CHAPTER II

POSITIVELY ORDERED 0-SKEWSEMIFIELDS

In this chapter, we shall give some fundamental theorems of a theory of positively ordered skewsemifields.

Definition 2.1. Let $\leq$ be a partial order on a semiring S with a multiplicative zero 0. $\leq$ is said to be compatible if and only if it satisfies the following property, for all $x, y, z \in S$, $x \leq y$ implies that 1) $x + z \leq y + z$ and $z + x \leq z + y$ and 2) $xz \leq yz$ and $zx \leq zy$, if $z \geq 0$.

Definition 2.2. A partial order $\leq$ on a semiring S with a mutiplicative zero 0 is said to be multiplicatively regular (M.R.) if ($xz \leq yz$ and $0 < z$ imply that $x \leq y$) and ($zx \leq zy$ and $0 < z$ imply that $x \leq y$) for all $x, y, z \in S$.

Definition 2.3. A system $(S , + , \bullet , \leq)$ is said to be an ordered semiring if and only if $(S , + , \bullet)$ is a semiring with a multiplicative zero 0 and $\leq$ is a compatible partial order on S. If $0 \leq x$, for every $x \in S$ then we say that S is a positively ordered semiring.

Remark 2.4. Let K be a positively ordered skewsemifield. Then the following statements clearly hold :

1) for all nonzero elements $x, y \in K$, $x \leq y$ implies that $y^{-1} \leq x^{-1}$.

2) for all $x, y, z \in K$, $zx \leq zy$ or $xz \leq yz$ implies that $z = 0$ or $x \leq y$.

Examples 2.5. 1) $\mathbf{Q}_0^+$, $\mathbf{R}_0^+$ are positively ordered skewsemifields.

2) Let $A = \{ f : \mathbf{R} \to \mathbf{R} \ / \ f(x) = ax + b, a > 0 \} \cup \{ 0 \}$. Let $K = (A, \circ, +)$ and $L = (A, \circ, \oplus)$ where $f \oplus g = \begin{cases} f \text{ if } f \neq 0 \\ g \text{ if } f = 0 \end{cases}$, for all $f, g \in L$. Then K and L are skewsemifields, so $\mathbb{K} = K^* \times L^* \cup \{ (0,0) \}$ which is a skewsemifiled. Define $\leq$ on A as follows : let $f(x) = ax + b$, $g(x) = cx + d \in A \backslash \{ 0 \}$. Define $f \leq g$ if 1) $a < c$ or 2) $a = c$ and $b \leq d$. And let $0 \leq h$ for every $h \in A$. Define $\leq^*$ on $\mathbb{K}$ by $(f_1,f_2) \leq^* (h_1,h_2)$ if 1) $f_1 < h_1$ or 2) $f_1 = h_1$ and $f_2 \leq h_2$ for all $(f_1,f_2), (h_1,h_2) \in \mathbb{K}$. Clearly, $\leq^*$ is a partial order and $F \leq^* G$ implies that $FH \leq^* GH$ and $HF \leq^* HG$ for all $F, G, H \in \mathbb{K}$. Let $F = (f_1,f_2)$, $H = (h_1,h_2)$ and $G = (g_1,g_2) \in \mathbb{K}$ be such that $(f_1,f_2) \leq^* (h_1,h_2)$. Consider $F + G = (f_1 + g_1, f_2 \oplus g_2)$, $H + G = (h_1 + g_1, h_2 \oplus g_2)$, $G + F = (g_1 + f_1, g_2 \oplus f_2)$ and $G + H = (g_1 + h_1, g_2 \oplus h_2)$. If $G = 0$ then $F + G = F \leq^* H = H + G$ and $G + F = F \leq^* H = G + H$. Suppose that $G \neq 0$. Then $g_2 \neq 0$.

Case 1 : $F = 0$. Then $f_1 = 0$. If $H = 0$ then $F + G = G \leq^* G = H + G$ and $G + F = G \leq^* G = G + H$. Suppose that $H \neq 0$. Then $h_1 > 0$. Since K is additively cencellative, $f_1 + g_1 = g_1 < h_1 + g_1$ and $g_1 + f_1 = g_1 < g_1 + h_1$.

Case 2 : $F \neq 0$. Then $H \neq 0$, so $f_2, h_2 \neq 0$.

Subcase 2.1 : $f_1 < h_1$. Since K is additively cencellative, $f_1 + g_1 < h_1 + g_1$ and $g_1 + f_1 < g_1 + h_1$.

Subcase 2.2 : $f_1 = h_1$. Then $f_2 \leq h_2$. Thus $f_2 \oplus g_2 = f_2 \leq h_2 = h_2 \oplus g_2$ and $g_2 \oplus f_2 = g_2 \leq g_2 = g_2 \oplus f_2$. Therefore $F + G \leq^* H + G$ and $G + F \leq^* G + H$, hence $\mathbb{K}$ is a positively ordered skewsemifield.

3) Let $K = \{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \ / \ a, b \in \mathbf{Q}^+[\mathbf{R}^+] \text{ and } c \in \mathbf{Q}[\mathbf{R}] \} \cup \{ 0 \}$.

Then K with the usual binary operation is a skewsemifield. Define a relation $\leq$ on K by $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \leq \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix}$ if and only if 1) $a < a'$ or 2) $a = a'$ and $b < b'$ or 3) $a = a'$, $b = b'$ and $c < c'$. To show that $\leq$ is a partial order, it is clear that

$\leq$ is reflexive. Let $M_1 = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$, $M_2 = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix} \in K$ be such that $M_1 \leq M_2$ and $M_2 \leq M_1$. Then $a \leq a'$ and $a' \leq a$, so $a = a'$. Hence $b \leq b'$ and $b' \leq b$, so $b = b'$. Thus $c \leq c'$ and $c' \leq c$, so $c = c'$. Therefore $M_1 = M_2$, so $\leq$ is anti-symmetric. Let $M_1 = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$, $M_2 = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix}$, $M_3 = \begin{bmatrix} a'' & c'' \\ 0 & b'' \end{bmatrix} \in K$ be such that $M_1 \leq M_2$ and $M_2 \leq M_3$. Then $a \leq a'$ and $a' \leq a''$.

Case 1 : $a < a'$ or $a' < a''$. Then $a < a''$.

Case 2 : $a = a' = a''$. Then $b \leq b'$ and $b' \leq b''$.

Subcase 2.1 : $b < b'$ or $b' < b''$. Then $b < b''$.

Subcase 2.2 : $b = b' = b''$. Then $c \leq c'$ and $c' \leq c''$, so $c \leq c''$.

Hence $M_1 \leq M_3$. Therefore $\leq$ is transitive. Next, to show that $\leq$ is compatible, Let $M_1 = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$, $M_2 = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix} \in K$ be such that $M_1 \leq M_2$. Let $W = \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \in K$. Since $M_1 \leq M_2$, $a < a'$ or $a = a'$. If $a < a'$ then $a + x < a' + x$, so done. Suppose that $a = a'$. Then $a + x = a' + x$ and $b \leq b'$.

Case 1 : $b < b'$. Then $b + y < b' + y$.

Case 2 : $b = b'$. Then $c \leq c'$, so $c + z = c' + z$. Therefore $M_1 + W \leq M_2 + W$.

Thus $M_1W = \begin{bmatrix} ax & az+cy \\ 0 & by \end{bmatrix}$, $WM_1 = \begin{bmatrix} xa & xc+zb \\ 0 & yb \end{bmatrix}$, $M_2W = \begin{bmatrix} a'x & a'z+c'y \\ 0 & b'y \end{bmatrix}$, and $WM_2 = \begin{bmatrix} xa' & xc'+zb' \\ 0 & yb' \end{bmatrix}$. If $W = 0$ then $M_1W = 0 \leq M_2W$ and $WM_1 = 0 \leq WM_2$. Suppose that $W \neq 0$. Then $x, y > 0$. If $a < a'$ then $ax < a'a$ and $xa < xa'$. Suppose that $a = a'$. Then $ax = a'x$, $xa = xa'$ and $b \leq b'$.

Case 1 : $b < b'$. Then $by < b'y$ and $yb < yb'$.

Case 2 : $b = b'$. Then $c \leq c'$, so $ax + cy \leq a'x + cy'$ and $xc + zb \leq xc' + zb'$. Therefore $WM_1 \leq WM_2$ and $M_1W \leq M_2W$. Hence K is a positively ordered skewsemifield.

4) Let K and L be positively ordered skewsemifields. Define a relation $\leq$ on $K^* \times L^* \cup \{ (0,0) \}$ by $(x,y) \leq (z,w)$ if and only if $x \leq z$ and $y \leq w$,

for all $(x,y), (z,w) \in K^* \times L^* \cup \{ (0,0) \}$. Then $K^* \times L^* \cup \{ (0,0) \}$ is a positively ordered skewsemifield.

5) Let K and L be positively ordered skewsemifields such that K is additively cancellative. Define a relation $\leq$ on $K^* \times L^* \cup \{ (0,0) \}$ by $(x,y) \leq (z,w)$ if and only if $x < z$ or $x = z$ and $y \leq w$, for all (x,y) and $(z,w) \in K^* \times L^* \cup \{ (0,0) \}$. Then $K^* \times L^* \cup \{ (0,0) \}$ is a positively ordered skewsemifield.

