CHAPTER II

RINGS OF ALL STRICTLY UPPER TRIANGULAR MATRICES

For a ring R and a positive integer n, let $SU_n(R)$ denote the ring of all strictly upper triangular $n \times n$ matrices under the usual addition and multiplication of matrices.

Let R be a ring and n a positive integer. If $n \leq 2$, then $SU_n(R)$ is a zero ring. It is clearly seen that if $|R| > 1$ and $n \geq 2$, $SU_n(R)$ has no left identity and no right identity and it is not a regular ring.

Assume R is not a zero ring and $n > 2$. Then there exist $a, b \in R$ such that $ab \neq 0$. Define the matrices $A, B \in SU_n(R)$ by

$$A = \begin{bmatrix} 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & b \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & \cdots & 0 & ab \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \neq [0]_{n \times n} = BA.$$

Therefore $SU_n(R)$ is not commutative.

We conclude that

(1) if $n \leq 2$, then $SU_n(R)$ is a zero ring, and hence $SU_n(R)$ has the intersection property of quasi-ideals,

(2) if $|R| > 1$ and $n \geq 2$, then $SU_n(R)$ has no left identity and no right identity,

(3) if $|R| > 1$ and $n \geq 2$, then $SU_n(R)$ is not regular and

(4) if R is not a zero ring and $n > 2$, then $SU_n(R)$ is not commutative.

It seems worthwhile to study the intersection property of quasi-ideals of $SU_n(R)$ for certain rings R. Rings with identity of characteristic $\neq 2$, division rings and the rings $\mathbf{Z}_{p^k}$ for all primes p and positive integers k are rings of our interest in this chapter.

We show in the following theorem that $SU_n(R)$ does not have the intersection property of quasi-ideals if R has an identity, $|R| > 1$, $char(R) \neq 2$ and $n \geq 4$.

Theorem 2.1. *Let R be a ring with identity, $|R| > 1$ and $char(R) \neq 2$. If n is a positive integer such that $SU_n(R)$ has the intersection property of quasi-ideals, then $n \leq 3$.*

Proof. Let e be the identity of R. Since $char(R) \neq 2$, $2e \neq 0$ and $-e \neq e$. Assume that $n \geq 4$. Let

$$A = \begin{bmatrix} 0 & \cdots & 0 & e & e & e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \cdots & 0 & e & -e & e \\ 0 & \cdots & 0 & 0 & 0 & 2e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $C, D \in SU_n(R)$,

$$CA = \begin{bmatrix} 0 & C_{12} & C_{13} & \cdots & C_{1n} \\ 0 & 0 & C_{23} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & e & e & e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & C_{12}+C_{13} \\ 0 & \cdots & 0 & C_{23} \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$DB = \begin{bmatrix} 0 & D_{12} & D_{13} & \cdots & D_{1n} \\ 0 & 0 & D_{23} & \cdots & D_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & D_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & e & -e & e \\ 0 & \cdots & 0 & 0 & 0 & 2e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & 2D_{12}+D_{13} \\ 0 & \cdots & 0 & D_{23} \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

and then

$$CA + DB = \begin{bmatrix} 0 & \cdots & 0 & C_{12}+C_{13}+2D_{12}+D_{13} \\ 0 & \cdots & 0 & C_{23}+D_{23} \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

For $C, D \in SU_n(R)$,

$$AC = \begin{bmatrix} 0 & \cdots & 0 & e & e & e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & C_{12} & C_{13} & \cdots & C_{1n} \\ 0 & 0 & C_{23} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & C_{n-2,n-1} & C_{n-2,n}+C_{n-1,n} \\ 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

and

$$BD = \begin{bmatrix} 0 & \cdots & 0 & e & -e & e \\ 0 & \cdots & 0 & 0 & 0 & 2e \\ 0 & \cdots & 0 & 0 & 0 & e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & D_{12} & D_{13} & \cdots & D_{1n} \\ 0 & 0 & D_{23} & \cdots & D_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & D_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & D_{n-2,n-1} & D_{n-2,n}-D_{n-1,n} \\ 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

and then

$$AC+BD =$$

$$\begin{bmatrix} 0 & \cdots & 0 & C_{n-2,n-1}+D_{n-2,n-1} & C_{n-2,n}+C_{n-1,n}+D_{n-2,n}-D_{n-1,n} \\ 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

