CHAPTER IV

TOTALLY POSITIVELY ORDERED 0-SKEWSEMIFIELDS

Lemma 4.1. Let K be a positively ordered skewsemifield. The following statements hold :

1) K is a totally positively ordered skewsemifield if and only if $K = P \cup P^{-1} \cup \{0\}$.

2) K is a totally positively ordered skewsemifield if and only if for every $x \in K$, $x \leq 1$ or $x \geq 1$.

3) Suppose that K is a totally positively ordered skewsemifield, A and B o-convex subsets of K. Then AB is an o-convex set of K.

4) Suppose that K is a totally positively ordered skewsemifield, C an o-convex normal subgroup of K. Then C is a convex normal subgroup of K.

Proof 1) and 2) are obvious.

3) Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Let $x \in K$ be such that $a_1b_1 \leq x \leq a_2b_2$. Then $a_1b_2 \geq x$ or $a_1b_2 \leq x$.

Case 1 : $a_1b_2 \geq x$. Then $a_1b_1 \leq x \leq a_1b_2$, so $b_1 \leq (a_1)^{-1}x \leq b_2$. By the o-convexity of B, $(a_1)^{-1}x \in B$, so $x = a_1(a_1)^{-1}x \in AB$.

Case 2 : $a_1b_2 \leq x$. Then $a_1b_2 \leq x \leq a_2b_2$, so $a_1 \leq x(a_2)^{-1} \leq a_1$. By the o-convexity of A, $x(b_2)^{-1} \in A$, so $x = x(b_2)^{-1}b_2 \in AB$. Thus AB is an o-convex set.

4) See [3], pp. 68. #

Examples 4.2. 1) $\mathbf{R}_0^+, \mathbf{Q}_0^+$ are totally positively ordered skewsemifields.

2) From Example 2.5., 3), we have that K is a positively ordered skewsemifield. To show that K is total, let $M_1 = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ and $M_2 = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix} \in K$.

We shall show that $M_1 \leq M_2$ or $M_2 \leq M_1$. Since **R** satisfies the Trichotomy law, $a < a'$ or $a = a'$ or $a > a'$. If $a < a'$ or $a > a'$ then done. So suppose that $a = a'$. Since **R** satisfies the Trichotomy law, $b < b'$ or $b = b'$ or $b > b'$. If $b < b'$ or $b > b'$ then done. So suppose that $b = b'$. Since **R** satisfies the Trichotomy law, $c \leq c'$ or $c' \leq c$, so $M_1 \leq M_2$ or $M_2 \leq M_1$. Hence K is a totally positively ordered skewsemifield.

3) From Example 2.5. 4), $K^* \times L^* \cup \{ (0,0) \}$ is a totally positively ordered skewsemifield where K and L are totally positive ordered skewsemifields.

Let K be a totally positively ordered skewsemifield and C a convex normal subgroup of K. Then $K/_C$ is a positively ordered skewsemifild.

To prove that $\leq$ on $K/_C$ is a total order, let $x \in K$.

Case 1 : $x \leq 1$. Then $xC \leq C$.

Case 2 : $x \geq 1$. Then $xC \geq C$.

By Lemma 4.1., 2) $K/_C$ is a totally positively ordered skewsemifield.

Theorem 4.3. Let S be a totally positively ordered semiring with multiplicative zero 0 having the M.C. property and such that $(S, \bullet)$ satisfies the right [left] Ore condition. If $\leq$ is M.R. then S can be embedded into a totally positively ordered skewsemifield.

Proof See [5], pp. 46. #

Theorem 4.4. Let $n \in \mathbf{Z}^+$ be such that $n \geq 2$. Let $K_n = \{ 0 \} \cup \{ A \in M_n(\mathbf{R}) [M_n(\mathbf{Q})] \ / \ A_{ij} > 0$ if $i = j$ and $A_{ij} = 0$ if $i = j \}$. Then there exists a positively total order on K_n.

Proof If $n = 2$ then done by Example 4.3., 2). Induction assumption, let $n \in \mathbf{Z}^+$ be such that $n > 2$. Let K_{n-1} with the following partial order is a totally

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

positively ordered skewsemifield. Let $P_n = \{ \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \in K_n$ / 1) $A_1 > 1$ or 2) $A_1 = 1$ and $A_2 > 1$ or 3) $A_1 = 1$, $A_2 = 1$ and $A_{n-1,n} > 0$ or 4) $A_1 = 1$, $A_2 = 1$ and there exists an $i \in \{1,\ldots,n-2\}$ such $A_{in} > 0$ and $A_{kn} = 0$ for all $n > k > i$ or 4) $A_1 = 1$, $A_2 = 1$ and $A_3 = 0$ where $A_1 \in K_{n-1}$ }. By Theorem 2.13., P is the positive cone of K.

