CHAPTER III

POSITIVE LATTICE 0-SKEWSEMIFIELDS

Definition 3.1. Let S be a positively ordered semiring. S is said to be a positive lattice semiring if and only if the partial order of S is a lattice, that is for all $x, y \in S$, $x \vee y$ and $x \wedge y$ exist.

Examples 3.2. 1) $\mathbf{Q}_0^+$, $\mathbf{R}_0^+$ are positive lattice skewsemifields.

2) Let G be a lattice group. Let $K = G \cup \{a\}$ where a is an element not representing in G. Define + on K by $x + y = x \vee y$ and $x + a = x = a + x$ for all $x, y \in K$ and define $ax = a = xa$ and $a \leq x$, for every $x \in K$. Then we have that K is a positive lattice skewsemifield.

3) Let K be a positive lattice skewsemifield. Then $(K, +^*, \bullet, \leq)$ is a positive lattice skewsemifield such that $x +^* x = x$ for all $x \in K$ if we define $x +^* y = x \vee y$ for all $x, y \in K$.

Remark 3.3. Let K be a positively ordered skewsemifield. Then the following statements hold :

1) For all $x, y \in K$, if $x \vee y$ exists then $xw \vee yw$ and $wx \vee wy$ exist for every $w \in K$. Morever, $(x \vee y)w = xw \vee yw$ and $w(x \vee y) = wx \vee wy$.

2) For all $x, y \in K$, if $x \wedge y$ exists then $xw \wedge yw$ and $wx \wedge wy$ exist for every $w \in K$. Morever, $(x \wedge y)w = xw \wedge yw$ and $w(x \wedge y) = wx \wedge wy$.

Proof 1) Let $x, y \in K$ be such that $x \vee y$ exists. Let $w \in K$. If $w = 0$ then done. So suppose that $w \neq 0$. Since $x \leq x \vee y$ and $y \leq x \vee y$, $xw \leq (x \vee y)w$ and $yw \leq (x \vee y)w$. Hence $(x \vee y)w$ is an upper bound of xw and yw. Let $z \in K$ be such that $xw \leq z$ and $yw \leq z$. Then $x \leq zw^{-1}$ and $y \leq zw^{-1}$, $x \vee y \leq zw^{-1}$, so $(x \vee y)w \leq z$.

Hence $(x \vee y)w = xw \vee yw$. Similarly, $w(x \vee y) = wx \vee wy$.

2) Dual to 1). #

Theorem 3.4. ([1]) Every positive lattice group G is distributive, that is for all $x, y, z \in G$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Proof See [1], pp. 294. #

Proposition 3.5. Let K be a positively ordered skewsemifield. Then the following statements are equivalent.

1) K is a lattice.

2) For every $x \in K$, $x \vee 1$ exists.

3) For every $x \in K$, $x \wedge 1$ exists.

4) P is a lattice where P is the positive cone of K.

Proof 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3) Let $x \in K$. If $x = 0$ then $x \wedge 1 = 0$. Suppose that $x \neq 0$. By 2), $x^{-1} \vee 1$ exists, say y. Then $x^{-1} \leq y$ and $1 \leq y$, so $y^{-1} \leq x$ and $y^{-1} \leq 1$. Let $w \in K^*$ be such that $w \leq 1$ and $w \leq x$. Then $1 \leq w^{-1}$ and $x^{-1} \leq w^{-1}$, so $y = x^{-1} \vee 1 \leq w$. Thus $w \leq y^{-1}$, hence $x \wedge 1 = y^{-1}$.

3) $\Rightarrow$ 4) Let $x, y \in P$. Then $xy^{-1} \in K$. By 3), $xy^{-1} \wedge 1$ exists. By Remark 3.3., 2), $x \wedge y = (xy^{-1} \wedge 1)y$. Since $x \geq 1$ and $y \geq 1$, $x \wedge y \geq 1$, so $x \wedge y \in P$. Next, we shall show that $x \vee y$ exists. By 3), $x^{-1} \wedge y^{-1}$ exists, say w. Since $(x+y)^{-1} \leq x^{-1}$ and $(x+y)^{-1} \leq y^{-1}$, $(x+y)^{-1} \leq w$, so $w \neq 0$. Since $x^{-1} \geq w$, $w^{-1} \geq x$, so $w^{-1} \in P$. Let $z \in K$ be such that $x \leq z$ and $y \leq z$. Then $z^{-1} \leq x^{-1}$ and $z^{-1} \leq y^{-1}$, so $z^{-1} \leq x^{-1} \wedge y^{-1} = w$. Therefore $w^{-1} \leq z$, so $x \vee y = w^{-1} \in P$. Hence P is a lattice.

4) $\Rightarrow$ 1) Let $x, y \in K$. If $x = 0$ or $y = 0$ then done. Suppose that $x \neq 0$ and $y \neq 0$. Then $xy^{-1} \in K^*$. By Remark 2.8., 6), there exist $p, q \in P$ such that $xy^{-1} = pq^{-1}$. By 4), $p \vee q, p \wedge q$ exist. By Remark 3.3., $xy^{-1} \vee 1 = pq^{-1} \vee 1$

$= (p \vee q)q^{-1}$ and $xy^{-1} \wedge 1 = pq^{-1}\ 1 = (p \wedge q)q^{-1}$. Hence $x \vee y = (xy^{-1} \vee 1)y$ and so we get that $x \wedge y = (xy^{-1} \wedge 1)y$. #

Proposition 3.6. Let K be a positive lattice skewsemifield. Then the following statements hold :

1) For every nonzero element $x \in K$, $x = pq^{-1}$ for some $p, q \in P$ such that $p \wedge q = 1$.

2) For all $x, y \in K^*$, $(x \vee y)^{-1} = x^{-1} \wedge y^{-1}$ and $(x \wedge y)^{-1} = x^{-1} \vee y^{-1}$.

3) For all $x, y \in K^*$, $x \vee y = x(x \wedge y)^{-1}y$ and $x \wedge y = x(x \vee y)^{-1}y$.

4) For every $x \in K$, $x = (x \vee 1)(x \wedge 1)$.

5) For all $x, y, z \in K$, if $z \neq 0$ then $[(x \vee y) \wedge z]z^{-1}[(x \wedge y) \vee z]$ $= [(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z]$.

Proof 1) Let $x \in K^*$. By Remark 2.8., 6), there exist $a, b \in P$ such that $x = ab^{-1}$. Let $p = a(a \wedge b)^{-1}$ and $q = b(a \wedge b)^{-1}$. Then $p, q \in P$ and $p \wedge q$ $= [a(a \wedge b)^{-1}] \wedge [b(a \wedge b)^{-1}] = (a \wedge b)(a \wedge b)^{-1} = 1$. Therefore $x = ab^{-1}$ $= [a(a \wedge b)^{-1}][(a \wedge b)^{-1}b^{-1}] = pq^{-1}$.