From these equalities, we obtain that

$$SU_n(R)\{A,B\} = \left\{ \begin{bmatrix} 0 & \cdots & 0 & x \\ 0 & \cdots & 0 & y \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \middle| \; x, y \in R \right\} \qquad \text{..........(a)}$$

and

$$\{A,B\}SU_n(R) = \left\{ \begin{bmatrix} 0 & \cdots & 0 & x' & y' \\ 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \middle| \; x', y' \in R \right\}. \qquad \text{..........(b)}$$

From (a) and (b),

$$SU_n(R)\{A,B\} \cap \{A,B\}SU_n(R) = \left\{ \begin{bmatrix} 0 & \cdots & 0 & x \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \middle| \; x \in R \right\}. \qquad \text{..........(c)}$$

Since $\mathbf{Z}\{A,B\} = \{nA + n'B \mid n, n' \in \mathbf{Z}\}$,

$\mathbf{Z}\{A,B\}$ =

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 & (n+n')e & (n-n')e & (n+n')e \\ 0 & \cdots & 0 & 0 & 0 & (n+2n')e \\ 0 & \cdots & 0 & 0 & 0 & (n+n')e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; n, n' \in \mathbf{Z} \right\}. \qquad \text{..........(d)}$$

Then from (b) and (d),

$\mathbf{Z}\{A,B\} + \{A,B\}SU_n(R)$ =

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 & (n+n')e & x'+(n-n')e & y'+(n+n')e \\ 0 & \cdots & 0 & 0 & 0 & (n+2n')e \\ 0 & \cdots & 0 & 0 & 0 & (n+n')e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; \begin{matrix} n, n' \in \mathbf{Z} \\ \text{and} \\ x', y' \in R \end{matrix} \right\}.$$

..........(e)

From (a) and (e), we have

$SU_n(R)\{A,B\} \cap (\mathbf{Z}\{A,B\} + \{A,B\}SU_n(R))$ =

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 & z \\ 0 & \cdots & 0 & n'e \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \middle| \; n' \in \mathbf{Z} \text{ and } z \in R \right\} \qquad \text{..........(f)}$$

and from (c) and (d),

$$\mathbf{Z}\{A,B\} + \big(SU_n(R)\{A,B\} \cap \{A,B\}SU_n(R)\big) =$$

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 & (n+n')e & (n-n')e & (n+n')e+x \\ 0 & \cdots & 0 & 0 & 0 & (n+2n')e \\ 0 & \cdots & 0 & 0 & 0 & (n+n')e \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \begin{matrix} n, n' \in \mathbf{Z} \\ \text{and} \\ x \in R \end{matrix} \right\}. \quad \text{..........(g)}$$

From (f), we have that

$$\begin{bmatrix} 0 & \cdots & 0 & e \\ 0 & \cdots & 0 & -e \\ 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{..........(*)}$$

is an element of $SU_n(R)\{A,B\} \cap (\mathbf{Z}\{A,B\} + \{A,B\}SU_n(R))$. We shall show that the matrix (*) is not an element of $\mathbf{Z}\{A,B\} + \big(SU_n(R)\{A,B\} \cap \{A,B\}SU_n(R)\big)$. Suppose on the contrary that it is. From (g), there exist integers $n, n' \in \mathbf{Z}$ and $x \in R$ such that

$$(n+n')e = 0 \quad \text{..........(1)}$$

$$(n-n')e = 0 \quad \text{..........(2)}$$

$$(n+2n')e = -e \quad \text{..........(3)}$$

$$(n+n')e+x = e \quad \text{..........(4)}$$

By (1) and (3),

$$n'e = -e \quad \text{..........(5)}$$

By (1) and (5),

$$ne = e \quad \text{..........(6)}$$

By (2), (5) and (6), we have $e = -e$ which is a contradiction since $char(R) \neq 2$. Therefore the matrix (*) is an element of $SU_n(R)\{A, B\} \cap (\mathbf{Z}\{A, B\} + \{A, B\}SU_n(R))$ but not of $\mathbf{Z}\{A, B\} + \big(SU_n(R)\{A, B\} \cap \{A,B\}SU_n(R)\big)$. Hence $SU_n(R)\{A,B\} \cap (\mathbf{Z}\{A,B\} + \{A,B\}SU_n(R)) \not\subseteq \mathbf{Z}\{A,B\} + \big(SU_n(R)\{A,B\} \cap$