To show that $K_n = P_n \cup (P_n)^{-1} \cup \{0\}$, let $X \in K_n$ be such that $X \neq 0$ and $X \notin P$. We must to show that $X^{-1} \in P$. By induction hypothesis, $X_1 \leq 1$ or $X_1 > 1$. Since $X \notin P$, $X_1 \leq 1$, so $X_1^{-1} \geq 1$. If $X_1^{-1} > 1$ then done Suppose that $X_1^{-1} = 1$. Since **R** satisfies the Trichotomy law, $X_2 \leq 1$ or $X_2 > 1$. Since $X \notin P$, $X \leq 1$, so $X_2^{-1} \geq 1$. If $X_2^{-1} > 1$ then done. Suppose that $X_2^{-1} = 1$. Since **R** satisfies the Trichotomy law, $X_{n-1,n} > 0$ or $X_{n-1,n} \leq 0$. If $X_{n-1,n} > 0$ then $X \in P$ which is a contradiction since $X \notin P$. Thus $X_{n-1,n} \leq 0$.

Case 1 : $X_{n-1,n} < 0$. Then $0 = (XX^{-1})_{n-1,n} = \sum_{i=1}^{n} X_{n-1,i}(X^{-1})_{in}$

$= \sum_{i=1}^{n-1} X_{n-1,i}(X^{-1})_{in} + X_{n-1,n}(X^{-1})_{nn} = (X^{-1})_{n-1,n} + X_{n-1,n}$. Hence $(X^{-1})_{n-1,n} = -(X_{n-1,n}) > 0$ and therefore $X^{-1} \in P$.

Case 2 : $X_{n-1,n} = 0$.

Claim that there exists an $i^* \in \{1,\ldots,n-2\}$ such $X_{i^*n} < 0$ and $X_{kn} = 0$ for all $n > k > i^*$. Suppose that (*), $X_{in} \geq 0$ for all $n-1 > i \geq 1$. If $X_{in} = 0$ for all $1 \leq i < n-1$ then $X = 1$ which is a contradiction since $X \notin P$. Then there is a $j \in \{1,\ldots,n-2\}$ such that $X_{jn} > 0$, so let $j^* = \max\{ j \ / \ 1 \leq j < n-1$ and $X_{jn} > 0 \}$. Let $n > k > j^*$. Then $X_{kn} \leq 0$. By (*), $X_{kn} = 0$ which is a contradiction since $X \notin P$. Thus (*) is not true, hence there is an $i \in \{1,\ldots,n-2\}$ such that $X_{in} < 0$. let $i^* = \max\{ i \ / \ 1 \leq i < n-1$ and $X_{in} < 0 \}$. Suppose that there exists a $i^* < k < n$ such that $X_{kn} \neq 0$. Then $X_{kn} > 0$. Let $j^{**} = \max\{ j \ / \ 1 \leq j < n$ and $X_{jn} > 0 \}$. Let $n > k > j^{**}$. Then $X_{kn} \leq 0$. Since $i^* < k$, $X_{kn} = 0$ which is a contradiction since $X \notin P$. Then $X_{kn} = 0$ for all $n > k > i^*$, so we have claim.

By claim, there exists an $i^* \in \{1,\ldots,n-2\}$ such $X_{i^*n} < 0$ and $X_{kn} = 0$ for all

$n > k > i^*$. Then $0 = I_{i^*,n} = (XX^{-1})_{i^*n} = \sum_{i=1}^{n} X_{i^*i}(X^{-1})_{in} = \sum_{i=1}^{n-1} X_{i^*i}(X^{-1})_{in} + X_{i^*n}(X^{-1})_{nn}$ $= (X^{-1})_{i^*n} + X_{i^*n}$. Thus $(X^{-1})_{i^*n} = -(X_{i^*n}) > 0$. Let $n > k > i^*$. Then $0 = (XX^{-1})_{kn}$ $= \sum_{i=1}^{n} X_{ki}(X^{-1})_{in} = \sum_{i=1}^{n-1} X_{ki}(X^{-1})_{in} + X_{kn}(X^{-1})_{nn} = (X^{-1})_{kn} + X_{kn}$. $(X^{-1})_{kn} = X_{kn} = 0$. Therefore $X^{-1} \in P$. Hence K_n is a totally positively ordered skewsemifield. #

<u>Proposition 4.5.</u> $\prod_{i \in I} K_i$ is a totally positively ordered skewsemifield if and only if either $I = \{ i \}$ and K_i is a totally positively ordered skewsemifield or there exists $i_0 \in I$ such that K_{i_0} is a totally positively ordered skewsemifiled and $| K_{i_0} | = 2$ for every $i \in I \setminus \{ i_0 \}$.

<u>Proof</u> See [4], pp. 46. #

Let K be a skewsemifield and $A \subseteq K^*$. Let $\mathcal{C} = \{ B \subseteq K$ / $A \subseteq B$ and B has the property that

1) $1 \in B$,
2) $B^2 \subseteq B$,
3) $B + K \subseteq K$ and $K + B \subseteq B$ and
4) B is and a-convex normal subset of K }.

Since $K^* \in \mathcal{C}$. $\mathcal{C} \neq \varnothing$. Then the smallest subset of K satisfying 1) – 4) exists.

