 6$, the ladder L_n is super edge-magic when n is odd and the generalized prism $C_m \times P_n$ is super edge-magic when m is odd and $n \ge 2$. R.M. Figueroa- Centeno, R. Ichishima and F.A. Muntaner-Batle [4] proved the following:

1. P_n^2 is super edge-magic when $n \ge 3$ with $k = 3n$.
2. Let $\bigcup_{i=1}^{l} P_{n_i}$, where n_i is an integer with $n_i \ge 2$ for all i, be a super edge-magic. If $m \ge 3$ is odd, then $m(\bigcup_{i=1}^{l} P_{n_i})$ is super edge-magic.
3. The disjoint union of m copies of C_n is super edge-magic when $m \ge 1$ and $n \ge 3$ are odd.

4. For positive integers m and n, the disjoint union of m copies of $K_{1,n}$ is super edge-magic when m is odd.
5. The disjoint union of $K_{1,n}$ and $K_{1,n+1}$ is super edge-magic when $n \geq 1$.
6. The disjoint union of P_2 and P_n is super edge-magic when $n \geq 3$.
7. For every positive integer n, the disjoint union of P_3 and n copies of P_2 is super edge-magic.
8. The disjoint union of n copies of P_2 is super edge-magic when n is odd.

J.Wijaya and E.T.Baskoro [9] proved that the disjoint union of k copies of P_n is super edge-magic when $n \geq 1$ and $k \geq 3$ is odd.

This thesis surveys, collects many classes of graphs that can admit a super edge-magic labeling on some graphs (connected and disconnected) and presents new classes of super edge-magic graphs. Also proofs of some theorems are rewritten for better understanding.

There are four chapters in this thesis. In Chapter I, we introduce some authors who have studied super edge-magic labelings on many classes of graphs.

In Chapter II, we give a definition of a super edge-magic graph, lemmas and corollaries that will be used in this thesis.

In Chapter III, super edge-magic labelings on many classes of connected graphs are discussed and super edge-magic labelings on connected graphs: an $(n,1)$-kite, an (n,m)-pineapple and an n-sun, are presented.

In Chapter IV, super edge-magic labelings on many classes of disconnected graphs are discussed and we also present super edge-magic labelings on the following disconnected graphs: the graph $n(n,1)$-kites, the disjoint union of n copies of $(n,1)$-kite when n is odd and the disjoint union of $K_{1,n}$ and $K_{1,n+1}$.

CHAPTER II

DEFINITIONS AND EXAMPLES

We introduce the definition, provide examples, lemmas and corollaries for a graph to be super edge-magic.

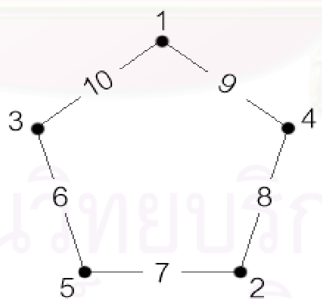
Definition 2.1. A *super edge-magic labeling* on a graph G is a bijective function

$$f : V(G) \cup E(G) \to \{1,2,\ldots, p+q\}$$

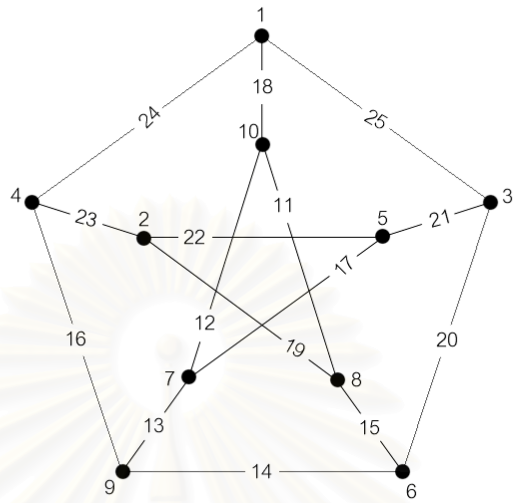
where $p = |V(G)|$ and $q = |E(G)|$ such that $f(u) + f(v) + f(uv) = k$ is a constant, called the *valence* of f, for any edge uv of G and $f(V(G)) = \{1,2,\ldots, p\}$.

A graph G is called *super edge-magic* if it admits a super edge-magic labeling.

Example 2.2. The cycle C_5 is super edge-magic with $k = 14$.



Example 2.3. The Petersen graph is super edge-magic with $k = 29$.



Lemma 2.4. [3] A $(p,q)-$graph G is super edge-magic if and only if there exists a bijective function $f:V(G)\to\{1,2,...,p\}$ such that the set

$$S=\{f(u)+f(v):uv\in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k=p+q+s$, where $s=\min(S)$ and

$$S=\{f(u)+f(v):uv\in E(G)\}$$
$$=\{k-(p+1),k-(p+2),...,k-(p+q)\}.$$

Proof. Let G be a $(p,q)-$graph.

$(\Rightarrow)$ Assume that G is super edge-magic. Then there exists a bijective function

$$f:V(G)\cup E(G)\to \{1,2,...,p+q\} \text{ and a constant } k$$

such that $f(u)+f(v)+f(uv)=k$ for any edge uv of G and $f(V(G))=\{1,2,...,p\}$.

Then $f(E(G))=\{p+1,p+2,...,p+q\}$.

Define $$S=\{k-f(uv):uv\in E(G)\}$$
$$=\{k-(p+1),k-(p+2),...,k-(p+q)\}.$$

Then $S=\{f(u)+f(v):uv\in E(G)\}$ consists of q consecutive integers.

$(\Leftarrow)$ Assume that there exists a bijective function $f: V(G) \to \{1,2,..., p\}$ such that the set

$$S = \{ f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers. Let xy be an edge in $E(G)$ such that $f(x) + f(y) = \min(S) = s$. Then $S = \{s, s+1, s+2,..., s+q-1\}$.

Define $f(uv) = p+q+s-(f(u)+f(v))$ for any edge uv of G.

Then $f(E(G)) = \{ p+q+s-(s), p+q+s-(s+1),..., p+q+s-(s+q-1)\}$

$$= \{ p+1, p+2,..., p+q\}.$$

Thus there exists a bijective function $f: V(G) \cup E(G) \to \{1,2,..., p+q\}$ such that $f(u) + f(v) + f(uv) = p+q+s$ is a constant for any edge uv of G and $f(V(G)) = \{1,2,..., p\}$.

Therefore G is super edge-magic. □

In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph.

Corollary 2.5. [3] If G is a super edge-magic $(p,q)-$graph and f is a super edge-magic labeling of G with a valence k, then

$$\sum_{v\in V(G)} f(v)\deg v = qs + \binom{q}{2},$$

where s is defined as in the previous lemma. In particular,

$$2\sum_{v\in V(G)} f(v)\deg v \equiv 0 (\bmod q).$$

Proof. Assume that G is a super edge-magic $(p,q)-$graph and f is a super edge-magic labeling of G with valence k. Let $u,v \in V(G)$.

Since $k = p+q+s$, $kq = (p+q+s)q$

$$\sum_{uv\in E(G)} f(u) + f(v) + f(uv) = pq + q^2 + qs$$

$$\sum_{v\in V(G)} f(v)\deg v + \sum_{i=p+1}^{p+q} i = pq + q^2 + qs$$

$$\sum_{v\in V(G)} f(v)\deg v + pq + \frac{q(q+1)}{2} = pq + q^2 + qs$$

$$\sum_{v\in V(G)} f(v)\deg v = qs + q^2 - \frac{q(q+1)}{2}$$
$$\sum_{v\in V(G)} f(v)\deg v = qs + \frac{2q^2 - q^2 - q}{2}$$
$$\sum_{v\in V(G)} f(v)\deg v = qs + \frac{q(q-1)}{2}$$
$$\sum_{v\in V(G)} f(v)\deg v = qs + \binom{q}{2}. \qquad \square$$

Corollary 2.6. [3] If G is a $(p,q)-$graph, where every vertex of G has even degree and $q \equiv 2(\bmod 4)$, then G is not super edge-magic.

Proof. Let G be a $(p,q)-$graph, where every vertex of G has even degree and $q \equiv 2(\bmod 4)$. Then $q = 4t+2$ for some $t \in \mathbb{Z}^+$.

Suppose G is super edge-magic. By corollary 2.5,

$$\sum_{v\in V(G)} f(v)\deg v = qs + \binom{q}{2}$$

$$f(v_1)\deg v_1 + f(v_2)\deg v_2 + ... + f(v_p)\deg v_p = (4t+2)s + \frac{(4t+2)(4t+1)}{2}$$

$$f(v_1)\deg v_1 + f(v_2)\deg v_2 + ... + f(v_p)\deg v_p = (4t+2)s + (2t+1)(4t+1).$$

Since $\deg v_i$ is even for all i, $\sum_{i=1} f(v_i)\deg v_i$ is even. Since $4t+2$ is even, $(4t+2)s$ is even. And since $2t+1$ and $4t+1$ are odd, $(2t+1)(4t+1)$ is odd.

So $(4t+2)s+(2t+1)(4t+1)$ is odd, a contradiction. Hence G is not super edge-magic. $\square$

Lemma 2.7. [3] If G is a $(p,q)-$graph, where $\deg v = r$ for every vertex v and $r>0$, then q is odd and the valence of any super edge-magic labeling of G is $\frac{1}{2}(4p+q+3)$.

Proof. Assume that G is a $(p,q)-$graph, where $\deg v = r$ for every vertex v and $r>0$.

The valence of any super edge-magic labeling of G is

$$kq = \sum_{uv\in E(G)} f(u) + f(v) + f(uv)$$

$$k=\frac{1}{q}(kq)=\frac{1}{q}(\sum_{uv\in E(G)} f(u)+f(v)+f(uv))$$
$$=\frac{1}{q}\left\{r\sum_{i=1}^{p} i+\sum_{i=p+1}^{p+q} i\right\}$$
$$=\frac{1}{q}\left\{\frac{2q}{p}\left(\frac{p(p+1)}{2}\right)+pq+\frac{q(q+1)}{2}\right\}$$
$$=\frac{1}{q}\left(2pq+\frac{q^2}{2}+\frac{3q}{2}\right)$$
$$=\frac{1}{2}(4p+q+3),$$

which implies that q is odd. □

In the following lemma, Enomoto et al. provided an upper bound for the size of super edge-magic graphs.

Lemma 2.8. [2] If a $(p,q)-$graph is super edge-magic, then $q\le 2p-3$.

Proof. Assume that a $(p,q)-$graph is super edge-magic.

By lemma 2.5, the valence of f is $k=p+q+\min(\{f(u)+f(v):uv\in E(G)\})$.

Thus $k\ge p+q+1+2$. Now, consider the extreme values of k :

$$p+q+\max(\{f(u)+f(v):uv\in E(G)\})\ge p+1+p+(p-1)\ge k.$$

Then $\quad 1+2+p+q\le k\le p+(p-1)+(p+1).$

Therefore $\quad q\le 2p-3.$ □

The following corollary implies that the minimum degree is at most 3 for every super edge-magic graph.

Corollary 2.9. [3] Every super edge-magic $(p,q)-$graph contains at least two vertices of degree less than 4.

Proof. Assume, to the contrary, that $p-1$ vertices of G have degree at least 4.

By the previous lemma,

$$4p-4=\sum_{i=1}^{p-1}4\le\sum_{v\in V(G)}\deg v=2q\le 2(2p-3)=4p-6,$$

which is a contradiction. □

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CHAPTER III

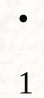
SUPER EDGE-MAGIC LABELINGS ON CONNECTED GRAPHS

In this chapter, we discuss some connected graphs which are or are not super edge-magic and give some examples of small cases.

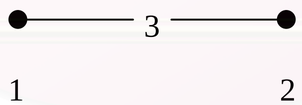
Theorem 3.1. [2] The complete graph K_n is super edge-magic when $n = 1,2,3$.

Proof. K_1, K_2 and K_3 are super edge-magic with valences 1, 6 and 9 respectively.

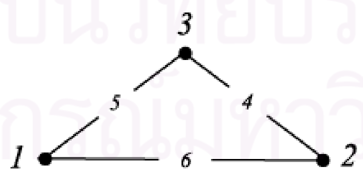
K_1:



K_2:



K_3:



□

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Theorem 3.2. [2] The complete graph K_n is not super edge-magic when $n \geq 4$.

Proof. Assume, to the contrary, that the complete graph K_n is super edge-magic.

By lemma 2.8, $$\frac{n(n-1)}{2} \leq 2n-3$$

$$n^2 - 5n + 6 \leq 0$$

$(n-2)(n-3) \leq 0$, which implies $2 \leq n \leq 3$.

Since $n \geq 4$, it is a contradiction.

Therefore the complete graph K_n is not super edge-magic when $n \geq 4$. □

Theorem 3.3. [2] A wheel W_n is not super edge-magic.

Proof. Assume, to the contrary, that W_n is super edge-magic.

By lemma 2.8,

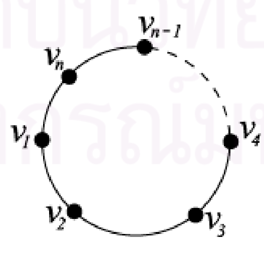
$$2n \leq 2(n+1) - 3 = 2n-1,$$

which is a contradiction.

Therefore W_n is not super edge-magic. □

Theorem 3.4. [2] Every odd cycle C_n is super edge-magic with $k = \dfrac{5n+3}{2}$.

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, ..., v_nv_1\}$.



Let $n = 2t + 1$ for some $t \in Z^+$ and define a vertex labeling f as follows:

$$f(v_i) = \begin{cases} \dfrac{i+1}{2} & if \quad i = 1,3,\ldots,2t+1, \\[2ex] t + \dfrac{i+2}{2} & if \quad i = 2,4,\ldots,2t. \end{cases}$$

The labeling f is shown in figure 3.1

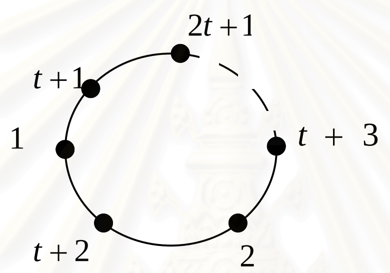


Figure3.1: A vertex labeling of C_{2t+1} for some $t \in Z^+$.

In order to show that f extends to a super edge-magic labeling of C_n, it suffices to verify the following by lemma 2.4:

a) $f(V(C_n)) = \{1,2,\ldots,n\}$

b) $S = \{f(x) + f(y) : xy \in E(C_n)\}$ is a set of n consecutive integers.

For a) the numbers $1,2,3,\ldots,t+1$ are labels of $v_1, v_3, v_5, \ldots, v_{2t+1}$. The numbers $t+2, t+3, t+4,\ldots,2t+1$ are labels of $v_2, v_4, v_6,\ldots, v_{2t}$. Thus, the set of vertex labelings is $\{1,2,\ldots,2t+1\}$.

For b) observe that

for $v_1 v_n \in E(C_{2t+1})$,

$$f(v_1) + f(v_n) = \frac{1+1}{2} + \frac{2t+1+1}{2} = t + 2,$$

for $v_i v_{i+1} \in E(C_{2t+1})$: $i = 1,3,...,2t-1$,

$$f(v_i) + f(v_{i+1}) = \frac{i+1}{2} + t + \frac{i+1+2}{2} = t + i + 2,$$

for $v_i v_{i+1} \in E(C_{2t+1})$: $i = 2,4,...,2t$,

$$f(v_i) + f(v_{i+1}) = t + \frac{i+2}{2} + t + \frac{i+1+1}{2} = t + i + 2.$$

Thus, $\{f(x) + f(y) : xy \in E(C_n)\} = \{t+2, t+3,...,3t+2\}$ is a set of $2t+1$ consecutive integers.

Then $k = (2t+1) + (2t+1) + \min\{f(u) + f(v) : uv \in C_{2t+1}\}$

$$= (2t+1) + (2t+1) + (t+2) = 5t + 4 = \frac{5n+3}{2}.$$

Therefore f extends to a super edge-magic labeling of C_n when n is odd with $k = \frac{5n+3}{2}$. □

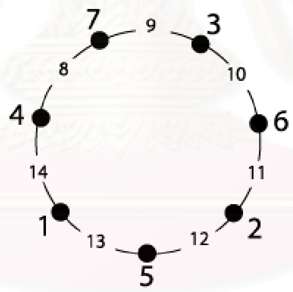
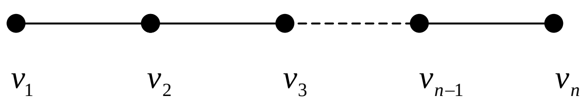


Figure 3.2: A super edge-magic labeling of C_7 with $k = 19$.

Theorem 3.5. [4] The path P_n is super edge-magic when n is even with $k = \frac{5n+2}{2}$ and when n is odd with $k = \frac{5n+3}{2}$.

Proof. Let $V(P_n) = \{v_1, v_2,..., v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3,..., v_{n-1}v_n\}$.



Case 1: n is even. Let $n = 2t$ for some $t \in \mathbb{Z}^+$. The labeling f is shown in figure 3.3.

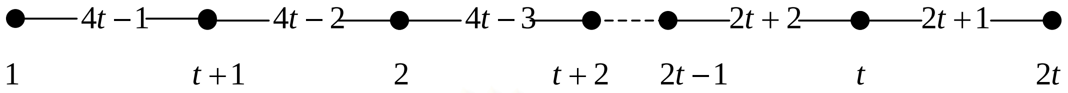


Figure 3.3: A super edge-magic labeling of P_n with $k = \dfrac{5n+2}{2}$.

Case 2: n is odd. Let $n = 2t+1$ for some $t \in \mathbb{Z}^+$. The labeling f is shown in figure 3.4.

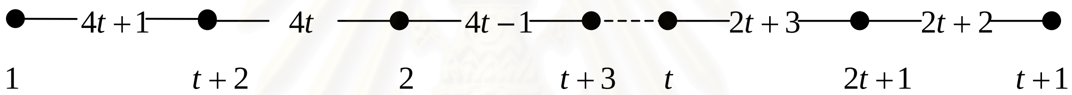


Figure 3.4: A super edge-magic labeling of P_n with $k = \dfrac{5n+3}{2}$.