Note that the partial order $\leq$ defined in Example 2.5., 2), 3) and 5) are called the lexicographic order.

Definition 2.6. Let C be a subset of a positively ordered skewsemifield K. Then C is called a convex subset of K if it is an o-convex subset and an a-convex subset of K.

Definition 2.7. Let K be a positively ordered skewsemifield. Then the set $P = \{ x \in K \ / \ x \geq 1 \}$ is called the positive cone of K.

Remark 2.8. Let P be the positive cone of a positively ordered skewsemifield K. Then the following statements hold :

1) If $P = \{ 1 \}$ then $K = \{ 0, 1 \}$.

2) P is a multiplicative subsemigroup of K.

3) For every $x \in K$, $1 + x \in P$ and $x + 1 \in P$. Hence P is an additive ideal of K, that is $K + P \subseteq P$ and $P + K \subseteq P$.

4) P is a conic subset of K.

5) P is a convex normal subset of K.

6) For every $x \in K^*$, $x = ab^{-1}$ for some $a, b \in P$.

7) For all $x, y \in P$, $xy = 1$ implies that $x = y = 1$.

8) If H is a subskewsemifield of K, then $P_H = P \cap H$ where $P_H = \{ x \in H \ / \ x \geq 1 \}$.

<u>Theorem 2.9.</u> Let K be a skewsemifield and $P \subseteq K^*$. Suppose that P satisfies the following conditions :

1) P is multiplicative subsemigroup of K^*,

2) P is a conic subset of K,

3) P is an additive ideal of K,

4) P is an a-convex normal subset of K.

Then there exists a unique compatible partial order $\leq$ on K such that P is the positive cone of $\leq$. $\leq$ is called the partial order induced by P.

<u>Proof</u> Define $\leq_P$ on K as follows : let $x, y \in K$, $x \leq_P y$ if and only if $x = 0$ or $x^{-1}y \in P$. To show that $\leq_P$ is a partial order, it is clear that $\leq_P$ is reflexive since $1 \in P$. Next, let $x, y \in K$ be such that $x \leq_P y$ and $y \leq_P x$.
Case 1 : $x = 0$. If $y \neq 0$ then $0 = y^{-1}(0) \in P$ which is a contradiction, so $y = 0 = x$.
Case 2 : $x \neq 0$. If $y = 0$ then $0 = x^{-1}y \in P$ which is a contradiction, so $y \neq 0$. Therefore $x^{-1}y \in P$ and $(x^{-1}y)^{-1} = y^{-1}x \in P$. By 2), $x^{-1}y = 1$, so $x = y$. Therefore $\leq_P$ is anti-symmetric. Let $x, y, z \in K$ be such that $x \leq_P y$ and $y \leq_P z$. If $x = 0$ then $x \leq_P z$. Suppose that $x \neq 0$. Then $y \neq 0$, so $x^{-1}y \in P$ and $y^{-1}z \in P$. By 1), $x^{-1}z = (x^{-1}y)(y^{-1}z) \in P$, so $x \leq_P z$. Therefore $\leq_P$ is transitive, hence $\leq_P$ is a partial order. Next, let $x, y \in K$ be such that $x \leq_P y$. Let $z \in K$. If $x = 0$ or $z = 0$ then $xz = 0 = yz$ and $zx = 0 = zy$. Suppose that $x \neq 0$ and $z \neq 0$. Then $(zx)^{-1}zy = (x^{-1}z^{-1})zy = x^{-1}y \in P$, so $zx \leq_P zy$. Since P is a normal set of K, $(xz)^{-1}yz = z^{-1}(x^{-1}y)z \in P$, so $xz \leq_P yz$.

Next, let $x, y \in K$ be such that $x \leq_P y$. Let $z \in K$.
Case 1 : $x = 0$. If $z = 0$ then $x + z = 0 \leq_P y + z$ and $z + x = 0 \leq_P z + y$, so done. Suppose that $z \neq 0$. By 3), $z^{-1}(z + y) = 1 + z^{-1}y \in P$ and $z^{-1}(y + z) = z^{-1}y + 1 \in P$, so $z + x = z \leq_P z + y$ and $x + z = z \leq_P y + z$.
Case 2 : $x \neq 0$. Then $x^{-1}y \in P$. By the a-convexity of P, $(x + z)^{-1}(y + z)$ $= (x + z)^{-1}y + (x + z)^{-1}z = [\,(x + z)^{-1}x\,]x^{-1}y + [\,(x + z)^{-1}z\,] \in P$, so $x + z \leq_P y + z$. Similarly, $z + x \leq_P z + y$. Therefore $\leq_P$ is a compatible partial order on K.

Clearly, for every $x \in K$, $0 \leq_P x$. Hence P is a positive cone of K. Thus K is a positively ordered skewsemifield having P as its positive cone.

To prove the uniqueness, let $\leq^*$ be a compatible partial order on K such that P is its positive cone. Let $x, y \in K$ be such that $x \leq^* y$. If $x = 0$ then $x \leq_P y$. Suppose that $x \neq 0$. Then $1 \leq^* x^{-1}y$, so $x^{-1}y \in P$. Then $x \leq_P y$. Hence $\leq^* \subseteq \leq_P$. Similarly, $\leq_P \subseteq \leq^*$. Therefore $\leq_P = \leq^*$. #

Corollary 2.10. Let K be a skewsemifield, $\mathbb{A}$ the set of all subsets of K^* which satisfy 1) – 4) in Theorem 2.9. and $\mathbb{B}$ the set of all positive compatible partial orders on K. Then there exists an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$.

Proof Define $\varphi : \mathbb{A} \to \mathbb{B}$ as follows : let $P \in \mathbb{A}$. By Theorem 2.10., P determines a unique positive compatible partial order $\leq_P$ induced by P on K. Define $\varphi(P) = \leq_P$. Clearly, φ is a bijection.

To prove that φ is isotone, let $P, Q \in \mathbb{A}$ be such that $P \subseteq Q$. Then $\varphi(P) = \leq_P$ and $\varphi(Q) = \leq_Q$. We must show that $\leq_P \subseteq \leq_Q$. Let $x, y \in K$ be such that $x \leq_P y$. Then $x = 0$ or $x^{-1}y \in P$.

Case 1 : $x = 0$. Then $x \leq_Q y$.

Case 2 : $x^{-1}y \in P$. Since $P \subseteq Q$, $x^{-1}y \in Q$, so $x \leq_Q y$.

Then $\varphi(P) = \leq_P \subseteq \leq_Q = \varphi(Q)$, so φ is isotone.

Next, to show that φ^{-1} is isotone, let $\leq, \leq^* \in \mathbb{B}$ be such that $\leq \subseteq \leq^*$. Then $\varphi^{-1}(\leq) = P_{\leq}$ and $\varphi^{-1}(\leq^*) = P_{\leq^*}$. We must to show that $P_{\leq} \subseteq P_{\leq^*}$. Let $x \in K$ be such that $x \in P$. Then $1 \leq x$. Since $\leq \subseteq \leq^*$, $1 \leq^* x$, so $x \in P^*$. Therefore $\varphi^{-1}(\leq) \subseteq \varphi^{-1}(\leq^*)$, so φ^{-1} is isotone. Hence φ is an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$. #

Proposition 2.11. Let K be a skewsemifield. Suppose that K has a compatible partial order. Then there exits maximal compatible partial order on K.

Proof Let $\mathbb{P} = \{ P \ / \ P$ is a positive cone of $K \}$. Since K has a compatible partial order, there exists a $P \in \mathbb{P}$, so $\mathbb{P} \neq \emptyset$. Let $\mathbb{D}$ be a nonempty chain of $\mathbb{P}$. Let $Q = \cup\mathbb{D}$. Then $Q \in \mathbb{P}$. By Zorn's Lemma, $\mathbb{P}$ has a maximal element. #

Theorem 2.12. Let $(S, \bullet)$ be a positively ordered semiring with multiplicative zero 0 having the M.C. property and satisfying the right [left] Ore condition. If $\leq$ is M.R. then S can be embedded into a positively ordered skewsemifield.

Proof Using the construction of Theorem 1.31., we have that $K = S \times (S \setminus \{ 0 \})/_{\sim}$ is the skewsemifield of a right quotients of S. Let i be a right quotient embedding of S into K. Let P $= \{ \alpha \in K \ / \ \alpha = i(x)i(y)^{-1}$ for some $x, y \in S\setminus\{ 0 \}$ such that $x \geq y \}$.

To show that P is a multiplicative subsemigroup of K, Let $\alpha = i(x)i(y)^{-1}$ and $\beta = i(z)i(w)^{-1} \in K$. Then $y \leq x$ and $w \leq z$. By the right Ore condition, there exist $a, b \in S$ such that $ya = zb$, so $\alpha\beta = i(x)i(y)^{-1}$. Since $wb \leq zb = ya \leq xa$, $\alpha\beta \in P$. Then P is a multiplicative subsemigroup of K.