<u>Definition 4.6.</u> Let K be a positively ordered skewsemifield and $A \subseteq K^*$. The <u>hull</u> of A, denoted by H(A) is the smallest subset of K satisfying 1) – 4). And A has property (*) if and only if for all $x_1,\ldots,x_n \in K^*$, there exist $\varepsilon_1,\ldots,\varepsilon_n \in \{-1, 1\}$ such that $H(P \cup \{ x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n} \})$ is a conic subset of K. From now on we shall use $H(P, x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ instead of $H(P \cup \{ x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n} \})$.

<u>Notation</u> Let K be a skewsemifield and $S \subseteq K$. Let (S) be the smallest multiplicative normal subsemigroup of K containing S. Then $(S) = \{ x(a_1 \ldots a_n)x^{-1}$ /

$n \in \mathbf{Z}^+, a_i \in S$ for all $1 \le i \le n$ and $x \in K^* \}$.

Proposition 4.7. Let K be a positively ordered skewsemifield and $A \subseteq K$. Then

$H(A) = \{ \sum_{i=1}^{n}[s_i + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) + t_i] \;/\; n, m \in \mathbf{Z}^+,\; s_i, t_i \in K$ for all $1 \le i \le n,\; \alpha_{j_i} \in K$

$m_{j_i} \in (A \cup \{1\})$ such that $\sum_{j_i=1}^{m_i} \alpha_{j_i} = 1$ for all $1 \le j_i \le m_i \}$.

Proof Let $B = \{ \sum_{i=1}^{n}[s_i + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) + t_i] \;/\; n, m \in \mathbf{Z}^+,\; s_i, t_i \in K$ for all $1 \le i \le n$,

$\alpha_{j_i} \in K$ $m_{j_i} \in (A \cup \{1\})$ such that $\sum_{j_i=1}^{m_i} \alpha_{j_i} = 1$ for all $1 \le j_i \le m_i \}$.

Clearly, B is an additive ideal of K.

Let $b_1 = \sum_{i=1}^{n}[s_i + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) + t_i]$ and $b_2 = \sum_{k=1}^{p}[u_k + (\sum_{l_k=1}^{q_k} \beta_{l_k} n_{l_k}) + v_k] \in K$.

Then $b_1 b_2 = \sum_{i=1}^{n}[s_i + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) + t_i] b_2 = \sum_{i=1}^{n}[u_i b_2 + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) b_2 + t_i b_2]$

$= \sum_{i=1}^{n}[s_i b_2 + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i})(\sum_{k=1}^{p}[u_k + (\sum_{l_k=1}^{q_k} \beta_{l_k} n_{l_k}) + v_k]) + t_i b_2]$

$= \sum_{i=1}^{n}[s_i b_2 + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i})(\sum_{k=1}^{p} u_k)] + [(\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i})(\sum_{l_k=1}^{q_k} \beta_{l_k} n_{l_k})] + [(\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) v_k + t_i b_2])$

$= \sum_{i=1}^{n}[s_i b_2 + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i})(\sum_{k=1}^{p} u_k)] + [(\sum_{j_i=1}^{m_i}(\sum_{l_k=1}^{q_k} (\alpha_{j_i} m_{j_i})(\beta_{l_k} n_{l_k}))] + [(\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) v_k + t_i b_2])$

$= \sum_{i=1}^{n}[s_i b_2 + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i})(\sum_{k=1}^{p} u_k)] + [\sum_{j_i=1}^{m_i}(\sum_{l_k=1}^{q_k} (\alpha_{j_i} m_{j_i})(\beta_{l_k}^{-1} m_{j_i} \beta_{l_k} n_{l_k}))]$

$+ [(\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) v_k + t_i b_2])$. Let $1 \le i \le n$ and $1 \le k \le p$. We must to show that

$\sum_{j_i=1}^{m_i}(\sum_{l_k=1}^{q_k} (\alpha_{j_i} m_{j_i})) = 1$. Therefore $\sum_{j_i=1}^{m_i}(\sum_{l_k=1}^{q_k} (\alpha_{j_i} m_{j_i})) = \sum_{j_i=1}^{m_i} \alpha_{j_i} (\sum_{l_k=1}^{q_k} b\beta_{l_k}) = \sum_{j_i=1}^{m_i} \alpha_{j_i} = 1$.

Thus $b_1 b_2 \in B$, so $B^2 \subseteq B$. Next, let $x \in K^*$. Then $x b_1 x^{-1}$

$= x(\sum_{i=1}^{n}[s_i + (\sum_{j_i=1}^{m_i} \alpha_{j_i} m_{j_i}) + t_i]) x^{-1} = \sum_{i=1}^{n}[x s_i x^{-1} + (\sum_{j_i=1}^{m_i} x(\alpha_{j_i} m_{j_i}) x^{-1}) + x t_i x^{-1}]$

$= \sum_{i=1}^{n}[xs_ix^{-1}+(\sum_{j_i=1}^{m_i} x\alpha_{j_i}x^{-1}(xm_{j_i}x^{-1}))+xt_ix^{-1}]$. Let $1 \le i \le n$. We must to show that $\sum_{j_i=1}^{m_i} x\alpha_{j_i}x^{-1} = 1$. Therefore $\sum_{j_i=1}^{m_i} x\alpha_{j_i}x^{-1} = x(\sum_{j_i=1}^{m_i}\alpha_{j_i})x^{-1} = 1$, so B is a normal set. Next, let $a, b \in K$ be such that $a + b = 1$. Then $ab_1 + bb_2$

$= a(\sum_{i=1}^{n}[s_i+(\sum_{j_i=1}^{m_i}\alpha_{j_i}m_{j_i})+t_i]) + b(\sum_{k=1}^{p}[u_k+(\sum_{l_k=1}^{q_k}\beta_{l_k}n_{l_k})+v_k])$

$= (\sum_{i=1}^{n}[as_i+(\sum_{j_i=1}^{m_i}a\alpha_{j_i}m_{j_i})+at_i]) + (\sum_{k=1}^{p}[bu_k+(\sum_{l_k=1}^{q_k}b\beta_{l_k}n_{l_k})+bv_k])$.