2) Define $f : K^* \to K^*$ by $f(x) = x^{-1}$ for every $x \in K^*$. Then f is a bijection. Let $a, b \in K^*$. Then $a \vee b \in K^*$. By the definition of f, $f(a \vee b) = (a \vee b)^{-1}$.
Claim that $f(a \vee b) = f(a) \wedge f(b) = a^{-1} \wedge b^{-1}$.
By $a \leq (a \vee b)$ and $b \leq (a \vee b)$, $f(a) \geq f(a \vee b)$ and $f(b) \geq f(a \vee b)$. Let $z \in K$ be such that $z \leq f(a)$ and $z \leq f(b)$. Since f is onto, there exists a $w \in K^*$ such that $z = f(w)$. Then $w = f^{-1}(f(w) = f^{-1}(z) \geq f^{-1}(f(a)) = a$. Similarly, $w \geq b$. Then $w \geq (a \vee b)$. Thus $z = f(w) \leq f(a \vee b)$, so we have the claim.
Therefore $(a \vee b)^{-1} = f(a \vee b) = a^{-1} \wedge b^{-1}$. Dually, $(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$.

3) Let $x, y \in K^*$, $x(x \wedge y)^{-1}y = x(x^{-1} \vee y^{-1})y = (1 \vee xy^{-1})y = y \vee x = x \vee y$.
Similarly, $x \wedge y = x(x \vee y)^{-1}y$.

4) Follows directly from 3).

5) Let $x, y, z \in K$ be such that $z \neq 0$. Suppose that $x \wedge y \neq 0$.

By 3), $[(x \vee y) \wedge z] = (x \vee y)[(x \vee y) \wedge z]^{-1}z \neq 0$. Then $(x \wedge y) \vee [(x \vee y) \wedge z]$

$= (x \wedge y)[(x \wedge y) \wedge ((x \vee y) \wedge z)]^{-1}[(x \vee y) \wedge z] = (x \wedge y)[(x \wedge y) \wedge z]^{-1}[(x \vee y) \wedge z]$

$= (x \wedge y)(x \wedge y)^{-1}[(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z] = [(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z]$.

Dually, $(x \vee y) \wedge [(x \wedge y) \vee z] = [(x \vee y) \wedge z]z^{-1}[(x \wedge y) \vee z]$.

By Remark 1,5., $(x \wedge y) \vee [(x \vee y) \wedge z] = (x \vee y) \wedge [(x \wedge y) \vee z]$.

Hence $[(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z] = [(x \vee y) \wedge z]z^{-1}[(x \wedge y) \vee z]$. #

Note that for all nonzero elements x, y in a positive lattice skewsemifileld K, $x \vee y$ and $x \wedge y$ are non zero.

Proposition 3.7. Let K be a positive lattice skewsemifield. Then the following statements hold : for all $x, y, z \in K$,

1) if $x \leq y$ then $x \vee z \leq y \vee z$ and $x \wedge z \leq y \wedge z$,

2) $x + (y \wedge z) \leq (x + y) \wedge (x + z)$ and $(y \wedge z) + x \leq (y + x) \wedge (z + x)$,

3) $x + (y \vee z) \geq (x + z) \vee (y + z)$ and $(y \vee z) + x \geq (y + x) \vee (z + x)$

4) $(x + y) \wedge z \leq (x \wedge z) + (y \wedge z)$ and $(x + y) \vee z \leq (x \vee z) + (y \vee z)$.

Proof Let $x, y, z \in K$.

1) Obvious.

2) By $y \wedge z \leq y$ and $y \wedge z \leq z$, $x + (y \wedge z) \leq x + y$ and $x + (y \wedge z) \leq x + z$. Hence $x + (y \wedge z) \leq (x + y) \wedge (x + z)$. Similarly, $(y \wedge z) + x \leq (y + x) \wedge (z + x)$.

3) Dual to 2.

4) Suppose that $x, y, z \neq 0$. Since $x \leq x + y$ and (by 1)), $x \vee z \leq (x + y) \vee z$, we get that $[(x + y) \vee z]^{-1} \leq (x \vee z)^{-1}$. Therefore $x[(x + z) \vee z]^{-1}z \leq x(x \vee z)^{-1}z$. Similarly, $y[(x + y) \vee z]^{-1}z \leq y(y \vee z)^{-1}z$. Then $(x + y) \wedge z = (x + y)[(x + y) \vee z]^{-1}z$

$= x[(x + y) \vee z]^{-1}z + y[(x + y) \vee z]^{-1}z \leq x[x \vee z]^{-1}z + y[y \vee z]^{-1}z$

$= x(x^{-1} \wedge z^{-1})z + y(y^{-1} \wedge z^{-1})z = (z \wedge x) + (z \wedge y) = (x \wedge z) + (y \wedge z)$.

Since $x \leq x \vee z$ and $y \leq y \vee z$, $(x + y) \leq (x \vee z) + (y \vee z)$.

Clearly, $z \leq (x \vee z) + (y \vee z)$, so $(x + y) \vee z \leq (x \vee z) + (y \vee z)$. #

Theorem 3.8. ([2]) Let K be a positive lattice skewsemifield and $a_1,\dots, a_m$, $b_1,\dots, b_n \in P$ such that $a_1 \dots a_m = b_1 \dots b_n$. Then there exist elements $c_{ij} \in P$ for all $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$ satisfying

1) $a_i = c_{i1} \dots c_{in}$ $i \in \{1,\dots,m\}$,
2) $b_j = c_{1j} \dots c_{mj}$ $j \in \{1,\dots,n\}$,
3) $c_{i+1,j} \dots c_{mj} \wedge c_{i,j+1} \dots c_{in} = 1$ for all $i < m$ and $j < n$.

Proof See [2], pp. 68. #

Corollary 3.9. If $a, b_1,\dots, b_n$ are in the positive cone of a positive lattice skewsemifield K such that $a \le b_1 \dots b_n$ then there exist $a_1,\dots, a_n \in P$ satisfying $a = a_1 \dots a_n$ with $a_i \le b_i$ for every $i \in \{1,\dots,n\}$.

Proof It follows from Theorem 3.8. #

Proposition 3.10. Let K be a skewsemifield and $P \subseteq K^*$ positive cone. Then the partial order on K induced by P is a positive lattice if and only if for every $x \in K^*$, there exists a $z \in P$ satisfying the following conditions:

1) $zx^{-1} \in P$ and
2) for every $w \in P$, $wx^{-1} \in P$ implies that $wz^{-1} \in P$.

Proof Let K be a positive lattice skewsemifield and let $x \in K^*$. Let $z = x \vee 1$. Then $z \ge x$ and $z \ge 1$, so $zx^{-1}, z \in P$. Let $w \in P$ be such that $wx^{-1} \in P$. Then $w \ge x$ and $w \ge 1$, so $w \ge (x \vee 1) = z$. Thus $wz^{-1} \in P$.

Conversely, assume that for all $x \in K^*$, there exists a $z \in P$ such that satisfying conditions 1) and 2). Let $a \in K$. If $a = 0$ then $a \vee 1 = 0$, so done. Suppose that $a \ne 0$. By assumption, there exists a $z \in P$ satisfying conditions 1) and 2). Then $z \ge x$ and $z \ge 1$. Let $w \in K$ be such that $w \ge x$ and $w \ge 1$. Then $wx^{-1}, w \in P$, so $wz^{-1} \in P$. Therefore $w \ge z$, so $x \vee 1 = z$. Hence K is a lattice. #

Theorem 3.11. Let S be a positive lattice semiring with multiplicative zero 0 satisfying the M.C. property and suppose that $(S, \bullet)$ satisfies the right [left] Ore condition. If $\leq$ is M.R. then S can be embedded into a positive lattice skewsemifield.