To show that P is an additive ideal of K, Let $\alpha = i(a)i(b)^{-1} \in K$. Then $\alpha + 1 = [(a,b)] + [(b,b)] = i(ab + bb)i(bb)] = i(ab + bb)i(bb)^{-1}$ and $1 + \alpha = i(b,)i(b) + i(a)i(a) = i(bb + ab)i(bb)$. Since $bb \leq ab + bb$ and $bb \leq bb + ab$, $\alpha + 1 \in P$ and $1 + \alpha \in P$. Hence P is an additive ideal.

Claim that for all $a, b \in S\setminus\{ 0 \}$, $i(a)i(b)^{-1} \in P$ implies that $a \geq b$. Let $a, b \in S\setminus\{ 0 \}$ be such that $i(a)i(b)^{-1} \in P$. Then there exist $p, q \in S\setminus\{ 0 \}$ such that $i(a)i(b)^{-1} = i(p)i(q)^{-1}$ and $p \geq q$. Then there exist $p', q' \in S\setminus\{ 0 \}$ such that $ap' = pq'$ and $bp' = qq'$. Then $ap' = pq' \geq qq' = bq'$. By the M.R. property, $a \geq b$, so we have the claim.

To show that P is a conic set of K, let $\alpha \in P \cap P^{-1}$. Then $\alpha \in P$ and $\alpha \in P^{-1}$. Then there exist $a, b \in S\setminus\{ 0 \}$ such that $\alpha = i(a)i(b)^{-1}$ and $a \geq b$. Since $i(b)i(a)^{-1} = (i(a)i(b)^{-1})^{-1} = \alpha^{-1} \in P$ and by the claim, $b \geq a$, we get that $a = b$. Thus $\alpha = [(a,a)] = 1$, so P is a conic set.

To show that P is a normal subset of K, Let $\alpha = i(x)i(y)^{-1} \in P$ and $\beta = i(z)i(w)^{-1} \in K$. Then $y \leq x$. By the right Ore condition, there exist $a, b \in S\setminus\{0\}$ such that $ya = zb$, so $\beta\alpha = i(x)i(y)^{-1}$. By the right Ore condition, there exist $c, d \in S\setminus\{0\}$ such that $ybc = wd$, so $\beta\alpha\beta^{-1} = i(x)i(y)^{-1}$. Since $wd = ybc \leq xbc = wac$, using the M.C. property, $d \leq ac$, so $\beta\alpha\beta^{-1} \in P$. Thus P is a normal set.

Claim that for all $\alpha, \beta, \gamma \in K^*$, $\beta\alpha^{-1} \in P$ implies that $(\beta + \gamma)(\alpha + \gamma)^{-1} \in P$. Let $\alpha = i(x)i(y)^{-1}$, $\beta = i(z)i(w)^{-1}$ and $\gamma = i(u)i(v)^{-1} \in K^*$ be such that $\beta\alpha^{-1} \in P$. By the right Ore condition, there exist $a, b \in S\setminus\{0\}$ such that $ya = vb$, so $\alpha + \gamma = i(xa + ub)i(ya)^{-1}$. By the right Ore condition, there exist $c, d \in S\setminus\{0\}$ such that $wc = vd$, so $\beta + \gamma = i(zc + ud)i(wc)^{-1}$. By the right Ore condition, there exist $e, f \in S\setminus\{0\}$ such that $wce = yaf$, so $(\beta + \gamma)(\alpha + \gamma)^{-1} = i[(zc + ud)e]i[(xa + ud)f]^{-1}$. Since $vbf = yaf = wce = vde$, $bf = de$. Since $i(zce)i(xaf) = \beta\alpha^{-1} \in P$, $xaf \leq zce$,so $(xa + ub)f = (xaf + ubf) = (xaf + ude) \leq (zce + ude) = (zc + ud)e$. Then $(\beta + \gamma)(\alpha + \gamma)^{-1} \in P$, so we have the claim.

To show the a-convexity of P, let $x, y \in P$ and $\alpha, \beta \in K$ be such that $\alpha + \beta = 1$. If $\alpha = 0$ then $x\alpha + y\beta = y \in P$. So suppose that $\alpha \neq 0$. By the claim, $(y + y\beta\alpha^{-1})(1 + y\beta\alpha^{-1})^{-1} \in P$. Since P is a normal subset of K, $(1 + y\beta\alpha^{-1})^{-1}(y + y\beta\alpha^{-1}) = (1 + y\beta\alpha^{-1})^{-1}(y + y\beta\alpha^{-1})(1 + y\beta\alpha^{-1})^{-1}(1 + y\beta\alpha^{-1}) \in P$. Then $\alpha + y\beta = (\alpha + y\beta)(\alpha + \beta)^{-1} = [\,(1 + y\beta\alpha^{-1})(y + y\beta\alpha^{-1})^{-1}\,]y \in P$. Thus $x\alpha + y\beta = (x\alpha + y\beta)(\alpha + y\beta)^{-1}(\alpha + y\beta) = [\,(x + y\beta\alpha^{-1})(1 + y\beta\alpha^{-1})^{-1}\,](\alpha + y\beta) \in P$. Hence P is an a-convex normal subset of K. By Theorem 2.9., P is the positive cone of K, so K is a positively ordered skewsemifield.

To show that i is an isotone map, let $a, b \in S$ be such that $b \geq a$. If $a = 0$ then $i(a) = 0 \leq i(b)$. Suppose that $a \neq 0$. Then $i(b)i(a)^{-1} \in P$, so $i(b)i(a)^{-1} \geq 1$. Thus $i(b) \geq i(a)$. Hence i is isotone. #

<u>Theorem 2.13.</u> Let $n \in \mathbf{Z}^+$ be such that $n \geq 2$. Let $K_n = \{0\} \cup \{\, A \in M_n(\mathbf{R})\ [\,M_n(\mathbf{Q})\,]\ /\ A_{ij} > 0$ if $i = j$ and $A_{ij} = 0$ if $i > j\,\}$. Then there exists a compatible positive partial order on K_n.

Proof If $n = 2$ then done by Example 2.5., 3). Induction assumption, let $n \in \mathbf{Z}^+$ be such that $n > 2$. Let K_{n-1} with the following partial order is a positively ordered skewsemifield. Let $P_n = \{ \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \in K_n$ / 1) $A_1 > 1$ or 2) $A_1 = 1$ and $A_2 > 1$ or 3) $A_1 = 1$, $A_2 = 1$ and $A_{n-1,n} > 0$ or 4) $A_1 = 1$, $A_2 = 1$ and there exists an $i \in \{1,\ldots,n-2\}$ such $A_{in} > 0$ and $A_{kn} = 0$ for all $n > k > i$ or 4) $A_1 = 1$, $A_2 = 1$ and $A_3 = 0$ where $A_1 \in K_{n-1}$ }.

To show that P_n is an additive ideal of K, let $X \in K_n$ If $X = 0$ then $1 + X = 1 \in P_n$. Suppose that $X \neq 0$. Then $X_1 \neq 0$, so $X_1 > 0$. Since K_{n-1} is additive cencellative, $1 + X_1 > 1$. Therefore $1 + X \in P_n$. Hence P_n is an additive ideal.

To show that P_n is a multiplicative subsemigroup of K, let $X, Y \in P_n$. Then $X_1 \geq 1$, $Y_1 \geq 1$ and $XY = \begin{bmatrix} X_1Y_1 & X_1Y_3 + X_3Y_2 \\ 0 & X_2Y_2 \end{bmatrix}$, so $X_1Y_1 \geq 1$. If $X_1Y_1 > 1$ then $XY \in P_n$. Suppose that $X_1Y_1 = 1$. Then $X_1 = (Y_1)^{-1} \in (P_{n-1}) \cap (P_{n-1})^{-1} = \{1\}$, so $X_1 = Y_1 = 1$. Thus $X_2 \geq 1$ and $Y_2 \geq 1$, so $X_2Y_2 \geq 1$. If $X_2Y_2 > 1$ then $XY \in P_n$. Suppose that $X_2Y_2 = 1$. Then $X_2 = Y_2 = 1$. Thus $X_{n-1,n} \geq 0$ and $Y_{n-1,n} \geq 0$.

Case 1 : $X_{n-1,n} > 0$ or $Y_{n-1,n} > 0$. Then $(XY)_{n-1,n} = \sum_{k=1}^{n} (X_{n-1,k})(Y_{kn})$
$= \sum_{k=1}^{n-1} (X_{n-1,k})(Y_{kn}) + (X_{n-1,n})(Y_{nn}) = (X_{n-1,n-1})(Y_{n-1,n}) + (X_{n-1,n})(Y_{n-1,n}) = Y_{n-1,n} + X_{n-1,n} > 0$.

Case 2 : $X_{n-1,n} = 0$ and $Y_{n-1,n} = 0$. If $X_3 = 0$ or $Y_3 = 0$ then $X = 1$ or $Y = 1$, so $XY \in P_n$. Suppose that $X_3 \neq 0$ and $Y_3 \neq 0$. Then there exist $i', i'' \in \{1,\ldots,n-2\}$ such that $X_{i'n} > 0$, $X_{kn} = 0$ for all $n > k > i'$ and $Y_{i''n} > 0$, $Y_{kn} = 0$ for all $n > k > i''$. Let $i = \max\{i', i''\}$. Then $i = i'$ or $i = i''$, so $(XY)_{in} = \sum_{k=1}^{n} (X_{ik})(Y_{kn}) = (X_{ii})(Y_{in}) + (X_{in})(Y_{nn})$
$= X_{in} + Y_{in} > 0$. Let $n > j > i$. Then so $(XY)_{jn} = \sum_{k=1}^{n} (X_{jk})(Y_{kn}) = (X_{jj})(Y_{jn}) + (X_{jn})(Y_{nn})$
$= X_{jn} + Y_{jn} = 0$. Therefore $XY \in P_n$, hence P_n is a multiplicative subsemigroup of K.

To show that P_n is a normal subset of K_n, let $A \in P_n$. Then $A_1 \geq 1$. Let

$X \in K_n^{\cdot}$. Then $XA = \begin{bmatrix} X_1A_1 & X_1A_3+X_3A_2 \\ 0 & X_2A_2 \end{bmatrix}$. Since $A \in P_n$, $A_1 \in P_n$, so there exists a $B_1 \in P_{n-1}$ such that $X_1A_1 = B_1X_1$. Since $X_2A_2 = A_2X_2$, let $B_2 = A_2$. Let $B_3 = (X_1A_3 + X_3A_2 - B_1X_3)(X_2)^{-1}$. Then $XA = BX$. We must show that $B \in P$. Since $A \in P_n$, $A_1 \geq 1$. If $A_1 > 1$ then $B_1 > 1$, so $B \in P_n$. Suppose that $A_1 = 1$. Then $B_1 = 1$ and $A_2 \geq 1$. If $A_2 > 1$ then $B_2 > 1$, so $B \in P_n$. Suppose that $A_2 = 1$.
If $A_3 = 0$ then $B = 1 = A \in P_n$. Suppose that $A_3 \neq 0$.
Case 1 : $A_{n-1,n} > 0$. Then $B_{n-1,n} = (\sum_{k=1}^{n-1} X_{n-1,k}A_{kn} + X_{n-1,n}A_{nn} - \sum_{k=1}^{n-1} B_{n-1,k}X_{kn})(X_{nn})^{-1}$
$= (X_{n-1,n-1}A_{n-1,n} + X_{n-1,n} - B_{n-1,n-1}X_{n-1,n})(X_{nn})^{-1} = (X_{n-1,n-1}A_{n-1,n})(X_{nn})^{-1} > 0$.
Case 2 : there exists an $i \in \{1,\ldots,n-2\}$ such that $A_{in} > 0$ and $A_{kn} = 0$ for all $n > k > i$. Then $B_{in} = (\sum_{k=1}^{n-1} X_{ik}A_{kn} + X_{in}A_{nn} - \sum_{k=1}^{n-1} B_{ik}X_{kn})(X_{nn})^{-1} = (X_{ii}A_{in} + X_{in} - B_{ii}X_{in})(X_{nn})^{-1}$
$= (X_{ii}A_{in})(X_{nn})^{-1} > 0$. Let $n > j > i$. Then $B_{jn} = (\sum_{k=1}^{n-1} X_{jk}A_{kn} + X_{jn}A_{nn} - \sum_{k=1}^{n-1} B_{jk}X_{kn})(X_{nn})^{-1}$
$= (X_{jj}A_{jn} + X_{jn} - B_{jj}X_{jn})(X_{nn})^{-1} = (X_{jn} - X_{jn})(X_{nn})^{-1} = 0$. Then $B \in P_n$. Hence $XP_n \subseteq P_nX$. Therefore P_n is a normal subset of K_n. Next, to show the a-convexity of P_n, let $X, Y \in P_n$ Let $A, B \in K_n$ be such that $A + B = \begin{bmatrix} A_1+B_1 & A_3+B_3 \\ 0 & A_2+B_2 \end{bmatrix} = 1$. Then $A_1 + B_1 = 1$, $A_2 + B_2 = 1$ and $A_3 + B_3 = 0$. Therefore $XA + YB =$
$\begin{bmatrix} X_1A_1+Y_1B_1 & X_1A_3+X_3A_2+Y_1B_3+Y_3B_2 \\ 0 & X_2A_2+Y_2B_2 \end{bmatrix}$. If $X_1 > 1$ or $Y_1 > 1$, then $X_1A_1 > A_1$ or $Y_1B_1 > B_1$, so $X_1A_1 + Y_1B_1 > A_1 + B_1 = 1$. Hence $XA + YB \in P_n$. Suppose that $X_1 = 1$ and $Y_1 = 1$. Then $X_1A_1 + Y_1B_1 = 1, X_2 \geq 1$ and $Y_2 \geq 1$. If $X_2 > 1$ or $Y_2 > 1$ then by using a proof similar to the above, we get that $XA + YB \in P_n$. Suppose that $X_2 = 1$ and $Y_2 = 1$. Then $X_2A_2 + Y_2B_2 = A_2 + B_2 = 1$.
Claim that for all $i \in \{1,\ldots,n-1\}$, $(XA + YB)_{in} = (X_{in})(A_{nn}) + (Y_{in})(B_{nn})$.
Let $i \in \{1,\ldots,n-1\}$. Then $(XA + YB)_{in} = \sum_{k=1}^{n} (X_{ik})(A_{kn}) + \sum_{k=1}^{n} (Y_{ik})(B_{kn})$
$= (X_{ii})(A_{in}) + (X_{in})(A_{nn}) + (Y_{ii})(B_{in}) + (Y_{in})(B_{nn}) = A_{in} + (X_{in})(A_{nn}) + B_{in} + (Y_{in})(B_{nn})$
$= (A_{in})(A_{nn}) + (Y_{in})(B_{nn})$, so we have the claim.

Case 1 : $X_3 = 0$ and $Y_3 = 0$. Then $X = 1, Y = 1$ and $XA + YB = A + B = 1 \in P_n$.

Case 2 : $X_3 = 0$ and $Y_3 \neq 0$.

Subcase 2.1 : $Y_{n-1,n} > 0$. Then $(XA + YB)_{n-1,n} = (X_{n-1,n})(A_{nn}) + (Y_{n-1,n})(B_{nn}) = (Y_{n-1,n})(B_{nn}) > 0$, so $XA + YB \in P_n$.

Subcase 2.2 : there exist an $i \in \{1,\ldots,n-2\}$ such that $Y_{in} > 0$ and $Y_{kn} = 0$ for all $n > k > i$. Then $(XA + YB)_{in} = (X_{in})(A_{nn}) + (Y_{in})(B_{nn}) = (Y_{in})(B_{nn}) > 0$. Let $n > j > i$. Then $(XA + YB)_{jn} = (X_{jn})(A_{nn}) + (Y_{jn})(B_{nn}) = 0$, so $XA + YB \in P_n$.

Case 3 : $X_3 \neq 0$ and $Y_3 = 0$. The proof is similar to the proof of case 2.

Case 4 : $X_3 \neq 0$ and $Y_3 \neq 0$.

Subcase 4.1 : $X_{n-1,n} > 0$ or $Y_{n-1,n} > 0$. Then $(XA + YB)_{n-1,n} = (X_{n-1,n})(A_{nn}) + (Y_{n-1,n})(B_{nn}) > 0$, so $XA + YB \in P_n$.

Subcase 4.2 : there exist $i', i'' \in \{1,\ldots,n-2\}$ such that $X_{i'n} > 0$ and $X_{kn} = 0$ for all $n > k > i'$ and $Y_{i''n} > 0$, $Y_{kn} = 0$ for all $n > k > i''$. Let $i = \max\{i', i''\}$. Then $i = i'$ or $i = i''$, so $(XA + YB)_{in} = (X_{in})(A_{nn}) + (Y_{in})(B_{nn}) > 0$. Let $n > j > i$. Then $(XA + YB)_{jn} = (X_{jn})(A_{nn}) + (Y_{jn})(B_{nn}) = 0$. Therefore $XA + YB \in P_n$.

Hence P_n is an a-convex normal subset of K_n.

To show that P_n is a conic set of K_n, let $X \in (P_n) \cap (P_n)^{-1}$. Then $X \in P_n$ and $X^{-1} \in P_n$, so $X_1 \geq 1$ and $X_1^{-1} \geq 1$. Hence $X_1 = 1$, so $X_2 \geq 1$ and $X_2^{-1} \geq 1$. Thus $X_2 = 1$.

Case 1 : $X_{n-1,n} > 0$. Then $0 = I_{n-1,n} = (XX^{-1})_{n-1,n} = \sum_{k=1}^{n} (X_{n-1,k})(X^{-1})_{kn}$
$= \sum_{k=1}^{n-1} (X_{n-1,k})(X^{-1})_{kn} + (X_{n-1,n})(X^{-1})_{nn} = (X_{n-1,n-1})(X^{-1})_{n-1,n} + (X_{n-1,n})(X^{-1})_{nn} = (x^{-1})_{n-1,n} + (x_{n-1,n})$, so $(X^{-1})_{n-1,n} = -(X_{n-1,n}) < 0$ which is a contradiction since $X^{-1} \in P_n$.

Case 2 : there exist an $i \in \{1,\ldots,n-2\}$ such that $x_{in} > 0$ and $x_{kn} = 0$ for all $n > k > i$. Then $0 = I_{in} = (XX^{-1})_{in} = \sum_{k=1}^{n} (X_{ik})(X^{-1})_{kn} = \sum_{k=1}^{n-1} (X_{ik})(X^{-1})_{kn} + (X_{in})(X^{-1})_{nn}$
$= (X_{ii})(X^{-1})_{in} + (X_{in})(X^{-1})_{nn} = (X^{-1})_{in} + (X_{in})$, so $(X^{-1})_{in} = -(X_{in}) < 0$. Let $n > j > i$. Then
$0 = I_{jn} = (XX^{-1})_{jn} = \sum_{k=1}^{n} (X_{jk})(X^{-1})_{kn} = \sum_{k=1}^{n-1} (X_{jk})(X^{-1})_{kn} + (X_{jn})(X^{-1})_{nn} = (X_{jj})(X^{-1})_{jn} + (X_{jn})(X^{-1})_{nn}$
$= (X^{-1})_{jn} + (X_{jn})$, so $(X^{-1})_{jn} = -(X_{jn}) = 0$ which is a contradiction since $X^{-1} \in P_n$.

Hence $X_3 = 0$, so $X = 1$. Therefore P_n is a conic subset of K_n.

By Theorem 2.9., P_n is the positive cone of K_n, hence K_n is a positively ordered skewsemifield. #

Definition 2.14. Let K and M be positively ordered skewsemifields. A function $f: K \to M$ is called an order homomorphism of K into M if f is an isotone homomorphism of skewsemifields.

An order honomorphism $f: K \to M$ is called an order monomorphism if f is an injection and $f(P_K) = P_{f(K)}$, an order epimorphism if f is onto and $f(P_K) = P_M$, and an order isomorphism if f is a bijection and f and f^{-1} are isotone. K and M are said to be order isomorphic if there exists an order isomorphism K onto M and we denote this by $K \cong_o M$.

Remark 2.15. Let $f: K \to M$ be an order homomorphism of positively ordered skewsemifields. Then the following statements hold :

1) $f(P_K) \subseteq P_M$.

2) m-ker f is a convex normal subgroup of K.

3) If C' is a convex normal subgroup of M then $f^{-1}(C')$ is a convex normal subgroup of K containing m-ker f.

Proof 1) Obvious.

2) By Remark 1.41., 2), m-ker f is an a-convex normal subgroup of K. To show the o-convexity of m-ker f, let $x, y \in$ m-ker f and $z \in K$ be such that $x \le z \le y$. Since f is isotone, $1 = f(x) \le f(z) \le f(y) = 1$, so $f(z) = 1$. Then $z \in$ m-ker f. Hence m-ker f is a convex normal subgroup of K.

3) By Remark 1.41., 3), $f^{-1}(C')$ is an a-convex normal subgroup of K containing m-ker f and by Remark 1.20., 2), $f^{-1}(C')$ is an o-convex subset of K. Hence $f^{-1}(C')$ is a convex normal subgroup of K containing m-ker f. #

Proposition 2.16. Let $f: K \to M$ be a homomorphism of positively ordered skewsemifields. Then the following statements hold :

1) f is isotone if and only if $f(P_K) \subseteq P_M$.

2) if f is a bijection then f^{-1} is isotone if and only if $P_M \subseteq f(P_K)$.

Proof 1) Obvious.

2) Assume that f^{-1} is isotone. Let $y \in P_M$. Then $y \geq 1$. Since f is onto, there exists an $x \in K$ be such that $f(x) = y$. Since f^{-1} is isotone, $1 = f^{-1}(1) \geq f^{-1}(y) = f^{-1}(f(x)) = x$, so $x \in P_K$. Then $y \in f(P_K)$, hence $P_M \subseteq f(P_K)$.

Conversely, assume that $P_M \subseteq f(P_K)$. Let $x, y \in M$ be such that $y \geq x$. If $x = 0$ then $f(y) \geq 0 = f(x)$. Suppose that $x \neq 0$. Then $x^{-1}y \in P_M$, so there exists a $p \in P_K$ such that $f(p) = x^{-1}y$. Since f is onto, there exist $a, b \in K$ such that $f(a) = x$ and $f(b) = y$, so $f(p) = x^{-1}y = f(a)^{-1}f(b) = f(a^{-1}b)$. Since f is an injection, $a^{-1}b = p \in P_K$, so $a^{-1}b \geq 1$. Thus $f^{-1}(y) = b \geq a = f^{-1}(x)$. Hence f^{-1} is isotone. #

Corollary 2.17. Let $f : K \to M$ be a isomorphism of positively ordered skewsemifields. Then f is an order isomorphism if and only if $f(P_K) = P_M$.

Let C be a convex normal subgroup of a positively ordered skewsemifield K. Then K/C is a skewsemifild. Define a relation $\leq$ on K/C as follows : for all $aC, bC \in K/C$, define $aC \leq bC$ if and only if there exist $c_1, c_2 \in C$ such that $ac_1 \leq bc_2$. To show that $\leq$ is a partial order on K/C, it is clear that $\leq$ is reflexive. Let $xC, yC \in K/C$ be such that $xC \leq yC$ and $yC \leq xC$. Then there exist $c_1, c_2, c_3, c_4 \in C$ such that $xc_1 \leq yc_2$ and $yc_3 \leq xc_4$.

Case 1 : $x = 0$. Then $0 \leq yc_2$ and $yc_3 \leq 0$, so $0 \leq y$ and $y \leq 0$. Therefore $y = 0$. Hence $xC = yC$.

Case 2 : $x \neq 0$. Then $c_1(c_2)^{-1} \leq x^{-1}y$ and $x^{-1}y \leq c_4(c_3)^{-1}$. By the o-convexity of C, $x^{-1}y \in C$, so $xC = yC$. Hence $\leq$ is anti-symmetric. Next, let $xC, yC, zC \in K/C$ be such that $xC \leq yC$ and $yC \leq zC$. Then there exist $c_1, c_2, c_3, c_4 \in C$ such that $xc_1 \leq yc_2$ and $yc_3 \leq xc_4$. Since $yc_2C = Cyc_2$, there exists a $c_5 \in C$ such that

$yc_2c_3 = c_5yc_3$. Since $c_5zc_4 \in c_5zC = zc_4C$, there exists a $c_6 \in C$ such that $c_5zc_4 = zc_4c_6$. Then $xc_1c_3 \le yc_2c_3 = c_5bc_3 \le c_5zc_4 = zc_4c_6$, so $xC \le zC$. Thus $\le$ is transitive, and hence $\le$ is a partial order. Next, to show that $\le$ is a compatible partial order on $K_{/C}$, let $xC, yC \in K_{/C}$ be such that $xC \le yC$. Then there exist $c_1, c_2 \in C$ such that $xc_1 \le yc_2$. Let $zC \in K_{/C}$. Then $zxc_1 \le zyc_2$, so $(zC)(xC) = zxC \le zyC = (zC)(yC)$. Since $zC = Cz$, there exist $c_3, c_4 \in C$ such that $zc_3 = c_1z$ and $zc_4 = c_2z$, so $xzc_3 = xc_1z \le yc_2z = yzc_3$. Therefore $(xC)(zC) = xzC \le yzC = (yC)(zC)$. Since C is an a-convex normal set, $(yc_2 + zc_1) \in (yC + zC) = (y + z)C$, so there exists a $c_5 \in C$ such that $yc_2 + zc_1 = (y + z)c_5$. Then $(x + z)c_1 = xc_1 + zc_1 \le yc_2 + zc_1 = (y + z)c_5$, so $xC + zC = (x + z)C \le (y + z)C = yC + zC$. Similarly, $zC + xC \le zC + yC$. Clearly, $[0] \le \alpha$ for every $\alpha \in K_{/C}$. Therefore $K_{/C}$ is a positively ordered skewsemifield.

From the above, we define $\le^*$ on $K_{/C}$ as follows: let $\alpha, \beta \in K_{/C}$, define $\alpha \le \beta$ if and only if for every $a \in \alpha$, there exists a $b \in \beta$ such that $a \le b$. Then we get that $\le^*$ is a positively compatible partial order on $K_{/C}$. To show that $\le = \le^*$, let $\alpha, \beta \in K_{/C}$ be such that $\alpha \le \beta$. If $\alpha = 0$ then $\alpha \le^* \beta$. Suppose that $\alpha \neq 0$. Then there exist an $a \in \alpha$ and $b \in \beta$ such that $a \le b$ and $a \neq 0$. Let $c \in C$. Then $ca^{-1} \in C$. Since $a \le b$, $1 \le a^{-1}b$, so $c \le c(a^{-1}b)$. Since $c(a^{-1}b) \in Cb = bC$, $\alpha \le^* \beta$, so $\le \subseteq \le^*$. Clearly, $\le^* \subseteq \le$.. Hence $\le = \le^*$.

<u>Proposition 2.18.</u> Let K be a positively ordered skewsemifield and $C \subseteq K^*$. Then C is a convex normal subgroup of K if and only if C the m-kernel of some order epimorphism.

<u>Proof</u> Assume that C is a convex normal subgroup of K. Define $\Pi : K \to K_{/C}$ by $\Pi(x) = xC$, for every $x \in K$. Then Π is an epimorphism and m-ker $\Pi = C$. To show that $\Pi(P) = P_{K/C}$, let $x \in P$. Then $x \ge 1$, so $\Pi(x) = xC \ge C$. Therefore $\Pi(x) \in P_{K/C}$, so $\Pi(P) \subseteq P_{K/C}$. Next, let $\alpha \in P_{K/C}$. Then $\alpha \ge C$, so there

exist $a \in \alpha$ and $c \in C$ such that $a \geq c$. Therefore $ac^{-1} \in P$, so $\alpha = aC = (ac^{-1})C = \Pi(ac^{-1}) \in \Pi(P)$. Then $P_{K/C} \subseteq \Pi(P)$, so $\Pi(P) = P_{K/C}$. Therefore Π is an order epimorphism.

The converse follows from Remark 2.16., 2). #

Theorem 2.19. (First Isomorphism Theorem).

Let $f : K \rightarrow M$ be an order epimorphism of positively ordered skewsemifields. Then $K/_{m\text{-}\ker f} \cong_o M$.

Proof Let φ be the isomorphism defined in the proof of Theorem 1.51. To show that φ is isotone, let $\alpha \in K/_{m\text{-}\ker f}$ be such that $\alpha \leq \beta$. There exist $x \in \alpha$, and $y \in \beta$ such that $x \leq y$. Since f is isotone, $\varphi(\alpha) = f(a) \leq f(b) = \varphi(\beta)$. Then φ is isotone. Next, to show that φ^{-1} is isotone, let $y \in P_M$. There exists a $p \in P_K$ such that $f(p) = y$, so $p(m\text{-}\ker f) \in K/_{m\text{-}\ker f}$. Then $y = f(p) = \varphi(p(m\text{-}\ker f)) \in \varphi(P_{K/_{m\text{-}\ker f}})$. Therefore $P_M \subseteq \varphi(P_{K/_{m\text{-}\ker f}})$. Therefore φ^{-1} is isotone, so φ is an order isomorphism. Hence $K/_{m\text{-}\ker f} \cong_o M$. #

Lemma 2.20. Let H be a subskewsemifield of a positively ordered skewsemifield K and C a convex normal subgroup of K. Then $H \cap C$ is a convex normal subgroup of H and HC is a subskewsemifield of K.

Proof By, Lemma 1.52., $H \cap C$ is an a-convex normal subgroup of H and HC is a subskewsemifield of K. To show the o-convexity of $H \cap C$, let $x, y \in H \cap C$ and $z \in H$ be such that $x \leq z \leq y$. By the o-convexity of C, $z \in C$, so $z \in H \cap C$. Therefore $H \cap C$ is a convex normal subgroup of H. #

Theorem 2.21. (Second Isomorphism Theorem).

Let H be a subskewsemifield of a positively ordered skewsemifields K and

C a convex normal subgroup of K such that $P_{HC} \subseteq P_H$. Then $H/_{H \cap C} \cong_o HC/_C$.

Proof Let φ be the epimorphism given in the proof of Theorem 1.53. Let $x \in H$ be such that $x \geq 1$. Then $f(x) = xC \geq C$, hence $\varphi(P_H) \subseteq P_{HC/C}$. To show that $P_{HC/C} \subseteq \varphi(P_H)$, let $\alpha \in P_{HC/C}$. Define $\Pi : HC \to HC/_C$ by $\Pi(x) = xC$. Then Π is an order epimorphism. Then $\Pi(P_{HC}) = P_{HC/C}$. Hence there exists an $x \in P_{HC}$ such that $\alpha = \Pi(x) = xC$. Since $P_{HC} \subseteq P_H$, $x \in P_{HC}$, so $\alpha = xC = \varphi(x) \in \varphi(P_H)$. Hence $P_{HC/C} \subseteq \varphi(P_H)$. Therefore $\varphi(P_H) = P_{HC/C}$, so φ is an order epimorphism and m-ker $\varphi = H \cap C$. Thus $H/_{H \cap C} \cong_o HC/_C$. #

Lemma 2.22. Let D and H be convex normal subgroups of a positively ordered skewsemifield K such that $H \subseteq D$. Then $D/_H$ is a convex normal subgroup of $K/_H$.

Proof By Lemma 1.54., $D/_H$ is a convex normal subgroup of $K/_H$. To show the o-convexity of $D/_H$, let $\alpha, \beta \in D/_H$ and $\gamma \in K/_H$ be such that $\alpha \leq \beta \leq \gamma$. Then there exist $a \in \alpha$, $b, c \in \beta$ and $d \in \gamma$ such that $a \leq b$ and $c \leq d$, so $a(b^{-1}c) \leq b(b^{-1}c) = c \leq d$. Since $bH = \gamma = cH$, $b^{-1}c \in H \subseteq D$, so $ab^{-1}c \in D$. By the o-convexity of D, $c \in D$, so $\gamma = cH \in D/_H$. Therefore $D/_H$ is a convex normal subgroup of $K/_H$. #

Theorem 2.23. (Third Isomorphism Theorem).

Let K be a positively ordered skewsemifield, D and C an a-convex normal subgroup of K such $H \subseteq D$. Then $K/_H/_{D/_H} \cong_o K/_D$.

Proof Let φ be the epimorphism given in the proof of Theorem 1.55. Let $\alpha, \beta \in K/_H$ be such that $\alpha \leq \beta$. Then there exist $a \in \alpha$, and $b \in \beta$ such that $a \leq b$. Then $\varphi(\alpha) = (aH) = aD \leq bD = (bH) = \varphi(\beta)$. Therefore f is isotone. Therefore

$\varphi(P_{K/H}) \subseteq (P_{K/D})$. To show that $P_{K/D} \subseteq \varphi(P_{K/H})$, let $\alpha \in P_{K/D}$. Then there exist $a \in \alpha$ and $b \in D$ such that $a \geq b$, so $ab^{-1} \geq 1$. Hence $ab^{-1}H \geq H$. Therefore $ab^{-1}H \in P_{K/H}$. Thus $\alpha = aD = (aD)(b^{-1}D) = (ab^{-1})D = (ab^{-1}H) \in \varphi(P_{K/H})$. Therefore $\varphi(P_{K/H}) = (P_{K/D})$. Hence φ is an order epimorphism and m-ker $\varphi = D_{/H}$. Then

$K_{/H}/_{D/H} \cong_o K_{/D}$. #

<u>Proposition 2.24.</u> Let $f : K \to M$ be an epimorphism of positive ordered skewsemifields. If C' is a convex normal subgroup of M then $K/f^{-1}(C') \cong_o M_{/D}$.

<u>Proof</u> By Remark 2.16., 3), $f^{-1}(C')$ is a convex normal subgroup of M. Let φ be the epimorphism defined in the proof of Proposition 1.56.

To show that φ is isotone, let $x, y \in K$ be such that $x \geq y$. Then $f(x) \geq f(y)$. So $\varphi(x) = f(x)C' \geq f(y)C' = \varphi(y)$. Therefore φ is isotone, so $\varphi(P_K) = P_{M/C'}$. Let $\alpha \in P_{M/C'}$. Define $\Pi : M \to M_{/C'}$ by $\Pi(x) = xC'$. for all $x \in M$. Then Π is an order isomorphism. Thus $\Pi(P_M) = P_{M/C'}$. Then there exists a $y \in P_M$ such that $\alpha = \Pi(y) = yC'$. Since $f(P_K) \subseteq P_M$, there exists an $x \in P_K$ such that $f(x) = y$, so $\alpha = yC' = f(x)C' = \varphi(x) \in \varphi(P_K)$. Hence $P_{M/C'} \subseteq \varphi(P_K)$. Therefore $\varphi(P_K) = P_{M/C'}$, so φ is an order epimorphism and ker $\varphi = f^{-1}(C')$. Then $K/f^{-1}(C') \cong_o M_{/D}$. #

<u>Theorem 2.25.</u> Let P be a semiring with 1. Then there exists a positively ordered skewsemifield having its positive cone isomorphic to P if and only if P satisfies the following conditions :

1) P is M.C.
2) For all $x, y \in P$, $xy = 1$ implies that $x = y = 1$.
3) For every $a \in P$, $aP = Pa$.
4) For all $a, b \in P$, $aP + bP = (a + b)P$.
5) For all $a, b \in P$, $a + b \in aP$ and $a + b \in bP$.

Proof Assume that P satisfies properties 1) – 5). By properties 1) and 3) of P, we get that for all $a, x \in P$, there exists a unique $x_a \in P$ such that $xa = ax_a$. Using the same proof as in [4], pp. 10, we get that

1) $a_a = a$.

2) $(xy)_a = x_a y_a$.

3) $(x_a)_b = x_{ab}$ and

4) $(x + y)_a = x_a + y_a$, for all $a, b, x, y \in P$.

Define a relation $\sim$ on $P \times P$ as follows : for all $a, b, c, d \in P$, $(a,b) \sim (c,d)$ if and only if $ad_b = cb$. In [4], pp. 10 it was shown that $\sim$ is an equivalence relation. Let $K = P \times P/_\sim \cup \{0\}$. Define the operations + and $\bullet$ on K by

$[(a,b)] \bullet [(c,d)] = [(ac_b, db)]$

$[(a,b)] + [(c,d)] = [(ad + cb_d, bd)]$, for all $a, b, c, d \in P$.

In [4], pp. 10, it was shown that $\bullet$ is well-defined and $(K^*, \bullet)$ is a group with $[(1,1)]$ as the identity and $[(b,a)]$ as the multiplicative inverse of $[(a,b)]$ for all $a, b \in P$.

In [4], pp. 53 it was shown that + is well-defined, associative and $\bullet$ is distributive over + in K. Therefore K is a skewsemifiled.

Define $i : P \to K$ by $i(x) = [(x,1)]$ for every $x \in P$. In [4], pp. 56. it was shown that i is a right quotient embedding of P into K. Then K is a skewsemifield of right quotients.

Since i is a homorphism, i(P) is a multiplicative subsemigroup of K. To show that i(P) is a normal set, let $a, b, x \in P$. Since $xa_b \in aP = Pa$, there exists a $y \in P$ such that $ax_b = ya$, so $ax_b b = yab$. By $(yab,ab) \sim (y,1)$,
$[(a,b)]i(x)[(a,b)]^{-1} = [(a,b)]i(x)[(b,a)] = [(a,b)][(x,1)][(b,a)] = [(ax_b,b)][(b,a)]$
$= [(ax_b b_b, ab)] = [(ax_b b, ab)] = [(yab,ab)] = [(y,1)] \in i(P)$, so i(P) is a normal subset of K. Next, to show the a-convexity, let $i(a), i(b) \in i(P)$ and $\alpha = [(x,y)]$ and $\beta = [(c,d)] \in K$ be such that $\alpha + \beta = 1$. Then $[(1,1)] = \alpha + \beta = [(x,y)] + [(c,d)]$
$= [(xd + cy_d, yd)]$, so $xd + cy_d = yd$. Hence $\alpha i(a) + \beta i(b) = [(x,y)][(a,1)] + [(c,d)][(b,1)]$
$= [(ax_y, y)] + [(cb_d, d)] = [(xa_y d + cb_d y_d, yd)] = [(xda_{yd} + cy_d b_{yd}, xd + cy_d)]$. By 4),

there exists a $p \in P$ such that $xda_{yd} + cy_d b_{yd} = p(xd + cy_d)$, so $(xda_{yd} + cy_d b_{yd}, xd + cy_d) \sim (p,1)$. Then $\alpha i(a) + \beta i(b) = [(p,1)] = i(p) \in i(P)$, hence $i(P)$ is an a-convex normal set.

To show that $i(P)$ is an additive ideal of K, let $\alpha = [(a,b)]$ K. Then $1 + \alpha = [(1,1)] + [(a,b)] = [(b + a,b)]$ By 5) and 4), there exists a $p \in P$ such that $b + a = pb$. Then $(b + a,b) \sim (p,1)$, so $1 + \alpha = [(p,1)] \in i(P)$. By 5) and 4), there exists a $p' \in P$ such that $a + b = p'b$. Then $\alpha + 1 = [(a,a)] + [(1,1)] = [(a + b,b)] = [(p',1)] \in i(P)$, so $i(P)$ is an additive ideal of K.

To show that $i(P)$ is a conic set of K, let $\alpha \in i(P) \cap i(P)^{-1}$. Then there exist $a, b \in P$ such that $i(a) = \alpha = i(b)^{-1}$. Then $[(ab,1)] = i(ab) = i(a)i(b) = i(b)^{-1}i(a) = [(1,1)]$, so $(ab,1) \sim (1,1)$ and therefore $ab = 1$. By 2), $a = b = 1$, so $\alpha = [(1,1)]$, hence $i(P)$ is a conic set.. By Theorem 2.9., $i(P)$ is a positive cone of K.

Conversely, let P be the positively cone of some positive ordered skewsemifield. Then 1), 2) and 3) clearly hold. Let $a, b \in P$. By Proposition 1.36., $aP + bP = (a + b)P$.

To prove 5), let $a, b \in P$. Then $ab^{-1} + 1 \in P$, so there exists a $p \in P$ such that $ab^{-1} + 1 = p$. Then $(a + b) = (ab^{-1} + 1)b = pb \in Pb = bP$. Since $1 + a^{-1}b \in P$, there exists a $p' \in P$ such that $1 + a^{-1}b = p'$. Then $(a + b) = a(a^{-1}b + 1)b = ap' \in aP$. #

Theorem 2.26. Let P be a semiring with 1 which satisfies 1) – 5) in Theorem 2.25. and K its skewsemifield of right quotients. Then K is the smallest positively ordered skewsemifield having P as its positive cone.

Proof Let i be a right quotients embedding of P into K. Let L be a skewsemifield and $j : P \to L$ a monomorphism. Define $f : K \to L$ by $f[(x,y)] = j(x)j(y)^{-1}$ for every $[(x,y)] \in K$. To prove that f is well-defined, let $(a,b) \sim (a',b')$. Then $ab'_b = a'b$, so $j(a)j(b'_b) = j(ab'_b) = j(a'b) = j(a')j(b)$.

Claim that for all $x, y \in P$, $j(x_y) = j(y)^{-1}j(x)j(y)$. Let $x, y \in P$. Since $xy = yx_y$,

$j(x)j(y) = j(y)j(x_y)$, so $j(x_y) = j(y)^{-1}j(x)j(y)$ and we have the claim.
By the claim, $j(a)j(b)^{-1}j(b')j(b) = j(a)j(b'_b) = j(a')j(b)$, so $j(a)j(b)^{-1} = j(a')j(b')^{-1}$.
Therefore f is well-defined.

Next, to show that f is a monomorphism, let $\alpha = [(a,b)]$ and $\beta = [(c,d)] \in K$. Then $\alpha = [(ac_b, db)]$, so $f(\alpha\beta) = j(ac_b)j(db)^{-1}$ $= j(a)j(c_b)j(d)j(b) = j(a)j(b)^{-1}j(c)j(b)j(b){-}1j(d)^{-1} = j(a)j(b)^{-1}j(c)j(d)^{-1} = f(\alpha)f(\beta)$.
Since $\alpha + \beta = [(ad + cb_d, bd)]$, $f(\alpha + \beta) = j(ad + cb_d)j(bd)^{-1}$ $= [j(ad) + j(cb_d)]j(d)^{-1}j(b)^{-1} = j(a)j(b)^{-1} + j(c)j(d)^{-1}j(b)j(d)j(d)^{-1}j(b)^{-1}$ $= j(a)j(b)^{-1} + j(c)j(d)^{-1} = f(\alpha) + f(\beta)$. Thus f is a homomorphism. Next, let $\alpha = [(a,b)] \in K$ be such that $j(a)j(b)^{-1} = f(\alpha) = 1$. Then $j(a) = j(b)$, so $a = b$. Therefore m-ker $f = \{ 1 \}$, hence f is a monomorpism.

To prove that $f \circ i = j$, let $x \in P$. Then $f \circ i(x) = f(i(x)) = f([(x,1)]) = j(x)j(1)^{-1}$ $= j(x)$. Next, to show the uniqueness, let $h : K \rightarrow L$ be such that $h \circ i = j$. Let $\alpha = [(a,b)] \in K$. Then $f(\alpha) = j(a)j(b)^{-1} = [h \circ i(a)][h \circ i(b)]^{-1} = h(i(a))h(i(b^{-1})) = h[(a,b)]$ $= h(\alpha)$.

To prove that f is isotone, let $\alpha = [(a,b)] \in K$ be such that $1 \leq \alpha$. Then $[(a,b)] \in i(P)$, so $f([(a,b)]) \in f((P)) = j(P)$. Therefore $f(P_K) \subseteq P_{f(K)}$.
By Proposition 2.17., 1), f is isotone.

Next, to show that $P_{f(K)} \subseteq f(P_K)$, let $\alpha \in P_{f(K)} = j(P)$. Then there exists a $p \in P$ such that $\alpha = j(p) = f(i(p)) \in f(P_K)$. Then $P_{f(K)} = f(P_K)$, so by Corollary 2.18., $K \cong_o f(K)$. Therefore K is the smallest positively ordered skewsemifield having P as its positive cone. #

<u>Definition 2.27.</u> Let G be a group. A <u>compatible partial order</u> $\leq$ on G is a partial order on G such that for all $x, y, z \in G$, $x \leq y$ implies that $xz \leq yz$ and $zx \leq zy$.

<u>Proposition 2.28.</u> Let C be an a-convex normal subgroup of skewsemifield K. Let $\leq$ be a compatible partial order on C and $\leq^*$ a compatible partial order on

the skewsemifield K/C. Suppose that

1) P_C is invariant under all inner automorphisms of K,

2) for all $x, y \in P_C$ and $a, b \in K$ such that $a + b = 1$, $ax + by$ P_C,

3) for every $x \in K$, $1 + x \in C$ implies that $1 + x \in P_C$ and $x + 1 \in C$ implies that $x + 1 \in P_C$, and

4) K/C is a left [right] additively cancellative skewsemifield.

Then there exists a compatible partial order $\leq$ on K such that $\leq$ is the restriction of the partial order on C and the projection map Π is an order epimorphism.

<u>Proof</u> Let $P = P_C \cup (\bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha)$

To show that P is a multiplicative subsemigroup of K, let $x, y \in P$.

Case 1 : $x, y \in P_C$. Then $xy \in P_C \subseteq P$.

Case 2 : $x \in P_C$ and $y \in \alpha$ where $\alpha \in P_{K/C-\{C\}}$. Then $xy \in (P_C)\alpha = \alpha$, so $xy \in P$.