Let $1 \le i \le n$ and $1 \le k \le p$. We must to show that $(\sum_{j_i=1}^{m_i}a\alpha_{j_i}) + (\sum_{l_k=1}^{q_k}b\beta_{l_k}) = 1$. Therefore $(\sum_{j_i=1}^{m_i}a\alpha_{j_i}) + (\sum_{l_k=1}^{q_k}b\beta_{l_k}) = a(\sum_{j_i=1}^{m_i}\alpha_{j_i}) + b(\sum_{l_k=1}^{q_k}\beta_{l_k}) = a + b = 1$. Hence $ab_1 + bb_2 \in B$, so B is an a-convex normal set of K.

Let S be an additively a-convex normal mutiplicative subsemigroup of K containing $A \cup \{1\}$. We shall show that $b_1 \in S$. Clearly, $m_{j_i} \in S$ for all $1 \le j \le m$. Since $\sum_{j_i=1}^{m_i}\alpha_{j_i} = 1$ for all $1 \le i \le n$ and by Remark 1.37., 3), $\sum_{j_i=1}^{m_i}\alpha_{j_i} \in S$ for all $1 \le i \le n$. Then $b_1 = (\sum_{i=1}^{n}[s_i+(\sum_{j_i=1}^{m_i}\alpha_{j_i}m_{j_i})+t_i]) = \in S$. Then B is the smallest additively a-convex normal multiplicative subsemigroup of K containing $A \cup \{1\}$, hence $H(A) = B$. #

Lemma 4.8. ([2]) Let K be a positively ordered skewsemifield and P the positive cone of K. Suppose that P satisfies property (*). Then for every $x \in K^*$ either $H(P, x)$ or $H(P, x^{-1})$ satisfies 1) – 4) of Theorem 2.9. and also satisfies property (*).

Proof The proof is similar to the one given in [3] pp. 70.

Theorem 4.9. Let K be a positively ordered skewsemifield and P the positive cone of K. Then $\leq_P$ (from Theorem 2.9.) can be extended to a total order on K if and only if P satisfies property (*).

Proof Assume that P can be extended to a total order of K, say Q. Let $x_1,\dots,x_n \in K^*$. Choose $\varepsilon_1,\dots,\varepsilon_n \in \{-1, 1\}$ such that $x_i^{\varepsilon_i} \in Q$ for all i. Then $H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n}) \subseteq Q$. To show that $H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$ is an conic subset of K, let $x \in H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n}) \cap [H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})]^{-1}$. Suppose that $x \neq 1$.
Case 1 : $x < 1$. Then $x \notin Q$, so $x \notin H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$ which is a contradiction.
Case 2 : $x > 1$. Then $x^{-1} < 1$. Therefore $x^{-1} \notin Q$, so $x \notin H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$ which is a contradiction. Hence $x = 1$, so $H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$ is a conic subset of K.

Conversely, let $\mathcal{C} = \{ Q \mid Q$ is a positive cone of K containing P and satisfies (*) $\}$. Since $P \in \mathcal{C}$, $\mathcal{C} \neq \varnothing$. Let $\{ Q_i \mid i \in I \}$ be a nonempty subset of $\mathcal{C}$. Suppose that $\bigcup_{i \in I} Q_i$ does not satisfy (*). Then there exist $x_1,\dots,x_n \in K^*$ such that $H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$ is not conic for all choices of $\varepsilon_1,\dots,\varepsilon_n$, so there exists an $x \in H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n}) \cap [H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})]^{-1}$ such that $x \neq 1$. Therefore $x, x^{-1} \in H(P, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$. By Proposition 4.7., we can choose k such that $x, x^{-1} \in H(Q_k, x_1^{\varepsilon_1},\dots, x_n^{\varepsilon_n})$ which is a contradiction to Q_k satisfying (*). Hence $\bigcup_{i \in I} Q_i$ satisfies (*). Clearly, $\bigcup_{i \in I} Q_i$ is a positive cone of K containing P which is an upper bound of $\{ Q_i \mid i \in I \}$, so $\bigcup_{i \in I} Q_i \in \mathcal{C}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element, say Q. Let $x \in K^*$. By Lemma 4.8., $H(Q, x) \in \mathcal{C}$ or $H(Q, x^{-1}) \in \mathcal{C}$. By the maximality of Q, either $x \in Q$ or $x^{-1} \in Q$. Hence Q defines a total order on K. #

Definition 4.10. Let K be a positively ordered skewsemifield. K is called a vector skewsemifield if and only if it is a subdirect product of a totally positively ordered skewsemifield.