Proof By Theorem 2.12., we have that $K = S \times (S\setminus\{0\})/_\sim$ is the positively ordered skewsemifield of a right quotients of S. Let $\alpha = [(a,b)] \in K^*$. Let z $= [(a \vee b,b)] = i(a \vee b)i(b)^{-1}$. Since $(a \vee b) \geq b$, $z \in P$. Since $z\alpha^{-1} = [(a \vee b,a)]$ $= i(a \vee b)i(a)^{-1}$, so $z\alpha^{-1} \in P$. Let $w = i(u)i(v)^{-1} \in P$ be such that $w\alpha^{-1} \in P$. Since $v, b \in S\setminus\{0\}$, there exist $x, y \in S\setminus\{0\}$ such that $vx = by$, so $w\alpha^{-1} = [(ux,ay)]$ $= i(ux)i(ay)^{-1}$ and $wz^{-1} = [(ux,(a \vee b)y)] = i(ux)i([(a \vee b)y]^{-1})$. Since $w\alpha^{-1} \in P$ and by claim in the proof of Theorem 2.12., we get that $ux \geq ay$. Since $wz^{-1} \in P$, $u \geq v$, so $ux \geq vx$. Therefore $ux \geq (ay \vee vx) = (ay \vee by) = (a \vee b)y$, so $wz^{-1} \in P$. Hence K is a lattice. #

Definition 3.12. Let K be a positive lattice skewsemifield and $x \in K^*$. The absolute value of x, denoted by $|x|$, is defined to be $x \vee x^{-1}$.

In [2], pp. 76 we have the following elementary properties of the absolute : for all $x, y \in K^*$,

1) $|x| \geq 1$ and $|x| = |x^{-1}|$,
2) $|x| = 1$ if and only if $x = 1$,
3) $|xy^{-1}| = (x \vee y)(x \wedge y)^{-1}$,
4) $|x| = (x \vee 1)(x \wedge 1)^{-1}$,
5) $|x^n| = |x|^n$ for all $n \in \mathbf{Z}^+$ and
6) $|xy| \leq |x|\,|y|\,|x|$.

Proposition 3.13. Let K be a positive lattice skewsemifield and $x, y, z \in K^*$. Then the following properties hold :

1) $|(x \vee z)(y \vee z)^{-1}||(x \wedge z)(y \wedge z)^{-1}| = |xy^{-1}|$,

2) $|(x \vee z)(y \vee z)^{-1}| \leq |xy^{-1}|$ and $|(x \wedge z)(y \wedge z)^{-1}| \leq |xy^{-1}|$,

3) $|x + y| \leq |x| + |y|$,

4) $|(x + z)(y + z)^{-1}| \leq |xy^{-1}|$.

Proof Let $x, y, z \in K^*$.

1) $|(x \vee z)(y \vee z)^{-1}||(x \wedge z)(y \wedge z)^{-1}|$

$= [(x \vee z) \vee (y \vee z)][(x \vee z) \wedge (y \vee z)]^{-1}[(x \wedge z) \vee (y \wedge z)][(x \wedge z) \wedge (y \wedge z)]^{-1}$

$= [(x \vee y) \vee z][(x \wedge y) \vee z]^{-1}[(x \vee y) \wedge z][z \wedge (x \wedge y)]^{-1}$

$= (x \vee y)[(x \vee y) \wedge z]^{-1}z[(x \wedge y) \vee z]^{-1}[(x \vee y) \wedge z]z^{-1}[(x \wedge y) \vee z)](x \wedge y)^{-1}$

$= (x \vee y)([(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z])^{-1}([(x \vee y) \wedge z]z^{-1}[(x \wedge y) \vee z])(x \wedge y)^{-1}$

$= (x \vee y)([(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z])^{-1}([(x \wedge y) \vee z]z^{-1}[(x \vee y) \wedge z])(x \wedge y)^{-1}$

$= (x \vee y)(x \wedge y)^{-1} = |xy^{-1}|$.

2) Since $1 \leq |(x \vee z)(y \vee z)^{-1}|$ and $1 \leq |(x \wedge z)(y \wedge z)^{-1}|$, by 1), we get that $|(x \vee z)(y \vee z)^{-1}| \leq |xy^{-1}|$ and $|(x \wedge z)(y \wedge z)^{-1}| \leq |xy^{-1}|$.

3) Since $x \leq |x|$ and $y \leq |y|$, $x + y \leq |x| + |y|$. Since $x \leq x + y$ and $y \leq x + y$, $(x + y)^{-1} \leq x^{-1}$ and $(x + y)^{-1} \leq y^{-1}$, so $(x + y)^{-1} \leq (x + y)^{-1} + (x + y)^{-1}$ $\leq x^{-1} + y^{-1} \leq |x| + |y|$. Hence $|x + y| = (x + y) \vee (x + y)^{-1} \leq |x| + |y|$.

4) $|(x + z)(y + z)^{-1}| = [(x + z) \vee (y + z)][(x + z) \wedge (y + z)]^{-1}$ $\leq [(x \vee y) + z][(x + z) \wedge (y + z)]^{-1} \leq [(x \vee y) + z)][(x \wedge y) + z)]^{-1}$.

Claim that $[(x \vee y) + z)][(x \wedge y) + z)]^{-1} \leq (x \vee y)(x \wedge y)^{-1}$.

Since $(x \wedge y) \leq (x \vee y)$, $(x \vee y)^{-1} \leq (x \wedge y)^{-1}$, we get that $(x \vee y)^{-1}z \leq (x \wedge y)^{-1}z$.

Hence $(x \vee y)^{-1}[(x \vee y) + z)] = 1 + (x \vee y)^{-1}z \leq 1 + (x \wedge y)^{-1}z = (x \wedge y)^{-1}[(x \wedge y) + z)]$ therefore $[(x \vee y) + z)][(x \wedge y) + z)]^{-1} \leq (x \vee y)(x \wedge y)^{-1}$, so we have the claim.

Thus $|(x + z)(y + z)^{-1}| \leq [(x \vee y) + z)][(x \wedge y) + z)]^{-1} \leq (x \vee y)(x \wedge y)^{-1} = |xy^{-1}|$. #

Proposition 3.14. Let K be a complete positive lattice skewsemifield. Then the following statements hold :

1) Let $x_\alpha \in K$ for all $\alpha \in I$, if $\bigvee_{\alpha \in I} x_\alpha$ exists then $\bigvee_{\alpha \in I} x_\alpha w$ and $\bigvee_{\alpha \in I} wx_\alpha$ exist

for all $w \in K$. Morever, $(\bigvee_{\alpha \in I} x_\alpha)w = \bigvee_{\alpha \in I} x_\alpha w$ and $w(\bigvee_{\alpha \in I} x_\alpha) = \bigvee_{\alpha \in I} wx_\alpha$.

2) Let $x_\alpha \in K$ for all $\alpha \in I$, if $\bigwedge_{\alpha \in I} x_\alpha$ exists then $\bigwedge_{\alpha \in I} x_\alpha w$ and $\bigwedge_{\alpha \in I} wx_\alpha$ exist for all $w \in K$. Morever, $(\bigwedge_{\alpha \in I} x_\alpha)w = \bigwedge_{\alpha \in I} x_\alpha w$ and $w(\bigwedge_{\alpha \in I} x_\alpha) = \bigwedge_{\alpha \in I} wx_\alpha$.