Case 3 : $x \in \alpha$ where $\alpha \in P_{K/C-\{C\}}$ and $y \in P_C$. The proof is similar to the proof of case 2.

Case 4 : $x \in \alpha$ and $y \in \beta$ where $\alpha, \beta \in P_{K/C-\{C\}}$. Then $xy \in \alpha\beta$ and $\alpha\beta \mathrel{*>} \alpha \mathrel{*>} C$, so $\alpha\beta \in P_{K/C-\{C\}}$. Therefore $xy \in P$, so P is a multiplicative subsemigroup.

To show that P is a conic set of K, let $x \in P \cap P^{-1}$. Then $x, x^{-1} \in P$.

Case 1 : $x \in P_C$ and $x^{-1} \in gC$ where $gC \in P_{K/C-\{C\}}$. Hence there exist $c_1, c_2 \in C$ such that $x = c_1$ and $x^{-1} = gc_2$, so $1 = x^{-1}x = gc_2c_1$. Then $g = (c_1)^{-1}(c_2)^{-1} \in C$, so $gC = C$ which is a contradiction.

Case 2 : $x^{-1} \in P_C$ and $x \in gC$ where $gC \in P_{K/C-\{C\}}$. Then there exist $c_1, c_2 \in C$ such that $x = gc_1$ and $x^{-1} = c_2$, so $1 = xx^{-1} = gc_1c_2$. Then $g = (c_2)^{-1}(c_1)^{-1} \in C$, so $gC = C$ which is a contradiction.

Case 3 : $x \in g_1C$ and $y \in g_2C$ where $g_1C, g_2C \in P_{K/C-\{C\}}$. Then there exist $c_1, c_2 \in C$ such that $x = g_1c_1$ and $x^{-1} = g_2c_2$, so $1 = x^{-1}x = g_2c_2g_1c_1$. Then $g_2c_2 = (c_1)^{-1}(g_1)^{-1} = (g_1c_1)^{-1}$, so $g_2C = ((g_1c_1)^{-1})C = (g_1)^{-1}C = (g_1C)^{-1}$. Therefore $g_2C \in P^* \cap (P^*)^{-1} = \{ C \}$, so $g_2C = C$ which is a contradiction. Then $x, x^{-1} \in P_C$, so $x \in P_C \cap (P_C)^{-1} = \{1\}$. Thus $x = 1$. Hence P is a conic subset of K.

To show that P is an additive ideal of K, let $x \in P$. Then $x + 1 \in xC + C$. Since $xC + C \in P_{K/C}$, $xC + C \,{}^*\!\geq C$.

Case 1 : $xC + C = C$. Then $x + 1 \in C$. By 3), $x + 1 \in P_C \subseteq P$.

Case 2 : $xC + C \,{}^*\!> C$. Then $x + 1 \in P$.

Similarly, $1+x \in P$. Hence P is an additive ideal of K.

Let $x \in P$ and $y \in K^*$.

Case 1 : $x \in P_C$. Let $i_y : K \to K$ be defined by $i_y(g) = ygy^{-1}$, for every $g \in K$. Then i_y is an inner automorphism of K. By 1), $yxy^{-1} \in yP_Cy^{-1} = i_y(P_C) = P_C$. Then $yxy^{-1} \in P$.

Case 2 : $x \in \alpha$ for some $\alpha \in P_{K/C-\{C\}}$. Then $yxy^{-1} \in (yxy^{-1})C = (yC)(xC)(yC)^{-1} = (yC)\alpha(yC)^{-1}$. Since $P_{K/C}$ is a normal set, $(yC)\alpha(yC)^{-1} \in P_{K/C}$, so $(yC)\alpha(yC)^{-1} \,{}^*\!\geq C$. If $(yC)\alpha(yC)^{-1} = C$ then $\alpha = C$ which is a contradiction. Thus $(yC)\alpha(yC)^{-1} \,{}^*\!> C$, so $yxy^{-1} \in P$. Hence P is a normal subset of K. Next, to show the a-convexity of P, let $x, y \in P$ and $a, b \in K$ be such that $a + b = 1$.

Case 1 : $x, y \in P_C$. By 2), $ax + by \in P_C \subseteq P$.

Case 2 : $x \in P_C$ and $y \in \alpha$ where $\alpha \in P_{K/C-\{C\}}$. Then $ax + by \in axC + byC = aC + b\alpha$. By 4), $aC + b\alpha > C$, so $ax + bx \in P$.

Case 3 : $y \in P_C$ and $x \in \alpha$ where $\alpha \in P_{K/C-\{C\}}$. The proof is similar to the proof of case 2.

Case 4 : $x \in \alpha$ and $y \in \beta$ where $\alpha, \beta \in P_{K/C-\{C\}}$. The proof is similar to the proof of case 2. By Theorem 2.9., P is the positive cone of K.

Let $\leq'$ be a positive compatible order induced by P. Next, to show that $\leq$ is the restriction of $\leq'$ on C, let $x, y \in C$ be such that $x \leq' y$. Then $1 \leq' x^{-1}y$, so $x^{-1}y \in P$. Therefore $x^{-1}y \in P_C$, so $x \leq y$. Hence $\leq$ is the restriction of $\leq'$ on C.

Finally, to prove that $\Pi(P) = P_{K/C}$, let $x \in P$.

Case 1 : $x \in P_C$. Then $x \in C$, so $\Pi(x) = xC = C \in P_{K/C}$.

Case 2 : $x \in \alpha$ for some $\alpha \in P_{K/C} - \{C\}$. Then $\Pi(x) = xC = \alpha \in P_{K/C}$.

Therefore $\Pi(P) \subseteq P_{K/C}$. Next, let $\alpha \in P_{K/C}$.

Case 1 : $\alpha = C$. Then $\alpha = C = \Pi(1) \in \Pi(P_C) \subseteq \Pi(P)$.

Case 2 : $\alpha \neq C$. Then $\alpha \in P_{K/C} - \{C\}$. Let $x \in \alpha$. Then $x \in P$, so $\alpha = xC = \Pi(\bigcup_{\alpha \in P_{K/C} - \{C\}} \alpha) \in \Pi(P)$. Hence $P_{K/C} \subseteq \Pi(P)$, so $\Pi(P) = P_{K/C}$. #

<u>Definition 2.29.</u> Let $\{ K_i \mid i \in I \}$ be a family of positively ordered skewsemifields. Define $\leq$ on $\prod_{i \in I} K_i$ by the natural partial order, that is for all $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} K_i$, $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ if and only if $x_i \leq y_i$ for every $i \in I$.

<u>Remark 2.30.</u> Let $\{ K_i \mid i \in I \}$ be a family of positively ordered a skewsemifields. Then $P_{\prod_{i \in I} K_i} = \prod_{i \in I} P_i$ where $P_i = \{ x \in K_i \mid x \geq 1_i \}$ for every $i \in I$.

<u>Proposition 2.31.</u> Let $\{ K_i \mid i \in I \}$ be a family of positively ordered skewsemifields and C_i a convex normal subgroup of K_i for all $i \in I$. Then $\prod_{i \in I} C_i$ is a convex normal subgroup of $\prod_{i \in I} K_i$ and $\prod_{i \in I} K_i / \prod_{i \in I} C_i \cong_o \prod_{i \in I} (K_i/C_i)$.

<u>Proof</u> Let φ be an epimorphism given in the proof of Proposition 1.61. To show that $\varphi(P_{\prod_{i \in I} K_i}) = P_{\prod_{i \in I} (K_i/C_i)}$, let $(x_i)_{i \in I} \in \prod_{i \in I} K_i$ be such that $(x_i)_{i \in I} \geq (1_i)_{i \in I}$. Then $x_i \geq 1_i$ for all $i \in I$, so $x_iC_i \geq C_i$ for all $i \in I$. Therefore $\varphi[(x_i)_{i \in I}] = (x_iC_i)_{i \in I} \in P_{\prod_{i \in I} (K_i/C_i)}$, so $\varphi(P_{\prod_{i \in I} K_i}) \subseteq P_{\prod_{i \in I} (K_i/C_i)}$.

Next, let $(x_i)_{i \in I} \in \prod_{i \in I} (K_i)$ be such that $(x_iC_i)_{i \in I} \geq (C_i)_{i \in I}$. Then $x_iC_i \geq C_i$ for all $i \in I$, so there exist $c_i, d_i \in C_i$ such that $x_ic_i \geq d_i$ for all $i \in I$. Therefore $(x_iC_i)_{i \in I} = ([x_ic_i(d_i)^{-1}]C_i)_{i \in I} = \varphi(x_ic_i(d_i)^{-1})_{i \in I} \in \varphi(P_{\prod_{i \in I} K_i})$, and hence $P(\prod_{i \in I} (K_i/C_i)) \subseteq \varphi(P_{\prod_{i \in I} K_i})$. Therefore $\varphi(P_{\prod_{i \in I} K_i}) = P_{\prod_{i \in I} (K_i/C_i)}$, so φ is an order epimorphism. Clearly, m-ker $\varphi = \prod_{i \in I} C_i$. By Theorem 2.20., $\prod_{i \in I} K_i / \prod_{i \in I} C_i \cong_o \prod_{i \in I} (K_i/C_i)$. #