Therefore P_n is super edge-magic. $\square$

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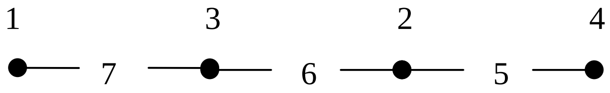
P_4:



P_5:

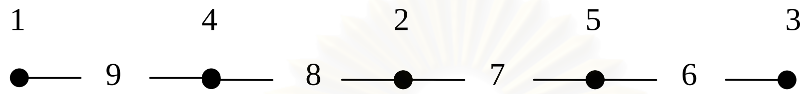
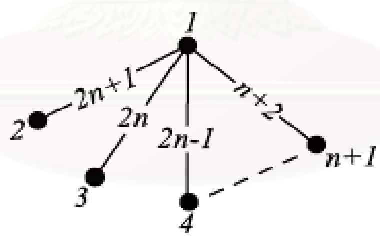


Figure 3.5: Super edge-magic labelings of P_4 and P_5 with $k=11$ and $k=14$, respectively.

Theorem 3.6. [4] Every star $K_{1,n}$ is super edge-magic with $k=2n+4$.

Proof. A super edge-magic labeling of $K_{1,n}$ is shown as follows:



□

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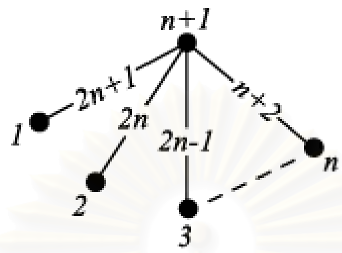
Theorem 3.7. [4] Every star $K_{1,n}$ is super edge-magic with $k = 3n+3$.

Proof. A super edge-magic labeling of $K_{1,n}$ is shown as follows:



$\square$

Theorem 3.8. [6] A caterpillar $CP_{n_1,n_2,\dots,n_t}$ is super edge-magic.

Proof. In fact, a caterpillar $CP_{n_1,n_2,\dots,n_t}$ is $K_{1,n_1} \cup K_{1,n_2} \cup \dots \cup K_{1,n_t}$ where K_{1,n_i} is a star with center c_i and n_i edges and in every case K_{1,n_i} shares an edge with $K_{1,n_{i+1}}$, then $CP_{n_1,n_2,\dots,n_t}$ has $q = \sum_{i=1}^{t} n_i - t + 1$ edges ($t-1$ edges are shared by two stars) and $p = \sum_{i=1}^{t} n_i - t + 2$ vertices. c_i will be a leaf in $K_{1,n_{i-1}}$ (unless $i=1$) and in $K_{1,n_{i+1}}$ (unless $i=n$) or equivalently the leaves of K_{1,n_i} will include c_{i-1} and c_{i+1}.

First the stars are ordered $K_{1,n_1}, K_{1,n_3}, K_{1,n_5}, \dots, K_{1,n_2}, K_{1,n_4}, K_{1,n_6}, \dots$. Define a vertex labeling f by mapping consecutively (we start at the number 1) the non-center vertices of the stars $K_{1,n_1}, K_{1,n_3}, K_{1,n_5}, \dots$ and then the non-center vertices of the stars $K_{1,n_2}, K_{1,n_4}, K_{1,n_6}, \dots$. Then all vertices are labeled.

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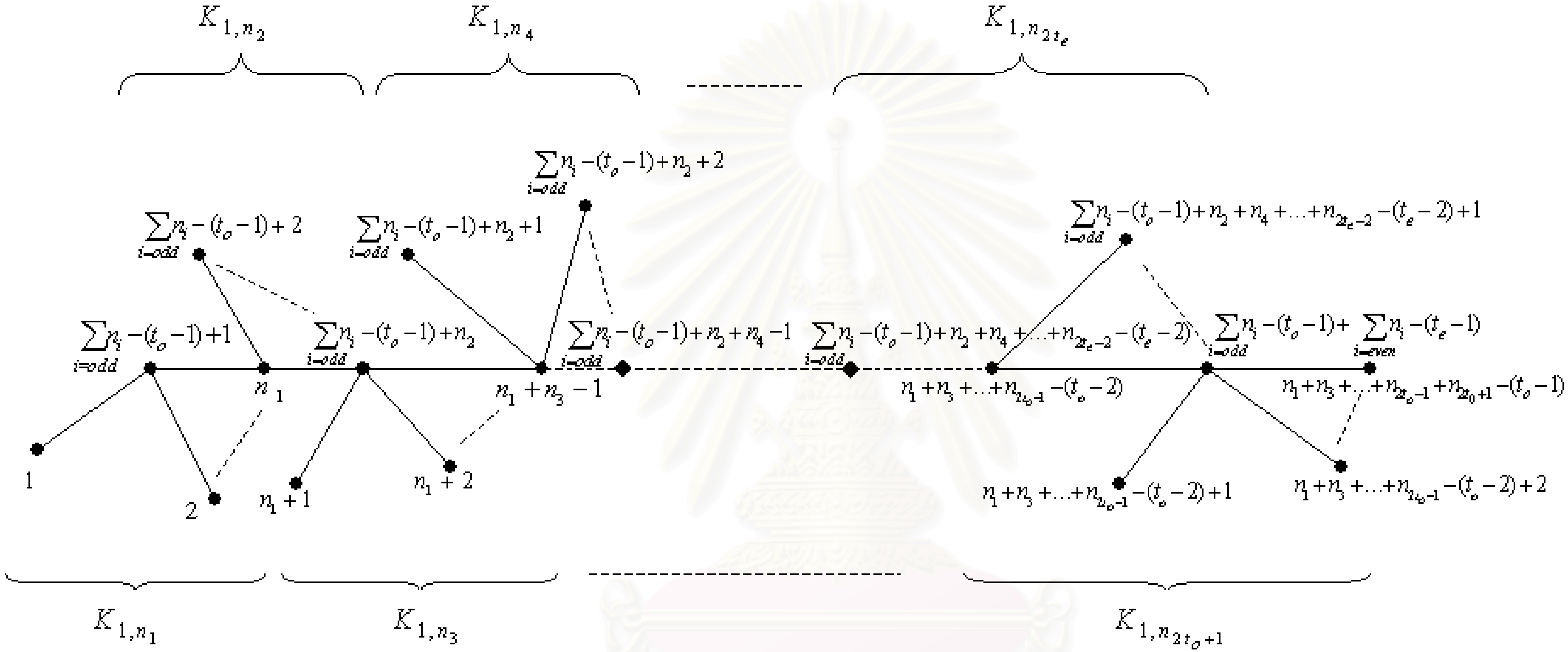

Note that : $t_o=\begin{cases}\dfrac{t}{2} & if \quad t \quad is \quad even,\\ \dfrac{t+1}{2} & if \quad t \quad is \quad odd.\end{cases}$ and $t_e=\begin{cases}\dfrac{t}{2} & if \quad t \quad is \quad even,\\ \dfrac{t-1}{2} & if \quad t \quad is \quad odd.\end{cases}$.

Figure3.5: A labeling of $CP_{n_1,n_2,\ldots,n_t}$

From the given labeling f, we consider the valence of the labeling,

$$k = \left(\sum_{i=1}^{t} n_i\right) - (t_o - 1) - (t_e - 1) + \left(\sum_{i=1}^{t} n_i\right) - (t_o - 1) - (t_e - 1) - 1 +$$

$$\min\{f(u) + f(v) : uv \in CP_{n_1,n_2,\ldots,n_t}\}$$

$$= \left(\sum_{i=1}^{t} n_i\right) - (t_o - 1) - (t_e - 1) + \left(\sum_{i=1}^{t} n_i\right) - (t_o - 1) - (t_e - 1) - 1 + 1 + \left(\sum_{i=1}^{2t_o+1} n_i\right) - (t_o - 1) + 1$$

$$= 6 - (3t_o + 2t_e) + 2\left(\sum_{i=1}^{t} n_i\right) + \sum_{i=1}^{2t_o+1} n_i$$

Therefore $CP_{n_1,n_2,\ldots,n_t}$ is super edge-magic with $k = 6 - (3t_o + 2t_e) + 2\left(\sum_{i=1}^{t} n_i\right) + \sum_{i=1}^{2t_o+1} n_i$.

□

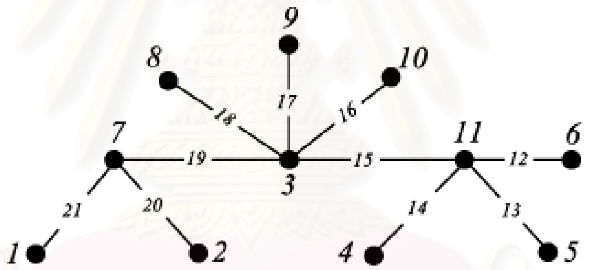

Figure3.6: A super edge-magic labeling of $CP_{3,5,4}$ with $k = 29$.

Conjecture 3.9. [2] Every tree is super edge-magic.

Theorem 3.10. [4] The graph P_n^2 is super edge-magic when $n \geq 3$ with $k = 3n$.

Proof. A super edge-magic labeling of P_n^2 is shown in figure 3.7.

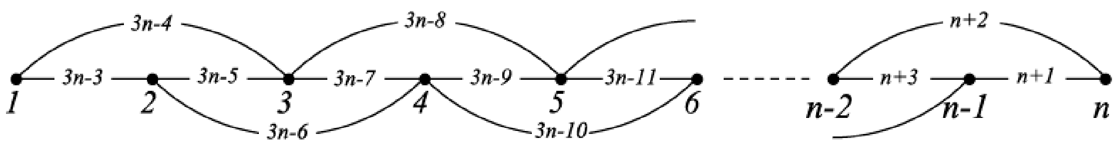

Figure 3.7: A super edge-magic labeling of P_n^2 with $k = 3n$. □

Theorem 3.11. [3] The fan F_n is super edge-magic when $1 \leq n \leq 6$ with $k = 3n + 3$.

Proof. The super edge-magic labelings of F_n when $1 \leq n \leq 6$ are shown as follows:

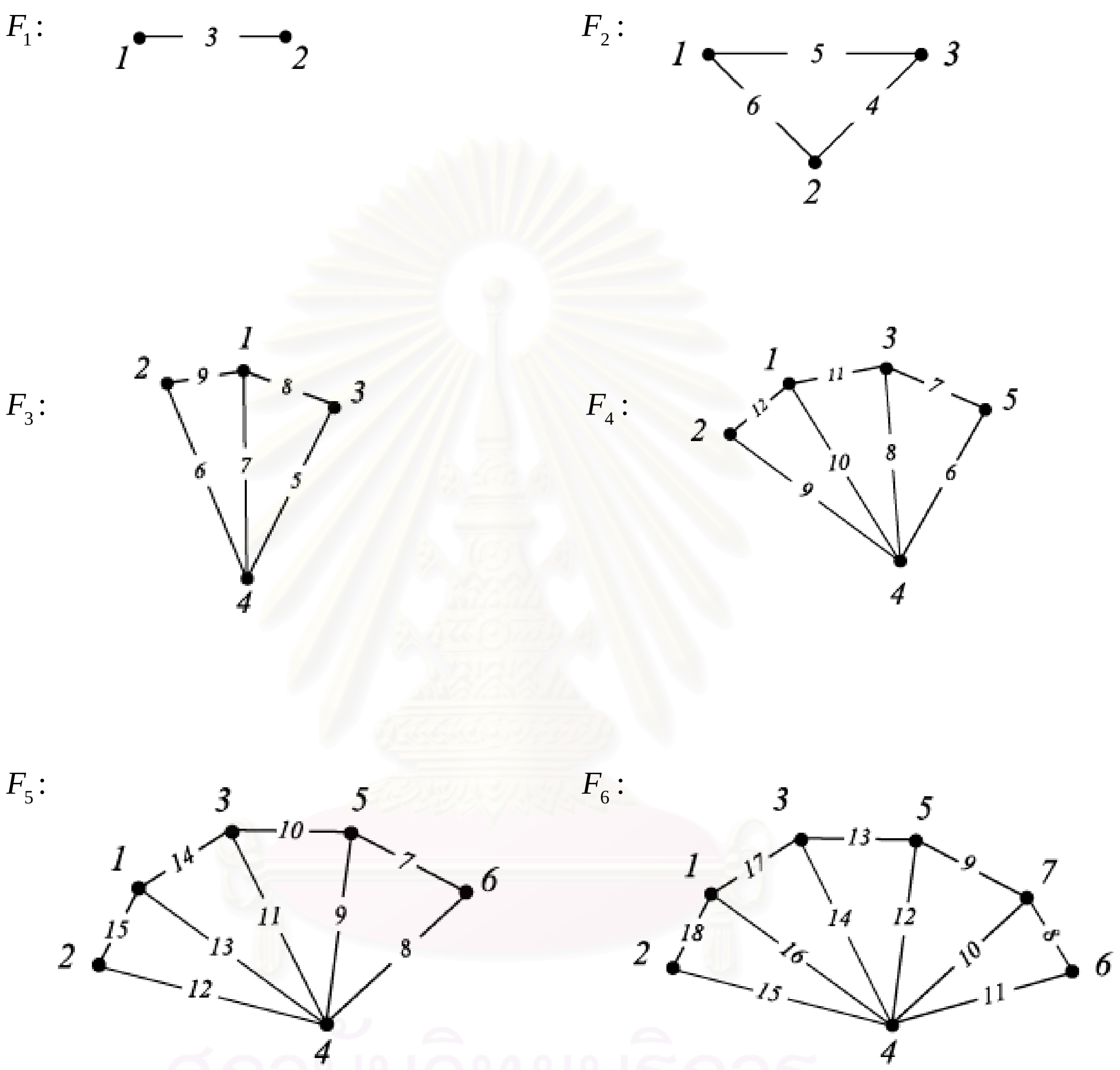

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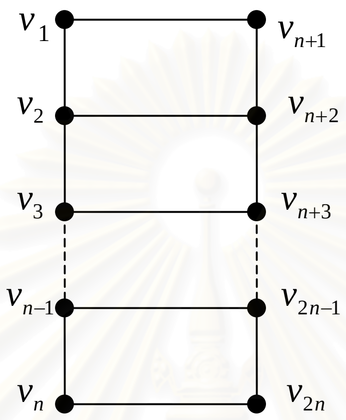
Figure 3.8: Super edge-magic labelings of F_n when $n = 1,2,...,6$ with $k = 6, 9, 12, 15, 18, 21$, respectively. □

Theorem 3.12. [3] The ladder L_n is super edge-magic when n is odd with $k = \frac{11n+1}{2}$.

Proof. Let $V(L_n) = \{v_1, v_2, ..., v_{2n}\}$ and
$E(L_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_{n+1}v_{n+2}, v_{n+2}v_{n+3}, ..., v_{2n-1}v_{2n}, v_iv_{n+i} : i = 1,2,...,n\}$.



Let $n = 2t+1$ for some $t \in Z^+$ and define a vertex labeling f as follows:

$$f(v_i) = \begin{cases} \frac{i+1}{2} & if \quad i = 1,3,...,2t+1, \\ \frac{2t+2+i}{2} & if \quad i = 2,4,...,2t, \\ \frac{4t+2+i}{2} & if \quad i = 2t+2, 2t+4, ..., 4t+2, \\ \frac{2t+1+i}{2} & if \quad i = 2t+3, 2t+5, ..., 4t+1. \end{cases}$$

In order to show that f extends to a super edge-magic labeling of L_n, it suffices to verify the following by lemma 2.4:

a) $f(V(L_n)) = \{1,2,...,2n\}$

b) $S = \{f(x) + f(y) : xy \in E(L_n)\}$ is a set of $n + 2(n-1) = 3n-2$ consecutive integers.

For a) the numbers $1,2,...,t+1$ are labels of $v_1, v_3,..., v_{2t+1}$. The numbers $t+2, t+3,...,2t+1$ are labels of $v_2, v_4,..., v_{2t}$. The numbers $2t+2, 2t+3,$ $...,3t+1$ are labels of $v_{2t+3}, v_{2t+5},..., v_{4t+1}$. The numbers $3t+2, 3t+3,...,4t+2$ are labels of $v_{2t+2}, v_{2t+4},..., v_{4t+2}$. Thus, the set of vertex labelings is $\{1,2,...,4t+2\}$.

For b) observe that

for $v_i v_{i+1} \in E(L_n):\ i=1,2,...,n-1,$

$$f(v_i)+f(v_{i+1})=\frac{i+1}{2}+\frac{2t+2+i+1}{2}=t+2+i\,,$$

for $v_i v_{i+1} \in E(L_n):\ i=n+1,n+2,...,2n-1,$

$$f(v_i)+f(v_{i+1})=\frac{4t+2+i}{2}+\frac{2t+1+i+1}{2}=3t+2+i\,,$$

for $v_i v_{i+n} \in E(L_n):\ i=1,2,...,n,$

$$f(v_i)+f(v_{i+n})=\frac{i+1}{2}+\frac{4t+2+i+2t+1}{2}=3t+2+i\,.$$

Thus, $\{f(x)+f(y): xy \in E(L_n)\}=\{t+3, t+4,...,7t+3\}$ is a set of $3n-2$ consecutive integers.

Then $k=(2n)+(n+2(n-1))+\min\{f(u)+f(v): uv \in E(L_n)\}$

$$=(2n)+(n+2(n-1))+(\frac{n+3}{2}+1)=\frac{11n+1}{2}\,.$$

Therefore f extends to a super edge-magic labeling of L_n when n is odd with $k=\dfrac{11n+1}{2}$. □

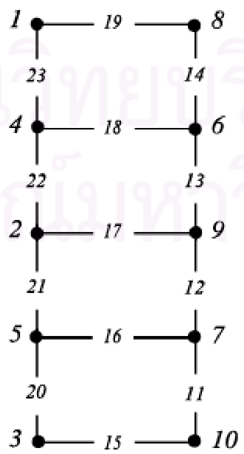


Figure 3.9: A super edge-magic labeling of L_5 with $k=28$.