3) Let $x_\alpha \in K$ for all $\alpha \in I$, if $\bigvee_{\alpha \in I} x_\alpha$ exists then $w + (\bigvee_{\alpha \in I} x_\alpha) \leq \bigvee_{\alpha \in I} (w + x_\alpha)$ and $(\bigvee_{\alpha \in I} x_\alpha) + w \leq \bigvee_{\alpha \in I} (x_\alpha + w)$ for all $w \in K$.

4) Let $x_\alpha \in K$ for all $\alpha \in I$, if $\bigwedge_{\alpha \in I} x_\alpha$ exists then $w + (\bigwedge_{\alpha \in I} x_\alpha) \leq \bigwedge_{\alpha \in I} (w + x_\alpha)$ and $(\bigwedge_{\alpha \in I} x_\alpha) + w \leq \bigwedge_{\alpha \in I} (x_\alpha + w)$ for all $w \in K$.

Proof Let $x_\alpha \in K$ for all $\alpha \in I$.

1) Assume that $\bigvee_{\alpha \in I} x_\alpha$ exists. Let $w \in K$. If $w = 0$ then done. Suppose that $w \neq 0$. Let $\alpha_0 \in I$. Then $x_{\alpha_0} \leq \bigvee_{\alpha \in I} x_\alpha$, so $wx_{\alpha_0} \leq w(\bigvee_{\alpha \in I} x_\alpha)$. Hence $w(\bigvee_{\alpha \in I} x_\alpha)$ is an upper bound of $\{wx_\alpha / \alpha \in I\}$. Therefore $\bigvee_{\alpha \in I} (wx_\alpha)$ exists and $\bigvee_{\alpha \in I} (wx_\alpha) \leq w(\bigvee_{\alpha \in I} x_\alpha)$. Let $z \in K$ be such that $\bigvee_{\alpha \in I} (wx_\alpha) \leq z$. Let $\alpha_0 \in I$. Then $wx_{\alpha_0} \leq \bigvee_{\alpha \in I} (wx_\alpha) \leq z$, so $x_{\alpha_0} \leq w^{-1}z$. Hence and $\bigvee_{\alpha \in I} x_\alpha \leq w^{-1}z$, so $w(\bigvee_{\alpha \in I} x_\alpha) \leq z$. Hence $w(\bigvee_{\alpha \in I} x_\alpha) = \bigvee_{\alpha \in I} (wx_\alpha)$. Similary, $(\bigvee_{\alpha \in I} x_\alpha)w = \bigvee_{\alpha \in I} (x_\alpha w)$.

2) Dual to 1.

3) Assume that $\bigvee_{\alpha \in I} x_\alpha$ exists. Let $w \in K$. Let $\alpha_0 \in I$. Then $x_{\alpha_0} \leq \bigvee_{\alpha \in I} x_\alpha$, so $w + x_{\alpha_0} \leq w + (\bigvee_{\alpha \in I} x_\alpha)$. Hence $w + (\bigvee_{\alpha \in I} x_\alpha)$ is an upper bound of $\{w + x_\alpha / \alpha \in I\}$. Therefore $\bigvee_{\alpha \in I} (w + x_\alpha)$ exists and $\bigvee_{\alpha \in I} (w + x_\alpha) \leq w + (\bigvee_{\alpha \in I} x_\alpha)$. Similarly, $\bigvee_{\alpha \in I} (x_\alpha + w) \leq (\bigvee_{\alpha \in I} x_\alpha) + w$.

4) Dual to 3. #

Definition 3.15. Let K be a positive lattice skewsemifield and A a convex normal subgroup of K. A is said to be an L-ideal if for every $x \in A$, $x \vee 1 \in A$ and $x \wedge 1 \in A$.

Remark 3.16. Let K be a positive lattice skewsemifield. Then following statements clearly hold :

1) { 1 } and K^* are trivial L-ideals of K.

2) The intersection of a family of L-ideals of K is an L-ideal of K. Also the union of an increasing chain of L-ideals is an L-ideal.

3) Let A be a convex normal subgroup of K. Then A is an L-ideal of K if and only if $x \vee 1 \in A$ for every $x \in A$.

Proposition 3.17. Let K be a positive lattice skewsemifield and $A \subseteq K$. Then A is an L-ideal if and only if it is an a-convex normal subgroup of K such that for all $a \in A$ and $x \in K$, if $|x| \leq |a|$ then $x \in A$.

Proof Let A be an ideal of K. Let $a \in A$ and $x \in K$ be such that $|x| \leq |a|$. Then $x, x^{-1} \leq |a|$. so $|a|^{-1} \leq x \leq |a|$. By the o-convexity of A, $x \in A$.

Conversely, to show the o-convexity of A, let $x, y \in A$ and $z \in K$ be such that $x \leq z \leq y$. Then $1 \leq zx^{-1} \leq yx^{-1}$, so $|zx^{-1}| = zx^{-1} \leq yx^{-1} = |yx^{-1}|$. By assumption, $zx^{-1} \in I$, so $z \in A$. Next, let $x \in A$. Since $1 \leq |x|$ and $x \leq |x|$, $|x \vee 1| = x \vee 1 \leq |x|$, so $x \vee 1 \in A$. Hence A is an L-ideal of K. #

Corollary 3.18. Let K be a positive lattice skewsemifield and A an L-ideal of K. Then for all $x, y, z \in K^*$, $xy^{-1} \in A$ implies that $(x \vee z)(y \vee z)^{-1} \in A$ and $(x \wedge z)(y \wedge z)^{-1} \in A$.

Proof Let $x, y, z \in K^*$ be such that $xy^{-1} \in A$. By Proposition 3.13., 2), $|(x \vee z)(y \vee z)^{-1}| \leq |xy^{-1}|$ and $|(x \wedge z)(y \wedge z)^{-1}| \leq |xy^{-1}|$. By Proposition 3.17., $(x \vee z)(y \vee z)^{-1} \in A$ and $(x \wedge z)(y \wedge z)^{-1} \in A$. #

Proposition 3.19. Let A and B be L-ideals of a positive lattice skewsemifield K. Then AB is an L-ideal of K which is the smallest L-ideal containing A and B.

Proof By Remark 1.37., 2), AB is an a-convex normal subgroup of K. Let $x \in A, y \in B, z \in K$ be such $|z| \leq |xy|$. Then $|xy| \leq |x||y||x|$. We must show that $z \in AB$. By Corollary 3.9., there exist $a, b, c \in P$ such that $a \leq |x|$, $b \leq |y|$, $c \leq |x|$ and $|z| = abc$. By Proposition 3.17., $a, c \in A$ and $b \in B$. Since B is a normal subset of K, there exists a $d \in B$ such that $bc = cd$, so $|z| = abc = acd \in AB$. Since $1 \leq (z \vee 1)$ and $1 \leq |z|$, $|z \vee 1| = z \vee 1 \leq |z| = |z| \vee |z^{-1}| = ||z||$. By using the same proof in a manner similar to the above, we get that $z \vee 1 \in AB$. Since $|z| = (z \vee 1)(z \wedge 1)^{-1}$, $(z \wedge 1)$ AB, so $z = (z \vee 1)(z \wedge 1) \in AB$. By Proposition 3.17., AB is an L-ideal of K.

Next, let D be an L-ideal of K such that $A, B \subseteq D$. Let $a \in A$ and $b \in B$. Then $ab \in D$, so $AB \subseteq D$. Therefore AB is an L-ideal of K which is the smallest L-ideal containing A and B. #

Let $\mathcal{C}$ be the set of all L-ideals of a positive lattice skewsemifield K. Let $A, A' \in \mathcal{C}$. Then $A \vee A' = AA'$ and $A \wedge A' = A \cap A'$. Hence $\mathcal{C}$ is a lattice. Morever, we shall show that $\mathcal{C}$ is a distributive lattice.

