CHAPTER III

REGRESSIVE GENERALIZED TRANSFORMATION SEMIGROUPS

Let S be a transformation semigroup on a set and $\theta \in S$. To make the difference between the nth power of $\alpha \in S$ in the transformation semigroup S and the nth power of $\alpha \in S$ in the generalized transformation semigroup (S,θ) where n is a positive integer, we use α^n and $(\alpha,\theta)^n$ to denote the nth power of α in S and (S,θ), respectively.

Through this chapter, let X denote a partially ordered set.

3.1 Regular Elements of Regressive Generalized Transformation Semigroups

Theorem 3.1.1. Let (S,θ) be a regressive generalized transformation semigroup on X. Then every regular element of (S,θ) is an idempotent of (S,θ).

Proof. Let $\alpha \in S$ be a regular element of (S,θ). Then there exists an element $\beta \in S$ such that $\alpha\theta\beta\theta\alpha = \alpha$. Since $\theta\beta\theta \in S$, α is a regular element of S. By Theorem 2.1.1, $\alpha^2 = \alpha$. Let $x \in \Delta\alpha$. Then $x\alpha = x\alpha\theta\beta\theta\alpha$. Since α, β and θ are regressive, $x\alpha = x\alpha\theta\beta\theta\alpha \le x\alpha\theta \le x\alpha$. It implies that $x\alpha = x\alpha\theta$, so we have that $x\alpha = x\alpha^2 = x\alpha\theta\alpha$. Now we have proved that $\Delta\alpha \subseteq \Delta\alpha\theta\alpha$ and $x\alpha = x\alpha\theta\alpha$ for all $x \in \Delta\alpha$. But $\Delta\alpha\theta\alpha \subseteq \Delta\alpha$, so $\alpha = \alpha\theta\alpha = (\alpha,\theta)^2$. Hence α is an idempotent in (S,θ). $\square$

Theorem 3.1.2. Let (S,θ) be a regressive generalized transformation semigroup on X. Then every idempotent of (S,θ) is an idempotent of S.

Proof. Let $\alpha \in S$ be an idempotent of (S,θ). Then $\alpha\theta\alpha = \alpha$. Therefore α is a regular element of S. By Theorem 2.1.1, α is an idempotent of S. $\square$

Corollary 3.1.3. Let (S,θ) be a regressive generalized transformation semigroup on X. If (S,θ) is a regular semigroup, then S is a regular semigroup.

Proof. Assume that (S,θ) is a regular semigroup. By Theorem 3.1.1, every element of (S,θ) is an idempotent of (S,θ). Then by Theorem 3.1.2, every element of S is an idempotent of S. Hence S is a regular semigroup. □

Theorem 3.1.4. Let (S,θ) be a regressive generalized transformation semigroup on X and $\alpha \in S$. Then α is an idempotent of (S,θ) if and only if

(i) for every $a \in \nabla\alpha$, $a = \min(a\alpha^{-1})$ and

(ii) $\nabla\alpha \subseteq \Delta\theta$ and $a\theta = a$ for all $a \in \nabla\alpha$.

Proof. Assume that α is an idempotent of (S,θ). By Theorem 3.1.2, α is an idempotent of S. Then by Theorem 2.1.2, we have that for every $a \in \nabla\alpha$, $a = \min(a\alpha^{-1})$. Hence (i) holds. To proves (ii), let $b \in \nabla\alpha$. Then $x\alpha = b$ for some $x \in \Delta\alpha$. Since $\alpha\theta\alpha = \alpha$, $x \in \Delta\alpha\theta\alpha$ which implies that $x\alpha \in \Delta\theta$. Then $b \in \Delta\theta$. By (i), $b = \min(b\alpha^{-1})$. Since α and θ are regressive, $b = x\alpha = x\alpha\theta\alpha = b\theta\alpha \le b\theta \le b$ which implies that $b\theta = b$. Hence (ii) holds.