Let $\{ K_i \mid i \in I \}$ be a family of totally positively ordered skewsemifields. Let K be a subskewsemifield of $\prod_{i \in I} K_i$. Then the j^{th} projection map from K into K_j is isotone for every $j \in I$ as will now be shown.

To prove this, Let $(x_i)_{i \in I}, (y_i)_{i \in I} \in K$ be such that $(x_i)_{i \in I} \leq (y_i)_{i \in I}$. Then $x_i \leq y_i$ for all $i \in I$. Let $j \in I$. Then $\Pi_j((x_i)_{i \in I}) = x_j \leq y_j = \Pi_j((y_i)_{i \in I})$, so Π_j is isotone.

Theorem 4.11. Let K be a positively ordered skewsemifield. Then the following statements hold :

1) if K is a subskewsemifield of $\prod_{i \in I} K_i$ where K_i is a totally positively ordered skewsemifield for every $i \in I$ then its positive cone can be represented as the intersection of T_i where

i) T_i are convex normal multiplicative subsemigroups of K^* containing P,

ii) for every $x \in K^*$, $x \notin T_i$ implies that $x^{-1} \in T_i$,

iii) $1 + K \subseteq T_i$ and $K + 1 \subseteq T_i$.

2) If $P = \bigcap_{i \in I} T_i$ where T_i satisfies i) – iii) as above for all $i \in I$ then K is a vector skewsemifield.

Proof 1) Assume K is a subskewsemifield of $\prod_{i \in I} K_i$ where K_i is a totally positively ordered skewsemifield for every $i \in I$. Let $i \in I$. Let $T_i = \Pi_i^{-1}(P_i) \cap K$ where P_i is a positive cone of K_i.

i) Let $x, y \in T_i$. Then $\Pi_i(x), \Pi_i(y) \in P_i$ and $x,y \in K$, so $\Pi_i(xy) = \Pi_i(x)\Pi_i(y) \in P_i$ and $xy \in K$. Then $xy \in \Pi_i^{-1}(P_i) \cap K = T_i$. Next, let $z \in K^*$. Then $zxz^{-1} \in K$ and $\Pi_i(zxz^{-1}) = \Pi_i(z)\Pi_i(x)\Pi_i(y)^{-1} \in P_i$, so $zxz^{-1} \in \Pi_i^{-1}(P_i) \cap K = T_i$. Next, let $a, b \in K$ be such that $a + b = 1$. Then $\Pi_i(a) + \Pi_i(b) = \Pi_i(a + b) = \Pi_i(1) = 1_i$. Thus $ax + by \in K$ and $\Pi_i(ax + by) = \Pi_i(a)\Pi_i(x) + \Pi_i(b)\Pi_i(y) \in P_i$, so $ax + by \in \Pi_i^{-1}(P_i) \cap K = T_i$. Next, let $u \in K$ be such that $x \leq u \leq y$. Since Π_i is isotone, $\Pi_i(x) \leq \Pi_i(u) \leq \Pi_i(y)$. By the o-convexity of P, $\Pi_i(u) \in P$, so $u \in \Pi_i^{-1}(P_i) \cap K = T_i$. Let $p \in P$. Then $1 \leq p$, so $1 = \Pi_i(1) \leq \Pi_i(p)$. Thus $p \in \Pi_i^{-1}(P_i) \cap K = T_i$. Clearly, $0 \in T_i$. Then T_i is convex normal multiplicative subsemigroup of K^* containing P.

ii) Let $x \in K^*$ be such that $x \notin T_i$. Then $\Pi_i(x) \notin P_i$. Since P_i is a total order, $\Pi_i(x) \in (P_i)^{-1}$, so $\Pi_i(x^{-1}) = (\Pi_i(x))^{-1} \in P_i$. Hence $x^{-1}\ \Pi_i^{-1}(P_i) = T_i$.

iii) Let $x \in K$. Then $x + 1 \in P$ and $1 + x \in P$. Since $\Pi_i(P) \subseteq P_i$, $\Pi_i(x + 1) \in P_i$ and $\Pi_i(1 + x) \in P_i$, we get that $1 + x, x + 1 \in \Pi_i^{-1}(P_i) \cap K = T_i$.

Finally, to show that $P = \bigcap_{i \in I} T_i$, let $x \in P$. Let $i \in I$. Since $\Pi_i(P) \subseteq P_i$, $\Pi_i(x) \in P_i$, we get that $x \in \Pi_i^{-1}(P_i) \cap K = T_i$. Hence $x \in \bigcap_{i \in I} T_i$, so $P \subseteq \bigcap_{i \in I} T_i$. Next, let $y \in \bigcap_{i \in I} T_i$. Let $i \in I$. Then $y \in T_i = \Pi_i^{-1}(P_i) \cap K$, so $\Pi_i(y) \in P_i$. Thus $y \in P$, so $\bigcap_{i \in I} T_i \subseteq P$. Hence $P = \bigcap_{i \in I} T_i$.