$\{A, B\}SU_n(R))$. By Theorem 1.4, $SU_n(R)$ does not have the intersection property of quasi-ideals. Hence the theorem is proved. □

We know that for any positive integer m, the characteristic of the ring $\mathbf{Z}_m$ is m. Then by Theorem 2.1, we have

Corollary 2.2. *If m and n are positive integers, $m > 2$ and $n \geq 4$, then $SU_n(\mathbf{Z}_m)$ does not have the intersection property of quasi-ideals.*

We shall prove in the next theorem that if R is a division ring, then every quasi-ideal of $SU_3(R)$ is an ideal of $SU_3(R)$. The following lemma is required and it is true for any ring.

Lemma 2.3. *Let R be a ring and Q a quasi-ideal of $SU_3(R)$. Then the following statements hold.*

(1) *If for every $A \in Q$, $A_{12} = 0$, then Q is a right ideal of $SU_3(R)$.*

(2) *If for every $A \in Q$, $A_{23} = 0$, then Q is a left ideal of $SU_3(R)$.*

Proof. First, we note that for $A, B \in SU_3(R)$,

$$AB = \begin{bmatrix} 0 & 0 & A_{12}B_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{..........(*)}$$

(1) Assume that for every $A \in Q$, $A_{12} = 0$. Then for $A \in Q$ and $B \in SU_3(R)$, by (*), $AB = [0]_{3\times 3} \in Q$. Hence Q is a right ideal of $SU_3(R)$.

(2) Assume that for every $A \in Q$, $A_{23} = 0$. Then by (*), $BA = [0]_{3\times 3}$ for all $A \in Q$ and $B \in SU_3(R)$. Hence Q is a left ideal of $SU_3(R)$. □

Theorem 2.4. *If R is a division ring, then every quasi-ideal of $SU_3(R)$ is a left ideal or a right ideal of $SU_3(R)$. Hence for any division ring R, $SU_3(R)$ has the intersection property of quasi-ideals.*

Proof. Let R be a division ring and Q a quasi-ideal of $SU_3(R)$. If for every $A \in Q$, $A_{12} = 0$, then by Lemma 2.3(1), Q is a right ideal of $SU_3(R)$. If for every $A \in Q$, $A_{23} = 0$, then by Lemma 2.3(2), Q is a left ideal of $SU_3(R)$. These both cases imply that Q has the intersection property.

Next, assume that there exist $A, B \in Q$ such that $A_{12} \neq 0$ and $B_{23} \neq 0$. Then

$$SU_3(R)B = \left\{ \begin{bmatrix} 0 & 0 & C_{12}B_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| C \in SU_3(R) \right\}$$

and

$$ASU_3(R) = \left\{ \begin{bmatrix} 0 & 0 & A_{12}C_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| C \in SU_3(R) \right\}.$$

Since for every $x \in R$, $\begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$ is an element of $SU_3(R)$,

$$SU_3(R)B = \left\{ \begin{bmatrix} 0 & 0 & xB_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\}$$

and

$$ASU_3(R) = \left\{ \begin{bmatrix} 0 & 0 & A_{12}x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\}.$$

Since $A_{12} \neq 0$, $B_{23} \neq 0$ and R is a division ring, it follows that $A_{12}R = R$ and $RB_{23} = R$. Consequently,

$$SU_3(R)B = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\}$$

and

$$ASU_3(R) = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\}.$$

We have that each element of $SU_3(R)Q$ and each element of $QSU_3(R)$ is of the form $\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where $a \in R$. This implies that

$$SU_3(R)Q \subseteq \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\} = SU_3(R)B$$

and

$$QSU_3(R) \subseteq \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\} = ASU_3(R).$$

Hence

$$SU_3(R)Q = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| x \in R \right\} = QSU_3(R).$$

Since Q is a quasi-ideal, $SU_3(R)Q \cap QSU_3(R) \subseteq Q$. It follows that $SU_3(R)Q \subseteq Q$ and $QSU_3(R) \subseteq Q$. Therefore Q is an ideal of $SU_3(R)$ and hence Q has the intersection property. □

Observe from the proof of Theorem 2.4 that if R is a division ring and Q is a quasi-ideal of $SU_3(R)$ such that $A_{12} \neq 0$ and $B_{23} \neq 0$ for some $A, B \in Q$, respectively, then Q is an ideal of $SU_3(R)$.

Since the ring $\mathbf{Z}_m$ is a field if m is a prime, by Theorem 2.4, we have

Corollary 2.5. *If p is a prime, then every quasi-ideal of $SU_3(\mathbf{Z}_p)$ is a left ideal or a right ideal of $SU_3(\mathbf{Z}_p)$. Hence for every prime p, $SU_3(\mathbf{Z}_p)$ has the intersection property of quasi-ideals.*

From Theorem 2.1 and Theorem 2.4, the two following corollaries are obtained.

Corollary 2.6. *Let F be a field of characteristic $\neq 2$. Then for a positive integer n, $SU_n(F)$ has the intersection property of quasi-ideals if and only if $n \leq 3$.*

Corollary 2.7. *Let p be a prime such that $p > 2$. Then for a positive integer n, $SU_n(\mathbf{Z}_p)$ has the intersection property of quasi-ideals if and only if $n \leq 3$.*

We have from Corollary 2.5 that every quasi-ideal of $SU_3(\mathbf{Z}_m)$ is a left ideal or a right ideal if m is a prime. It is natural to ask whether or not this property holds if m is not a prime. The negative answer is given by $SU_3(\mathbf{Z}_6)$. We shall show that there exists a quasi-ideal in $SU_3(\mathbf{Z}_6)$ which is neither a left nor a right ideal.

First, we give a general fact of the ring $\mathbf{Z}_m$ as follows : If m and n are integers such that m and n are relatively prime, then in $\mathbf{Z}_{mn}$, $\mathbf{Z}\overline{m} \cap \mathbf{Z}\overline{n}$ = $\{\overline{0}\}$. To prove this, let $x\overline{m} = y\overline{n}$ for some $x, y \in \mathbf{Z}$. Then $mn \mid (xm - yn)$. Then there exists $z \in \mathbf{Z}$ such that $mnz = xm - yn$, so $yn = xm - mnz = m(x - nz)$. Since $x - nz \in \mathbf{Z}$, $m \mid yn$. Since m and n are relative prime, $m \nmid n$, so we have $m \mid y$. Then there exists $k \in \mathbf{Z}$ such that $y = mk$. Thus in $\mathbf{Z}_{mn}$, $x\overline{m} = y\overline{n} = (mk)\overline{n}$ = $k(\overline{mn}) = \overline{0}$.

Example. Let Q be the subset of $SU_3(\mathbf{Z}_6)$ defined by

$$Q = \left\{ \begin{bmatrix} 0 & m\overline{2} & 0 \\ 0 & 0 & n\overline{3} \\ 0 & 0 & 0 \end{bmatrix} \middle| m, n \in \mathbf{Z} \right\}.$$

Then Q is an additive subgroup of $SU_3(\mathbf{Z}_6)$. Since for $A, B \in SU_3(\mathbf{Z}_6)$, $m, n \in \mathbf{Z}$,

$$A\begin{bmatrix} 0 & m\overline{2} & 0 \\ 0 & 0 & n\overline{3} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & nA_{12}\overline{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & m\overline{2} & 0 \\ 0 & 0 & n\overline{3} \\ 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & mB_{23}\overline{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

it follows that

$$SU_3(\mathbf{Z}_6)Q = \left\{ \begin{bmatrix} 0 & 0 & n\overline{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| n \in \mathbf{Z} \right\}$$

and

$$QSU_3(\mathbf{Z}_6) = \left\{ \begin{bmatrix} 0 & 0 & n\overline{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| n \in \mathbf{Z} \right\}.$$

Then $\begin{bmatrix} 0 & 0 & \overline{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & \overline{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are elements of $SU_3(\mathbf{Z}_6)Q$ and $QSU_3(\mathbf{Z}_6)$, respectively. But these matrices do not belong to Q, so Q is neither a left nor a right ideal of $QSU_3(\mathbf{Z}_6)$.

Let $m, n \in \mathbf{Z}$ be such that

$$\begin{bmatrix} 0 & 0 & m\overline{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & n\overline{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $m\overline{3} = n\overline{2} \in \mathbf{Z}\overline{2} \cap \mathbf{Z}\overline{3}$ in $\mathbf{Z}_{2\times 3}$. Since 2 and 3 are relatively prime, $\mathbf{Z}\overline{2} \cap \mathbf{Z}\overline{3} = \{\overline{0}\}$ which implies that $m\overline{3} = n\overline{2} = \overline{0}$. Hence $SU_3(\mathbf{Z}_6)Q \cap QSU_3(\mathbf{Z}_6) = \{\overline{0}\} \subseteq Q$. Therefore Q is a quasi-ideal of $SU_3(\mathbf{Z}_6)$. □

Observe that 6 is not a prime power. In the next theorem, we shall show that if p is a prime and n is a positive integer, then $SU_3(\mathbf{Z}_{p^n})$ has the property that each of its quasi-ideals is a left or a right ideal. The proof of theorem requires the fact that the ideals of $\mathbf{Z}_{p^n}$ form a chain under set inclusion.

We note that in a ring $\mathbf{Z}_m$ where m is a positive integer, the following statements hold.

(1) If I is an ideal of $\mathbf{Z}_m$, then $I = a\mathbf{Z}_m$ for some $a \in \mathbf{Z}$.

(2) If $a \in \mathbf{Z}$ is such that a and m are relatively prime, then $a\mathbf{Z}_m = \mathbf{Z}_m$.

To prove (1), let I be an ideal of $\mathbf{Z}_m$ and $I \neq \{\overline{0}\}$. We have that $\{x \in \mathbf{Z} \mid \overline{x} \in I \text{ and } x > 0\} \neq \varnothing$ since for every $x \in \mathbf{Z}$, $\overline{x} \in I$ implies $\overline{-x} \in I$. Let

$$a = \min \{x \in \mathbf{Z} \mid \overline{x} \in I \text{ and } x > 0\}.$$

Then $\overline{a} \in I$, so $a\mathbf{Z}_m = \overline{a}\,\mathbf{Z}_m \subseteq I$. Let $b \in \mathbf{Z}$ be such that $\overline{b} \in I$. Then there exist q and r in $\mathbf{Z}$ such that $b = qa + r$, $0 \leq r < a$. Therefore $\overline{b} = q\overline{a} + \overline{r}$. It follows that $\overline{r} = \overline{b} - q\overline{a} \in I$. By the property of a, $r = 0$. Then $\overline{b} = q\overline{a} = a\overline{q} \in a\mathbf{Z}_m$.

Next, we shall prove (2). Since a and m are relatively prime, $ax + my = 1$ for some x and y in $\mathbf{Z}$. Then $\overline{1} = \overline{ax + my} = a\overline{x} \in a\mathbf{Z}_m$. Hence $a\mathbf{Z}_m = \mathbf{Z}_m$.

Lemma 2.8. *If p is a prime and k is a positive integer, then $\{p^k\mathbf{Z}_{p^n} \mid k \in \{0, 1, \ldots, n\}\}$ is the set of all ideals of the ring $\mathbf{Z}_{p^n}$ and $p^k\mathbf{Z}_{p^n} \supsetneq p^{k+1}\mathbf{Z}_{p^n}$ for all $k \in \{0, 1, \ldots, n-1\}$.*

Proof. Let I be an ideal of $\mathbf{Z}_{p^n}$ and $I \neq \{\overline{0}\}$. Then $I = a\mathbf{Z}_{p^n}$ for some $a \in \mathbf{Z}$ and $a > 0$. Then $a = p^\ell b$ for some $\ell, b \in \mathbf{Z}$ such that $\ell \geq 0$ and $p \nmid b$. Therefore p^n and b are relatively prime since p is a prime. Consequently, $I = p^\ell b\mathbf{Z}_{p^n} = p^\ell(b\mathbf{Z}_{p^n}) = p^\ell\mathbf{Z}_{p^n}$. If $\ell \geq n$, then $I = \{\overline{0}\}$, a contradiction. Then $\ell < n$, and so we are done. If $k \in \{0, 1, \ldots, n-1\}$, then $p^{k+1}\mathbf{Z}_{p^n} = p^k(p\mathbf{Z}_{p^n}) \subsetneq p^k\mathbf{Z}_{p^n}$.

□

Theorem 2.9. *Let k be a positive integer and p a prime. Then every quasi-ideal of $SU_3(\mathbf{Z}_{p^k})$ is a left ideal or a right ideal. Hence $SU_3(\mathbf{Z}_{p^k})$ has the intersection property of quasi-ideals.*

Proof. Let Q be a quasi-ideal of $SU_3(\mathbf{Z}_{p^k})$. If for every $A \in Q$, $A_{12} = \overline{0}$, then by Lemma 2.3(1), Q is a right ideal of $SU_3(\mathbf{Z}_{p^k})$. If for every $A \in Q$, $A_{23} = \overline{0}$, then by Lemma 2.3(2), Q is a left ideal of $SU_3(\mathbf{Z}_{p^k})$.

Next, assume that there exist $A, B \in Q$ such that $A_{12} \neq \overline{0}$ and $B_{23} \neq \overline{0}$. Then $\{x \in \mathbf{Z} \mid x > 0$ and $\overline{x} = C_{12}$ for some $C \in Q\} \neq \varnothing$ and $\{x \in \mathbf{Z} \mid x > 0$ and $\overline{x} = C_{23}$ for some $C \in Q\} \neq \varnothing$. Let

$$a = \min\{x \in \mathbf{Z} \mid x > 0 \text{ and } \overline{x} = C_{12} \text{ for some } C \in Q\}$$

and

$$b = \min\{x \in \mathbf{Z} \mid x > 0 \text{ and } \overline{x} = C_{23} \text{ for some } C \in Q\}.$$

Then there exist $\hat{A}$, $\hat{B} \in Q$ such that $\hat{A}_{12} = \bar{a}$ and $\hat{B}_{23} = \bar{b}$. Let $C \in Q$ and let $c, d \in \mathbf{Z}$ be such that $C_{12} = \bar{c}$ and $C_{23} = \bar{d}$. Since $a, b, c, d \in \mathbf{Z}$, $a \neq 0$ and $b \neq 0$, there exist $q, r, s, t \in \mathbf{Z}$ such that

$$c = qa + r \text{ where } 0 \le r < a \quad \text{and} \quad d = sb + t \text{ where } 0 \le t < b.$$

Then $r = c - qa$ and $t = d - sb$ which imply that $\bar{r} = \bar{c} - q\bar{a}$ and $\bar{t} = \bar{d} - s\bar{b}$. Since $\hat{A}$, $\hat{B}, C \in Q$ and Q is an additive subgroup of $SU_3(\mathbf{Z}_{p^k})$, it follows that $C - q\hat{A}, C - s\hat{B} \in Q$. But $(C - q\hat{A})_{12} = C_{12} - q\hat{A}_{12} = \bar{c} - q\bar{a} = \bar{r}$ and $(C - s\hat{B})_{23} = C_{23} - s\hat{B}_{23} = \bar{d} - s\bar{b} = \bar{t}$, so by the properties of a and b, $r = 0$ and $t = 0$. Consequently, $C_{12} = q\hat{A}_{12} = q\bar{a}$ and $C_{23} = s\hat{B}_{23} = s\bar{b}$. Hence the following statement is proved.

(*) For every $C \in Q$, there exist $q, s \in \mathbf{Z}$ such that

$$C_{12} = q\bar{a} \text{ and } C_{23} = s\bar{b}.$$

Since for $n \in \mathbf{Z}$, $\begin{bmatrix} \bar{0} & \bar{n} & \bar{0} \\ \bar{0} & \bar{0} & \bar{n} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \in SU_3(\mathbf{Z}_{p^k})$,

$$\begin{bmatrix} \bar{0} & \bar{n} & \bar{0} \\ \bar{0} & \bar{0} & \bar{n} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \hat{B} = \begin{bmatrix} \bar{0} & \bar{0} & \bar{n}\hat{B}_{23} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} & n\bar{b} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}$$

and

$$\hat{A} \begin{bmatrix} \bar{0} & \bar{n} & \bar{0} \\ \bar{0} & \bar{0} & \bar{n} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} & \hat{A}_{12}\bar{n} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} & n\bar{a} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix},$$

we have that

$$SU_3(\mathbf{Z}_{p^k})Q \supseteq \left\{ \begin{bmatrix} \bar{0} & \bar{0} & n\bar{b} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| \, n \in \mathbf{Z} \right\} \qquad \text{..........(1)}$$

and

$$QSU_3(\mathbf{Z}_{p^k}) \supseteq \left\{ \begin{bmatrix} \bar{0} & \bar{0} & n\bar{a} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| n \in \mathbf{Z} \right\}. \qquad \text{..........(2)}$$

If $D \in Q$, then by (*), $D_{12} = k\bar{a}$ and $D_{23} = \ell\bar{b}$ for some $k, \ell \in \mathbf{Z}$ and hence for every $E \in SU_3(\mathbf{Z}_{p^k})$,

$$ED = \begin{bmatrix} \bar{0} & \bar{0} & \ell E_{12}\bar{b} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \text{ and } DE = \begin{bmatrix} \bar{0} & \bar{0} & kE_{23}\bar{a} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix}.$$

This implies that

$$SU_3(\mathbf{Z}_{p^k})Q \subseteq \left\{ \begin{bmatrix} \bar{0} & \bar{0} & n\bar{b} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| n \in \mathbf{Z} \right\} \qquad \text{..........(3)}$$

and

$$QSU_3(\mathbf{Z}_{p^k}) \subseteq \left\{ \begin{bmatrix} \bar{0} & \bar{0} & n\bar{a} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| n \in \mathbf{Z} \right\}. \qquad \text{..........(4)}$$

From (1) and (3), we have

$$SU_3(\mathbf{Z}_{p^k})Q = \left\{ \begin{bmatrix} \bar{0} & \bar{0} & n\bar{b} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| n \in \mathbf{Z} \right\}$$

and (2) and (4) give

$$QSU_3(\mathbf{Z}_{p^k}) = \left\{ \begin{bmatrix} \bar{0} & \bar{0} & n\bar{a} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| n \in \mathbf{Z} \right\}.$$

Hence

$$SU_3(\mathbf{Z}_{p^k})Q = \left\{ \begin{bmatrix} \bar{0} & \bar{0} & b\bar{n} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| \, n \in \mathbf{Z} \right\} = \left\{ \begin{bmatrix} \bar{0} & \bar{0} & \bar{x} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| \, \bar{x} \in b\mathbf{Z}_{p^k} \right\}$$

..........(5)

and

$$QSU_3(\mathbf{Z}_{p^k}) = \left\{ \begin{bmatrix} \bar{0} & \bar{0} & a\bar{n} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| \, n \in \mathbf{Z} \right\} = \left\{ \begin{bmatrix} \bar{0} & \bar{0} & \bar{x} \\ \bar{0} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} \end{bmatrix} \middle| \, \bar{x} \in a\mathbf{Z}_{p^k} \right\}.$$

..........(6)

By Lemma 2.8, $b\mathbf{Z}_{p^k} \subseteq a\mathbf{Z}_{p^k}$ or $a\mathbf{Z}_{p^k} \subseteq b\mathbf{Z}_{p^k}$. Since Q is a quasi-ideal of $SU_3(\mathbf{Z}_{p^k})$, $SU_3(\mathbf{Z}_{p^k})Q \cap QSU_3(\mathbf{Z}_{p^k}) \subseteq Q$.

Case 1 : $b\mathbf{Z}_{p^k} \subseteq a\mathbf{Z}_{p^k}$. By (5) and (6), $SU_3(\mathbf{Z}_{p^k})Q \subseteq QSU_3(\mathbf{Z}_{p^k})$. Then $SU_3(\mathbf{Z}_{p^k})Q = SU_3(\mathbf{Z}_{p^k})Q \cap QSU_3(\mathbf{Z}_{p^k}) \subseteq Q$. Therefore Q is a left ideal of $SU_3(\mathbf{Z}_{p^k})$.

Case 2 : $a\mathbf{Z}_{p^k} \subseteq b\mathbf{Z}_{p^k}$. By (5) and (6), $QSU_3(\mathbf{Z}_{p^k}) \subseteq SU_3(\mathbf{Z}_{p^k})Q$, so $QSU_3(\mathbf{Z}_{p^k}) = SU_3(\mathbf{Z}_{p^k})Q \cap QSU_3(\mathbf{Z}_{p^k}) \subseteq Q$. Therefore Q is a right ideal of $SU_3(\mathbf{Z}_{p^k})$. □