Conversely, assume that (i) and (ii) hold. By Theorem 2.1.2 and (i), $\alpha^2 = \alpha$. Therefore $\nabla\alpha \subseteq \Delta\alpha$ and $a\alpha = a$ for all $a \in \nabla\alpha$. Let $x \in \Delta\alpha$. Then $x\alpha \in \nabla\alpha$, so by (ii), $x\alpha \in \Delta\theta$ and $x\alpha\theta = x\alpha$ which implies that $x\alpha\theta\alpha = x\alpha^2 = x\alpha$. This proves that $\Delta\alpha \subseteq \Delta\alpha\theta\alpha$ and $x\alpha = x\alpha\theta\alpha$ for all $x \in \Delta\alpha$. But $\Delta\alpha\theta\alpha \subseteq \Delta\alpha$, so $\alpha = \alpha\theta\alpha$. Hence α is an idempotent of (S,θ). □

Corollary 3.1.5. Let (S,θ) be a regressive generalized transformation semigroup on X and $\alpha \in S$. Then α is a regular element of (S,θ) if and only if

(i) for every $a \in \nabla\alpha$, $a = \min(a\alpha^{-1})$ and

(ii) $\nabla\alpha \subseteq \Delta\theta$ and $a\theta = a$ for all $a \in \nabla\alpha$.

Proof. It follows directly from Theorem 3.1.1 and Theorem 3.1.4. □

Theorem 3.1.6. Let S be $PT_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$ or $W_{RE}(X)$. Then (S,θ) is a regular semigroup if and only if

(i) X is isolated and

(ii) $\theta = 1_X$.

Proof. Assume that (S,θ) is a regular semigroup. By Corollary 3.1.3, S is a regular semigroup. By Theorem 2.1.6, X is isolated. Hence (i) holds. Since $\theta \in S$ and X is isolated, by Lemma 2.1.5, $\theta = 1_{\Delta\theta}$. To show that $\theta = 1_X$, suppose not. Then there exists an element $a \in X \setminus \Delta\theta$. Let $\alpha \in PT(X)$ be such that $\Delta\alpha = \{a\} = \nabla\alpha$. Then $\alpha \in S$ and $\alpha\theta\alpha = 0 \neq \alpha$. Thus α is not an idempotent of (S,θ). By Theorem 3.1.1, α is not a regular element of (S,θ) which implies that (S,θ) is not a regular semigroup, a contradiction. Hence $\theta = 1_X$, and so (ii) holds.

Conversely, assume that (i) and (ii) hold. Since X is isolated, by Lemma 2.1.5, S is a regular semigroup. Since $\theta = 1_X$, we have that (S,θ) and S are the same semigroup. Hence (S,θ) is a regular semigroup. □

Theorem 3.1.7. Let S be $T_{RE}(X)$ or $V_{RE}(X)$ and $\theta \in S$. Then (S,θ) is a regular semigroup if and only if

(i) for every chain C of X, $|C| \leq 2$ and

(ii) $\theta = 1_X$.

Proof. Assume that (S,θ) is a regular semigroup. By Corollary 3.1.3, S is a regular semigroup. By Theorem 2.1.7, (i) holds. To prove that $\theta = 1_X$, suppose not. Then there exists an element $a \in X$ such that $a\theta < a$. Since $1_X \in S$ and $a1_X\theta 1_X = a\theta < a = a1_X$, we have that $1_X\theta 1_X \neq 1_X$, so 1_X is not an idempotent of (S,θ). By Theorem 3.1.1, 1_X is not a regular element of (S,θ) which is a contradiction.

Conversely, assume that (i) and (ii) hold. By Theorem 2.1.7, S is a regular semigroup. Since $\theta = 1_X$, we have that $(S,\theta) = S$ as semigroups. Hence (S,θ) is a regular semigroup. □

3.2 Eventual Regularity of $(PT_{RE}(X),\theta)$, $(T_{RE}(X),\theta)$ and $(I_{RE}(X),\theta)$

In this section, we give necessary and sufficient conditions for X and θ such that (S,θ) is eventually regular where S is $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S$.

We begin this section by giving a general fact of infinite chains which is used later.

Proposition 3.2.1. If X is an infinite chain, then there exist $x_1, x_2, x_3, \ldots$ in X such that

$$x_1 < x_2 < x_3 < \ldots$$

or there exist $x_{-1}, x_{-2}, x_{-3}, \ldots$ in X such that

$$x_{-1} > x_{-2} > x_{-3} > \ldots.$$

Proof. Assume that X is an infinite chain.

Case 1 : X does not have a maximum element. Let $x_1 \in X$. Then x_1 is not the maximum element of X, so there exists an element x_2 in X such that $x_1 < x_2$. Since X has no a maximum element, x_2 is not the maximum element of X. Then $x_2 < x_3$ for some x_3 in X. By continuing this process inductively, we can obtain $x_1, x_2, x_3, \ldots$ in X such that $x_1 < x_2 < x_3 < \ldots$.