To prove 2), assume that $P = \bigcap_{i \in I} T_i$. Let $i \in I$ and $N_i = T_i \cap (T_i)^{-1}$. To show that N_i is a convex normal subgroup of K, let $x, y \in N_i$. Then $x, y, x^{-1}, y^{-1} \in T_i$, so $xy^{-1} \in T_i \cap (T_i)^{-1} = N_i$. Let $z \in K^*$. Then $zxz^{-1} \in T_i$ and $(zxz^{-1})^{-1} = zx^{-1}z^{-1} \in T_i$, so $zxz^{-1} \in T_i \cap (T_i)^{-1} = N_i$. Next, let $a, b \in K$ be such that $a + b = 1$. Then $ax + by \in T_i$ and $(ax + by)^{-1} = (ax + by)^{-1}(a + b)$
$= [(ax + by)^{-1}ax]x^{-1} + [(ax + by)^{-1}by]y^{-1} \in T_i$, so $ax + by \in T_i \cap (T_i)^{-1} = N_i$. Let $u \in K$ be such that $x \le u \le y$. Then $y^{-1} \le u^{-1} \le x^{-1}$. By the o-convexity of T_i, $u, u^{-1} \in T_i$, so $u \in T_i \cap (T_i)^{-1} = N_i$. Thus N_i is a convex normal subgroup of K.

Let $K_i = K/N_i$. Then K_i is a skewsemifield for all $i \in I$. Define $f : K \to \prod_{i \in I} K_i$ by $f(x) = (xN_i)_{i \in I}$ for all $x \in K$. Then f is a homomorphism. Since $\bigcap_{i \in I} N_i$
$= \bigcap_{i \in I} [T_i \cap (T_i)^{-1}] = (\bigcap_{i \in I} T_i) \cap (\bigcap_{i \in I} T_i)^{-1} = P \cap P^{-1} = \{1\}$, f is a monomorpism.

Let $i \in I$. Let $P_i = \Pi_i \circ f(T_i)$.

To show that P_i is a multiplicative subsemigroup, let $\alpha, \beta \in P_i$. Then there exist $a, b \in T_i$ such that $\Pi_i \circ f(a) = \alpha$ and $\Pi_i \circ f(b) = \beta$, so $aN_i = \alpha$ and $bN_i = \alpha\beta$. Thus $\alpha\beta = (aN_i)(aN_i) = (abN_i) = \Pi_i \circ f(ab) \in \Pi_i \circ f(T_i)$.

To show that P_i is a conic set, let $\alpha \in P_i \cap (P_i)^{-1}$. Then $\alpha, \alpha^{-1} \in P_i$, so there exist $a, b \in T_i$ such that $\alpha = aN_i$ and $\alpha^{-1} = bN_i$. Then $N_i = \alpha\alpha^{-1} = (aN_i)(bN_i)$

$= (abN_i)$, so $ab \in N_i$. Thus $ab, (ab)^{-1} \in T_i$. Therefore $a^{-1} = b(b^{-1}a^{-1}) = b(ab)^{-1} \in T_i$, so $a \in T_i \cap (T_i) = N_i$. Hence $\alpha = (aN_i) = N_i$, so P_i is a conic set.

To show that P_i is an a-convex normal set, let $\alpha \in P_i$ and $\beta \in K_i \setminus \{ 0 \}$. Then there exist $a \in T_i$ and $b \in K$ such that $\alpha = aN_i$ and $\alpha\beta = bN_i$, so $b \neq 0$. Therefore $\beta\alpha\beta^{-1} = (bN_i)(aN_i)(bN_i)^{-1} = (bab^{-1})N_i = \Pi_i \circ f(bab^{-1}) \in \Pi_i \circ f(T_i)$. Next, to show the a-convexity of P_i, let $\alpha, \beta \in P_i$ and $C, D \in K_i$ be such that $C + D = N_i$. Then there exist $a, b \in T_i$, $c \in C$ and $d \in D$ such that $\alpha = aN_i$, $\beta = bN_i$ and $c + d = 1$. By the a-convexity of T_i, $ca + db \in T_i$. Thus $C\alpha + D\beta = (cN_i)(aN_i) + (dN_i)(bN_i) =(ca + db)N_i = \Pi_i \circ f(ca + db) \in \Pi_i \circ f(T_i)$, so P_i is an a-convex normal set.

To show that P_i is an additive ideal of K_i, let $\alpha \in K_i$. Let $x \in \alpha$. Then $x + 1 \in T_i$ and $1 + x \in Ti$, so $\alpha + N_i = xN_i + N_i = (x + 1)N_i = \Pi_i \circ f(x + 1) \in \Pi_i \circ f(T_i)$ and $N_i + \alpha = N_i + xN_i = (1 + x)N_i = \Pi_i \circ f(1 + x) \in \Pi_i \circ f(T_i)$.

By Theorem 2.9., P_i is a positive cone of K_i. Next, to show that $K_i = P_i \cup (P_i)^{-1} \cup \{ 0 \}$, let $\alpha \in K_i \setminus \{ 0 \}$. Let $x \in \alpha$. Then $x \neq 0$.