To prove this, let $A, B, C \in \mathcal{C}$. Let $a \in A \cap BC$. Then $a \in A$ and $a \in BC$, so $|a| \in A$ and $|a| \in BC$. Thus there exist $b \in B$ and $c \in C$ such that $|a| = bc$. Let $x = |a| \wedge (1 \vee bc)$, $y = |a| \wedge (1 \vee c)$ and $z = |a| \wedge 1$. Then $x = |a|$, so $1 \leq y \leq |a|$ and $z = 1$. By the o-convexity of A, $x, y, z \in A$. Since $(bc)c^{-1} \in B$ and by Corollary 3.18., $(1 \vee bc)(1 \vee c)^{-1} \in B$, we get that $xy^{-1} = [|a| \wedge (1 \vee bc)][|a| \wedge (1 \vee c)]^{-1} \in B$. Then $xy^{-1} \in A \cap B$. Since $1 \vee c \in C$ and (by Corollary 3.18.), $yz = yz^{-1}$ $= [|a| \wedge (1 \vee c)][|a| \wedge 1]^{-1} \in C$, we get that $yz \in A \cap C$. Therefore $|a| = x = xz$ $= (xy^{-1})(yz) \in (A \cap B)(A \cap C)$. Since $a^{-1} \leq |a|$, $|a|^{-1} \leq a \leq |a|$, we get that $a \in (A \cap B)(A \cap C)$, so $A \cap BC \subseteq (A \cap B)(A \cap C)$. Clearly, $(A \cap B)(A \cap C)$ $\subseteq A \cap BC$. Therefore $A \wedge (B \vee C) = A \cap BC = (A \cap B)(A \cap C) = (A \wedge B) \vee (A \wedge C)$, hence $\mathcal{C}$ is a distributive lattice.

Definition 3.20. Let K and M be positive lattice skewsemifields. A function

$f: K \rightarrow M$ is called an <u>L-homomorphism</u> of K into M if and only if f is a homomorphism and for all $x, y \in K$, $f(x \vee y) = f(x) \vee f(y)$.

The definitions of <u>L-monomorphisms</u>, <u>L-epimorphisms</u> and <u>L-isomorphisms</u> are defined as one would expect. If there exists an <u>L-isomorphism</u> K onto M, we denote this by $K \cong_L M$.

<u>Remark 3.21.</u> Let $f: K \rightarrow M$ be an L-homomorphism of positive lattice skewsemifields. Then the following statements hold :

1) f is isotone.

2) m-ker f is an L-ideal of K.

3) $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in K$.

4) If A' is an L-ideal of M then $f^{-1}(A')$ is an L-ideal of K.

<u>Proof</u> 1) Obvious.

2) By Remark 2.16. 2), m-ker f is a convex normal subgroup of K. Let $x \in$ m-ker f. Then $f(x \vee 1) = f(x) \vee f(1) = 1 \vee 1 = 1$, so $x \vee 1 \in$ m-ker f. Hence m-ker f is an L-ideal of K.

3) Let $x, y \in K$. If $x = 0$ or $y = 0$ then done. So assume that $x, y \neq 0$. Then $[f(x) \vee f(y)] \neq 0$. By Proposition 3.6., 3) $f(x)[f(x) \wedge f(y)]^{-1}f(y) = f(x) \vee f(y)$ $= f(x(x \wedge y)^{-1}y)) = f(x)[f(x \wedge y)]^{-1}f(y)$, so $f(x \wedge y) = f(x) \wedge f(y)$.

4) By Remark 2.16. 3), $f^{-1}(A')$ is a convex normal subgroup of K containing m-ker f. Let $x \in f^{-1}(A')$. Then $f(x) \in A'$, Since A' is an L-ideal of K, $f(x \vee 1) = f(x) \vee f(1) = f(x) \vee 1 \in A'$, so $x \vee 1 \in f^{-1}(A')$. By Proposition 3.4., 2), $f^{-1}(A')$ is an L-ideal of K. #

Let K be a positive lattice skewsemifield and A an L-ideal of K. Then $K_{/A}$ is a positively ordered skewsemifild.

To prove that $K_{/A}$ is a lattice, let $x \in K$. Claim that $xA \vee A = (x \vee 1)A$. If $x = 0$ then $xA \vee A = A = (x \vee 1)A$, so done. Suppose that $x \neq 0$. Choose $a \in xA$

and $b \in A$. Then there exist an $i \in A$ such that $a = xi$. Since $ix(bx)^{-1} = ib^{-1} \in A$ and by Corollary 3.18., $(a \vee b)[b(x \vee 1)]^{-1} = (ix \vee b)(bx \vee b)^{-1} \in A$. Since $b \in A$, $(a \vee b)(x \vee 1)^{-1} = (a \vee b)[b(x \vee 1)]^{-1}b \in A$. Hence $\vee$ is well-defined. Clearly, $A \leq (x \vee 1)A$ and $xA \leq (x \vee 1)A$. Let $\alpha \in K_{/A}$ be such that $xA, A \leq \alpha$. Then there exist $a_1, a_2 \in A$ and $y, z \in \alpha$ such that $xa_1 \leq y$ and $a_2 \leq z$, so $(a_1 \wedge a_2)(x \vee 1)$ $= (a_1 \wedge a_2)x \vee (a_1 \wedge a_2) \leq a_1x \vee a_2 \leq y \vee z = (1 \vee zy^{-1})y$. Since $y, z \in \alpha$, $zy^{-1} \in A$, so $y^{-1} \vee 1 \in A$. Thus $(x \vee 1)I \leq Ay = yA = \alpha$. Hence $xA \vee A = (x \vee 1)A$, so we have claim. By Proposition 3.5., 2), $K_{/A}$ is a positive lattice skewsemifiled.

Note that the projection map Π defined by $\Pi(x) = xC$, for every $x \in K$ is an L-epimorphism of K onto $K_{/A}$.

<u>Theorem 3.22. (First Isomorphism Theroem).</u>

Let $f : K \rightarrow M$ be an L-epimorphism of positive lattice skewsemifields. Then $K_{/m\text{-}ker\, f} \cong_L M$.

<u>Proof</u> Let φ be the order isomorphism defined in the proof of Theorem 2.19. To show that φ is an L-isomorphism, let $x, y \in K$. Then $\varphi(x(m\text{-}ker\, f) \vee y(m\text{-}ker\, f))$ $= f(x \vee y) = f(x) \vee f(y) = \varphi(x(m\text{-}ker\, f)) \vee \varphi(y(m\text{-}ker\, f))$. Then φ^{-1} is an L-isomorphism, so $K_{/m\text{-}ker\, f} \cong_L M$. #

<u>Lemma 3.23.</u> Let H be a subskewsemifield of a positive lattice skewsemifield K and A an L-ideal of K. Then $H \cap A$ is an L-ideal of H and HA is a subskewsemifield of K.

<u>Proof</u> This proof is similar to the proof of Lemma 2.20. #

Theorem 3.24. (Second Isomorphism Theorem).

Let H be a subskewsemifield of a positive lattice skewsemifield K and A an L-ideal of K such that $P_{HA} \subseteq P_H$. Then $H/_{H \cap A} \cong_L HA/_A$.

Proof This proof is similar to the proof of Theorem 2.21. #

Lemma 3.25. Let A and B be L-ideals of a positive lattice skewsemifield K such that $A \subseteq B$. Then $B/_A$ is a convex normal subgroup of $K/_A$.

Proof This proof is similar to the proof of Lemma 2.22. #

Theorem 3.26. (Third Isomorphism Theorem).

Let K be a positive lattice skewsemifield, A and B L-ideals of K such that $B \subseteq A$. Then $K/_B/A/_B \cong_L K/_A$.

Proof This proof is similar to the proof of Theorem 2.23. #

Proposition 3.27. Let $f : K \to M$ be an L-epimorphism of positive lattice skewsemifields. If A' is an L-ideal of M then $K/_{f^{-1}(A')} \cong_L M/_{A'}$.

Proof This proof is similar to the proof of Proposition 2.24. #

Proposition 3.28. Let $\{ K_i \mid i \in I \}$ be a family of positively ordered skewsemifields. Then $\prod_{i \in I} K_i$ is a lattice if and only if K_i is a lattice, for all $i \in I$.

Proof See [4], pp. 46. #

Definition 3.29. Let K be a positive lattice skewsemifield. A congruence ρ on K is said to be an L-congruence if and only if for all $x, y, z \in K$, $x \rho y$ implies that $(x \vee z) \rho (y \vee z)$.

Remark 3.34. Let K be a positive lattice skewsemifield and ρ an L-congruence. Then the following statements hold :

1) $x \rho y$ implies that $x^{-1} \rho y^{-1}$ for all $x, y \in K^*$.

2) $x \rho y$ implies that $(x \wedge z) \rho (y \wedge z)$ for all $x, y, z \in K$.

Examples 3.30. 1) Every positive lattice skewsemifield has the trivial L-congruence, that is for all $x, y \in K$, $x \rho y$ if and only if $x = y$.

2) Let A be an L-ideal of positive lattice skewsemifield K. Define a relation ρ_A on K by $x \rho_A y$ if and only if $xy^{-1} \in A$ or $x = y = 0$ for all $x, y \in K$. Then ρ_A is a congruence on K. Next, let $x, y, z \in K$ be such that $x \rho_A y$. If $x = y = 0$ then $x \vee z = z = y \vee z$ and $x \wedge z = 0 = y \wedge z$, so $(x \vee z) \rho_A (y \vee z)$ and $(x \wedge z) \rho_A (y \wedge z)$. Suppose that $y \neq 0$. Then $xy^{-1} \in A$. By Corollary 3.18., $(x \vee z)(y \vee z)^{-1}$, $(x \wedge z)(y \wedge z)^{-1} \in A$, so $(x \vee z) \rho_A (y \vee z)$ and $(x \wedge z) \rho_A (y \wedge z)$. Therefore ρ_A is an L-congruence on K induced by A.

Note that A is an equivalence class of K/ρ_A and ρ_A is a unique L-congruence on K such that $A \in K/\rho_A$. To prove uniqueness, let ρ^* be an L-congruence on K such that $A \in K/\rho^*$. Let $x, y \in K$ be such that $x \rho^* y$. If $y = 0$ then done. Suppose that $y \neq 0$. Then $xy^{-1} \in A$, so $x \rho_A y$. Therefore $\rho^* \subseteq \rho_A$. Obviously, $\rho_A \subseteq \rho^*$, so $\rho_A = \rho^*$.

Let C be the set of all L-congruences on a positive lattice skewsemifield K. Let $\rho, \rho' \in C$. Clearly, $\rho \wedge \rho' = \rho \cap \rho'$.

Define $x \rho^* y$ if and only if there exists a $u \in [1]_\rho$ such that $x \rho' uy$, for all $x, y \in K$. Then we have that ρ^* is a congruence and $\rho^* = \rho' \circ \rho$.

To show that ρ^* is an L-congruence, let $x, y, z \in K$ be such that $x \rho^* y$. Case 1 : $z = 0$. Then $x \vee z = x$ and $y \vee z = 0$. Therefore $(x \vee z) \rho^* x$ and $(y \vee z) \rho^* y$. Hence $(x \vee z) * (y \vee z)$.

Case 2 : $z \neq 0$. Then $(y \vee z) \neq 0$. Since $x \rho^* y$, there exists a $u \in [1]_\rho$ such that $x \rho' uy$, so $uy \rho y$. Then $(uy \vee z) \rho (y \vee z)$, so $(uy \vee z)(y \vee z)^{-1} \rho 1$. Therefore $(uy \vee z)(y \vee z)^{-1} \in [1]_\rho$. Since $x \rho' uy$, $(x \vee z) \rho' (uy \vee z)$. Therefore $(x \vee z) \rho' (uy \vee z)(y \vee z)^{-1}(y \vee z)$, so $(x \vee z) \rho^* (y \vee z)$. Thus ρ^* is an L-congruence, hence $\rho^* \in \mathbb{C}$. So we get that $\rho \vee \rho' = \rho^* = \rho' \circ \rho$. Therefore $\mathbb{C}$ is a lattice.

Let ρ be an L-congruence on a positive lattice skewsemifield K. Let $A_\rho = \{ x \in K \ / \ x \rho 1 \}$. Then we have that A_ρ is an a-convex normal subgroup of K.

To show the o-convexity of A_ρ, let $x, y \in A_\rho$ and $z \in K$ be such that $x \leq z \leq y$. Then $x \rho 1$ and $y \rho 1$, so $z = (x \vee z) \rho (1 \vee z)$ and $y = (y \vee z) \rho (1 \vee z)$. Therefore $z \rho y$, so $z \rho 1$. Thus $z \in A_\rho$ and hence A_ρ is an o-convex set of K. Next, let $x \in A_\rho$. Then $x \rho 1$, so $(x \vee 1) \rho (1 \vee 1) = 1$. Therefore $x \vee 1 \in A_\rho$ and hence A_ρ is an L-ideal of K.

Proposition 3.31. Let K be a positive lattice skewsemifiled, $\mathbb{A}$ the set of all L-congruences on K and $\mathbb{B}$ the set of all L-ideals of K. Then there exists an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$.

Proof This proof is similar to the proof of Proposition 1.43. #

Definition 3.32. A positive lattice skewsemifield K is said to be completely integrally closed if for every $a \in K$, if there exists a $b \in K$ such that $a^n \leq b$ for every $n \in \mathbb{Z}^+$ implies that $a \leq 1$.

Theorem 3.33. A positive lattice skewsemifield K can be embedded into a complete positive lattice skewsemifield if and only if it is completely integrally closed.

Proof Assume that a positive lattice skewsemifield K can be embedded into a complete positive lattice skewsemifield K′. Then there exists an

L-monomorphism $i : K \to K'$. Then $K \cong_L i(K)$. Consider K as a subset of K′. To prove that K is completely integrally closed, let $a, b \in K$ be such that $a^n \leq b$ for all $n \in \mathbf{Z}^+$ Let $A_a = \{ a \vee a^2 \vee \ldots \vee a^n \;/\; n \in \mathbf{Z}^+ \}$. Clearly, b is an upper bound of A_a. By assumption, $\sup A_a$ exists, say c. Then $ac = a(a \vee a_2 \vee \ldots) = a^2 \vee a^3 \vee \ldots \leq c$.

Case 1 : $c = 0$. Since $a \in A_a$, $0 \leq a \leq c = 0$, so $a = 0$. Then $a \leq 1$.

Case 2 : $c \neq 0$. Then $a \leq 1$.

Conversely, assume that K is completely integrally closed. Let $X \subseteq K$. Define $X^\# = L(U(X))$. By Remark 1.2., we have that for all subsets X, Y of K,

1) $X \subseteq X^\#$,
2) $X^{\#\#} = X^\#$,
3) $X \subseteq Y$ implies that $X^\# \subseteq Y^\#$,
4) $U(X) = U(X^\#)$ and $L(X) = L(X^\#)$.

Let $K' = \{ \varnothing \neq C \subseteq K \;/\; U(C) \neq \varnothing$ and $C = C^\# \}$. Define $\bullet$ on K′ as follows : let X, Y be nonempty subsets of K such that $U(X), U(Y) \neq \varnothing$. Then there exist $a \in U(X)$ and $b \in U(Y)$. Clearly, ab is an upper bound of XY. By 4), $U[(XY)^\#]$ $= U(XY) \neq \varnothing$. By 1), $(XY)^{\#\#} = (XY)^\#$, so $(XY)^\# \in K'$. Define $X^\# Y^\# = (XY)^\#$. Hence $AB = (AB)^\#$ for all $A, B \in K'$, for every $C \in K'$ and $a \in K$, $\{a\}^\# C = (aC)^\#$ and $C\{a\}^\# = (Ca)^\#$. Clearly, $\{a\}^\# = L(U(\{a\})) = L(\{a\})$ for all $a \in K$. Hence $\{1\}^\#$ is the multiplicative identity and $\{0\} = L(\{0\}) = \{0\}^\#$ which is the multiplicative zero 0.

To show that $\bullet$ is associative, let $X, Y, Z \in K'$. Then $(XY)Z = (XY)^\# Z = [(XY)Z]^\#$ $= [X(YZ)]^\# = X(YZ)^\# = X(YZ)$, so $\bullet$ is associative.

Let $C \in K'$ be such that $C \neq \{0\}$. Let $C^{-1} = \{ x^{-1} \;/\; x \in C$ and $x \neq 0 \}$. Then $C^{-1} \neq \varnothing$. Since $0 \in L(C^{-1})$, $L(C^{-1}) \neq \varnothing$. By Remark 1.2., $U(L(C^{-1})) \supseteq C^{-1} \neq \varnothing$ and $[L(C^{-1})]^\# = L(U(L(C^{-1}))) = L(C^{-1})$, so $L(C^{-1}) \in K'$. We shall show that $L(C^{-1})$ is the multiplicative inverse of C.

Claim 1), for every $x \in K$, $U(C)x \subseteq U(C)$ implies that $x \in P$.

Let $x \in K$ be such that $U(C)x \subseteq U(C)$. By induction, $U(C)x^n \subseteq U(C)$ for all $n \in \mathbf{Z}^+$. Let $u \in U(C)$. Then $ux^n \in U(C)$ for all $n \in \mathbf{Z}^+$. Since $C \neq \{0\}$, there exists a $c \in C$ such that $c \neq 0$. Then $ux^n \geq c$ for all $n \in \mathbf{Z}^+$, so $c^{-1}u \geq (x^{-1})^n$ for all $n \in \mathbf{Z}^+$. Since

K is completely integrally closed, $x^{-1} \leq 1$, so $x \geq 1$. Then $x \in P$, so we have claim 1.

Claim 2), $U(L(C^{-1})) = P$.

Let $x \in U(L(C^{-1}))$. To show that $U(C)x \subseteq U(C)$, let $u \in U(C)$. Let $y \in C^{-1}$. Then $y^{-1} \in C$, so $u \geq y^{-1}$. Thus $u^{-1} \leq y$, so $u^{-1} \in L(C^{-1})$. Let $c \in C$. Then $x \geq u^{-1}c$, so $ux \geq c$. Thus $ux \in U(C)$, so $U(C)x \subseteq U(C)$. By claim 1., $x \in P$, so $U(L(C^{-1})) \subseteq P$.

Let $x \in P$. Let $y \in L(C^{-1})$ and $c \in C$.

Case 1 : $c = 0$. Then $x \geq 0 = yc$.

Case 2 : $c \neq 0$. Then $c^{-1} \in C^{-1}$, so $c^{-1} \geq y$. Then $x \geq 1 \geq yc$, so $x \in U(L(C^{-1}))$.

Thus $P \subseteq U(L(C^{-1}))$. Hence $U(L(C^{-1})) = P$, so we have claim 2.

Now $L(C^{-1})C = [L(C^{-1})C]^{\#} = L(U[L(C^{-1})C]) = L(P) = L(\{1\}) = \{1\}^{\#}$, so $L(C^{-1})$ is the inverse of C. Hence K′ is a group with the multiplicative zero 0.

Define $\oplus$ on K′ as follows : let X, Y be nonempty subsets of K such that $U(X), U(Y) \neq \varnothing$. Then there exist $a \in U(X)$ and $b \in U(Y)$. Clearly, $a + b$ is an upper bound of $X + Y$. By 4), $U[(X + Y)^{\#}] = U(X + Y) \neq \varnothing$.. By 1), $(X + Y)^{\#\#} = (X + Y)^{\#}$, so $(X + Y)^{\#} \in K'$. Define $X^{\#} \oplus Y^{\#} = (X + Y)^{\#}$. Hence $A \oplus B = (A + B)^{\#}$ for all $A, B \in K'$.

To show that $\oplus$ is associative, let $X, Y, Z \in K'$. Then $(X \oplus Y) \oplus Z = (X + Y)^{\#} \oplus Z = [(X + Y) + Z]^{\#} = [X + (Y + Z)]^{\#} = X \oplus (Y + Z)^{\#} = X \oplus (Y \oplus Z)$, so $\oplus$ is associative.

To show that $\bullet$ is distributive over $\oplus$ in K′, let $X, Y, Z \in K'$. Then $(X \oplus Y)Z = (X + Y)^{\#}Z = [(X + Y)Z]^{\#} = [XZ + YZ]^{\#} = (XZ)^{\#} \oplus (YZ)^{\#} = (XZ) \oplus (YZ)$ and $Z(X \oplus Y) = Z(X + Y)^{\#} = [Z(X + Y)]^{\#} = [ZX + ZY]^{\#} = (ZX)^{\#} \oplus (ZY)^{\#} = (ZX) \oplus (ZY)$, so $\bullet$ is distributive over $\oplus$ in K′. Clearly, $\{0\} \oplus A = A = A \oplus \{0\}$ for every $A \in K'$. Hence K′ is a skewsemifield.

Define $\leq$ on K′ by $A \leq B$ if $A \subseteq B$ for all $A, B \in K'$. Then $\leq$ is a partial order. Next, to show that $\leq$ is a compatible order, let $A, B, C \in K'$ be such that $A \leq B$. Then $A \subseteq B$, so $AC \subseteq BC$, $CA \subseteq CB$, $A + C \subseteq B + C$ and $C + A \subseteq B + C$, so we have that :

1) $AC = (AC)^{\#} \subseteq (BC)^{\#} = BC$,

2) $CA = (CA)^{\#} \subseteq (CB)^{\#} = CB$,

3) $A \oplus C = (A + C)^{\#} \subseteq (B + C)^{\#} = B \oplus C$ and

4) $C \oplus A = (C + A)^{\#} \subseteq (C + B)^{\#} = C \oplus B$.