Case 2 : X does not have a minimum element. By similar proof to Case 1, we can get $x_{-1}, x_{-2}, x_{-3}, \ldots$ in X such that $x_{-1} > x_{-2} > x_{-3} > \ldots$.

Case 3 : X has a maximum element and a minimum element. Let M and m be the maximum element and the minimum element of X, respectively.

Subcase 3.1 : There exists an element a in $X \setminus \{M\}$ such that for $b \in X$, $a < b$ implies that $a < x < b$ for some $x \in X$. Since $a < M$, there exists an element $x_{-1} \in X$ such that $a < x_{-1} < M$. By assumption, $a < x_{-2} < x_{-1}$ for some $x_{-2} \in X$. Continue this process inductively, we have $x_{-1}, x_{-2}, x_{-3}, \ldots$ in X such that $x_{-1} > x_{-2} > x_{-3} > \ldots$.

Subcase 3.2 : For every $a \in X \setminus \{M\}$, there exists an element b in X such that $a < b$ and for $x \in X$, $a \le x \le b$ implies that $x = a$ or $x = b$. Since X is infinite, $m \in X \setminus \{M\}$. Let $x_1 = m$. Then there exists $x_2 \in X$ such that $x_1 < x_2$ and there is no $x \in X$ such that $x_1 < x < x_2$. Then $x_2 \ne M$ since X is infinite. By assumption, $x_2 < x_3$ for some $x_3 \in X$ such that there is no $x \in X$ with $x_2 < x < x_3$. Then $\{x \in X / x \le x_3\} = \{x_1, x_2, x_3\}$. Since X is infinite, $x_3 \in X \setminus \{M\}$. Again, there exists an element $x_4 \in X$ such that $x_3 < x_4$ and there is no $x \in X$ with $x_3 < x < x_4 \ldots$. By this process, we can obtain $x_1, x_2, x_3, \ldots$ in X such that $x_1 < x_2 < x_3 < \ldots$. □

Lemma 3.2.2. Let $\theta \in PT_{RE}(X)$ have the property that for x, y in the domain of θ, $x < y$ implies $x \le y\theta \le y$. Then for all x, y, z in the domain of θ, $x < y < z$ implies that $x\theta < z\theta$.

Proof. Let $x, y, z \in \Delta\theta$ be such that $x < y < z$. By assumption, $x\theta \le x \le y\theta \le y \le z\theta \le z$. Since $x < y$, we have that $x\theta < z\theta$. □

Lemma 3.2.3. Let $\theta \in PT_{RE}(X)$. If C is a finite chain of X contained in the domain of θ such that $x \le y\theta \le y$ for all $x, y \in C$ with $x < y$, then $|C\theta| \ge \frac{|C|}{2}$.

Proof. Let $C = \{x_1, x_2, \ldots, x_n\}$ and $x_1 < x_2 < \ldots < x_n$. Then

$x_1\theta \le x_1 \le x_2\theta \le x_2 \le \ldots \le x_n\theta \le x_n$. By Lemma 3.2.2, $x_1\theta < x_3\theta < x_5\theta < \ldots < x_n\theta$ if n is odd and $x_1\theta < x_3\theta < x_5\theta < \ldots < x_{n-1}\theta$ if n is even. This proves that $|\{x_1\theta, x_2\theta, \ldots, x_n\theta\}| \ge \frac{n}{2}$. Hence $|C\theta| \ge \frac{|C|}{2}$. □

Lemma 3.2.4. Let $\theta \in PT(X)$. If $x_1, x_2, \ldots, x_n$ are in the domain of θ such that $n > 1$, $x_1 < x_2 < \ldots < x_n$, $x_1\theta \le x_1 \le x_2\theta \le x_2 \le \ldots \le x_n\theta \le x_n$ and $x_1\theta < x_2\theta < \ldots < x_n\theta$, then the partial transformation α of X defined by $(x_i\theta)\alpha = x_{i-1}$ for all $i \in \{2,3,\ldots,n\}$ belongs to $I_{RE}(X)$.

Proof. Since $x_{i-1} < x_{j-1}$ for all $i, j \in \{2,3,\ldots,n\}$ such that $i < j$, we have that α is one-to-one. Since $(x_i\theta)\alpha = x_{i-1} \le x_i\theta$ for all $i \in \{2,3,\ldots,n\}$, α is regressive. Hence $\alpha \in I_{RE}(X)$. □

Theorem 3.2.5. Let S be $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S$. If the domain of θ contains a sequence of disjoint finite chains $C_1, C_2, C_3, \ldots$ such that

(i) $|C_1| < |C_2| < |C_3| < \ldots$,

(ii) for $i \in \mathbf{N}$ and $x, y \in C_i$, $x < y$ implies that $x \le y\theta \le y$ and

(iii) for distinct $i, j \in \mathbf{N}$, $C_i\theta \cap C_j\theta = \varnothing$,

then (S, θ) is not an eventually regular semigroup.

Proof. We know that if $(x_1, x_2, x_3, \ldots)$ is a strictly increasing sequence of positive integers, then it has a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ such that $x_{k_1} > 2$ and $2x_{k_i} < x_{k_{i+1}}$ for all $i \in \mathbf{N}$. Then we may assume that $|C_1| > 2$ and for every $i \in \mathbf{N}$, $2|C_i| < |C_{i+1}|$. For each $i \in \mathbf{N}$, let $C_i = \{x_1^{(i)}, x_2^{(i)}, \ldots, x_{n_i}^{(i)}\}$ and $x_1^{(i)} < x_2^{(i)} < \ldots < x_{n_i}^{(i)}$. Then $n_1 > 2$ and $2n_i < n_{i+1}$ for all $i \in \mathbf{N}$. From (ii) and Lemma 3.2.3, we have that for every $i \in \mathbf{N}$, there exist $k_{i1}, k_{i2}, \ldots, k_{im_i}$ in $\{1,2,\ldots,n_i\}$ such that

$$k_{i1} < k_{i2} < \ldots < k_{im_i},$$

$$m_i \geq \frac{|C_i|}{2}$$

and

$$x_{k_{i1}}^{(i)}\theta < x_{k_{i2}}^{(i)}\theta < \ldots < x_{k_{im_i}}^{(i)}\theta \quad \ldots\ldots(*)$$

Since for every $i \in \mathrm{N}$, $\frac{|C_i|}{2} \leq m_i \leq |C_i| < \frac{|C_{i+1}|}{2} \leq m_{i+1} \leq |C_{i+1}|$, it follows that $(m_1, m_2, m_3, \ldots)$ is a strictly increasing sequence of positive integers.

Define the partial transformation α of X by

$$(x_{k_{ij}}^{(i)}\theta)\alpha = x_{k_{i,j-1}}^{(i)} \text{ for all } i \in \mathrm{N} \text{ and } j \in \{2, \ldots, m_i\}.$$

Because of the assumption (*iii*) and (*), we have that α is well-defined. It follows from the assumption (*ii*), (*) and Lemma 3.2.4 that restriction of α to $\{x_{k_{i1}}^{(i)}\theta, x_{k_{i2}}^{(i)}\theta, \ldots, x_{k_{i,m_i-1}}^{(i)}\theta\}$ is one-to-one and regressive. But $\{x_{k_{i1}}^{(i)}\theta, x_{k_{i2}}^{(i)}\theta, \ldots, x_{k_{i,m_i-1}}^{(i)}\theta\}\alpha \subseteq C_i$ for all $i \in \mathrm{N}$ and $C_1, C_2, C_3, \ldots$ are all disjoint, so $\alpha \in I_{RE}(X)$. It is obtained inductively that

$$\left.\begin{array}{c}\text{for all } n, i \in \mathrm{N},\ n < m_i \text{ and } j > n \text{ implies that} \\ \left(x_{k_{ij}}^{(i)}\theta\right)(\alpha,\theta)^n = x_{k_{i,j-n}}^{(i)}.\end{array}\right\} \quad \ldots\ldots(**)$$

Extend α to $\overline{\alpha} : X \to X$ by

$$x\overline{\alpha} = \begin{cases} x\alpha & \text{if } x = x_{k_{ij}}^{(i)}\theta \text{ for some } i \in N \text{ and } j \in \{2, \ldots, m_i\}, \\ x & \text{otherwise.} \end{cases}$$

Then $\overline{\alpha} \in T_{RE}(X)$. Let

$$\beta = \begin{cases} \alpha & \text{if } S = PT_{RE}(X) \text{ or } I_{RE}(X), \\ \overline{\alpha} & \text{if } S = T_{RE}(X). \end{cases}$$

Let $n \in \mathrm{N}$. Since $(m_1, m_2, m_3, \ldots)$ is a strictly increasing sequence of positive integers, $m_p > 2n$ for some $p \in \mathrm{N}$. Then by (**) and the definition of β, we have that

$$\left(x_{k_{pm_p}}^{(p)}\theta\right)(\beta,\theta)^n = \left(x_{k_{pm_p}}^{(p)}\theta\right)(\alpha,\theta)^n = x_{k_{p,m_p-n}}^{(p)}$$

and

$$\left(x^{(p)}_{k_{pm_p}}\theta\right)(\beta,\theta)^{2n}=\left(x^{(p)}_{k_{pm_p}}\theta\right)(\alpha,\theta)^{2n}=x^{(p)}_{k_{p,m_p-2n}}.$$

Since $k_{p,m_p-n}\neq k_{p,m_p-2n}$, $x^{(p)}_{k_{p,m_p-n}}\neq x^{(p)}_{k_{p,m_p-2n}}$. Then $(\beta,\theta)^n\neq(\beta,\theta)^{2n}$. This proves that $(\beta,\theta)^n$ is not an idempotent of (S,θ) for every $n\in\mathbf{N}$. Hence β is not an eventually regular element of S, and so (S,θ) is not an eventually regular semigroup. □

Lemma 3.2.6. If $\theta\in PT(X)$ and the domain of θ contains an infinite chain C such that for $x,y\in C$, $x<y$ implies that $x\leq y\theta\leq y$, then there exists a sequence of disjoint finite chains $C_1,C_2,C_3,\ldots$ such that

(i) $|C_1|<|C_2|<|C_3|<\ldots$,

(ii) for $i\in\mathbf{N}$ and $x,y\in C_i$, $x<y$ implies that $x\leq y\theta\leq y$ and

(iii) for distinct $i,j\in\mathbf{N}$, $C_i\theta\cap C_j\theta=\varnothing$.

Proof. By Proposition 3.2.1, there exist $x_1,x_2,x_3,\ldots$ in C such that

$$x_1<x_2<x_3<\ldots$$

or there exist $x_{-1},x_{-2},x_{-3},\ldots$ in C such that

$$x_{-1}>x_{-2}>x_{-3}>\ldots$$

Case 1 : There exist $x_1,x_2,x_3,\ldots$ in C such that $x_1<x_2<x_3<\ldots$. It follows from the assumption that

$$x_1\theta\leq x_1\leq x_2\theta\leq x_2\leq x_3\theta\leq x_3\leq\ldots.$$

Using Lemma 3.2.2, we get that

$$x_1\theta<x_3\theta<x_5\theta<\ldots.$$

For each $i\in\mathbf{N}$, let $y_i=x_{2i-1}$. Then

$$y_1\theta<y_2\theta<y_3\theta<\ldots.$$

For each $i\in\mathbf{N}$, let

$$C_i = \left\{ y_{\frac{(i-1)i}{2}+1}, y_{\frac{(i-1)i}{2}+2}, \ldots, y_{\frac{(i-1)i}{2}+i} \right\},$$

that is,

$$C_1 = \{y_1\}$$
$$C_2 = \{y_2, y_3\}$$
$$C_3 = \{y_4, y_5, y_6\}$$
$$C_4 = \{y_7, y_8, y_9, y_{10}\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then $C_1, C_2, C_3, \ldots$ are all disjoint finite chains satisfying (i),(ii) and (iii), as required.

Case 2 : There exist $x_{-1}, x_{-2}, x_{-3}, \ldots$ in C such that $x_{-1} > x_{-2} > x_{-3} > \ldots$. Then

$$x_{-1} \geq x_{-1}\theta \geq x_{-2} \geq x_{-2}\theta \geq x_{-3} \geq x_{-3}\theta \geq \ldots.$$

By Lemma 3.2.2, $x_{-1}\theta > x_{-3}\theta > x_{-5}\theta > \ldots$. For each $i \in \mathbf{N}$, let $y_{-i} = x_{-2i+1}$. Then $y_{-1}\theta > y_{-2}\theta > y_{-3}\theta > \ldots$. For each $i \in \mathbf{N}$, let

$$C_i = \left\{ y_{\frac{(1-i)i}{2}-1}, y_{\frac{(1-i)i}{2}-2}, \ldots, y_{\frac{(1-i)i}{2}-i} \right\},$$

that is,

$$C_1 = \{y_{-1}\}$$
$$C_2 = \{y_{-2}, y_{-3}\}$$
$$C_3 = \{y_{-4}, y_{-5}, y_{-6}\}$$
$$C_4 = \{y_{-7}, y_{-8}, y_{-9}, y_{-10}\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then $C_1, C_2, C_3, \ldots$ are all disjoint finite chains which satisfy (i),(ii) and (iii). $\square$

Theorem 3.2.7. Let S be $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S$. If X contains an infinite chain C such that for $x, y \in C$, $x < y$ implies $x \leq y\theta \leq y$, then (S, θ) is not an eventually regular semigroup.