Case 1 : $x \in T_i$. Then $\alpha = xN_i = \Pi_i \circ f(x) \in \Pi_i \circ f(T_i) = P_i$.

Case 2 : $x \notin T_i$. Then $x^{-1} \in T_i$, so $\alpha^{-1} = (xN_i)^{-1} = x^{-1}N_i = \Pi_i \circ f(x^{-1}) \in \Pi_i \circ f(T_i) = P_i$.

Hence K_i is a totally positively ordered skewsemifield.

Finally, to show that $f(P) = P_{f(K)}$, let $x \in P = \bigcap_{i \in I} T_i$. Then $x \in T_i$ for every $i \in I$. Let $j \in I$. Then $xN_j = f(x) = f(T_j) = P_j$, so $xN_j \geq N_j$. Therefore $f(x) = (xN_i)_{i \in I} \geq (N_i)_{i \in I}$, so $f(x) \in P_{f(K)}$. Next, let $(xN_i)_{i \in I} \in P_{f(K)}$. Then $(xN_i)_{i \in I} \geq (N_i)_{i \in I}$, so $xN_i \geq N_i$ for all $i \in I$. Hence $xN_i \in P_i = \Pi_i \circ f(T_i)$ for all $i \in I$. Let $j \in I$. Then there exists a $y \in T_j$ such that $xN_j = \Pi_j \circ f(y) = yN_j$, so $y^{-1}x \in N_j = T_j \cap (T_j)^{-1}$. Thus $x = y(y^{-1}x) \in T_j$. Hence $x \in \bigcap_{i \in I} T_i = P$, so $x \geq 1$. Therefore $(xN_i)_{i \in I} = f(x) \in f(P)$, so $f(P) = P_{f(K)}$. Hence $K \cong_o f(K)$, so K is a vector skewsemifield. #

Corollary 4.12. Let K be a positively ordered skewsemifield. If K is a vector skewsemifield then its positive cone P satisfies the property that for every $x_1, \ldots, x_n \in K^*$, $\cap H(P, x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) = P$ where the intersection is to be extended over all possible choices of signs $\varepsilon_1,\ldots,\varepsilon_n \in \{-1, 1\}$.

Proof Assume that K is a vector skewsemifield. By Theorem 4.12., $P = \bigcap_{i\in I} T_i$ where T_i satisfies i) – iii) for all $i \in I$. Let $x_1,\ldots,x_n \in K^*$. By Property ii) of T_i, we can choose $\varepsilon_{1_i},\ldots,\varepsilon_{n_i} \in \{1,-1\}$ such that $x_1^{\varepsilon_{1_i}},\ldots,x_n^{\varepsilon_{n_i}} \in T_i$ for all $i \in I$. Hence $H(P, x_1^{\varepsilon_{1_i}},\ldots,x_n^{\varepsilon_{n_i}}) \subseteq T_i$ for all $i \in I$. So we get that $P \subseteq \cap H(P, x_1^{\varepsilon_1},\ldots,x_n^{\varepsilon_n}) \subseteq \bigcap_{i\in I} H(P, x_1^{\varepsilon_{1_i}},\ldots,x_n^{\varepsilon_{n_i}}) \subseteq \bigcap_{i\in I} T_i = P$ and therefore $\cap H(P, x_1^{\varepsilon_1},\ldots,x_n^{\varepsilon_n}) = P$. #

Lemma 4.13. Let A and B be subskewsemifields of a totally positively ordered skewsemifield K, A_1 and B_1 convex normal subgroup of A and B, respectively. Then $(A_1 \cap B)(A \cap B_1)$ is a convex normal subgroup of $A \cap B$ and $(A \cap B)A_1$, $(A \cap B)B_1$ are subskewsemifields of K.

Proof By Lemma 1.57., $(A_1 \cap B)(A \cap B_1)$ is an a-convex normal subgroup of $(A \cap B)$. Since A_1 is a convex subset of A, by Lemma 2.21., $A_1 \cap B$ is also a convex subset of $A \cap B$ therefore $A_1 \cap B$ is an o-convex subset of $A \cap B$. Similarly, $A \cap B_1$ is an a-convex subset of $A \cap B$. By Remark 4.2., 3), $(A_1 \cap B)(A \cap B_1)$ is an o-convex subset of $A \cap B$. Then $(A_1 \cap B)(A \cap B_1)$ is a convex normal subgroup of $A \cap B$. #

Proposition 4.14. Let A and B be subskewsemifields of a totally positively ordered skewsemifield K, A_1 and B_1 convex normal subgroups of A and B, respectively. Then $(A \cap B)A_1/(A \cap B_1)A_1 \cong_o (A \cap B)B_1/(A_1 \cap B)B_1$.

Proof Let f be the epimorphism defined in the proof of Proposition 1.58. To show that $f(P_{(A \cap B_1)A_1}) \subseteq P_{(A \cap B)/(A_1 \cap B)(A \cap B_1)}$, let $c \in A \cap B$ and $a_1 \in A_1$ be such that $ca_1 \geq 1$.

Case 1 : $c \le 1$. Then $f(ca_1) = c[(A_1 \cap B)(A \cap B_1)] \ge (A_1 \cap B)(A \cap B_1)$.