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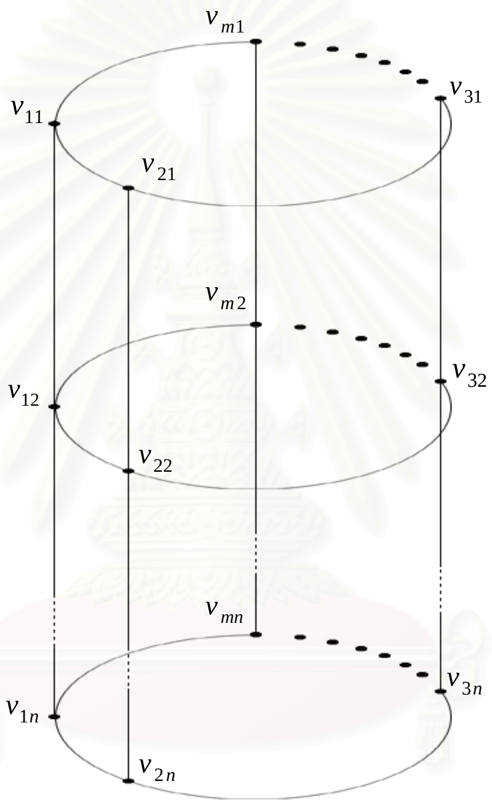
Theorem 3.13. [3] The generalized prism $C_m \times P_n$ is super edge-magic with $k = \dfrac{6mn - m + 3}{2}$ when m is odd and $n \geq 2$.

Proof. Let $V(C_m \times P_n) = \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(C_m \times P_n) = \{v_{ij}v_{i+1j} : 1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{v_{1j}v_{mj} : 1 \leq j \leq n\}$$
$$\cup \{v_{ij}v_{ij+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}.$$



Define a vertex labeling f as follows:

Define a vertex labeling f as follows:

$$f(v_{ij}) = \begin{cases} \dfrac{i+1}{2} & if \quad i = 1,3,...,m \quad and \quad j = 1, \\ \dfrac{i+m+1}{2} & if \quad i = 2,4,...,m-1 \quad and \quad j = 1, \\ \dfrac{i+m(2j-2)}{2} & if \quad i = 2,4,...,m-1 \quad and \quad j \quad is \quad even, \\ \dfrac{i+m(2j-1)}{2} & if \quad i = 1,3,...,m \quad and \quad j \quad is \quad even, \\ mj & if \quad i = 1 \quad and \quad j \neq 1 \quad is \quad odd, \\ \dfrac{i-1+m(2j-1)}{2} & if \quad i = 2,4,...,m-1 \quad and \quad j \neq 1 \quad is \quad odd, \\ \dfrac{i-1+m(2j-2)}{2} & if \quad i = 3,5,...,m \quad and \quad j \neq 1 \quad is \quad odd. \end{cases}$$

In order to show that f extends to a super edge-magic labeling of L_n, it suffices to verify the following by lemma 2.4:

a) $f(V(C_m \times P_n)) = \{1,2,...,mn\}$

b) $S = \{f(x) + f(y) : xy \in E(C_m \times P_n)\}$ is a set of $m(n-1) = mn - m$ consecutive integers.

We consider when n is odd.

For a) the numbers $1,2,...,\frac{m+1}{2},\frac{3m+1}{2},\frac{3m+3}{2},...,2m,...,mn,m(n-1)+1,$ $m(n-1)+2,...,\frac{m(2n-1)-1}{2}$ are labels of $v_{11},v_{31},...,v_{m1},v_{12},v_{32},...,v_{m2},...,v_{1n},v_{3n},$ $v_{5n},...,v_{mn}$. The numbers $\frac{m+3}{2},\frac{m+5}{2},...,m,m+1,m+2,...,\frac{3m-1}{2},...,\frac{m(2n-1)+1}{2},$ $\frac{m(2n-1)+3}{2},...,mn-1$ are labels of $v_{21},v_{41},...,v_{(m-1)1},v_{22},v_{42},...,v_{(m-1)2},...,v_{2n},v_{4n},$ $....,v_{(m-1)n}$. Thus, the set of vertex labelings is $\{1,2,...,mn\}$.

For b) observe that

for $v_{11}v_{m1} \in E(C_m \times P_n)$,

$$f(v_{11}) + f(v_{m1}) = 1 + \frac{m+1}{2} = \frac{m+3}{2},$$

for $j = 3,5,...,n$,

for $v_{1j}v_{mj} \in E(C_m \times P_n)$,

$$f(v_{1j}) + f(v_{mj}) = mj + \frac{m-1+m(2j-2)}{2} = \frac{m(4j-1)-1}{2},$$

for $i = 1,3,...,m-2$,

for $v_{i1}v_{(i+1)1} \in E(C_m \times P_n)$,

$$f(v_{i1}) + f(v_{(i+1)1}) = \frac{i+1}{2} + \frac{i+1+m+1}{2} = \frac{m+3}{2} + i,$$

for $v_{ij}v_{(i+1)j} \in E(C_m \times P_n)$: $j = 3,5,...,n$,

$$f(v_{ij}) + f(v_{(i+1)j}) = \frac{i-1+m(2j-1)}{2} + \frac{i+1-1+m(2j-2)}{2} = \frac{m(4j-3)-1}{2} + i,$$

for $v_{i1}v_{i2} \in E(C_m \times P_n)$,

$$f(v_{i1}) + f(v_{i2}) = \frac{i+1}{2} + \frac{i+m(2(2)-1)}{2} = \frac{3m+1}{2} + i,$$

for $v_{ij}v_{i(j+1)} \in E(C_m \times P_n)$: $j = 3,5,...,n-2$,

$$f(v_{ij}) + f(v_{i(j+1)}) = \frac{i-1+m(2j-1)}{2} + \frac{i+m(2(j+1)-1)}{2} = \frac{4mj-1}{2} + i,$$

for $i = 2,4,...,m-1$ and $j = 2,4,...,n-1$, we can verify similarly.

Next, consider when n is even, we can verify similarly to the previous case.

So $\{f(x) + f(y) : xy \in E(C_m \times P_n)\} = \{\frac{2mn+m+3}{2} + \frac{2mn+m+5}{2},...,\frac{4mn-m+1}{2}\}$ is a set of $m(n-1) = mn - m$ consecutive integers.

Therefore f extends to a super edge-magic labeling of $C_m \times P_n$ with $k = \frac{6mn-m+3}{2}$ when m is odd and $n \geq 2$. $\square$

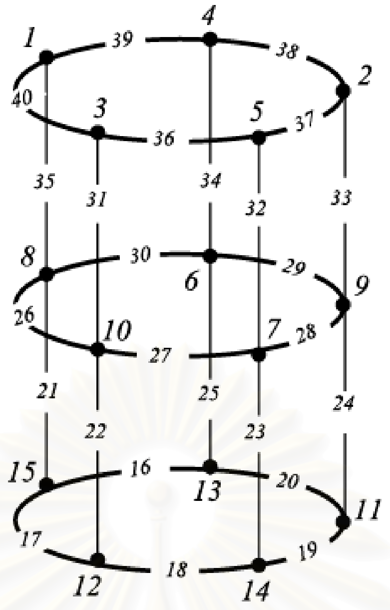
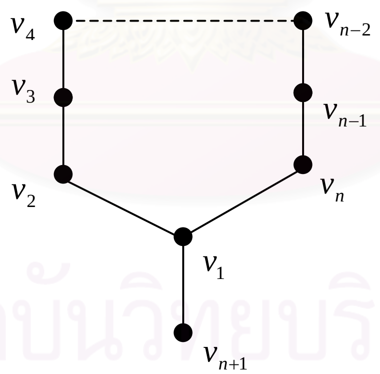


Figure 3.10: A super edge-magic labeling of $C_5 \times P_3$ with $k = 44$.

Theorem 3.14. The $(n,1)-kite$ is super edge-magic with $k = \dfrac{5n+9}{2}$ when n is odd.

Proof. Let $V((n,1)-kite) = \{v_1, v_2, ..., v_n, v_{n+1}\}$ and
$E((n,1)-kite) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_1v_n, v_1v_{n+1}\}$.



Let $n = 2t+1$ for some $t \in Z^+$ and define a vertex labeling f as follows:

$$f(v_i) = \begin{cases} t+1 & if \quad i = 2t+2, \\ 2t+2 & if \quad i = 1, \\ \dfrac{i-1}{2} & if \quad i = 3,5,...,2t+1, \\ t+1+\dfrac{i}{2} & if \quad i = 2,4,...,2t. \end{cases}$$

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The labeling f is shown in figure 3.11.

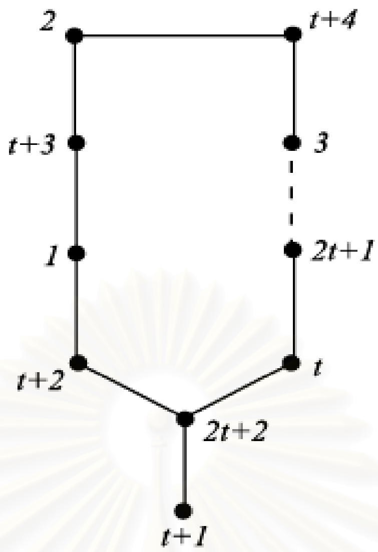


Figure 3.11: A vertex labeling of $(n,1)-kite$.

In order to show that f extends to a super edge-magic labeling of $(n,1)-kite$, it suffices to verify the following by lemma 2.4:

a) $f(V((n,1)-kite)) = \{1,2,...,n+1\}$

b) $S = \{f(x)+f(y): xy \in E((n,1)-kite)\}$ is a set of $n+1$ consecutive integers.

For a) $f(V((n,1)-kite)) = \{1,2,...,n+1\}$.

For b) observe that

for $v_1v_2 \in E((n,1)-kite)$, $f(v_1)+f(v_2) = 2t+2+t+1+\frac{2}{2} = 3t+4$,

for $v_1v_n \in E((n,1)-kite)$, $f(v_1)+f(v_n) = 2t+2+\frac{2t+1-1}{2} = 3t+2$,

for $v_1v_{n+1} \in E((n,1)-kite)$, $f(v_1)+f(v_{n+1}) = 2t+2+t+1 = 3t+3$,

for $v_iv_{i+1} \in E((n,1)-kite)$: $i = 3,5,...,2t-1$,

$f(v_i)+f(v_{i+1}) = \frac{i-1}{2}+t+1+\frac{i+1}{2} = t+i+1$,

for $v_iv_{i+1} \in E((n,1)-kite)$: $i = 2,4,...,2t$,

$f(v_i)+f(v_{i+1}) = t+1+\frac{i}{2}+\frac{i+1-1}{2} = t+i+1$.

Thus, $\{f(x)+f(y): xy \in E((n,1)-kite)\} = \{t+3, t+4, ..., 3t+4\}$ is a set of $2t+2$ consecutive integers.

Then $k = (n+1)+(n+1)+\min\{f(u)+f(v): uv \in E((n,1)-kite)\}$

$$= (n+1)+(n+1)+(t+3) = \frac{5n+9}{2}.$$

Therefore f extends to a super edge-magic labeling of $(n,1)-kite$ with $k = \frac{5n+9}{2}$ when n is odd. □

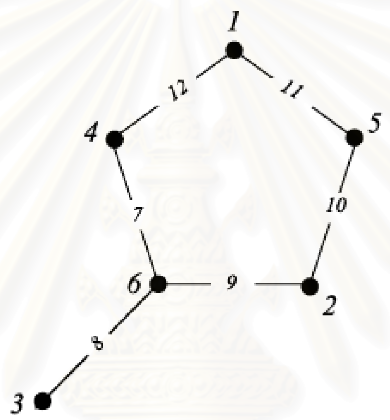


Figure 3.12: A super edge-magic labeling of (5,1)-*kite* with k =17.

Theorem3.15. The $(n,1)-kite$ is super edge-magic with $k = \frac{5n+7}{2}$ when n is odd.

Proof. Let $n = 2t+1$ for some $t \in Z^+$ and define a vertex labeling f as follows:

$$f(v_i) = \begin{cases} 2t+2 & if \quad i = 2t+2, \\ t+1-\frac{i-1}{2} & if \quad i = 1,3,...,2t+1, \\ 2t+2-\frac{i}{2} & if \quad i = 2,4,...,2t. \end{cases}$$

We can verify similarly to the previous theorem that

a) $f(V((n,1)-kite))=\{1,2,\ldots,n+1\}$

b) $S=\{f(x)+f(y):xy\in E((n,1)-kite)\}=\{\frac{n+3}{2},\frac{n+5}{2},\ldots,\frac{3n+3}{2}\}$ is a set of $n+1$ consecutive integers.

Therefore f extends to a super edge-magic labeling of $(n,1)-kite$ with $k=\frac{5n+7}{2}$ when n is odd. □

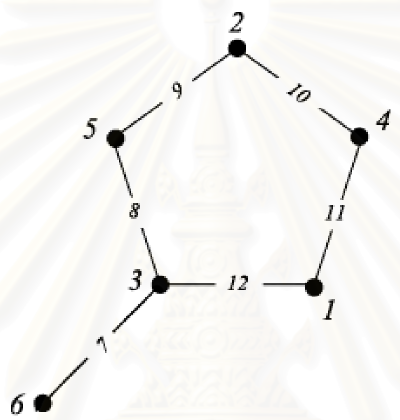
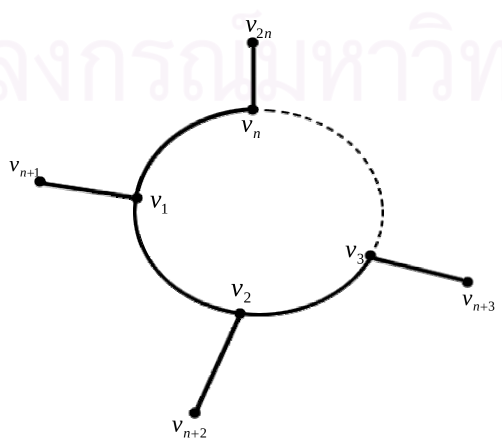


Figure 3.13: A super edge-magic labeling of (5,1)-*kite* with k=16.

Theorem 3.16. Every $n-sun$ is super edge-magic with $k=\frac{9n+3}{2}$ when n is odd.

Proof. Let $V(n-sun)=\{v_1,v_2,\ldots,v_{2n}\}$ and

$E(n-sun)=\{v_1v_2,v_2v_3,\ldots,v_{n-1}v_n,v_1v_n\}\cup\{v_iv_{i+1}:i=1,2,\ldots,n\}$.



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Let $n = 2t+1$ for some $t \in Z^+$ and define a vertex labeling f as follows:

$$f(v_i) = \begin{cases} \dfrac{i+1}{2} & if \quad i = 1,3,...,2t+1, \\ \dfrac{i}{2}+1+t & if \quad i = 2,4,...,2t, \\ 6t-i+4 & if \quad i = 2t+2, 2t+3,...,4t+2. \end{cases}$$

In order to show that f extends to a super edge-magic labeling of $n-sun$, it suffices to verify the following by lemma 2.4:

a) $f(V(n-sun)) = \{1,2,...,2n\}$

b) $S = \{f(x)+f(y) : xy \in E(n-sun)\}$ is a set of $2n = 4t+2$ consecutive integers.

For a) the numbers $1,2,...,t+1$ are labels of $v_1, v_3,..., v_{2t+1}$. The numbers $t+2, t+3,...,2t+1$ are labels of $v_2, v_4,..., v_{2t}$. The numbers $2t+2, 2t+3,..., 4t+2$ are labels of $v_{4t+2}, v_{4t+1},..., v_{2t+2}$. Thus, the set of vertex labelings is $\{1,2,...,4t+2\}$.

For b) observe that

for $v_1v_n \in E(n-sun)$,

$$f(v_1)+f(v_n) = \frac{1+1}{2}+\frac{n+1}{2} = \frac{n+3}{2} = t+2,$$

for $v_iv_{i+1} \in E(n-sun)$: $i = 1,3,5,...,2t-1$,

$$f(v_i)+f(v_{i+1}) = \frac{i+1}{2}+\frac{i+1}{2}+1+t = t+2+i,$$

for $v_iv_{i+1} \in E(n-sun)$: $i = 2,4,...,2t$,

$$f(v_i)+f(v_{i+1}) = \frac{i}{2}+1+t+\frac{i+1+1}{2} = t+2+i,$$

for $v_iv_{i+n} \in E(n-sun)$: $i = 1,3,5,...,2t+1$,

$$f(v_i)+f(v_{i+n}) = \frac{i+1}{2}+6t-(i+n)+4 = 4t+\frac{7-i}{2},$$

for $v_iv_{i+n} \in E(n-sun)$: $i = 2,4,...,2t$,

$$f(v_i)+f(v_{i+n})=\frac{i}{2}+1+t+3n-(i+n)+1=5t+4-\frac{i}{2}.$$

Thus, $\{f(x)+f(y):xy\in E(n-sun)\}=\{\frac{n+3}{2},\frac{n+5}{2},...,\frac{5n+1}{2}\}$ is a set of $2n$ consecutive integers.

Then $k=2n+2n+\min\{f(u)+f(v):uv\in E(n-sun)\}=2n+2n+\frac{n+3}{2}=\frac{9n+3}{2}$.

Therefore f extends to a super edge-magic labeling of $n-sun$ with $k=\frac{9n+3}{2}$ when n is odd. □

Figure 3.14: A super edge-magic labeling of 5-*sun* with k=24.

Theorem 3.17. [8] Every n-sun is super edge-magic with $k=\frac{11n+3}{2}$ when n is odd.

Proof. Let $n=2t+1$ for some $t\in Z^{+}$ and define a vertex labeling f as follows:

$$f(v_i) = \begin{cases} \dfrac{i+4t+3}{2} & if \quad i = 1,3,\ldots,2t+1, \\ \dfrac{i+4}{2}+3t & if \quad i = 2,4,\ldots,2t, \\ \dfrac{i+2}{2} & if \quad i = 2t+2, 2t+4,\ldots,4t, \\ \dfrac{i-2t+1}{2} & if \quad i = 2t+3, 2t+5,\ldots,4t+1, \\ 1 & if \quad i = 4t+2. \end{cases}$$

We can verify similarly to the previous theorem that

a) $f(V(n-sun)) = \{1,2,\ldots,2n\}$

b) $S = \{f(x)+f(y) : xy \in E(n-sun)\} = \{\frac{3n+3}{2}, \frac{3n+5}{2},\ldots,\frac{7n+1}{2}\}$ is a set of $2n = 4t+2$ consecutive integers.

Therefore f extends to a super edge-magic labeling of $n-sun$ with $k = \dfrac{11n+3}{2}$ when n is odd. $\square$

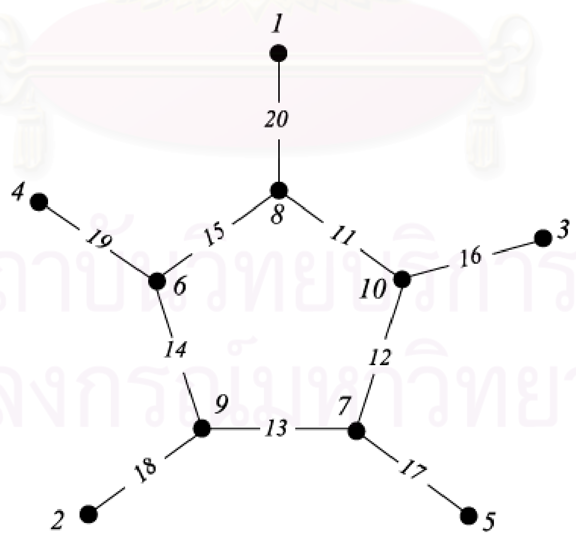
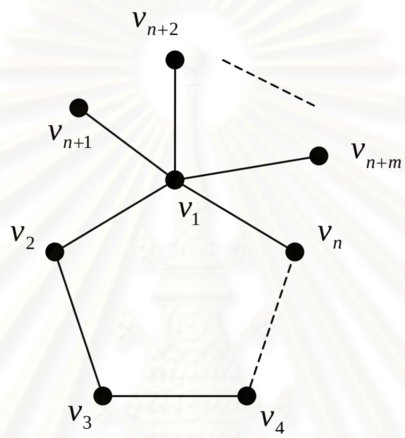


Figure 3.15: A super edge-magic labeling of 5-*sun* with $k = 29$.

Theorem 3.18. The $(n,m)-pineapple$ is super edge-magic with $k=2m+\frac{5n+3}{2}$ when n is odd.

Proof. Let $V((n,m)-pineapple)=\{v_1,v_2,...,v_{m+n}\}$ and
$E((n,m)-pineapple)=\{v_1v_2,v_2v_3,...,v_{n-1}v_n,v_1v_n\}\cup\{v_1v_i : i=n+1,n+2,...,n+m\}$.



Define a vertex labeling f as follows:

$$f(v_i)=\begin{cases}\frac{n+i}{2} & if \quad i=1,3,...,n,\\ \frac{i}{2} & if \quad i=2,4,...,n-1,\\ i & if \quad i=n+1,n+2,...,n+m.\end{cases}$$

The labeling f is shown in figure 3.16.

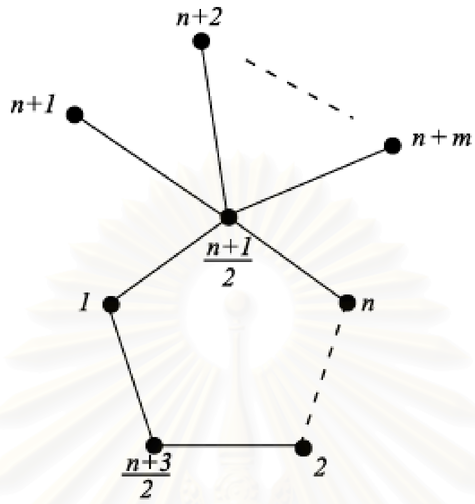


Figure 3.16: A vertex labeling of $(n,m)-pineapple$.

In order to show that f extends to a super edge-magic labeling of $(n,m)-pineapple$, it suffices to verify the following by lemma 2.4:

a) $f(V((n,m)-pineapple))=\{1,2,...,n+m\}$

b) $S=\{f(x)+f(y): xy\in E((n,m)-pineapple)\}$ is a set of $n+m$ consecutive integers.

For a) $f(V((n,m)-pineapple))=\{1,2,...,n+m\}$.

For b) observe that

for $v_1v_n\in E((n,m)-pineapple)$,

$$f(v_1)+f(v_n)=\frac{n+1}{2}+\frac{n+n}{2}=\frac{3n+1}{2},$$

for $v_iv_{i+1}\in E((n,m)-pineapple)$: $i=1,3,5,...,n-2$,

$$f(v_i)+f(v_{i+1})=\frac{n+i}{2}+\frac{i+1}{2}=i+\frac{n+1}{2},$$

for $v_iv_{i+1}\in E((n,m)-pineapple)$: $i=2,4,...,n-1$,

$$f(v_i)+f(v_{i+1})=\frac{i}{2}+\frac{n+i+1}{2}=i+\frac{n+1}{2},$$

for $v_1 v_i \in E((n,m)-pineapple)$: $i = n+1, n+2, ..., n+m$,

$f(v_1) + f(v_i) = \frac{n+1}{2} + i = \frac{n+1}{2} + i$.

Thus, $\{f(x)+f(y) : xy \in E((n,m)-pineapple)\} = \{\frac{n+3}{2}, \frac{n+5}{2}, ..., \frac{3n+1}{2} + m\}$ is a set of $n+m$ consecutive integers.

Then $k = (n+m) + (n+m) + \min\{f(u)+f(v) : uv \in E((n,m)-pineapple)\}$

$$= (n+m) + (n+m) + \frac{n+3}{2} = 2m + \frac{5n+3}{2}.$$

Therefore f extends to a super edge-magic labeling of $(n,m)-pineapple$ with $k = 2m + \frac{5n+3}{2}$ when n is odd. □

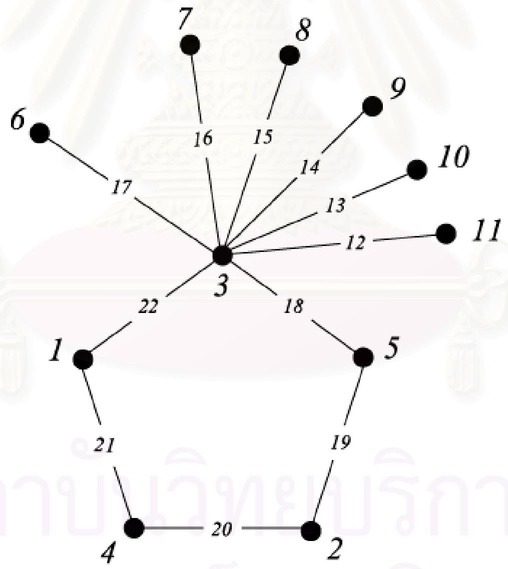


Figure 3.17: A super edge-magic labeling of $(5,6)-pineapple$ with $k = 26$.

CHAPTER IV

SUPER EDGE-MAGIC LABELINGS ON DISCONNECTED GRAPHS

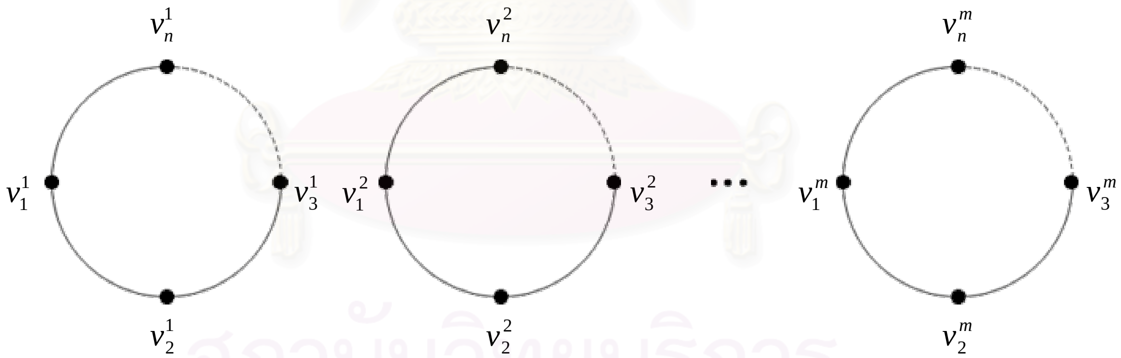
In this chapter, we consider the disconnected graphs which are super edge-magic. Moreover, we show some examples.

Theorem 4.1. [4] The graph mC_n is super edge-magic with $k = \dfrac{5nm+3}{2}$ when $m > 1$ and $n \geq 3$ are odd.

Proof. Let $m > 1$ and $n \geq 3$ are odd.

Let $V(mC_n) = V_1 \cup V_2 \cup ... \cup V_m$ where $V_i = \{v_1^i, v_2^i, ..., v_n^i\}$ and

$E(mC_n) = E_1 \cup E_2 \cup ... \cup E_m$ where $E_i = \{v_1^i v_2^i, v_2^i v_3^i, ..., v_{n-1}^i v_n^i, v_1^i v_n^i\}$.



Define a vertex labeling $f : V(G) \to \{1, 2, ..., mn\}$ as follows:

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$$f(v_j^i)=\begin{cases} i & \text{if} \quad 1\le i\le m \quad \text{and} \quad j=1,\\ m\left(\dfrac{n+j-1}{2}\right)+\dfrac{2i+1+m}{2} & \text{if} \quad 1\le i\le \dfrac{m-1}{2} \quad \text{and} \quad j=2,4,\ldots,n-1,\\ m\left(\dfrac{n+j-1}{2}\right)+\dfrac{2i+1-m}{2} & \text{if} \quad \dfrac{m+1}{2}\le i\le m \quad \text{and} \quad j=2,4,\ldots,n-1,\\ m\left(\dfrac{j-1}{2}+1\right)+1-2i & \text{if} \quad 1\le i\le \dfrac{m-1}{2} \quad \text{and} \quad j=3,5,\ldots,n,\\ m\left(\dfrac{j-1}{2}+2\right)+1-2i & \text{if} \quad \dfrac{m+1}{2}\le i\le m \quad \text{and} \quad j=3,5,\ldots,n. \end{cases}$$

In order to show that f extends to a super edge-magic labeling of mC_n, it suffices to verify the following by lemma 2.4:

a) $f(V(mC_n))=\{1,2,\ldots,mn\}$

b) $S=\{f(x)+f(y):xy\in E(mC_n)\}$ is a set of mn consecutive integers.

For a), let $n=2t+1$ and $m=2t'+1$ for some $t,t'\in Z^+$. The numbers $1,2,\ldots,2t'+1$ are labels of $v_1^1,v_1^2,\ldots,v_1^m$. The numbers $2t'+2,2t'+3,2t'+4,\ldots,$ $4t'+1,4t'+2$, $4t'+3,4t'+4,4t'+5,\ldots,6t'+2,6t'+3,\ldots,2tt'+t+1,2tt'+t+2,2tt'+t+3,$ $\ldots,2tt'+t+2t'$, $2tt'+t+2t'+1$ are labels of $v_3^{2t'+1},v_3^{t'},v_3^{2t'},\ldots,v_3^1,v_3^{t'+1},v_5^{2t'+1},v_5^{t'},v_5^{2t'},\ldots,$ $v_5^1,v_5^{t'+1},\ldots,v_n^{2t'+1},v_n^{t'},v_n^{2t'},\ldots,v_n^1,v_n^{t'+1}$. The numbers $2tt'+t+2t'+2,2tt'+t+2t'+3,\ldots,$ $2tt'+t+3t'+2,2tt'+t+3t'+3,2tt'+t+3t'+4,\ldots,2tt'+t+4t'+2$, $\ldots$, $4tt'+2t+1,$ $4tt'+2t+2$, $\ldots$, $4tt'+2t+t'+1,$ $4tt'+2t+t'+2,4tt'+2t+t'+3,\ldots,4tt'+2t+2t'+1$ are labels of $v_2^{t'+1},v_2^{t'+2},\ldots,v_2^{2t'+1},v_2^1,v_2^2\ldots,v_2^{t'},\ldots,v_{n-1}^{t'+1},v_{n-1}^{t'+2},\ldots,v_{n-1}^{2t'+1}$, $v_{n-1}^1,v_{n-1}^2\ldots,v_{n-1}^{t'}$. Thus, the set of vertex labelings is $\{1,2,\ldots,mn\}$.

For b), observe that, for $i=1,2,\ldots,\dfrac{m-1}{2}$,

for $v_1^iv_2^i\in E(mC_n)$,

$$f(v_1^i)+f(v_2^i)=i+m\left(\frac{n+2-1}{2}\right)+\frac{2i+1+m}{2}=2tt'+3t'+t+2+2i\,,$$

for $v_j^iv_{j+1}^i\in E(mC_n)$: $j=2,4,\ldots,n-1$,

$$\begin{aligned} f(v_j^i)+f(v^i{}_{j+1}) &= m\left(\frac{n+j-1}{2}\right)+\frac{2i+1+m}{2}+m\left(\frac{j}{2}+1\right)+1-2i\\ &= 2tt'+3t'+t+3+2jt'+j-i\,, \end{aligned}$$

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for $v_j^i v_{j+1}^i \in E(mC_n)$: $j = 3,5,...,n-2$,

$$f(v_j^i) + f(v_{j+1}^i) = m\left(\frac{j-1}{2}+1\right)+1-2i+m\left(\frac{n+j}{2}\right)+\frac{2i+1+m}{2}$$

$$= 2tt' + 3t' + t + 3 + 2jt' + j - i,$$

for $v_1^i v_n^i \in E(mC_n)$, $f(v_1^i) + f(v_n^i) = i + m\left(\frac{n-1}{2}+1\right)+1-2i = 2tt' + 2t' + t + 2 - i$,

for $i = \frac{m+1}{2}, \frac{m+3}{2}, ..., m$, we can verify similarly.

Thus, $\{f(x) + f(y) : xy \in E(mC_n)\} = \{\frac{mn+3}{2}, \frac{mn+5}{2}, ..., \frac{3mn+1}{2}\}$ is a set of mn consecutive integers.

Then $k = mn + mn + \min\{f(u) + f(v) : uv \in E(mC_n)\}$

$$= mn + mn + \frac{mn+3}{2} = \frac{5nm+3}{2}.$$

Therefore f extends to a super edge-magic labeling of mC_n with $k = \frac{5nm+3}{2}$ when $m > 1$ and $n \geq 3$ are odd. □

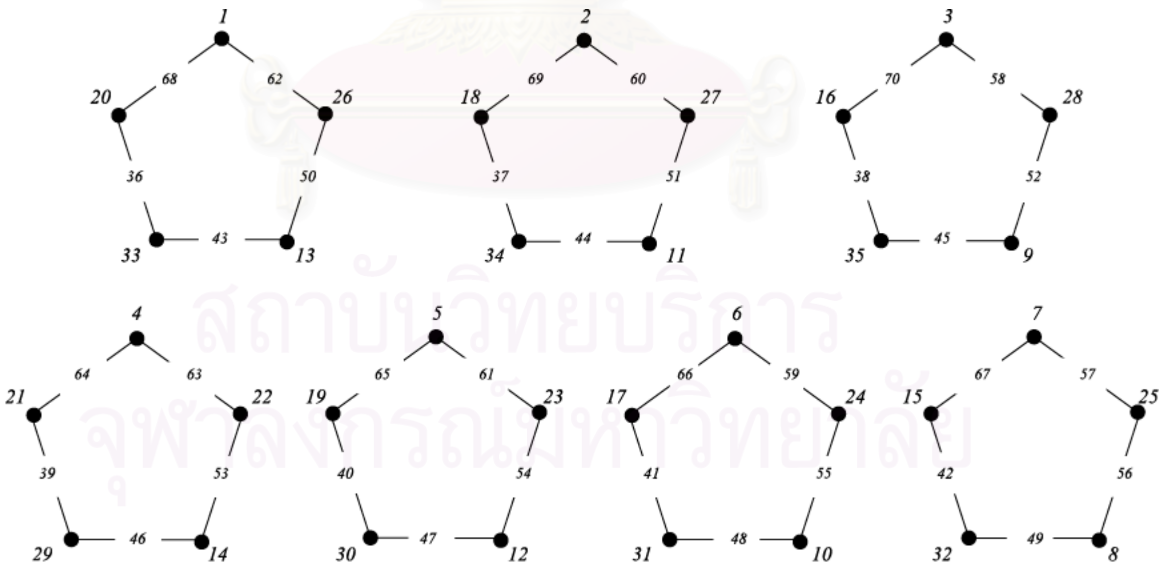


Figure 4.1: A super edge-magic labeling of $7C_5$ with $k = 89$.

Theorem 4.2. [9] The graph mC_n is super edge-magic with $k=\dfrac{5nm+3}{2}$ when $m>1$ and $n\geq 3$ are odd.

Proof. Define a vertex labeling f as follows:

$$f(v_j^i)=\begin{cases} i & \text{if} \quad 1\leq i\leq m \quad \text{and} \quad j=1,\\ \dfrac{m(j-1)}{2}+i & \text{if} \quad 1\leq i\leq m \quad \text{and} \quad j=3,5,\ldots,n-3,\\ \dfrac{m(n+j)+1}{2}+i & \text{if} \quad 1\leq i\leq \dfrac{m-1}{2} \quad \text{and} \quad j=2,4,\ldots,n-1,\\ \dfrac{m(n+j-2)+1}{2}+i & \text{if} \quad \dfrac{m+1}{2}\leq i\leq m \quad \text{and} \quad j=2,4,\ldots,n-1,\\ \dfrac{m(n+1)}{2}+1-2i & \text{if} \quad 1\leq i\leq \dfrac{m-1}{2} \quad \text{and} \quad j=n,\\ \dfrac{m(n+3)}{2}+1-2i & \text{if} \quad \dfrac{m+1}{2}\leq i\leq m \quad \text{and} \quad j=n. \end{cases}$$

We can verify similarly to the previous theorem.

Therefore f extends to a super edge-magic labeling of mC_n with $k=\dfrac{5nm+3}{2}$ when $m>1$ and $n\geq 3$ are odd. □

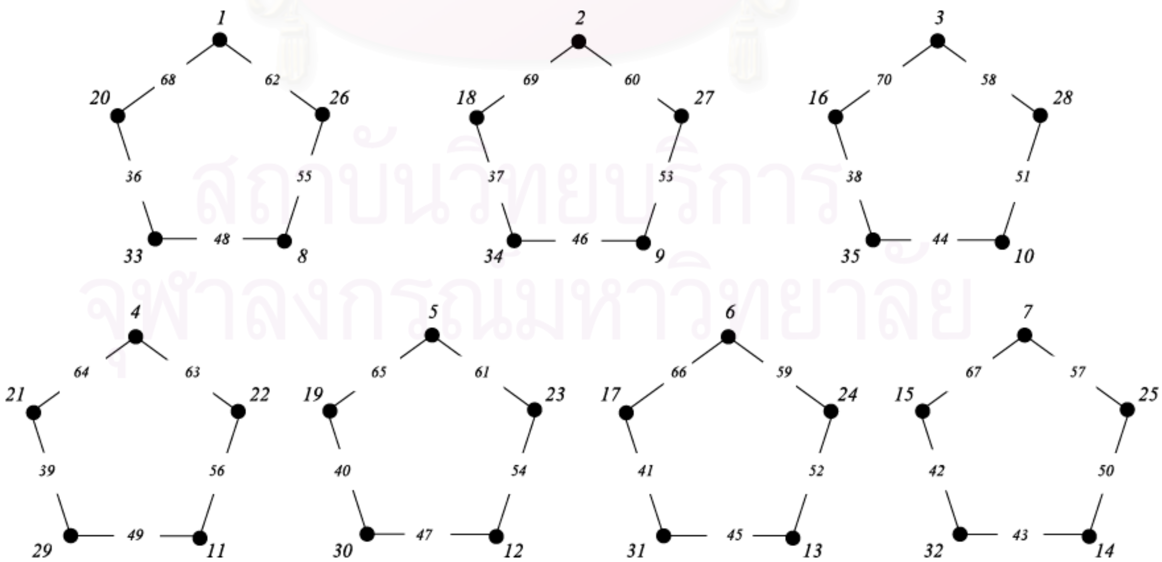
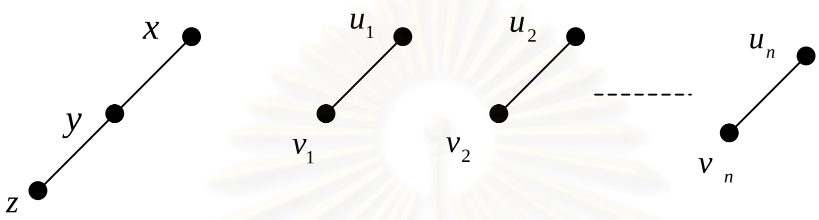


Figure 4.2: A super edge-magic labeling of $7C_5$ with $k=89$.

Theorem 4.3. [4] The linear forest $P_3 \cup nP_2$ is super edge-magic for every positive integer n.

Proof. Let $V(P_3 \cup nP_2) = \{x, y, z\} \cup \{u_i, v_i : i = 1,2,...,n\}$ and

$E(P_3 \cup nP_2) = \{xy, yz\} \cup \{u_i v_i : i = 1,2,...,n\}$.



We consider four possible cases.

Case1: $n = 1$. The labeling f is shown in figure 4.3.

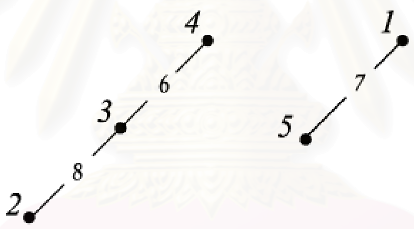


Figure 4.3: A super edge-magic labeling of $P_3 \cup P_2$ with $k = 13$.

Case2: $n = 3$. The labeling f is shown in figure 4.4.

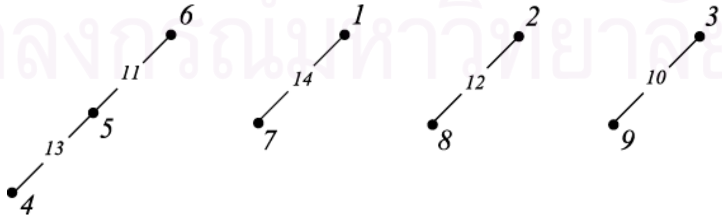


Figure 4.4: A super edge-magic labeling of $P_3 \cup 3P_2$ with $k = 22$.

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Case 3: $n = 2m+1$ for some $m \in Z^{+}$ and $m \geq 2$.

Define a vertex labeling $f : V(P_3 \cup nP_2) \to \{1,2,...,2n+3\}$ as follows:

$$f(w) = \begin{cases} 3m+3 & \text{if} \quad w = x, \\ 2m+3 & \text{if} \quad w = y, \\ m+3 & \text{if} \quad w = z, \\ i & \text{if} \quad w = u_i \quad \text{and} \quad i = 1,2,...,m+2, \\ i+3m+3 & \text{if} \quad w = v_i \quad \text{and} \quad i = 1,2,...,m+2, \\ i+1 & \text{if} \quad w = u_i \quad \text{and} \quad i = m+3,...,2m+1, \\ i+m+1 & \text{if} \quad w = v_i \quad \text{and} \quad i = m+3,...,2m+1. \end{cases}$$

The labeling f is shown in figure 4.5.

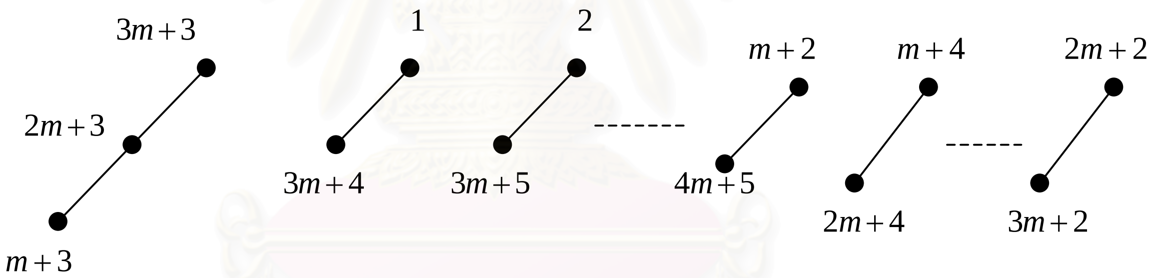

Figure 4.5: A vertex labeling of $P_3 \cup nP_2$.

In order to show that f extends to a super edge-magic labeling of $P_3 \cup nP_2$, it suffices to verify the following by lemma 2.4:

a) $f(V(P_3 \cup nP_2)) = \{1,2,...,2n+3\}$

b) $S = \{f(x) + f(y) : xy \in E(P_3 \cup nP_2)\}$ is a set of $n+2$ consecutive integers.

For a) $f(V(P_3 \cup nP_2)) = \{1,2,...,2n+3\}$

For b) observe that,

for $xy \in E(F)$, $f(x) + f(y) = 3m+3+2m+3 = 5m+6$,

for $yz \in E(F)$, $f(y) + f(z) = 2m+3+m+3 = 3m+6$,

for $u_i v_i \in E(F) : i = 1,2,...,m+2$,

$f(u_i) + f(v_i) = i + i + 3m + 3 = 2i + 3m + 3$, for $u_i v_i \in E(F) : i = m+3, m+4, ..., 2m+1$,

$f(u_i) + f(v_i) = i+1+i+m+1 = 2i+m+2$.

Thus, $\{f(x) + f(y) : xy \in E(P_3 \cup nP_2)\} = \{3m+5, 3m+6, ..., 5m+7\}$ is a set of $2m+3 = n+2$ consecutive integers.

Then $k = (2n+3) + (n+2) + \min\{f(u) + f(v) : uv \in E(F)\}$

$$= (2n+3) + (n+2) + (3m+5) = \frac{9n+17}{2}.$$

Then f extends to a super edge-magic labeling of $P_3 \cup (2m+1)P_2$ with $k = \dfrac{9n+17}{2}$.

Case 4: $n = 2m$ for some $m \in \mathbb{Z}^+$ and $m \geq 1$.

Define a vertex labeling $f : V(P_3 \cup nP_2) \to \{1, 2, ..., 2n+3\}$ as follows:

$$f(w) = \begin{cases} 2m+2 & \text{if} \quad w = x, \\ m+1 & \text{if} \quad w = y, \\ 2m+3 & \text{if} \quad w = z, \\ i & \text{if} \quad w = u_i \quad \text{and} \quad i = 1, 2, ..., m, \\ i+3m+3 & \text{if} \quad w = v_i \quad \text{and} \quad i = 1, 2, ..., m, \\ i+1 & \text{if} \quad w = u_i \quad \text{and} \quad i = m+1, ..., 2m, \\ i+m+3 & \text{if} \quad w = v_i \quad \text{and} \quad i = m+1, ..., 2m, \end{cases}$$

The proof is similar to the previous case that

a) $f(V(P_3 \cup nP_2)) = \{1, 2, ..., 2n+3\}$

b) $S = \{f(x) + f(y) : xy \in E(P_3 \cup nP_2)\} = \{3m+3, 3n+4, ..., 5m+4\}$ is a set of $n+2$ consecutive integers.

Then f extends to a super edge-magic labeling of $P_3 \cup (2m)P_2$ with $k = \dfrac{9n+16}{2}$.

Therefore the linear forest $P_3 \cup nP_2$ is super edge-magic for every positive integer n. $\square$

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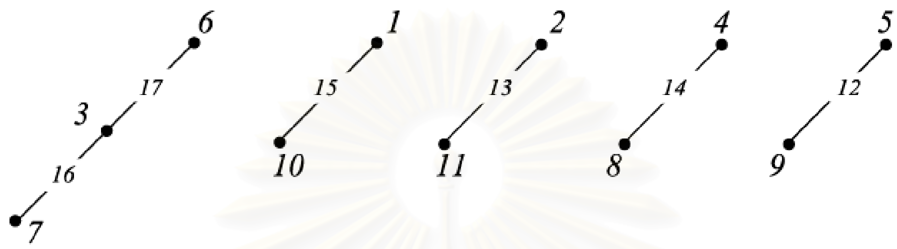
$P_3 \cup 4P_2$:



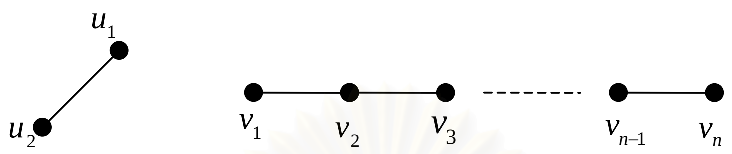
$P_3 \cup 5P_2$:

Figure 4.6: Super edge-magic labelings of $P_3 \cup 4P_2$ and $P_3 \cup 5P_2$ with $k = 26$ and $k = 31$, respectively.

Theorem 4.4. [4] The linear forest $P_2 \cup P_n$ is super edge-magic for every integer $n \geq 3$.

Proof. Let $V(P_2 \cup P_n) = \{u_1, u_2\} \cup \{v_i : i = 1,2,...,n\}$ and

$E(P_2 \cup P_n) = \{u_1u_2\} \cup \{v_iv_{i+1} : i = 1,2,...,n-1\}$.



Define a vertex labeling $f : V(P_2 \cup P_n) \to \{1,2,...,n+2\}$ as follows:

We now proceed by cases.

Case 1: $n = 3$. The labeling f is shown in figure 4.7.

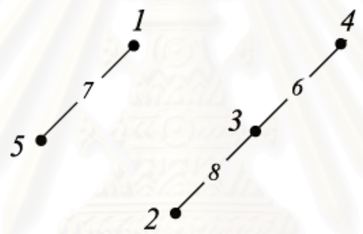


Figure 4.7: A super edge-magic labeling of $P_2 \cup P_3$ with $k = 13$.

Case 2: $n = 4$. The labeling f is shown in figure 4.8.

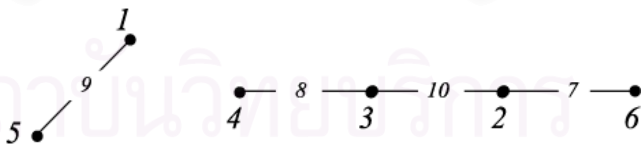


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Figure 4.8: A super edge-magic labeling of $P_2 \cup P_4$ with $k = 15$.

Case3: $n = 6$. The labeling f is shown in figure 4.9.

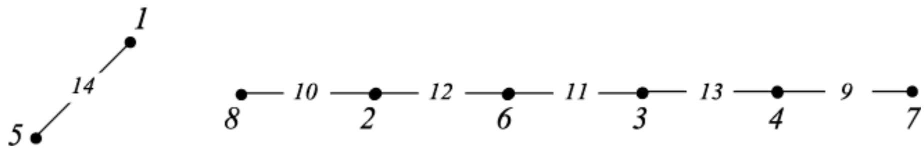

Figure 4.9: A super edge-magic labeling of $P_2 \cup P_6$ with $k = 20$.

Case 4: $n \equiv 0 \pmod 4$, where $n \geq 8$. Let $f(u_1) = 1, f(u_2) = \frac{n}{2} + 3,$ and

$$f(v_j) = \begin{cases} \frac{n}{2} + 2 & \text{if} \quad j = 1, \\ \frac{n}{2} + 4 & \text{if} \quad j = 3, \\ 2i & \text{if} \quad j = 4i \quad \text{and} \quad 1 \leq i \leq \frac{n}{4}, \\ \frac{n}{2} + 2i + 4 & \text{if} \quad j = 4i + 1 \quad \text{and} \quad 1 \leq i \leq \frac{n-4}{4}, \\ 2i + 3 & \text{if} \quad j = 4i + 2 \quad \text{and} \quad 0 \leq i \leq \frac{n-4}{4}, \\ \frac{n}{2} + 2i + 3 & \text{if} \quad j = 4i + 3 \quad \text{and} \quad 1 \leq i \leq \frac{n-4}{4}. \end{cases}$$

The proof is similar to the previous theorem that

a) $f(V(P_2 \cup P_n)) = \{1, 2, \ldots, n+2\}$

b) $S = \{f(x) + f(y) : xy \in E(P_2 \cup P_n)\} = \{\frac{n+8}{2}, \frac{n+10}{2}, \ldots, \frac{3n+6}{2}\}$ is a set of n consecutive integers.

Then f extends to a super edge-magic labeling of $P_2 \cup P_n$ with $k = \frac{5n+12}{2}$ where $n \equiv 0 \pmod 4$, when $n \geq 8$.

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Case 5: $n \equiv 1 (\bmod 4)$. Let $f(u_1) = 1, f(u_2) = n+2,$ and

$$f(v_j) = \begin{cases} \dfrac{2j+n+5}{4} & if \quad j \quad is \quad odd, \\ \dfrac{2j+3n+5}{4} & if \quad j \quad is \quad even \quad and \quad 2 \le j \le \dfrac{n-1}{2}, \\ \dfrac{2j-n+5}{4} & if \quad j \quad is \quad even \quad and \quad \dfrac{n+3}{2} \le j \le n-1. \end{cases}$$

The proof is similar to the previous theorem that

a) $f(V(P_2 \cup P_n)) = \{1,2,\ldots,n+2\}$

b) $S = \{f(x)+f(y) : xy \in E(P_2 \cup P_n)\} = \{\dfrac{n+7}{2}, \dfrac{n+9}{2}, \ldots, \dfrac{3n+5}{2}\}$ is a set of n consecutive integers.

Then f extends to a super edge-magic labeling of $P_2 \cup P_n$ with $k = \dfrac{5n+11}{2}$ where $n \equiv 1 (\bmod 4)$.

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Case 6: $n \equiv 2 (\bmod 4)$, where $n \geq 10$. Let $f(u_1) = 1$, $f(u_2) = \frac{n}{2} + 2$, and

$$f(v_j) = \begin{cases} n+2 & if \quad j = 1, \\ n & if \quad j = 3, \\ n+1 & if \quad j = n, \\ \frac{n}{2} - 2i + 2 & if \quad j = 4i \quad and \quad 1 \leq i \leq \frac{n-2}{4}, \\ n - 2i & if \quad j = 4i+1 \quad and \quad 1 \leq i \leq \frac{n-2}{4}, \\ \frac{n}{2} - 2i - 1 & if \quad j = 4i+2 \quad and \quad 0 \leq i \leq \frac{n-6}{4}, \\ n - 2i + 1 & if \quad j = 4i+3 \quad and \quad 1 \leq i \leq \frac{n-6}{4}. \end{cases}$$

The proof is similar to the previous theorem that

a) $f(V(P_2 \cup P_n)) = \{1, 2, \ldots, n+2\}$

b) $S = \{f(x) + f(y) : xy \in E(P_2 \cup P_n)\} = \{\frac{n+6}{2}, \frac{n+8}{2}, \ldots, \frac{3n+4}{2}\}$ is a set of n consecutive integers.

Then f extends to a super edge-magic labeling of $P_2 \cup P_n$ with $k = \frac{5n+10}{2}$ where $n \equiv 2 (\bmod 4)$, when $n \geq 10$.

Case 7: $n \equiv 3(\mathrm{mod}\,4)$, where $n \geq 7$. Let $f(u_1) = 1,\ f(u_2) = n+2,$ and

$$f(v_j) = \begin{cases} \dfrac{j+3}{2} & \text{if } j \text{ is odd and } 1 \leq j \leq \dfrac{n-1}{2}, \\[2mm] \dfrac{j+n+2}{2} & \text{if } j \text{ is odd and } \dfrac{n+3}{2} \leq j \leq n, \\[2mm] \dfrac{j+n+3}{2} & \text{if } j \text{ is even and } 2 \leq j \leq \dfrac{n-3}{2}, \\[2mm] \dfrac{j+4}{2} & \text{if } j \text{ is even and } \dfrac{n+1}{2} \leq j \leq n-1. \end{cases}$$

The proof is similar to the previous theorem that

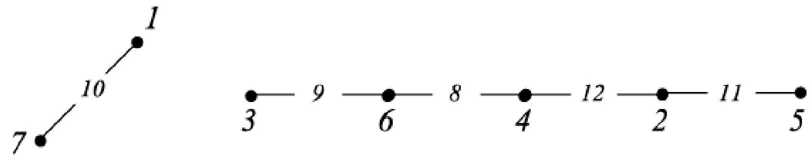
a) $f(V(P_2 \cup P_n)) = \{1,2,\ldots,n+2\}$

b) $S = \{f(x) + f(y) : xy \in E(P_2 \cup P_n)\} = \{\frac{n+7}{2}, \frac{n+9}{2}, \ldots, \frac{3n+5}{2}\}$ is a set of n consecutive integers.

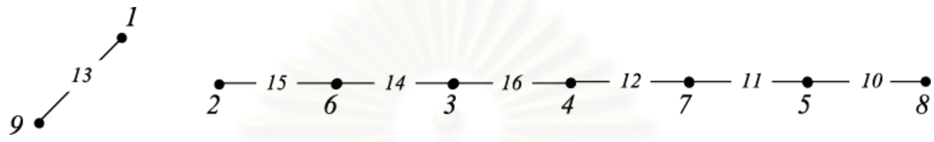
Then f extends to a super edge-magic labeling of $P_2 \cup P_n$ with $k = \dfrac{5n+11}{2}$ where $n \equiv 3(\mathrm{mod}\,4)$, when $n \geq 7$.

Therefore the linear forest $P_2 \cup P_n$ is super edge-magic for every integer $n \geq 3$. □

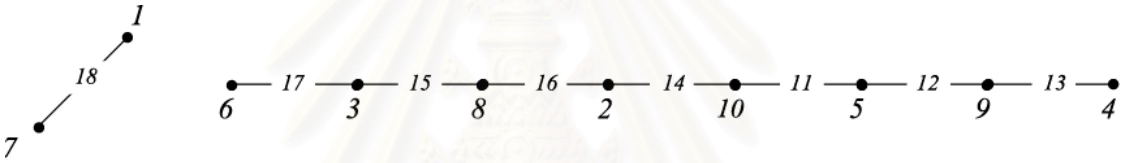
$P_2 \cup P_5$:



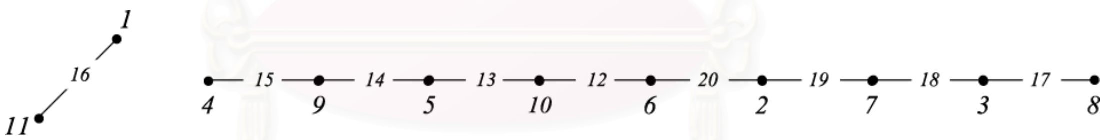
$P_2 \cup P_7$:



$P_2 \cup P_8$:



$P_2 \cup P_9$:



$P_2 \cup P_{10}$:

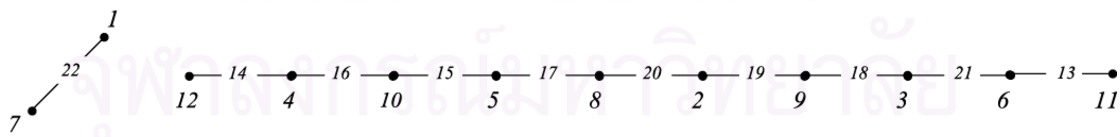
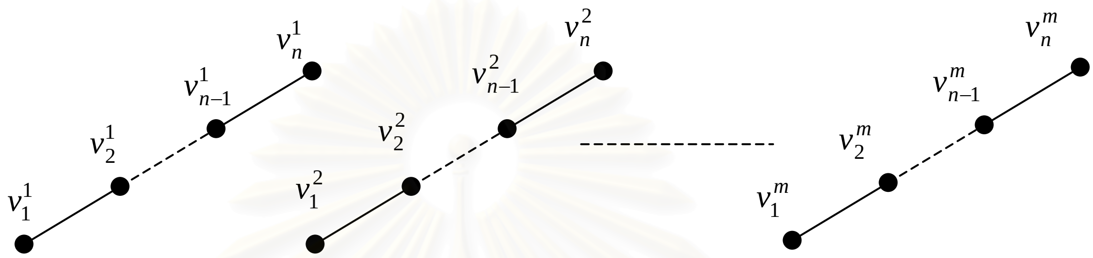


Figure 4.10: Super edge-magic labelings of $P_2 \cup P_5$, $P_2 \cup P_7$, $P_2 \cup P_8$, $P_2 \cup P_9$, $P_2 \cup P_{10}$ with $k = 18, 23, 26, 28, 30,$ respectively.

Theorem 4.5. [9] The linear forest mP_n is super edge-magic with $k = \dfrac{5mn-2m+3}{2}$ when $m(>1)$ and n are odd.

Proof. Let $V(mP_n) = V_1 \cup V_2 \cup ... \cup V_m$ where $V_i = \{v_1^i, v_2^i, ..., v_n^i\}$ and $E(mP_n) = E_1 \cup E_2 \cup ... \cup E_m$ where $E_i = \{v_1^i v_2^i, v_2^i v_3^i, ..., v_{n-1}^i v_n^i\}$.



Define a vertex labeling f as follows:

$$f(v_j^i) = \begin{cases} \dfrac{nm+1}{2} + i & \text{if} \quad i = 1,2,...,\dfrac{m-1}{2} \quad \text{and} \quad j = 1, \\ \dfrac{m(n-2)+1}{2} + i & \text{if} \quad i = \dfrac{m+1}{2}, \dfrac{m+3}{2}, ..., m \quad \text{and} \quad j = 1, \\ m\left(\dfrac{j}{2} - 1\right) + i & \text{if} \quad i = 1,2,...,m \quad \text{and} \quad j = 2,4,...,n-1, \\ \dfrac{m(n+j-1)+1}{2} + i & \text{if} \quad i = 1,2,...,\dfrac{m-1}{2} \quad \text{and} \quad j = 3,5,...,n, \\ \dfrac{m(n+j-3)+1}{2} + i & \text{if} \quad i = \dfrac{m+1}{2}, \dfrac{m+3}{2}, ..., m \quad \text{and} \quad j = 3,5,...,n. \end{cases}$$

In order to show that f extends to a super edge-magic labeling of mP_n, it suffices to verify the following by lemma 2.4:

a) $f(V(mP_n)) = \{1,2,...,mn\}$

b) $S = \{f(x) + f(y) : xy \in E(mP_n)\}$ is a set of $mn - m$ consecutive integers.

For a) $f(V(mP_n)) = \{1,2,...,mn\}$

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For b) observe that, for $i = 1,2,\ldots,\frac{m-1}{2}$,

for $v_1^i v_2^i \in E(mP_n)$,

$$f(v_1^i) + f(v_2^i) = \frac{nm+1}{2} + i + m\left(\frac{2}{2} - 1\right) + i = \frac{nm+1}{2} + 2i,$$

for $v_j^i v_{j+1}^i \in E(mP_n) : j = 2,4,\ldots,n-1$,

$$f(v_j^i) + f(v_{j+1}^i) = m\left(\frac{j}{2} - 1\right) + i + \frac{m(n+j)+1}{2} + i = \frac{m(n+2j-2)+1}{2} + 2i,$$

for $v_j^i v_{j+1}^i \in E(mP_n) : j = 3,5,\ldots,n$,

$$f(v_j^i) + f(v_{j+1}^i) = \frac{m(n+j-1)+1}{2} + i + m\left(\frac{j+1}{2} - 1\right) + i = \frac{m(n+2j-2)+1}{2} + 2i,$$

for $i = \frac{m+1}{2}, \frac{m+3}{2}, \ldots, m$, we can verify similarly.

Thus, $\{f(x) + f(y) : xy \in E(mP_n)\} = \{\frac{mn+3}{2}, \frac{mn+5}{2}, \ldots, \frac{3mn-2m+1}{2}\}$ is a set of $m(n-1)$ consecutive integers.

Then $k = mn + m(n-1) + \min\{f(u) + f(v) : uv \in E(mP_n)\}$

$$= mn + m(n-1) + \frac{nm+3}{2} = \frac{5mn-2m+3}{2}.$$

Therefore f extends to a super edge-magic labeling of the linear forest mP_n with $k = \frac{5mn-2m+3}{2}$ when $m(>1)$ and n are odd. □

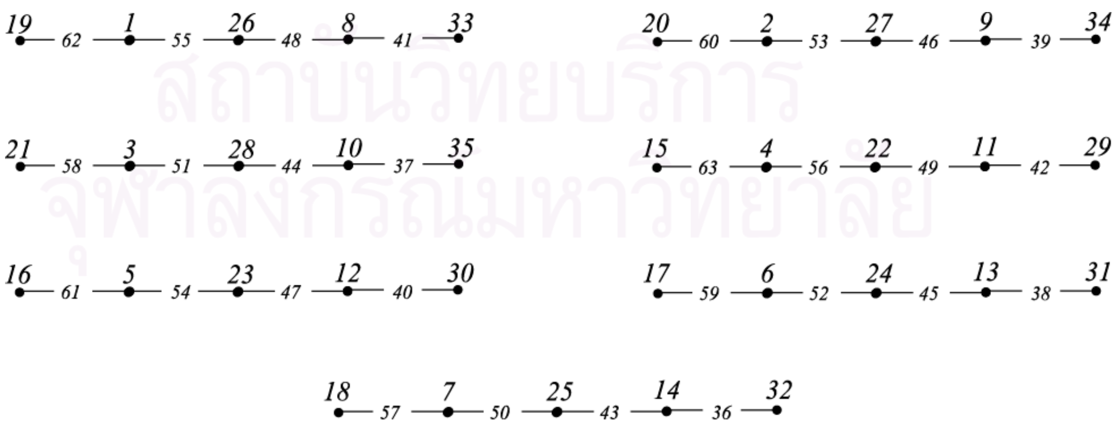

Figure 4.11: A super edge-magic labeling of $7P_5$ with $k = 82$.

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Theorem 4.6. The linear forest mP_n is super edge-magic with $k=\dfrac{5mn+3}{2}$ when $m(>1)$ and n are odd.

Proof. Define a vertex labeling f as follows:

$$f(v_j^i)=\begin{cases} i & \text{if} \quad 1\le i\le m \quad \text{and} \quad j=1,\\ m\left(\dfrac{n+j-1}{2}\right)+\dfrac{2i+1+m}{2} & \text{if} \quad 1\le i\le \dfrac{m-1}{2} \quad \text{and} \quad j=2,4,\dots,n-1,\\ m\left(\dfrac{n+j-1}{2}\right)+\dfrac{2i+1-m}{2} & \text{if} \quad \dfrac{m+1}{2}\le i\le m \quad \text{and} \quad j=2,4,\dots,n-1,\\ m\left(\dfrac{j-1}{2}+1\right)+1-2i & \text{if} \quad 1\le i\le \dfrac{m-1}{2} \quad \text{and} \quad j=3,5,\dots,n,\\ m\left(\dfrac{j-1}{2}+2\right)+1-2i & \text{if} \quad \dfrac{m+1}{2}\le i\le m \quad \text{and} \quad j=3,5,\dots,n. \end{cases}$$

We can verify similarly to theorem 4.1 that

a) $f(V(mP_n))=\{1,2,\dots,mn\}$

b) $S=\{f(x)+f(y):xy\in E(mP_n)\}=\{\dfrac{mn+2m+3}{2},\dfrac{mn+2m+5}{2},\dots,\dfrac{3mn+1}{2}\}$

is a set of $mn-m$ consecutive integers.

Therefore f extends to a super edge-magic labeling of the linear forest mP_n with $k=\dfrac{5mn+3}{2}$ when $m(>1)$ and n are odd. □

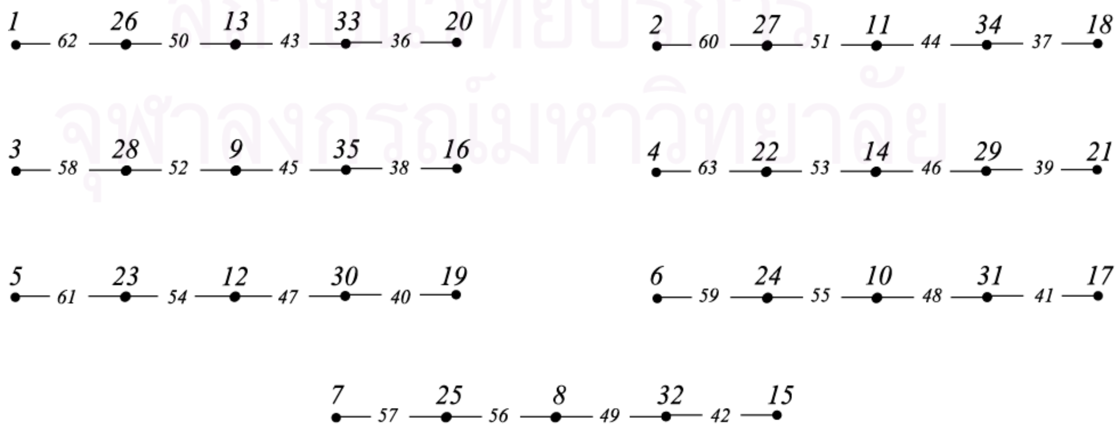

Figure 4.12: A super edge-magic labeling of $7P_5$ with $k=89$.

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Theorem 4.7. [9] The linear forest mP_n is super edge-magic with $k=\frac{5mn+3}{2}$ when $m(>1)$ and n are odd.

Proof. Define a vertex labeling f as follows:

$$f(v_j^i)=\begin{cases} i & \text{if} \quad 1\le i\le m \quad \text{and} \quad j=1,\\ \frac{m(j-1)}{2}+i & \text{if} \quad 1\le i\le m \quad \text{and} \quad j=3,5,\dots,n-3,\\ \frac{m(n+j)+1}{2}+i & \text{if} \quad 1\le i\le \frac{m-1}{2} \quad \text{and} \quad j=2,4,\dots,n-1,\\ \frac{m(n+j-2)+1}{2}+i & \text{if} \quad \frac{m+1}{2}\le i\le m \quad \text{and} \quad j=2,4,\dots,n-1,\\ \frac{m(n+1)}{2}+1-2i & \text{if} \quad 1\le i\le \frac{m-1}{2} \quad \text{and} \quad j=n,\\ \frac{m(n+3)}{2}+1-2i & \text{if} \quad \frac{m+1}{2}\le i\le m \quad \text{and} \quad j=n. \end{cases}$$

We can verify similarly to theorem 4.1 that

a) $f(V(mP_n))=\{1,2,\dots,mn\}$

b) $S=\{f(x)+f(y):xy\in E(mP_n)\}=\{\frac{mn+2m+3}{2},\frac{mn+2m+5}{2},\dots,\frac{3mn+1}{2}\}$ is a set of $mn-m$ consecutive integers.

Therefore f extends to a super edge-magic labeling of the linear forest mP_n with $k=\frac{5mn+3}{2}$ when $m(>1)$ and n are odd. □

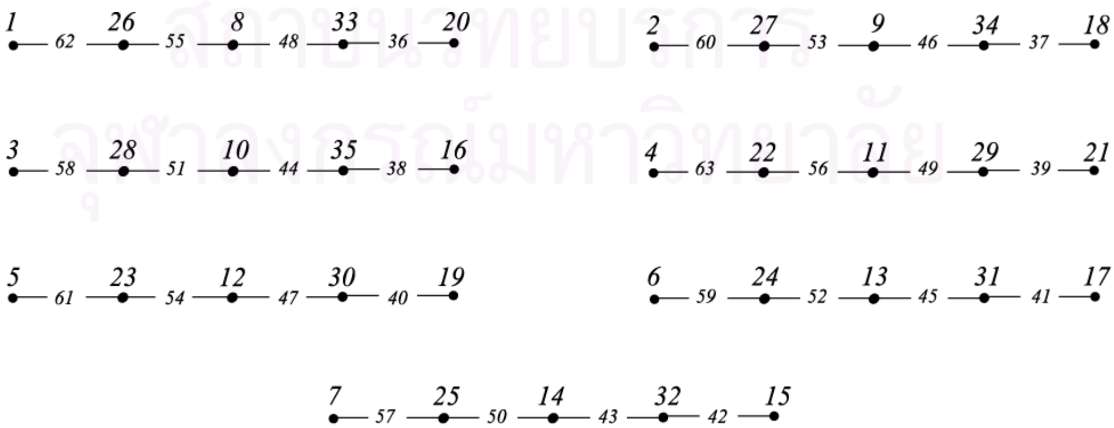

Figure 4.13: A super edge-magic labeling of $7P_5$ with $k=89$.

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Theorem 4.8. [9] The linear forest mP_n is super edge-magic with $k = \dfrac{5mn - m + 3}{2}$ when $m(>1)$ is odd and n is even.

Proof. Define a vertex labeling f as follows:

$$f(v_j^i) = \begin{cases} \dfrac{m(n+1)+1}{2} + i & \text{if} \quad i = 1,2,\ldots,\dfrac{m-1}{2} \quad \text{and} \quad j = 1, \\ \dfrac{m(n-1)+1}{2} + i & \text{if} \quad i = \dfrac{m+1}{2}, \dfrac{m+3}{2}, \ldots, m \quad \text{and} \quad j = 1, \\ m\left(\dfrac{j}{2} - 1\right) + i & \text{if} \quad i = 1,2,\ldots,m \quad \text{and} \quad j = 2,4,\ldots,n-1, \\ \dfrac{m(n+j)+1}{2} + i & \text{if} \quad i = 1,2,\ldots,\dfrac{m-1}{2} \quad \text{and} \quad j = 3,5,\ldots,n, \\ \dfrac{m(n+j-2)+1}{2} + i & \text{if} \quad i = \dfrac{m+1}{2}, \dfrac{m+3}{2}, \ldots, m \quad \text{and} \quad j = 3,5,\ldots,n. \end{cases}$$

We can verify similarly to theorem 4.5 that

a) $f(V(mP_n)) = \{1,2,\ldots,mn\}$

b) $S = \{f(x) + f(y) : xy \in E(mP_n)\} = \{\dfrac{mn+m+3}{2}, \dfrac{mn+m+5}{2}, \ldots, \dfrac{3mn-m+1}{2}\}$ is a set of $mn - m$ consecutive integers.

Therefore f extends to a super edge-magic labeling of the linear forest mP_n with $k = \dfrac{5mn - m + 3}{2}$ when $m(>1)$ is odd and n is even. □

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$3P_4$:

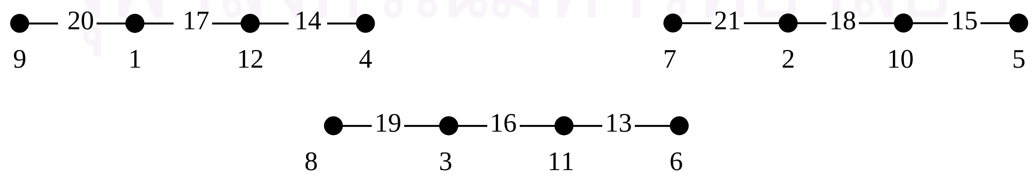
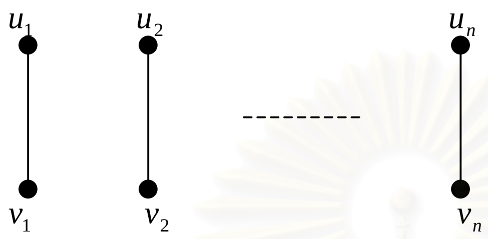


Figure 4.14: A super edge-magic labeling of $3P_4$ with $k = 30$.

Theorem 4.9. [4] The linear forest nP_2 is super edge-magic when n is odd with $k = \dfrac{9n+3}{2}$.

Proof. Let $V(nP_2) = \{u_i, v_i : i = 1,2,...,n\}$ and $E(nP_2) = \{u_i v_i : i = 1,2,...,n\}$.



Let $n = 2t+1$ for some $t \in Z^+$ and define a vertex labeling f as follows:

$$f(x) = \begin{cases} \dfrac{i+1}{2} & \text{if} \quad x = u_i \quad \text{and} \quad i = 1,3,...,2t+1, \\ t+1+\dfrac{i}{2} & \text{if} \quad x = u_i \quad \text{and} \quad i = 2,4,...,2t, \\ \dfrac{6t+3+i}{2} & \text{if} \quad x = v_i \quad \text{and} \quad i = 1,3,...,2t+1, \\ 2t+1+\dfrac{i}{2} & \text{if} \quad x = v_i \quad \text{and} \quad i = 2,4,...,2t. \end{cases}$$

We can verify similarly to theorem 4.5 that

a) $f(V(2P_n)) = \{1,2,...,2n\}$

b) $S = \{f(x) + f(y) : xy \in E(2P_n)\} = \{\frac{3n+3}{2}, \frac{3n+5}{2}, ..., \frac{5n+1}{2}\}$ is a set of n consecutive integers.

Therefore f extends to a super edge-magic labeling of nP_2 when n is odd with $k = \dfrac{9n+3}{2}$. □

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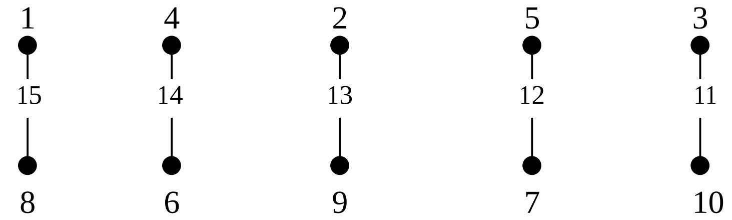

Figure 4.15: The linear forest $5P_2$ is super edge-magic with $k = 24$.

Theorem 4.10. [4] Let $\bigcup_{i=1}^{l} P_{n_i}$, where n_i is an integer with $n_i \geq 2$ for all i, be super edge-magic. If $m \geq 3$ is odd, then $m(\bigcup_{i=1}^{l} P_{n_i})$ is super edge-magic.

Proof. Let $V(\bigcup_{i=1}^{l} P_{n_i}) = \bigcup_{i=1}^{l}\{v_{ij} : 1 \leq j \leq n_i\}$ and $E(\bigcup_{i=1}^{l} P_{n_i}) = \bigcup_{i=1}^{l}\{v_{ij}v_{ij+1} : 1 \leq j \leq n_i - 1\}$.
Suppose that $f : V(\bigcup_{i=1}^{l} P_{n_i}) \to \{1,2,\ldots,\sum_{i=1}^{l} n_i\}$ is a vertex labeling that extends to a super edge-magic labeling of $\bigcup_{i=1}^{l} P_{n_i}$ with valence k and $S = \{f(x) + f(y) : xy \in E(\bigcup_{i=1}^{l} P_{n_i})\}$ is a set of q consecutive integers. If $\min(S)$ is $f(v_{ab}) + f(v_{ab+1})$ for some $v_{ab}v_{ab+1} \in E(\bigcup_{i=1}^{l} P_{n_i})$, then $k = \sum_{i=1}^{l} n_i + q + f(v_{ab}) + f(v_{ab+1})$.
We assume that $m \geq 3$.

Now, let $m(\bigcup_{i=1}^{l} P_{n_i})$ be the linear forest with

$$V(m(\bigcup_{i=1}^{l} P_{n_i})) = \bigcup_{t=1}^{m}\bigcup_{i=1}^{l}\{v_{ij}^{t} : 1 \leq j \leq n_i\} \text{ and}$$

$$E(m(\bigcup_{i=1}^{l} P_{n_i})) = \bigcup_{t=1}^{m}\bigcup_{i=1}^{l}\{v_{ij}^{t}v_{ij+1}^{t} : 1 \leq j \leq n_i - 1\}.$$

Define a vertex labeling $g : V(m(\bigcup_{i=1}^{l} P_{n_i})) \to \{1,2,\ldots,m\sum_{i=1}^{l} n_i\}$ as follows:

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$$g(v_{ij}^t)=\begin{cases} mf(v_{ij})-m+t & if \quad j \quad is \quad even \quad and \quad 1\le t\le m,\\ mf(v_{ij})+\dfrac{1-m}{2}+t & if \quad j \quad is \quad odd \quad and \quad 1\le t\le \dfrac{m-1}{2},\\ mf(v_{ij})+\dfrac{1-3m}{2}+t & if \quad j \quad is \quad odd \quad and \quad \dfrac{m+1}{2}\le t\le m. \end{cases}$$

In order to show that g extends to a super edge-magic labeling of $m(\bigcup_{i=1}^{l}P_{n_i})$, it suffices to verify the following by lemma 2.4:

a) $g(V(m(\bigcup_{i=1}^{l}P_{n_i})))=\{1,2,...,m\sum_{i=1}^{l}n_i\}$

b) $S'=\{g(x)+g(y):xy\in E(m(\bigcup_{i=1}^{l}P_{n_i}))\}$ is a set of mq consecutive integers.

For a) observe that for $i=1,2,...,l$ and odd j,

the numbers $mf(v_{ij})+\dfrac{1-m}{2}+1, mf(v_{ij})+\dfrac{1-m}{2}+2,...,mf(v_{ij}),$ $mf(v_{ij})+1-m, mf(v_{ij})+2-m,...,mf(v_{ij})+\dfrac{1-m}{2}$ are labels of $v_{ij}^1, v_{ij}^2,...,v_{ij}^{\frac{m-1}{2}}$, $v_{ij}^{\frac{m+1}{2}}, v_{ij}^{\frac{m+3}{2}},...,v_{ij}^m$.

for $i=1,2,...,l$ and even j,

the numbers $mf(v_{ij})-m+1, mf(v_{ij})-m+2,...,mf(v_{ij})$ are labels of $v_{ij}^1, v_{ij}^2,...,v_{ij}^m$.

Since $f(V(\bigcup_{i=1}^{l}P_{n_i}))=\{1,2,...,\sum_{i=1}^{l}n_i\}$, $g(V(m(\bigcup_{i=1}^{l}P_{n_i})))=\{1,2,...,m\sum_{i=1}^{l}n_i\}$.

For b) observe first that the minimum element in S' is

$$g(v_{ab}^{\frac{m+1}{2}})+g(v_{ab+1}^{\frac{m+1}{2}})=mf(v_{ab})+\frac{1-3m}{2}+\frac{m+1}{2}+mf(v_{ab+1})-m+\frac{m+1}{2}$$

$$=mf(v_{ab})+mf(v_{ab+1})+\frac{3-3m}{2},$$

and the maximum element in S' is

$$g(v_{cd}^m)+g(v_{cd+1}^m)=mf(v_{cd})+\frac{1-3m}{2}+m+mf(v_{cd+1})-m+m$$

$$=mf(v_{cd})+mf(v_{cd+1})+\frac{1-m}{2},$$

where $f(v_{cd})+f(v_{cd+1})$ is the maximum element in S.

Now, $g(v_{ij}^{t})+g(v_{ij+1}^{t})\neq g(v_{gh}^{t'})+g(v_{gh+1}^{t'})$ if and only if $i\neq g$ or $j\neq h$ or $t\neq t'$.

Thus, $S'=\{g(x)+g(y):xy\in E(m(\bigcup_{i=1}^{l}P_{n_i}))\}$

$$=\{mf(v_{ab})+mf(v_{ab+1})+\frac{3-3m}{2},\ldots,mf(v_{cd})+mf(v_{cd+1})+\frac{1-m}{2}\}$$ is a set of

mq consecutive integers.

The valence of $g = m\sum_{i=1}^{l}n_i + mq + \min\{g(x)+g(y):xy\in E(m(\bigcup_{i=1}^{l}P_{n_i}))\}$

$$= m\sum_{i=1}^{l}n_i + mq + mf(v_{ab}) + mf(v_{ab+1}) + \frac{3-3m}{2}\}$$

$$= m\{\sum_{i=1}^{l}n_i + q + f(v_{ab}) + f(v_{ab+1})\} + \frac{3-3m}{2}$$

$$= mk + \frac{3}{2}(1-m).$$

Therefore g extends to a super edge-magic labeling of $m(\bigcup_{i=1}^{l}P_{n_i})$ with valence

$mk+\frac{3}{2}(1-m)$. □

$P_3 \cup P_2$ is a super edge-magic.

Figure 4.16: A super edge-magic labeling of $3(P_3 \cup P_2)$ with $k = 36$.

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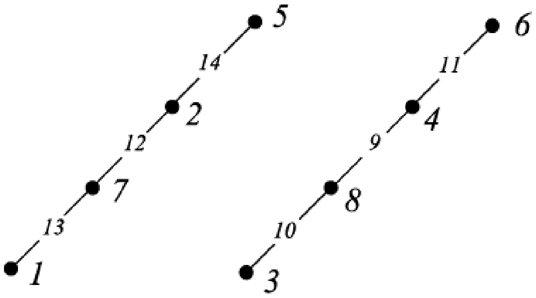
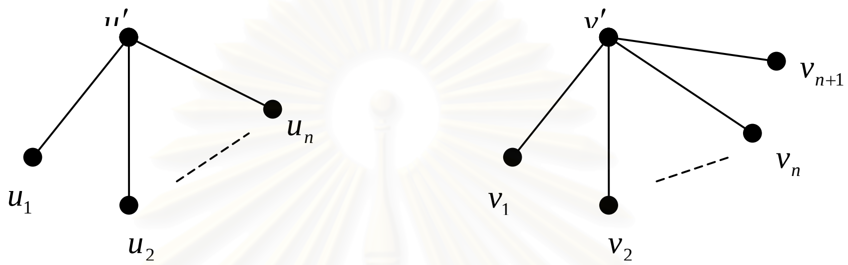
The converse of the previous theorem is not true. For example, the linear forest $2P_4$ is super edge-magic with $k = 21$.



Theorem 4.11. [4] The galaxy $K_{1,n} \cup K_{1,n+1}$ is super edge-magic when $n \geq 1$ with $k = 4n+9$.

Proof. Let $V(K_{1,n} \cup K_{1,n+1}) = \{u', v'\} \cup \{u_i : i = 1,2,...,n\} \cup \{v_i : i = 1,2,...,n+1\}$ and $E(K_{1,n} \cup K_{1,n+1}) = \{u'u_i : i = 1,2,...,n\} \cup \{v'v_i : i = 1,2,...,n+1\}$.



Define a vertex labeling f as follows:

$$f(x) = \begin{cases} 1 & \text{if} \quad x = u', \\ 3 & \text{if} \quad x = v', \\ 2i+3 & \text{if} \quad x = u_i \quad \text{and} \quad i = 1,2,...,n, \\ 2i & \text{if} \quad x = v_i \quad \text{and} \quad i = 1,2,...,n+1. \end{cases}$$

The labeling f is shown in figure 4.17.

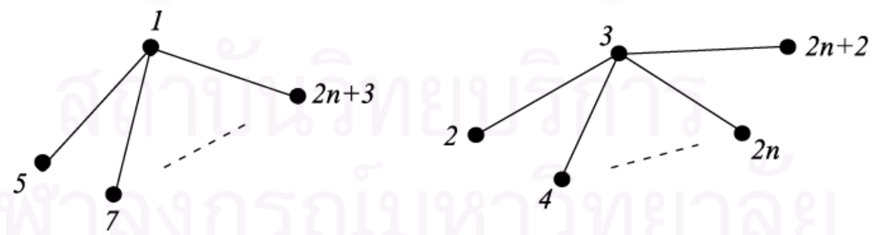


Figure 4.17: A vertex labeling of $K_{1,n} \cup K_{1,n+1}$.

In order to show that f extends to a super edge-magic labeling of $K_{1,n} \cup K_{1,n+1}$, it suffices to verify the following by lemma 2.4:

a) $f(V(K_{1.n} \cup K_{1,n+1})) = \{1,2,...,2n+3\}$

b) $S = \{f(x) + f(y) : xy \in E(K_{1,n} \cup K_{1,n+1})\}$ is a set of $n + (n+1) = 2n+1$ consecutive integers.

For a) $f(V(K_{1.n} \cup K_{1,n+1})) = \{1,2,...,2n+3\}$

For b) observe that,

for $u'u_i \in E(K_{1,n} \cup K_{1,n+1}) : i = 1,2,...,n$, $f(u') + f(u_i) = 1 + 2i + 3 = 2i + 4$,

for $v'v_i \in E(K_{1,n} \cup K_{1,n+1}) : i = 1,2,...,n+1$, $f(v') + f(v_i) = 3 + 2i = 2i + 3$.

Thus, $\{f(x) + f(y) : xy \in E(K_{1,n} \cup K_{1,n+1})\} = \{5,6,...,2n+5\}$ is a set of $2n+1$ consecutive integers.

Then $k = (2n+3) + (2n+1) + \min\{f(u) + f(v) : uv \in E(K_{1,n} \cup K_{1,n+1})\}$

$= (2n+3) + (2n+1) + 5 = 4n + 9$.

Therefore f extends to a super edge-magic labeling of $K_{1,n} \cup K_{1,n+1}$ with $k = 4n+9$. $\square$

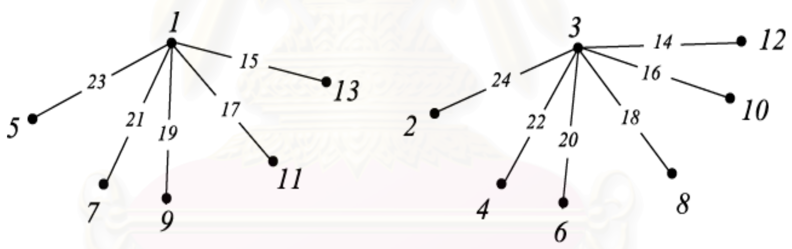
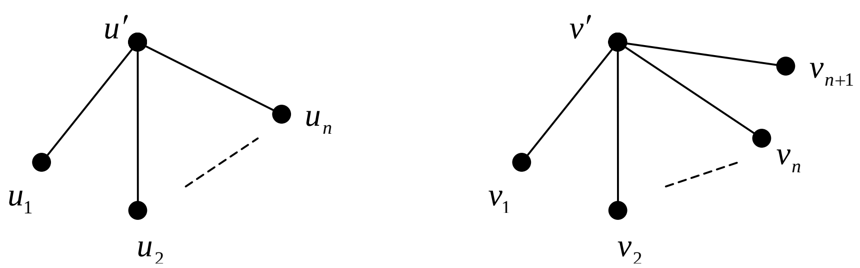


Figure 4.18: A super edge-magic labeling of $K_{1,5} \cup K_{1,6}$ with $k = 29$.

Theorem 4.12. The galaxy $K_{1,n} \cup K_{1,n+1}$ is super edge-magic when $n \geq 1$ with $k = 6n + 7$.

Proof. Let $V(K_{1,n} \cup K_{1,n+1}) = \{u', v'\} \cup \{u_i : i = 1,2,...,n\} \cup \{v_i : i = 1,2,...,n+1\}$ and $E(K_{1,n} \cup K_{1,n+1}) = \{u'u_i : i = 1,2,...,n\} \cup \{v'v_i : i = 1,2,...,n+1\}$.



Define a vertex labeling f as follows:

$$f(x) = \begin{cases} 2n+3 & if \quad x = u', \\ 2n+1 & if \quad x = v', \\ 2i-1 & if \quad x = u_i \quad and \quad i = 1,2,...,n, \\ 2i & if \quad x = v_i \quad and \quad i = 1,2,...,n+1. \end{cases}$$

The labeling f is shown in figure 4.19.

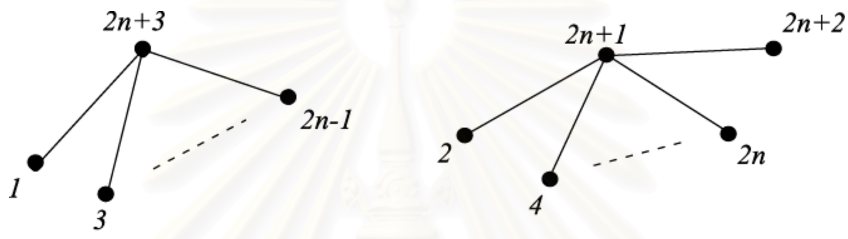


Figure 4.19: A vertex labeling of $K_{1,n} \cup K_{1,n+1}$.

The proof is similar to the previous theorem that

a) $f(V(K_{1.n} \cup K_{1,n+1})) = \{1,2,...,2n+3\}$

b) $S = \{f(x) + f(y) : xy \in E(K_{1,n} \cup K_{1,n+1})\} = \{2n+3, 2n+4,...,4n+3\}$ is a set of $2n+1$ consecutive integers.

Therefore f extends to a super edge-magic labeling of $K_{1,n} \cup K_{1,n+1}$ with $k = 6n+7$.

$\square$

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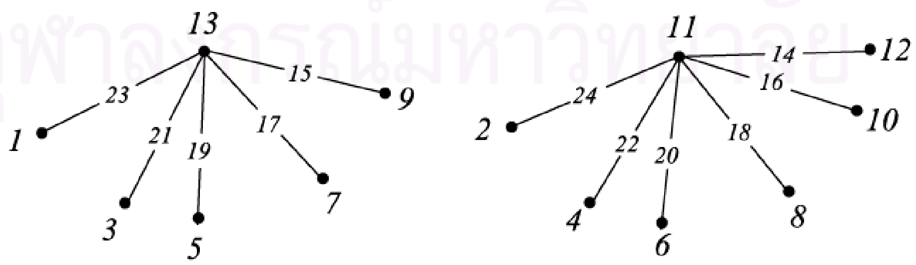
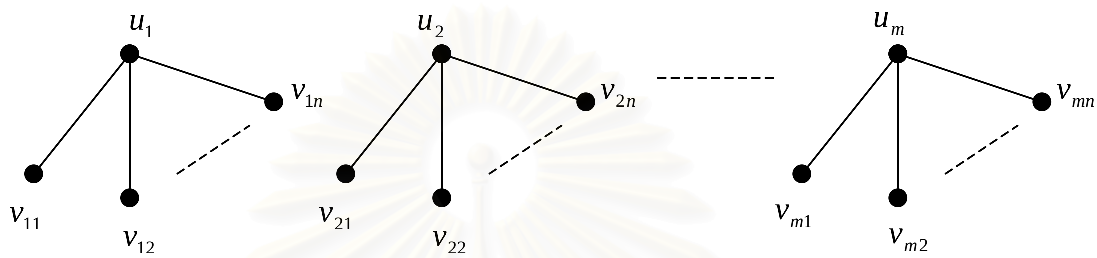


Figure 4.20: A super edge-magic labeling of $K_{1,5} \cup K_{1,6}$ with $k = 37$.

Theorem 4.13. [4] The galaxy $mK_{1,n}$ is super edge-magic when $n \geq 1$ and m is odd with $k = 2mn + 2m + 3$.

Proof. Let $V(mK_{1,n}) = \{u_i : i = 1,2,...,m\} \cup \{v_{ij} : i = 1,2,...,m,\ j = 1,2,...,n\}$ and $E(mK_{1,n}) = \{u_i v_{ij} : i = 1,2,...,m,\ j = 1,2,...,n\}$.



Let $n \geq 1$ and m be odd and define a vertex labeling f as follows:

$$f(x) = \begin{cases} i & \text{if} \quad x = u_i \quad \text{and} \quad i = 1,2,...,m, \\ i + \dfrac{3m+1}{2} & \text{if} \quad x = v_{i1} \quad \text{and} \quad i = 1,2,...,\dfrac{m-1}{2}, \\ i + \dfrac{m+1}{2} & \text{if} \quad x = v_{i1} \quad \text{and} \quad i = \dfrac{m+1}{2}, \dfrac{m+3}{2},...,m, \\ i + m(j-1) + \dfrac{3m+1}{2} & \text{if} \quad x = v_{ij} : j \neq 1 \quad \text{and} \quad i = 1,2,...,\dfrac{m-1}{2}, \\ i + m(j-1) + \dfrac{m+1}{2} & \text{if} \quad x = v_{ij} : j \neq 1 \quad \text{and} \quad i = \dfrac{m+1}{2}, \dfrac{m+3}{2},...,m. \end{cases}$$

The proof is similar to theorem 4.10 that

a) $f(V(mK_{1,n})) = \{1,2,...,mn+m\}$

b) $S = \{f(x) + f(y) : xy \in E(mK_{1,n})\} = \{m+3, m+4,..., mn+m+2\}$ is a set of mn consecutive integers.

Therefore f extends to a super edge-magic labeling of $mK_{1,n}$ when $n \geq 1$ and m is odd with $k = 2mn + 2m + 3$. □

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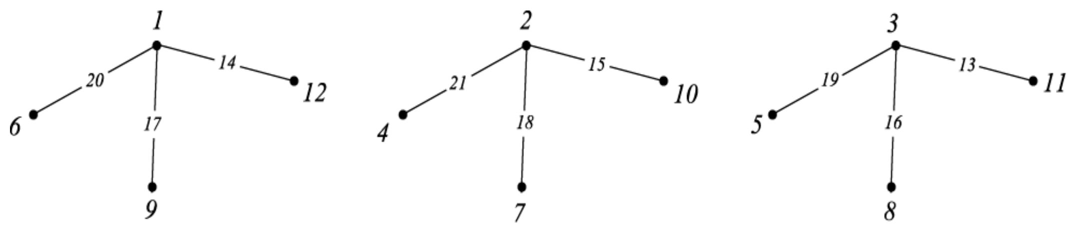
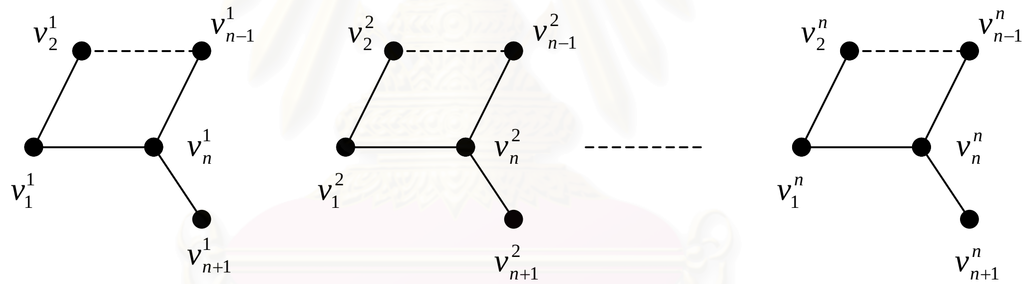


Figure 4.21: A super edge-magic labeling of $3K_{1,3}$ with $k = 27$.

Theorem 4.14. The graph $n(n,1)-kite$ is super edge-magic when n is odd with $k = \dfrac{5n^2+4n+3}{2}$.

Proof. Let $V(n(n,1)-kite) = V_1 \cup V_2 \cup ... \cup V_n$ where $V_i = \{v_1^i, v_2^i, ..., v_n^i\}$ and $E(n(n,1)-kite) = E_1 \cup E_2 \cup ... \cup E_n$ where $E_i = \{v_1^i v_2^i, v_2^i v_3^i ..., v_n^i v_1^i, v_n^i v_{n+1}^i\}$.



Define a vertex labeling f as follows:

$$f(v_j^i) = \begin{cases} i & if \quad 1 \le i \le n \quad and \quad j = 1, \\ n\left(\dfrac{n+j-1}{2}\right) + \dfrac{2i+1+n}{2} & if \quad 1 \le i \le \dfrac{m-1}{2} \quad and \quad j = 2,4,...,n-1, \\ n\left(\dfrac{n+j-1}{2}\right) + \dfrac{2i+1-n}{2} & if \quad \dfrac{m+1}{2} \le i \le m \quad and \quad j = 2,4,...,n-1, \\ n\left(\dfrac{j-1}{2}+1\right) + 1 - 2i & if \quad 1 \le i \le \dfrac{m-1}{2} \quad and \quad j = 3,5,...,n, \\ n\left(\dfrac{j-1}{2}+2\right) + 1 - 2i & if \quad \dfrac{m+1}{2} \le i \le m \quad and \quad j = 3,5,...,n. \end{cases}$$

The proof is similar to theorem 4.1 that

a) $f(V(n(n,1)-kite)) = \{1,2,\ldots,n^2+n\}$

b) $S = \{f(x)+f(y) : xy \in E(n(n,1)-kite)\} = \{\frac{n^2+3}{2}, \frac{n^2+5}{2}, \ldots, \frac{3n^2+3n+1}{2}\}$

is a set of $n(n+1) = n^2+n$ consecutive integers.

Therefore f extends to a super edge-magic labeling of $n(n,1)-kite$ when n is odd with $k = \frac{5n^2+4n+3}{2}$. □

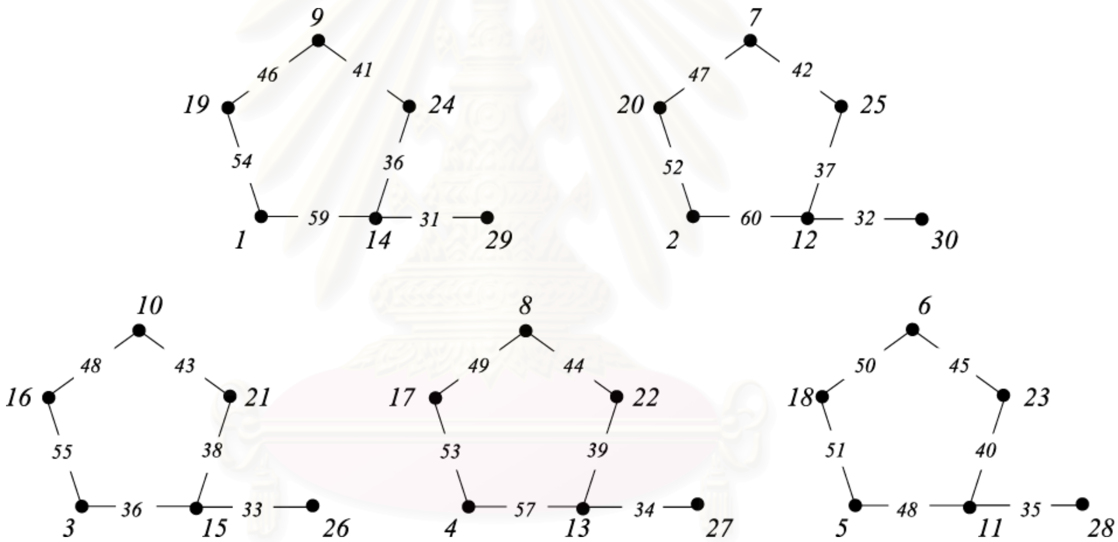


Figure 4.22: A super edge-magic labeling of $5(5,1)-kite$ with $k = 74$.

REFERENCES

[1] Chen, Z. A generalization of the Bodendiek conjecture about graceful graphs. In R. Bodendiek; and R. Henn, (eds.), Topics in Combinatorics and Graph Theory, pp. 737-746. Heidelberg: Physica-Verlag, 1990.

[2] Enomoto, H.; Llado, A.S.; Nakamigawa, T.; and Ringel, G. Super edge-magic graphs. SUT J. Math. 34 (1998): 105-109.

[3] Figueroa-Centeno, R.M.; Ichishima,R.; and Muntaner-Batie,F.A. The place of super edge-magic labeling among other classes of labelings, Discrete Math. 231 (2001): 153-168.

[4] Figuerroa-Centeno, R.M.; Ichishima, R.; and Muntaner-Batle, F.A. On super edge-magic graphs. ARS Combin. (To appear).

[5] Gallian, J. A dynamic survey of graph labeling. The Electronic Journal of Combinatorics 5 (1998).

[6] Ringel, G; and Llado, A.S. Another tree conjecture. Bull. of ICA 18(1996): 83-85.

[7] Tsuchiya, M.; and Yokomura, K. Some families of edge-magic graphs. The Proceedings of the Eight International Conference on Graph theory, Combinatorics, Algorithms and Applications (To appear).

[8] Wallis, W. D.; Baskoro, T; Miller, M; and Slamin, M. Edge-magic total labelings. Austral. J. Combin. 22 (2000): 177-190.

[9] Wijaya, K.; and Basloro, T. Edge-magic total labelings on disconnected graphs. Proc. Eleventh Australasian Workshop on Combinatorial Algorithms (2000): 139-144.

[10] Wilson, J.; and Watkins, J. Graphs an introductory approach. Canada: JohnWiley & Sons, 1990.

APPENDIX

Definition 1. A *graph* G consists of a finite nonempty set $V(G)$ of elements, called *vertices*, and a set $E(G)$ of 2-element subsets of $V(G)$, called *edges.* We call $V(G)$ as the *vertex – set* of G and $E(G)$ as the *edge – set* of G. If $\{x, y\}$ is an edge in a graph G, then an edge $\{x, y\}$ *joins* x and y, or x and y are *adjacent*, or an edge $\{x, y\}$ is *incident* to x (or y). We usually write $\{x, y\}$ as xy.

Definition 2. A *path of length n* in a graph G is a finite sequence of distinct vertices and edges of the form v_{i_0}, e_{i_1}, v_{i_1}, ..., e_{i_n}, v_{i_n} where $e_{i_1} = v_{i_0}v_{i_1}, e_{i_2} = v_{i_1}v_{i_2}, \ldots,$ $e_{i_n} = v_{i_{n-1}}v_{i_n}$.

Definition 3. A graph G is *connected* if every pair of vertices is joined by a path and *disconnected* otherwise.

Definition 4. A *component* of a graph G is a connected subgraph of G that is not contained in any larger connected subgraph of G.

Definition 5. The *degree* of a vertex v in a graph G, denoted by $\deg v$, is the number of edges incident to v.

Definition 6. The *distance* $d(u, v)$ between two points u and v in G is the length of a shortest path joining them.

Definition 7. Let G_1 and G_2 be graphs with disjoint vertex-sets $V(G_1)$ and $V(G_2)$ and edge-sets $E(G_1)$ and $E(G_2)$ respectively. The *join* of G_1 and G_2, denoted by $G_1 + G_2$, is a graph with the vertex-set $V(G_1) \cup V(G_2)$ and the edge-set $E(G_1) \cup E(G_2)$ and all edges joining vertices in $V(G_1)$ and $V(G_2)$.

Definition 8. A *cycle* C_n, $n \geq 3$, is a graph which the vertex-set is $\{v_1, v_2, \ldots, v_n\}$ and the edge-set is $\{e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_n, e_n = v_nv_1\}$.

Definition 9. A *path* P_n is a cycle with an edge deleted.

Definition 10. A graph P_n^k, the *k*th power of P_n, is obtained from P_n by adding edges that join all vertices u and v with $d(u, v) \leq k$.

Definition 11. A *complete graph* K_n is a graph of n vertices which any two distinct vertices are adjacent.

Definition 12. The *wheel* W_n, $n \geq 4$, is the graph $K_1 + C_n$.

Definition 13. The *fan* F_n is the graph $P_n + K_1$.

Definition 14. A star $K_{1,n}$ is a graph whose the vertex-set can be partitioned into two subsets V_1 and V_2 where $|V_1| = 1$ and $|V_2| = n$ and two vertices are adjacent if they lie in different sets.

Definition 15. Let G_1 and G_2 be graphs with disjoint vertex-sets $V(G_1)$ and $V(G_2)$ and edge-sets $E(G_1)$ and $E(G_2)$ respectively. The *product* of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with the vertex-set $V(G_1) \times V(G_2)$ and specified by putting (u_1, u_2) adjacent to (v_1, v_2) if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$.

Definition 16. The *ladder* L_n is the graph $P_n \times P_2$.

Definition 17. The *generalized prism* is the graph $C_m \times P_n$.

Definition 18. A *tree* is a connected graph with n vertices and $n-1$ edges.

Definition 19. Let $G_1, G_2, ..., G_m$ be graphs with disjoint vertex-sets $V(G_1), V(G_2), ..., V(G_m)$ and edge-sets $E(G_1), E(G_2), ..., E(G_m)$ respectively. The *disjoint union* of $G_1, G_2, ..., G_m$, denoted by $G_1 \cup G_2 \cup ... \cup G_m$, is a graph with the vertex-set $V(G_1) \cup V(G_2) \cup ... \cup V(G_m)$ and the edge-set $E(G_1) \cup E(G_2) \cup ... \cup E(G_m)$.

If $G_1 = G_2 = ... = G_m = G$ then $G_1 \cup G_2 \cup ... \cup G_m$ is denoted by mG and is called the disjoint union of m copies of G.

Definition 20. A *caterpillar* $CP_{n_1,...,n_t}$ is the graph $K_{1,n_1} \cup ... \cup K_{1,n_t}$ in which each K_{1,n_i} shares exactly one edge with $K_{1,n_{i+1}}$ and $t-1$ is the length of the skeleton path.

Definition 21. An $(n,t)-kite$ is a graph which consists of a cycle C_n and a path P_{t+1} (the tail) attached to one vertex.

Definition 22. An $n-sun$ is cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex.

Definition 23. An $(n,m)-pineapple$ is a graph which consists of a cycle C_n and m copies of P_2 attached to one vertex.

Definition 24. A linear forest is a graph whose connected components are paths.

Definition 25. A galaxy is a graph whose connected components are stars.

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