Thus $AC \le BC, CA \le CB$, $A + C \le B + C$ and $C + A \le B + C$, so we get that $\le$ is a compatible order. Clearly, $\{0\} \subseteq L(U(A) = A^{\#} = A$ for every $A \in K'$, hence K' is a positively ordered skewsemifield. Next, to show that $\le$ is a lattice, let $A, B \in K'$. Let $x \in U(A)$ and $y \in U(B)$. Then $x \vee y \in U(A \cup B)$. By 4), $\varnothing \ne U(A \cup B)$ $= U([A \cup B]^{\#})$. By 2), $(A \cup B)^{\#\#} = (A \cup B)^{\#}$, so $(A \cup B)^{\#} \in K'$. Next, we shall show that $A \vee B = (A \cup B)^{\#}$. Since $A \subseteq A \cup B$ and (by using 1)), we get that $A = A^{\#}$ $\subseteq (A \cup B)^{\#}$. Similarly, $B \subseteq (A \cup B)^{\#}$. Let $C \in K'$ be such that $A, B \le C$. Then $A \cup B$ $\subseteq C$. By 1), $(A \cup B)^{\#} \subseteq C^{\#} = C$, so $A \vee B = (A \cup B)^{\#}$. Hence K' is a lattice.

Next, to show that K' is complete, let $\mathcal{C}$ be a nonempty subset of K' which has an upper bound. Let $\mathbf{B} = \{ C \in K' \ / \ C$ is an upper bound of $\mathcal{C} \}$. We shall show that $\bigcap_{C \in \mathbf{B}} C = \sup \mathcal{C}$. By assumption, $\mathbf{B} \ne \varnothing$, so there exists a $C' \in \mathbf{B}$. Then $\bigcap_{C \in \mathbf{B}} C \subseteq C'$. By Remark 1.2., $\varnothing \ne U(C') \subseteq U(\bigcap_{C \in \mathbf{B}} C)$.

Claim 3, $L(\bigcup_{C \in \mathbf{B}} U(C)) \subseteq \bigcap_{C \in \mathbf{B}} [L(U(C)]$.

Let $C' \in \mathbf{B}$. Then $U(C') \subseteq \bigcup_{C \in \mathbf{B}} (U(C))$, so $L[U(C')] \supseteq L[\bigcup_{C \in \mathbf{B}} (U(C))]$. Then $L(\bigcup_{C \in \mathbf{B}} U(C)) \subseteq \bigcap_{C \in \mathbf{B}} [L(U(C)]$, so we have claim 3.

Claim 4, $L(U(\bigcap_{C \in \mathbf{B}} C)) \subseteq L(\bigcup_{C \in \mathbf{B}} [U(C)])$.

Let $C' \in \mathbf{B}$. By $\bigcap_{C \in \mathbf{B}} C \subseteq C'$, $U[\bigcap_{C \in \mathbf{B}} C] \supseteq U(C)$, so $U[\bigcap_{C \in \mathbf{B}} C] \supseteq \bigcup_{C \in \mathbf{B}} [U(C)]$. Then $L(U(\bigcap_{C \in \mathbf{B}} C)) \subseteq L(\bigcup_{C \in \mathbf{B}} [U(C)])$, so we have claim 4.

Thus $\bigcap_{C \in \mathbf{B}} C = \bigcap_{C \in \mathbf{B}} C^{\#} = \bigcap_{C \in \mathbf{B}} L(U(C)) \supseteq L(\bigcup_{C \in \mathbf{B}} U(C)) \supseteq L(U(\bigcap_{C \in \mathbf{B}} C)) = (\bigcap_{C \in \mathbf{B}} C)^{\#}$.

Hence $\bigcap_{C \in \mathbf{B}} C \in K'$. Clearly, $\bigcap_{C \in \mathbf{B}} C = \sup \mathcal{C}$. Hence K' is complete.

Define $f : K \to K'$ by $f(x) = \{x\}^{\#}$ for every $x \in K$. To show that f is an L-homomorphism, let $a, b \in K$. Then $f(ab) = \{ab\}^{\#} = (\{a\}\{b\})^{\#} = \{a\}^{\#}\{b\}^{\#} = f(a)f(b)$,

$f(a+b) = \{a+b\}^{\#} = (\{a\}+\{b\})^{\#} = \{a\}^{\#} \oplus \{b\}^{\#} = f(a) \oplus f(b)$ and $f(a \vee b) = \{a \vee b\}^{\#}$ $= L(\{a \vee b\})$. Since $f(a) = \{a\}^{\#}$ and $f(b) = \{b\}^{\#}$, $f(a) \vee f(b) = \{a\}^{\#} \vee \{b\}^{\#} = (\{a\}^{\#} \cup \{b\}^{\#})^{\#}$ $= [L(\{a\}) \cup L(\{b\})]^{\#}$. We shall show that $L(\{a \vee b\}) = [L(\{a\} \cup L(\{b\})]^{\#}$. Since $L(\{a\}) \subseteq L(\{a \vee b\})$ and $L(\{b\}) \subseteq L(\{a \vee b\})$, $L(\{a\}) \cup L(\{b\}) \subseteq L(\{a \vee b\})$, so $[L(\{a\}) \cup L(\{b\})]^{\#} \subseteq [L(\{a \vee b\})]^{\#}$. Next, let $x \in L(\{a \vee b\})$ and $y \in U[L(\{a\}) \cup L(\{b\})]$. Then $y \geq z$ for all $z \in L(\{a\}) \cup L(\{b\})$. Since $a \in L(\{a\})$ and $b \in L(\{b\})$, $y \geq a$ and $y \geq b$, so $y \geq x$. Hence x is a lower bound of $U[L(\{a\}) \cup L(\{b\})]$, so $x \in L(U[L(\{a\}) \cup L(\{b\})]) = [L(\{a\}) \cup L(\{b\})]^{\#}$. So we get that $L(\{a \vee b\})$ $\subseteq [L(\{a\} \cup L(\{b\})]^{\#}$. Therefore $f(a \vee b) = L(\{a \vee b\}) = [L(\{a\} \cup L(\{b\})]^{\#} = f(a) \vee f(b)$, so f is an L-homomorphism.

To show that f is an injection, let $x, y \in K$ be such that $f(x) = f(y)$. Since $x \in \{x\}^{\#} = \{y\}^{\#} = L(\{y\})$, $x \leq y$. Since $y \in \{y\}^{\#} = \{x\}^{\#} = L(\{x\})$, $y \leq x$, so $x = y$.

To show that $f(P) = P_{f(K)}$, let $x \in P$. Then $f(x) = \{x\}^{\#} = L(\{x\}) \supseteq L(\{1\}) = \{1\}^{\#}$, so $f(x) \in P_{f(K)}$. Next, let $x \in K$ be such that $f(x) \in P_{f(K)}$. Then $L(\{x\}) = \{x\}^{\#} = f(x) \supseteq \{1\}^{\#}$. Since $1 \in \{1\}^{\#} \subseteq L(\{x\})$, $1 \leq x$, so $x \in P$. Therefore $f(P) = P_{f(K)}$, hence f is an L-monomorphism. Hence $K \cong_L f(K)$, so K can be embedded into a complete positive lattice skewsemifield K'. #