Proof. It follows directly from Theorem 3.2.5 and Lemma 3.2.6. $\square$

Theorem 3.2.8. Let S be $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S$. Then (S,θ) is an eventually regular semigroup if and only if there exists a positive integer n such that $|C| \le n$ for every chain C of the domain of θ having the property that for $x,y \in C$, $x < y$ implies $x \le y\theta \le y$.

Proof. Assume that there exists a positive integer n such that $|C| \le n$ for every chain C of the domain of θ having the property that for $x,y \in C$, $x < y$ implies $x \le y\theta \le y$. To show that (S,θ) is eventually regular, let $\alpha \in S$. Let $x \in \Delta(\alpha,\theta)^n$. Then $x \in \Delta(\alpha,\theta)^i$ for all $i \in \{1,2,\ldots,n\}$. Since α and θ are regressive, $x \ge x(\alpha,\theta) \ge x(\alpha,\theta)^2 \ge \ldots \ge x(\alpha,\theta)^n$. Then $\{x, x(\alpha,\theta)^1, x(\alpha,\theta)^2, \ldots, x(\alpha,\theta)^n\}$ is a chain of X. It follows that $\left|\{x, x(\alpha,\theta)^1, x(\alpha,\theta)^2, \ldots, x(\alpha,\theta)^n\}\right| \le n$ which implies that $x(\alpha,\theta)^j = x(\alpha,\theta)^{j+1}$ for some $j \in \{0,1,\ldots,n-1\}$ where $x(\alpha,\theta)^0 = x$. Consequently, $x(\alpha,\theta)^{n-1} = x(\alpha,\theta)^n$. Since $x \in \Delta(\alpha,\theta)^n$, $x(\alpha,\theta)^{n-1} \in \Delta(\alpha,\theta)^1$. Hence $\left(x(\alpha,\theta)^{n-1}\right)(\alpha,\theta)^1 = \left(x(\alpha,\theta)^n\right)(\alpha,\theta)^1$, so $x(\alpha,\theta)^n = x(\alpha,\theta)^{n+1}$. Then $x \in \Delta(\alpha,\theta)^{n+1}$. This proves that $\Delta(\alpha,\theta)^n \subseteq \Delta(\alpha,\theta)^{n+1}$ and $x(\alpha,\theta)^n = x(\alpha,\theta)^{n+1}$ for every $x \in \Delta(\alpha,\theta)^n$. But $\Delta(\alpha,\theta)^{n+1} \subseteq \Delta(\alpha,\theta)^n$, so $(\alpha,\theta)^{n+1} = (\alpha,\theta)^n$. Hence $(\alpha,\theta)^n$ is an idempotent of (S,θ), so α is an eventually regular element of (S,θ).

Conversely, suppose that for every positive integer n, there exists a chain C in $\Delta\theta$ such that $|C| > n$ and for $x,y \in C$, $x < y$ implies $x \le y\theta \le y$. Let C_1 be a finite chain in $\Delta\theta$ such that for $x,y \in C_1$, $x < y$ implies $x \le y\theta \le y$. Let $|C_1| = k_1$. If $\Delta\theta \setminus C_1$ does not contain a chain C such that $|C| > 3k_1$ and for $x,y \in C$, $x < y$ implies $x \le y\theta \le y$, then for every chain C of $\Delta\theta$ having the property that for $x,y \in C$, $x < y$ implies $x \le y\theta \le y$, $|C| \le 4k_1$ which contradicts the assumption. Then there exists a finite chain A_2 in $\Delta\theta \setminus C_1$ such that $|A_2| > 3k_1$ and for $x,y \in A_2$, $x < y$ implies $x \le y\theta \le y$. Let $C_2 = \{x \in A_2 / x\theta \notin C_1\theta\}$ and

$k_2 = |C_2|$. By Lemma 3.2.3, $|\{x \in A_2 / x\theta \in C_1\theta\}| \leq 2|\{x \in A_2 / x\theta \in C_1\theta\}\theta|$. But $\{x \in A_2 / x\theta \in C_1\theta\}\theta \subseteq C_1\theta$, so $|\{x \in A_2 / x\theta \in C_1\theta\}| \leq 2|C_1\theta| \leq 2|C_1| \leq 2k_1$. Then

$$
\begin{aligned}
|C_2| &= |A_2 \setminus \{x \in A_2 / x\theta \in C_1\theta\}| \\
&= |A_2| - |\{x \in A_2 / x\theta \in C_1\theta\}| \\
&\geq |A_2| - 2k_1 \\
&> 3k_1 - 2k_1 \\
&= k_1 \\
&= |C_1|.
\end{aligned}
$$

Then $C_1 \cap C_2 = \varnothing$ and $C_1\theta \cap C_2\theta = \varnothing$.

If $\Delta\theta \setminus (C_1 \cup C_2)$ does not contain a chain C such that $|C| > 5k_2$ and for $x, y \in C$, $x < y$ implies $x \leq y\theta \leq y$, then for every chain C of $\Delta\theta$ having the property that for $x, y \in C$, $x < y$ implies $x \leq y\theta \leq y$, $|C| \leq 5k_2 + k_2 + k_1 < 7k_2$ which contradicts the assumption. Then there exists a finite chain A_3 in $\Delta\theta \setminus (C_1 \cup C_2)$ such that $|A_3| > 5k_2$ and for $x, y \in A_3$, $x < y$ implies $x \leq y\theta \leq y$. Let $C_3 = \{x \in A_3 / x\theta \notin C_1\theta \cup C_2\theta\}$ and $k_3 = |C_3|$. By Lemma 3.2.3, $|\{x \in A_3 / x\theta \in C_1\theta \cup C_2\theta\}| \leq 2|\{x \in A_3 / x\theta \in C_1\theta \cup C_2\theta\}\theta|$. But $\{x \in A_3 / x\theta \in C_1\theta \cup C_2\theta\}\theta \subseteq C_1\theta \cup C_2\theta = (C_1 \cup C_2)\theta$, so $|\{x \in A_3 / x\theta \in C_1\theta \cup C_2\theta\}| \leq 2|(C_1 \cup C_2)\theta| \leq 2|C_1 \cup C_2| \leq 2|C_1| + 2|C_2| = 2k_1 + 2k_2 < 4k_2$. Then

$$
\begin{aligned}
|C_3| &= |A_3 \setminus \{x \in A_3 / x\theta \in C_1\theta \cup C_2\theta\}| \\
&= |A_3| - |\{x \in A_3 / x\theta \in C_1\theta \cup C_2\theta\}| \\
&\geq |A_3| - 4k_2 \\
&> 5k_2 - 4k_2 \\
&= k_2 \\
&= |C_2|.
\end{aligned}
$$

Then $C_1 \cap C_3 = \varnothing$, $C_2 \cap C_3 = \varnothing$, $C_1\theta \cap C_3\theta = \varnothing$ and $C_2\theta \cap C_3\theta = \varnothing$.

By continuing this process inductively, we obtain a sequence of disjoint finite chain $C_1, C_2, C_3, \ldots$ such that

(i) $|C_1| < |C_2| < |C_3| < \ldots$,

(ii) for $i \in \mathbf{N}$ and $x, y \in C_i$, $x < y$ implies that $x \le y\theta \le y$ and

(iii) for distinct $i, j \in \mathbf{N}$, $C_i\theta \cap C_j\theta = \varnothing$.

Hence by Theorem 3.2.5, (S, θ) is not eventually regular. □

Corollary 3.2.9. Let X be a partially ordered set and let S be $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S$. If $\nabla\theta$ is finite, then (S, θ) is eventually regular.

Proof. Assume that $\nabla\theta$ is finite. Let C be a chain in $\Delta\theta$ such that for $x, y \in C$, $x < y$ implies $x \le y\theta \le y$. By Lemma 3.2.3, $|C| \le 2|C\theta|$. Then $|C| \le 2|\nabla\theta|$. Hence by Theorem 3.2.8, (S, θ) is eventually regular. □

3.3 Eventual Regularity of $(U_{RE}(X), \theta), (V_{RE}(X), \theta)$ and $(W_{RE}(X), \theta)$

We use Lemma 2.3.1 to show in this section that each of these generalized transformation semigroups is eventually regular.

Lemma 3.3.1. Let S be a regressive transformation semigroup on X such that for every $\alpha \in S$, α is almost identical. Then for $\theta \in S$, (S, θ) is eventually regular.

Proof. By Lemma 2.3.1, S is an eventually regular semigroup. Let $\alpha \in S$. Then $\alpha\theta \in S$. Since S is eventually regular, there exists a positive integer n such that $(\alpha\theta)^n = (\alpha\theta)^{2n}$. Then $(\alpha\theta)^n\alpha = (\alpha\theta)^{2n}\alpha$ which implies that $(\alpha, \theta)^{n+1} = (\alpha, \theta)^{2n+1}$. Since $n+1 < 2n+1$, it follows that $(\alpha, \theta)^m$ is an idempotent for some positive integer

m. Hence α is eventually regular in (S,θ). Therefore (S,θ) is an eventually regular semigroup. □

Theorem 3.3.2. If S is $(U_{RE}(X),\theta)$, $(V_{RE}(X),\theta)$ or $(W_{RE}(X),\theta)$ and $\theta \in S$, then (S,θ) is eventually regular.

Proof. It follows from Lemma 3.3.1. □

3.4 Eventual Regularity of $(M_{RE}(X),\theta)$ and $(E_{RE}(X),\theta)$

We give necessary and sufficient conditions for these generalized transformation semigroups to be eventually regular by using the results from Section 2.4.

Theorem 3.4.1. Let $\theta \in M_{RE}(X)$. Then the following statements are equivalent.

(1) Every chain of X has a minimum element.

(2) $M_{RE}(X) = \{1_X\}$.

(3) $(M_{RE}(X),\theta)$ is regular.

(4) $(M_{RE}(X),\theta)$ is eventually regular.

Proof. (1)$\Rightarrow$(2). Assume (1). By Theorem 2.4.3, $M_{RE}(X) = \{1_X\}$.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(4). Trivial.

(4)$\Rightarrow$(1). Assume that (4) holds. Since $\theta \in M_{RE}(X)$, by Theorem 3.1.1, $(\theta,\theta)^n$ is an idempotent of $(M_{RE}(X),\theta)$ for some positive integer n. Then $(\theta,\theta)^{2n} = (\theta,\theta)^n$ which implies that $\theta^{4n-1} = \theta^{2n-1}$. But $4n-1 < 2n-1$, so $\theta^m = \theta^{2m}$ for some positive integer m. Then for every $x \in X$, $x\theta^m = x\theta^{2m}$. Since θ is regressive, for every $x \in X$,

$$x \geq x\theta \geq x\theta^2 \geq \ldots \geq x\theta^m \geq x\theta^{m+1} \geq \ldots \geq x\theta^{2m}.$$

Since θ is one-to-one, for $x \in X$, $x > x\theta$ implies that

$$x > x\theta > x\theta^2 > \ldots > x\theta^m > x\theta^{m+1} > \ldots > x\theta^{2m}$$

which is a contradiction since $x\theta^m = x\theta^{2m}$. Then $x\theta = x$ for all $x \in X$. Hence $\theta = 1_X$. Then $M_{RE}(X)$ and $(M_{RE}(X),\theta)$ are the same semigroup. Thus $M_{RE}(X)$ is eventually regular. By Theorem 2.4.3, (1) holds. □

Theorem 3.4.2. Let $\theta \in E_{RE}(X)$. Then the following statements are equivalent.

(1) Every chain of X has a maximum element.

(2) $E_{RE}(X) = \{1_X\}$.

(3) $(E_{RE}(X),\theta)$ is regular.

(4) $(E_{RE}(X),\theta)$ is eventually regular.

Proof. (1)$\Rightarrow$(2) follows from Theorem 2.4.4.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(4). Trivial.

(4)$\Rightarrow$(1). Assume that (4) holds. Since $\theta \in E_{RE}(X)$, by Theorem 3.1.1, $(\theta,\theta)^n$ is an idempotent of $(E_{RE}(X),\theta)$ for some positive integer n. Then $(\theta,\theta)^{2n} = (\theta,\theta)^n$ which implies that $\theta^m = \theta^{2m}$ some positive integer m. Thus θ^m is an idempotent in $E_{RE}(X)$. Therefore $x\theta^m = x$ for all $x \in \nabla\theta^m$. But $\nabla\theta^m = X$, so $\theta^m = 1_X$. Then θ is one-to-one. Now we have that $\theta^m = \theta^{2m}$ and θ is one-to-one. By the proof of (4)$\Rightarrow$(1) of Theorem 3.4.1, we have that $\theta = 1_X$. Hence $E_{RE}(X)$ and $(E_{RE}(X),\theta)$ are the same semigroup. Thus $E_{RE}(X)$ is eventually regular. By Theorem 2.4.4, (1) holds. □