Case 2 : $c \le 1$. Then $1 \le ca_1 \le a_1$, so $(a_1)^{-1} \le c \le 1$. By the o-convexity of A_1, $c \in A_1$, so $c \in A_1 \cap B$. Therefore $f(ca_1) = c[(A_1 \cap B)(A \cap B_1)] = (A_1 \cap B)(A \cap B_1)$. Hence $f(P_{(A \cap B_1)A_1}) \subseteq P_{(A \cap B)/(A_1 \cap B)(A \cap B_1)}$.

Next, to show that $P_{(A \cap B)/(A_1 \cap B)(A \cap B_1)} \subseteq f(P_{(A \cap B_1)A_1})$, let $c[(A_1 \cap B)(A \cap B_1)] \in P_{(A \cap B)/(A_1 \cap B)(A \cap B_1)}$. We must to show that there exist $x \in A \cap B$ and $y \in A_1$ such that $xy \ge 1$ and $f(xy) = c[(A_1 \cap B)(A \cap B_1)]$. Since $c[(A_1 \cap B)(A \cap B_1)] \ge (A_1 \cap B)(A \cap B_1)$, there exist $a_1, a_2 \in A_1 \cap B$ and $b_1, b_2 \in A \cap B_1$ such that $ca_1b_1 \ge a_2b_2$., so $(a_2)^{-1}ca_1b_1(b_2)^{-1} \ge 1$. Since A_1 is a normal subset of A, there exists a $a_3 \in A_1$ such that $(a_1)^{-1}c = ca_3$. Let $x = cb_1(b_2)^{-1}$ and $y = [b_1(b_2)^{-1}]^{-1}a_3a_1[b_1(b_2)^{-1}]$. Then $x \in A \cap B$. Since A_1 is a normal subset of A, $y = [b_1(b_2)^{-1}]^{-1}a_3a_1[b_1(b_2)^{-1}] \in A_1$. Then xy $=[cb_1(b_2)^{-1}][b_1(b_2)^{-1}]^{-1}a_3a_1[b_1(b_2)^{-1}] = [ca_3a_1[b_1(b_2)^{-1}] = (a_2)^{-1}ca_1[b_1(b_2)^{-1}] \ge 1$ and $f(xy) = x[(A_1 \cap B)(A \cap B_1)] = cb_1(b_2)^{-1}[(A_1 \cap B)(A \cap B_1)]$ $= c[(A_1 \cap B)(A \cap B_1)]b_1(b_2)^{-1}[(A_1 \cap B)(A \cap B_1)] = c[(A_1 \cap B)(A \cap B_1)]$. Thus $f(P_{(A \cap B_1)A_1}) \supseteq P_{(A \cap B)/(A_1 \cap B)(A \cap B_1)}$, so $f(P_{(A \cap B_1)A_1})$ $= P_{(A \cap B)/(A_1 \cap B)(A \cap B_1)}$. By Proposition 1.58., $(A \cap B_1)A_1 = \ker f$. Then $(A \cap B)A_{1/(A \cap B_1)A_1} \cong_o (A \cap B)_{/(A_1 \cap B)(A \cap B_1)}$. Similarly, we get that $(A \cap B)B_{1/(A_1 \cap B)B_1} \cong_o (A \cap B)_{/(A_1 \cap B)(A \cap B_1)}$. Hence $(A \cap B)A_{1/(A \cap B_1)A_1} \cong_o (A \cap B)B_{1/(A_1 \cap B)B_1}$. #

<u>Definition 4.15.</u> Let K be a positively ordered skewsemifield. K is said to be Archimedian if and only if for all $x, y \in K^*$, if $x < y$ then

1) there exists an $n \in \mathbf{Z}^+$ such that $y < nx$ and
2) there exists an $n \in \mathbf{Z}$ such that $y < x^n$ if $x \ne 1$.

<u>Theorem 4.16.</u> Let K be an Archimedian totally positively ordered skewsemifield such that $1+1 \ne 1$ and $K_0 \subseteq K$, the prime skewsemifield of K is order isomophic

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

to $\mathbf{Q}_0^+$. Then K can be embedded into a complete totally positively ordered skewsemifield.

Proof The proof is exactly the same as the proof for semifield given in [3] pp. 80 – 85.

Corollary 4.17. A complete totally positively ordered skewsemifield is multiplicative commutative and addition is either commutative or for every $x \in K^*$, there exists a unique $y \in K$ such that $x + y \neq y + x$.

REFERENCES

Birkhoff, G. Lattice Theory. American Mathematical Society Colloquium Publications Vol. 25, 1940.

Fuchs, L. Partially Ordered Algebraic Systems. Oxford : Pergamon Press, 1963.

Namnak, C. "Positive Ordered 0-Semifields" Master' s thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 1996.

Phayakul, J. "Foundations of Some Partially Ordered Semirings" Master' s thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 1991.

Satravaha, P. "Extensions and Decompositions of Semirings" Master' s thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 1989.

Wisetmongkolchai, P. "Classification of Some Complete Ordered Semirings" Master' s thesis, Department of Mathematics, Graduate School, Chulalongkorn University, 1990.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย