สาทิสสัณฐานหลายค่าของกึ่งกรุปและการเป็นปกติของกึ่งกรุปของฟังก์ชันหลายค่า

นายวัชระ เทพารส

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2549

ISBN 974-14-2047-1

MULTI–VALUED HOMOMORPHISMS OF SEMIGROUPS AND
REGULARITY OF SEMIGROUPS OF MULTI–VALUED FUNCTIONS

Mr. Watchara Teparos

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics

Department of Mathematics
Faculty of Science
Chulalongkorn University

Academic Year 2006
ISBN 974-14-2047-1

Thesis Title	MULTI-VALUED HOMOMORPHISMS OF SEMIGROUPS AND REGULARITY OF SEMIGROUPS OF MULTI-VALUED FUNCTIONS
By	Mr. Watchara Teparos
Field of Study	Mathematics
Thesis Advisor	Professor Yupaporn Kemprasit, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science
(Professor Piamsak Menasveta, Ph.D.)

THESIS COMMITTEE

P. Udomk.. Chairman
(Asscociate Professor Patanee Udomkavanich, Ph.D.)

.......Yupaporn Kemprasit........ Thesis Advisor
(Professor Yupaporn Kemprasit, Ph.D.)

.......Sajee Pianskool.................... Member
(Assistant Professor Sajee Pianskool, Ph.D.)

วัชระ เทพารส : สาทิสสัณฐานหลายค่าของกึ่งกรุปและการเป็นปกติของกึ่งกรุปของฟังก์ชันหลายค่า(MULTI-VALUED HOMOMORPHISMS OF SEMIGROUPS AND REGULARITY OF SEMIGROUPS OF MULTI-VALUED FUNCTONS) อ.ที่ปรึกษา : ศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์, 33 หน้า ISBN 974-14-2047-1.

เราเรียกสมาชิก x ของกึ่งกรุป S ว่า *สมาชิกปกติ* ถ้า $x = xyx$ สำหรับบางสมาชิก $y \in S$ และเรียก S ว่าเป็น*กึ่งกรุปปกติ* ถ้าทุกสมาชิกของ S เป็นสมาชิกปกติ

เราเรียกฟังก์ชันหลายค่า f จากกึ่งกรุป S ไปยังกึ่งกรุป S' ว่า*สาทิสสัณฐานหลายค่า* เมื่อ

$$f(xy) = f(x)f(y)\ (\{st \mid s \in f(x) \text{ และ } t \in f(y)\}) \text{ สำหรับทุก } x, y \in S$$

สำหรับกึ่งกรุป S ให้ $\mathrm{MHom}(S)$ เป็นกึ่งกรุปของสาทิสสัณฐานหลายค่าของ S ทั้งหมดภายใต้การประกอบ และให้ $\mathrm{SMHom}(S)$ เป็นกึ่งกรุปย่อยของ $\mathrm{MHom}(S)$ ที่ประกอบด้วย $f \in \mathrm{MHom}(S)$ ทั้งหมด ซึ่งสอดคล้องเงื่อนไข $\bigcup_{x \in S} f(x) = S$ ให้ $(\mathbb{Z}, +)$ และ $(\mathbb{Z}_n, +)$ เป็นกรุปการบวกของจำนวนเต็มและกรุปการบวกของจำนวนเต็มมอดุโล n ตามลำดับ ได้มีการให้ลักษณะของสมาชิกของ $\mathrm{MHom}(\mathbb{Z}, +)$, $\mathrm{MHom}(\mathbb{Z}_n, +)$, $\mathrm{SMHom}(\mathbb{Z}, +)$ และ $\mathrm{SMHom}(\mathbb{Z}_n, +)$ ไว้แล้ว

ในการวิจัยนี้ เราให้ลักษณะของสมาชิกปกติของกึ่งกรุป $\mathrm{MHom}(\mathbb{Z}, +)$, $\mathrm{MHom}(\mathbb{Z}_n, +)$, $\mathrm{SMHom}(\mathbb{Z}, +)$ และ $\mathrm{SMHom}(\mathbb{Z}_n, +)$ และให้ลักษณะที่บอกว่า เมื่อใด $\mathrm{MHom}(\mathbb{Z}_n, +)$ และ $\mathrm{SMHom}(\mathbb{Z}_n, +)$ เป็นกึ่งกรุปปกติ เราให้ลักษณะของสมาชิกของ $\mathrm{MHom}(S)$ เมื่อ S เป็นหนึ่งในกึ่งกรุปเหล่านี้ด้วย : กึ่งกรุปศูนย์ซ้าย กึ่งกรุปศูนย์ขวา กึ่งกรุปศูนย์ กึ่งกรุปครอนเนคเกอร์ นอกจากนี้ เรายังให้เงื่อนไขที่เพียงพอบางอย่างสำหรับ $f \in \mathrm{MF}(X)$ ที่จะเป็นสมาชิกปกติ เมื่อ $\mathrm{MF}(X)$ เป็นกึ่งกรุปของฟังก์ชันหลายค่าทั้งหมดของเซตไม่ว่าง X

ภาควิชา ...คณิตศาสตร์... ลายมือชื่อนิสิต..
สาขาวิชา ...คณิตศาสตร์... ลายมือชื่ออาจารย์ที่ปรึกษา..
ปีการศึกษา ......2549......

##4772462923 : MAJOR MATHEMATICS

KEY WORDS : MULTI-VALUED FUNCTIONS / MULTI-VALUED HOMOMORPHISMS / REGULAR ELEMENTS / REGULAR SEMIGROUPS

WATCHARA TEPAROS : MULTI-VALUED HOMOMORPHISMS OF SEMI GROUPS AND REGULARITY OF SEMIGROUPS OF MULTI-VALUED FUNCTIONS. THESIS ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 33 pp. ISBN 974-14-2047-1.

An element x of a semigroup S is said to be *regular* if $x = xyx$ for some $y \in S$, and S is called a *regular semigroup* if every element of S is regular.

A multi-valued function f from a semigroup S into a semigroup S' is called a *multi−valued homomorphism* if

$$f(xy) = f(x)f(y)(= \{\ st \mid s \in f(x) \ and \ f(y)\ \})\ for\ all\ x, y \in S.$$

For a semigroup S, let $MHom(S)$ be the semigroup of all multi-valued homomorphisms of S under composition and let $SMHom(S)$ be the subsemigroup of $MHom(S)$ consisting of all $f \in MHom(S)$ satisfying the condition $\bigcup_{x \in S} f(x) = S$. Let $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ be the additive group of integers and the additive group of integers modulo n, respectively. Elements of $MHom(\mathbb{Z}, +)$, $MHom(\mathbb{Z}_n, +)$, $SMHom(\mathbb{Z}, +)$ and $SMHom(\mathbb{Z}_n, +)$ have been already characterized.

In this research, we characterize the regular elements of the semigroups $MHom(\mathbb{Z}, +)$, $MHom(\mathbb{Z}_n, +)$, $SMHom(\mathbb{Z}, +)$ and $SMHom(\mathbb{Z}_n, +)$ and give a characterization determining when $MHom(\mathbb{Z}_n, +)$ and $SMHom(\mathbb{Z}_n, +)$ are regular semigroups. We also characterize the elements of $MHom(S)$ where S is one of the following semigroups : a left zero semigroup, a right zero semigroup, a zero semigroup and a Kronecker semigroup. In addition, some sufficient conditions for $f \in \mathrm{MF}(X)$ to be regular are given where $\mathrm{MF}(X)$ is the semigroup, under composition, of all multi-valued functions of a nonempty set X.

Department : ...Mathematics... Student's Signature : Watchara Teparos

Field of Study : ...Mathematics... Advisor's Signature : Yupaporn Kemprasit

Academic Year : ....2006............

ACKNOWLEDGEMENTS

I am very grateful to my supervisor, Professor Dr. Yupaporn Kemprasit, for her helpfulness, suggestions and encouragement throughout the preparation of this thesis. I am also thankful to my thesis committee and all the lecturers during my study.

Finally, I wish to express my gratitude to my parents for their kind encouragement throughout my study.

CONTENTS

page

INTRODUCTION

Whyburn [7], Smithson [5] and Feichtinger [1] presented characterizations of semi-continuity of multi-valued functions between topological spaces. Their works motivated Triphop, Harnchoowong and Kemprasit [6] to study multi-valued functions in an algebraic sense. They defined *multi-valued homomorphisms* between groups naturally and characterized multi-valued homomorphisms between cyclic groups. That is, they characterized the elements of $\text{MHom}(\mathbb{Z},+)$, $\text{MHom}((\mathbb{Z},+),(\mathbb{Z}_n,+))$, $\text{MHom}((\mathbb{Z}_n,+),(\mathbb{Z},+))$ and $\text{MHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,+))$ where $\text{MHom}(G,G')$ is the set of all multi-valued homomorphisms from a group G into a group G', $\text{MHom}(G) = \text{MHom}(G,G)$, $(\mathbb{Z},+)$ is the additive group of integers and $(\mathbb{Z}_n,+)$ is the additive group of integers modulo n. These sets were also counted in [6]. Nenthein and Lertwichitsilp [4] called an element $f \in \text{MHom}(G,G')$ a *surjective multi-valued homomorphism* if $f(G) = G'$ where $f(G) = \bigcup_{x \in G} f(x)$ and let $\text{SMHom}(G,G')$ denote the set of all surjective multi-valued homomorphisms from G into G'. The elements of $\text{SMHom}(\mathbb{Z},+)$, $\text{SMHom}((\mathbb{Z},+),(\mathbb{Z}_n,+))$, $\text{SMHom}((\mathbb{Z}_n,+),(\mathbb{Z},+))$ and $\text{SMHom}((\mathbb{Z}_m,+),(\mathbb{Z}_n,+))$ were characterized in [4] and these sets were also counted. Youngkhong and Savettaraserance [8] furthered the study of $\text{MHom}(G,G')$ where G' is either an additive group of real numbers or a multiplicative group of real numbers.

The semigroup, under composition, of all multi-valued functions from a nonempty set X into itself is denoted by $\text{MF}(X)$. Then $\text{MHom}(\mathbb{Z},+)$ and $\text{MHom}(\mathbb{Z}_n,+)$ are subsemigroups of $\text{MF}(\mathbb{Z})$ and $\text{MF}(\mathbb{Z}_n)$, respectively.

We organized this thesis as follows:

Chapter I contains basic definitions, known results and notations which will be used in the remaining chapters. For more details, see [2] and [3].

In Chapter II, we characterize the regular elements of the semigroups $\mathrm{MHom}(\mathbb{Z},+)$ and $\mathrm{SMHom}(\mathbb{Z},+)$.

Chapter III gives a characterization determining the regular elements of $\mathrm{MHom}(\mathbb{Z}_n,+)$. We prove that $\mathrm{MHom}(\mathbb{Z}_n,+)$ is a regular semigroup if and only if n is square-free. Moreover, it is shown that $\mathrm{SMHom}(\mathbb{Z}_n,+)$ is always a regular semigroup.

Multi-valued homomorphisms between semigroups are defined the same as that for groups in [6]. In Chapter IV, we determine the regular elements of $\mathrm{MHom}(S)$ where S is any of the following semigroups: a left zero semigroup, a right zero semigroup, a zero semigroup and a Kronecker semigroup. Here $\mathrm{MHom}(S)$ is also denoted the set of all multi-valued homomorphisms from S into itself.

In the last chapter, regular elements of the semigroup $\mathrm{MF}(X)$ are considered where X is a nonempty set. We provide some remarkable sufficient conditions for the elements f of the semigroup $\mathrm{MF}(X)$ to be regular in terms of the relationship among the values of f at points in X.

CHAPTER I

PRELIMINARIES

We adopt the following notations:

$\lvert X\rvert$	: the cardinality of a set X,
$\mathcal{P}(X)$	: the power set of a set X and $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}$,
$\mathbb{Z}$	: the set of integers,
$\mathbb{N}$ or $\mathbb{Z}^+$	: the set of positive integers and $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,
$\mathbb{R}$	: the set of real numbers,
$\mathbb{R}^+$	: the set of positive real numbers,
$\mathbb{Z}_n$	: the set of integers modulo n.

For a nonempty set X, let $T(X)$ be the full transformation semigroup on X, that is, the semigroup, under composition, of all functions $f : X \to X$. The semigroup of binary relations on X under composition is denoted by $\mathcal{B}(X)$, then

$$\begin{aligned}\mathcal{B}(X) &= \{\rho \mid \rho \subseteq X \times X\},\\ \sigma \circ \rho &= \{(x, y) \mid (x, z) \in \rho \text{ and } (z, y) \in \sigma \text{ for some } z \in X\}\\ &\qquad \text{for all } \rho, \sigma \in \mathcal{B}(X),\end{aligned}$$

and we have that $T(X)$ is a subsemigroup of $\mathcal{B}(X)$.

By a *multi-valued function* from a nonempty set X into a nonempty set Y we mean a function from X into $\mathcal{P}^*(Y)$. Let $\mathrm{MF}(X)$ denote the set of all multi-valued functions from X into itself. Therefore, we have

$$\mathrm{MF}(X) = \{\rho \in \mathcal{B}(X) \mid \text{for every } x \in X,\ (x, y) \in \rho \text{ for some } y \in X\}.$$

It is clearly seen that $\mathrm{MF}(X)$ is a subsemigroup of $\mathcal{B}(X)$ containing $T(X)$. Also

1_X, the identity map on X, is the identity of $\mathrm{MF}(X)$. For $f \in \mathrm{MF}(X)$ and $A \subseteq X$, let

$$f(A) = \bigcup_{a \in A} f(a).$$

It follows that

$$(g \circ f)(x) = g(f(x)) = \bigcup_{t \in f(x)} g(t) \quad \text{for all } x \in X.$$

The *range* of $f \in \mathrm{MF}(X)$ is defined to be $f(X)(= \bigcup_{x \in X} f(x))$ and it is denoted by $\mathrm{ran} f$.

Example 1.1. Let $\rho, \sigma \in \mathcal{B}(\mathbb{R})$ be defined by

$$\rho = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x > 0\},$$
$$\sigma = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > 0\}.$$

Then $\rho \in \mathcal{B}(\mathbb{R}) \smallsetminus \mathrm{MF}(\mathbb{R})$ and $\sigma \in \mathrm{MF}(\mathbb{R}) \smallsetminus T(\mathbb{R})$. Notice that

$$\sigma(x) = \mathbb{R}^+ \quad \text{for all } x \in \mathbb{R},$$

$$\rho \circ \sigma = \mathbb{R} \times \mathbb{R},\ \sigma \circ \rho = \mathbb{R}^+ \times \mathbb{R}^+.$$

A *multi-valued homomorphism* from a group G into a group G' is a multi-valued function f from G into G' such that

$$f(xy) = f(x)f(y)\ (= \{\ tr \mid t \in f(x) \text{ and } r \in f(y)\}\ )$$
$$\text{for all } x, y \in G.$$

A *surjective multi-valued homomorphism* from a group G into a group G' is a multi-valued homomorphism f from G into G' such that

$$\bigcup_{x \in G} f(x) = G'.$$

For groups G and G', let $\mathrm{MHom}(G, G')$ be the set of all multi-valued homomorphisms from G into G', and we write $\mathrm{MHom}(G)$ for $\mathrm{MHom}(G, G)$. Similarly, let

$\mathrm{SMHom}(G, G')$ be the set of all surjective multi-valued homomorphisms from G into G', and we write $\mathrm{SMHom}(G)$ for $\mathrm{SMHom}(G, G)$.

Characterizations of multi-valued homomorphisms and surjective multi-valued homomorphisms between cyclic groups were provided in [6] and [4], respectively. If $f, g \in \mathrm{MHom}(G)$, then for all $x, y \in G$,

$$\begin{aligned}
(g \circ f)(xy) &= g(f(xy)) \\
&= g(f(x)f(y)) \\
&= g(\{st \mid s \in f(x) \text{ and } t \in f(y)\}) \\
&= \bigcup_{\substack{s\in f(x)\\ t\in f(y)}} g(st) \\
&= \bigcup_{\substack{s\in f(x)\\ t\in f(y)}} g(s)g(t) \\
&= g(f(x))g(f(y)) \\
&= (g \circ f)(x)(g \circ f)(y).
\end{aligned}$$

and $g, f \in \mathrm{SMHom}(G)$ implies that $(g \circ f)(G) = g(f(G)) = g(G) = G$. This shows that $\mathrm{MHom}(G)$ and $\mathrm{SMHom}(G)$ is closed under composition. Hence $\mathrm{MHom}(G)$ is a subsemigroup of $\mathrm{MF}(G)$ and $\mathrm{SMHom}(G)$ is a subsemigroup of $\mathrm{MHom}(G)$. Observe that 1_G is the identity of the semigroup $\mathrm{MHom}(G)$ and $\mathrm{SMHom}(G)$. In addition, $\mathrm{Hom}(G)$ is a subsemigroup of $T(G)$ and $\mathrm{MHom}(G)$ where $\mathrm{Hom}(G)$ is the semigroup, under composition, of all homomorphisms of G into itself.

In this thesis, we also define *multi-valued homomorphisms* between semigroups analogously, that is, a *multi-valued homomorphism* from a semigroup S into a semigroup S' is a multi-valued function f from S into S' such that

$$f(xy) = f(x)f(y) \ (= \{ \ tr \mid t \in f(x) \text{ and } r \in f(y)\} \ )$$
$$\text{for all } x, y \in S.$$

For semigroups S and S', let $\mathrm{MHom}(S, S')$ be the set of all multi-valued homomorphisms of S into S', and we write $\mathrm{MHom}(S)$ for $\mathrm{MHom}(S, S)$. We can see from the above proof that $\mathrm{MHom}(S)$ is a subsemigroup of $\mathrm{MF}(S)$ containing the identity 1_S. Also, $\mathrm{Hom}(S)$ is a subsemigroup of both $T(S)$ and $\mathrm{MHom}(S)$ where $\mathrm{Hom}(S)$ is the semigroup, under composition, of all homomorphisms from S into itself.

Example 1.2. For $a \in \mathbb{R}$, let f_a be the multi-valued function from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f_a(x) = (a, \infty) \quad \text{for all } x \in \mathbb{R}.$$

It is clear that f_a is a multi-valued homomorphism from the group $(\mathbb{R}, +)$ into itself if and only if $a = 0$. We also have that f_a is a multi-valued homomorphism from the semigroup $(\mathbb{R}, \cdot)$ into itself if and only if $a = 0$ or $a = 1$. Hence

$$\{\ f_a \mid a \in \mathbb{R} \smallsetminus \{0\}\ \} \subseteq \mathrm{MF}(\mathbb{R}) \smallsetminus \mathrm{MHom}(\mathbb{R}, +),$$

$$\{\ f_a \mid a \in \mathbb{R} \smallsetminus \{0, 1\}\ \} \subseteq \mathrm{MF}(\mathbb{R}) \smallsetminus \mathrm{MHom}(\mathbb{R}, \cdot).$$

A semigroup S with zero 0 is called a *zero semigroup* if $xy = 0$ for all $x, y \in S$. A semigroup S is called a *left* [*right*] *zero semigroup* if

$$xy = x \ [xy = y] \quad \text{for all } x, y \in S.$$

A *Kronecker semigroup* S is a semigroup with zero 0 such that

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

An element a of a semigroup S is said to be *regular* if $a = axa$ for some $x \in S$, and S is called a *regular semigroup* if every element of S is regular. It is well-known that $T(X)$ is a regular semigroup for every set X ([2], page 4 and [3], page 63). The set of all regular elements of a semigroup is denoted by $\mathrm{Reg}(S)$.

Example 1.3. From Example 1.2, $f_a \circ f_a = f_a$ for every $a \in \mathbb{R}$. Then f_a is a regular element in the semigroup $\mathrm{MF}(\mathbb{R})$ for every $a \in \mathbb{R}$. In particular, f_0 is a regular element of $\mathrm{MHom}(\mathbb{R}, +)$ and f_0 and f_1 are regular elements of $\mathrm{MHom}(\mathbb{R}, \cdot)$.

If $g(x) = \{x, x+1\}$ for all $x \in \mathbb{R}$, then $g \in \mathrm{MF}(\mathbb{R})$ which is not regular. To see this, suppose that $g = g \circ h \circ g$ for some $h \in \mathrm{MF}(\mathbb{R})$. Then for every $x \in \mathbb{R}$,

$$\begin{aligned}\{x, x+1\} &= g(x)\\ &= g \circ h \circ g(x)\\ &= g \circ h(\{x, x+1\})\\ &= g(h(\{x, x+1\}))\\ &= g(h(x)) \cup g(h(x+1)),\end{aligned}$$

which implies that $g(h(x)) \subseteq \{x, x+1\}$ for every $x \in \mathbb{R}$. But $|g(h(x))| \geq 2$ for every $x \in \mathbb{R}$, so $g(h(x)) = \{x, x+1\}$ for all $x \in \mathbb{R}$. Hence for any $x \in \mathbb{R}$, $g(h(x)) \cup g(h(x+1)) = \{x, x+1\} \cup \{x+1, x+2\} = \{x, x+1, x+2\}$ which contradicts the above equalities.

An integer a is called *square-free* if for every $x \in \mathbb{Z} \smallsetminus \{0\}$, $x^2 \mid a$ (x^2 divides a) implies that $x = \pm 1$.

The congruence class modulo n of $x \in \mathbb{Z}$ will be denoted by $\overline{x}$ and let $\mathbb{Z}_n$ be the set of all congruence classes modulo n. Then

$$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} = \{\ \overline{x} \mid x \in \mathbb{Z}\ \} \text{ and } |\mathbb{Z}_n| = n.$$

For $k_1, \ldots, k_r \in \mathbb{Z}$, not all zero, let $(k_1, \ldots, k_r)$ denote the greatest common divisor of $k_1, \ldots, k_r$.

We recall the following basic facts.

(1) For $a, b \in \mathbb{Z}$, a and b are relatively prime (or $(a, b) = 1$) if and only if $ax + by = 1$ for some $x, y \in \mathbb{Z}$.

(2) For $a, b, k, l \in \mathbb{Z}$, $k \neq 0$ and $l \neq 0$, if $k \mid (a+b)$, $l \mid k$ and $l \mid a$, then $l \mid b$.

(3) For $a, b, k \in \mathbb{Z}$ and $k \neq 0$, if $k \mid ab$, then $\dfrac{k}{(k,a)} \mid b$.

(4) For $k, l \in \mathbb{Z}$, not both zero,

$$k\mathbb{Z} + l\mathbb{Z} = (k,l)\mathbb{Z} \quad \text{ and } \quad k\mathbb{Z}_n + l\mathbb{Z}_n = (k,l)\mathbb{Z}_n.$$

(5) For $k \in \mathbb{Z}$, $k\mathbb{Z}_n = (k, n)\mathbb{Z}_n$.

CHAPTER II

REGULAR ELEMENTS OF SEMIGROUPS OF MULTI-VALUED HOMOMORPHISMS OF $(\mathbb{Z},+)$

In this chapter, we give characterizations of the regular elements of the semigroups $\text{MHom}(\mathbb{Z},+)$ and $\text{SMHom}(\mathbb{Z},+)$.

For a subsemigroup H of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}$, define the multi-valued function from $\mathbb{Z}$ into itself by

$$F_{H,a}(x) = ax + H \quad \text{for all } x \in \mathbb{Z}.$$

The following known results will be referred.

Theorem 2.1 ([6]). *The following statements hold.*

(i) *If H is a subsemigroup of $(\mathbb{Z},+)$ containing 0, then $H \subseteq \mathbb{Z}_0^+$, $H \subseteq \mathbb{Z}_0^-$ or $H = k\mathbb{Z}$ for some $k \in \mathbb{Z}$.*

(ii) *$MHom(\mathbb{Z},+) = \{F_{H,a} \mid H$ is a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}\}$.*

(iii) $|MHom(\mathbb{Z},+)| = \aleph_0$.

Theorem 2.2 ([4]). *Let H be a subsemigroup of $(\mathbb{Z},+)$ containing 0. Then $F_{H,a} \in SMHom(\mathbb{Z},+)$ if and only if*

(i) *a is relatively prime to some $h \in H$ and*

(ii) *$a = 0$ implies $H = \mathbb{Z}$.*

Theorem 2.3 ([4]). *For $k, a \in \mathbb{Z}$, $F_{k\mathbb{Z},a} \in SMHom(\mathbb{Z},+)$ if and only if k and a are relatively prime.*

Theorem 2.4 ([4]). $|SMHom(\mathbb{Z},+)| = \aleph_0$.

Lemma 2.5. *For $k, l, a, b \in \mathbb{Z}$,*

$$F_{k\mathbb{Z},a}F_{l\mathbb{Z},b} = \begin{cases} F_{(k,al)\mathbb{Z},ab} & \textit{if } k \neq 0 \textit{ or } al \neq 0, \\ F_{0\mathbb{Z},ab} & \textit{if } k = 0 = al. \end{cases}$$

Proof. We have that for $x \in \mathbb{Z}$,

$$\begin{aligned} F_{k\mathbb{Z},a}F_{l\mathbb{Z},b}(x) &= F_{k\mathbb{Z},a}(bx + l\mathbb{Z}) \\ &= a(bx + l\mathbb{Z}) + k\mathbb{Z} \\ &= abx + al\mathbb{Z} + k\mathbb{Z} \\ &= \begin{cases} abx + (k, al)\mathbb{Z} & \text{if } k \neq 0 \text{ or } al \neq 0, \\ abx + 0\mathbb{Z} & \text{if } k = al = 0, \end{cases} \\ &= \begin{cases} F_{(k,al)\mathbb{Z},ab} & \text{if } k \neq 0 \text{ or } al \neq 0, \\ F_{0\mathbb{Z},ab} & \text{if } k = al = 0. \end{cases} \end{aligned}$$

□

Lemma 2.6. *If H is a subsemigroup of $(\mathbb{Z}, +)$ containing 0. Then $F_{H,0}$, $F_{H,1}$ and $F_{H,-1}$ are regular elements of $MHom(\mathbb{Z}, +)$.*

Proof. Note that $H + H = H$ and $-H - H = -H$. Since for every $x \in H$,

$$F_{H,0}F_{H,0}(x) = F_{H,0}(0 + H) = F_{H,0}(H) = 0H + H = H = F_{H,0}(x),$$

$$F_{H,1}F_{H,1}(x) = F_{H,1}(x + H) = 1(x + H) + H = x + H = F_{H,1}(x),$$

$$\begin{aligned} F_{H,-1}F_{-H,-1}F_{H,-1}(x) &= F_{H,-1}F_{-H,-1}(-x + H) \\ &= F_{H,-1}((-1)(-x + H) - H) \\ &= F_{H,-1}(x - H - H) \\ &= (-1)(x - H - H) + H \\ &= -x + H \\ &= F_{H,-1}(x), \end{aligned}$$

it follows that $F_{H,0}F_{H,0} = F_{H,0}$, $F_{H,1}F_{H,1} = F_{H,1}$ and $F_{H,-1}F_{-H,-1}F_{H,-1} = F_{H,-1}$. Hence $F_{H,0}$, $F_{H,1}$ and $F_{H,-1}$ are regular elements of $\mathrm{MHom}(\mathbb{Z},+)$. $\square$

Lemma 2.7. *Let $k, a \in \mathbb{Z}$ and $k \neq 0$. If $\left(a, \frac{k}{(k,a)}\right) = 1$, then $F_{k\mathbb{Z},a}$ is regular in $MHom(\mathbb{Z},+)$.*

Proof. Since $\left(a, \frac{k}{(k,a)}\right) = 1$, there are $b, c \in \mathbb{Z}$ such that $ab + \frac{kc}{(k,a)} = 1$. Then $\frac{k}{(k,a)} \mid (ab-1)$, so $k \mid (k,a)(ab-1)$ which implies that $k \mid a(ab-1)$. Thus $a^2b - a \in k\mathbb{Z}$. Hence for every $x \in \mathbb{Z}$, $a^2bx - ax \in k\mathbb{Z}$. Therefore

$$\text{for every } x \in \mathbb{Z},\ a^2bx + k\mathbb{Z} = ax + k\mathbb{Z},$$

that is, $F_{k\mathbb{Z},a^2b} = F_{k\mathbb{Z},a}$. By Lemma 2.5,

$$\begin{aligned} F_{k\mathbb{Z},a}F_{k\mathbb{Z},b}F_{k\mathbb{Z},a} = F_{k\mathbb{Z},a}F_{(k,bk)\mathbb{Z},ba} &= F_{(k,a(k,bk))\mathbb{Z},a^2b} \\ &= F_{k\mathbb{Z},a^2b} = F_{k\mathbb{Z},a}. \end{aligned}$$

Hence $F_{k\mathbb{Z},a}$ is regular in $\mathrm{MHom}(\mathbb{Z},+)$. $\square$

Theorem 2.8. *Let H be a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}$. Then $F_{H,a}$ is a regular element of $MHom(\mathbb{Z},+)$ if and only if one of the following two statements holds.*

(i) $a \in \{0, 1, -1\}$.

(ii) *$H = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$ and a and $\frac{k}{(k,a)}$ are relatively prime.*

Proof. By Theorem 2.1(i), $H \subseteq \mathbb{Z}_0^+$, $H \subseteq \mathbb{Z}_0^-$ or $H = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$. Assume that $F_{H,a}$ is a regular element of $\mathrm{MHom}(\mathbb{Z},+)$. By Theorem 2.1(ii), there are a subsemigroup K of $(\mathbb{Z},+)$ containing 0 and $b \in \mathbb{Z}$ such that $F_{H,a}F_{K,b}F_{H,a} = F_{H,a}$. Then for every $x \in \mathbb{Z}$,

$$\begin{aligned} ax + H &= F_{H,a}(x) \\ &= F_{H,a}F_{K,b}F_{H,a}(x) \\ &= F_{H,a}F_{K,b}(ax+H) \\ &= F_{H,a}(b(ax+H)+K) \end{aligned}$$

$$= a(b(ax + H) + K) + H$$
$$= a^2bx + abH + aK + H.$$

In particular,

$$H = a0 + H = a^2b0 + abH + aK + H = abH + aK + H.$$

Hence for every $x \in H$, $ax + H = a^2bx + H$, so $a + H = a^2b + H$. Since $0 \in H$, we have

$$a^2b - a \in H \text{ and } a - a^2b \in H. \tag{1}$$

Cases 1: $H \subseteq \mathbb{Z}_0^+$ or $H \subseteq \mathbb{Z}_0^-$. Then by (1), $a^2b = a$. Thus $a(ab - 1) = 0$. Since $a, b \in \mathbb{Z}$, it follows that $a = 0$, $a = b = 1$ or $a = b = -1$. Hence $a \in \{0, 1, -1\}$.

Cases 2: $H = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$. From (1), we have $a^2b - a \in k\mathbb{Z}$. Thus $k \mid (a^2b - a)$, hence $\frac{k}{(k,a)} \mid (ab - 1)$. It follows that $ab - 1 = \left(\frac{k}{(k,a)}\right)c$ for some $c \in \mathbb{Z}$. Therefore

$$ab + \left(\frac{k}{(k,a)}\right)(-c) = 1,$$

so we deduce that $\left(a, \frac{k}{(k,a)}\right) = 1$. Note that if $a \in \{0, 1, -1\}$, then $\left(a, \frac{k}{(k,a)}\right) = 1$.

The converse follows directly from Lemma 2.6 and Lemma 2.7. □

Corollary 2.9. $|Reg(MHom(\mathbb{Z}, +))| = |MHom(\mathbb{Z}, +) \setminus Reg(MHom(\mathbb{Z}, +))|$
$= |MHom(\mathbb{Z}, +)| = \aleph_0.$

Proof. Since for all distinct $k, l \in \mathbb{Z}^+$, $k\mathbb{Z} \neq l\mathbb{Z}$, we have that for $a, b \in \mathbb{Z}$,

$$F_{k\mathbb{Z},a}(0) = k\mathbb{Z} \neq l\mathbb{Z} = F_{l\mathbb{Z},b}(0).$$

Thus $F_{k\mathbb{Z},a} \neq F_{l\mathbb{Z},b}$ for all distinct $k, l \in \mathbb{Z}^+$ and for all $a, b \in \mathbb{Z}$. Since $\left(1, \frac{k}{(k,1)}\right) = 1$ and $\left(k, \frac{k^2}{(k^2,k)}\right) = k$ for all $k \in \mathbb{Z}^+$, by Theorem 2.8, we have

$$\{F_{k\mathbb{Z},1} \mid k \in \mathbb{Z}^+\} \subseteq \mathrm{Reg}(\mathrm{MHom}(\mathbb{Z}, +)),$$
$$\{F_{k^2\mathbb{Z},k} \mid k \in \mathbb{Z}^+ \text{ and } k > 1\} \subseteq \mathrm{MHom}(\mathbb{Z}, +) \setminus \mathrm{Reg}(\mathrm{MHom}(\mathbb{Z}, +)).$$

Consequently,

$$\aleph_0 = |\{F_{k\mathbb{Z},1} \mid k \in \mathbb{Z}^+\}| \leq |\mathrm{Reg}(\mathrm{MHom}(\mathbb{Z},+))|, \tag{1}$$

$$\aleph_0 = |\{F_{k^2\mathbb{Z},k} \mid k \in \mathbb{Z}^+ \text{ and } k > 1\}| \leq |\mathrm{MHom}(\mathbb{Z},+) \smallsetminus \mathrm{Reg}(\mathrm{MHom}(\mathbb{Z},+))|. \tag{2}$$

Theorem 2.1(iii), (1) and (2) yield the fact that

$$|\mathrm{Reg}(\mathrm{MHom}(\mathbb{Z},+))| = |\mathrm{MHom}(\mathbb{Z},+) \smallsetminus \mathrm{Reg}(\mathrm{MHom}(\mathbb{Z},+))| = |\mathrm{MHom}(\mathbb{Z},+)| = \aleph_0.$$

□

Theorem 2.10. *Reg(SMHom($\mathbb{Z}$,+))*

$$\begin{aligned} &= \{F_{H,1} \mid H \textit{ is a subsemigroup of } (\mathbb{Z},+) \textit{ containing } 0\} \\ &\quad \cup \{F_{H,-1} \mid H \textit{ is a subsemigroup of } (\mathbb{Z},+) \textit{ containing } 0\} \\ &\quad \cup \{F_{k\mathbb{Z},a} \mid k, a \in \mathbb{Z}, k \neq 0 \textit{ and } (k,a) = 1\}. \end{aligned}$$

Proof. Let H be a subsemigroup of $(\mathbb{Z},+)$ containing 0. By Theorem 2.2, we have that $F_{H,1}, F_{H,-1}, F_{-H,-1} \in \mathrm{SMHom}(\mathbb{Z},+)$. From the proof of Lemma 2.6,

$$F_{H,1}F_{H,1} = F_{H,1} \text{ and } F_{H,-1}F_{-H,-1}F_{H,-1} = F_{H,-1},$$

so $F_{H,1}, F_{H,-1} \in \mathrm{Reg}(\mathrm{SMHom}(\mathbb{Z},+))$.

Next, let $k, a \in \mathbb{Z}$ be such that $k \neq 0$ and $(k,a) = 1$. Then by Theorem 2.2, $F_{k\mathbb{Z},a} \in \mathrm{SMHom}(\mathbb{Z},+)$. Let $b, c \in \mathbb{Z}$ be such that $ab + kc = 1$. Then $k \mid (ab-1)$, so $k \mid (a^2b - a)$. Hence $a^2bx - ax \in k\mathbb{Z}$ for all $x \in \mathbb{Z}$, so $a^2bx + k\mathbb{Z} = ax + k\mathbb{Z}$ for all $x \in \mathbb{Z}$. Hence $F_{k\mathbb{Z},a^2b} = F_{k\mathbb{Z},a}$. From the proof of Lemma 2.7, we have

$$F_{k\mathbb{Z},a}F_{k\mathbb{Z},b}F_{k\mathbb{Z},a} = F_{k\mathbb{Z},a}.$$

Since $ab + kc = 1$, we have $(b,k) = 1$. Thus $F_{k\mathbb{Z},b} \in \mathrm{SMHom}(\mathbb{Z},+)$ by Theorem 2.2. Hence $F_{k\mathbb{Z},a}$ is a regular element of $\mathrm{SMHom}(\mathbb{Z},+)$.

For the reverse inclusion, let H be a subsemigroup of $(\mathbb{Z},+)$ containing 0 and $a \in \mathbb{Z}$ such that $F_{H,a} \in \mathrm{Reg}(\mathrm{SMHom}(\mathbb{Z},+))$. Then $F_{H,a} \in \mathrm{Reg}(\mathrm{MHom}(\mathbb{Z},+))$. By

Theorem 2.8, H and a satisfy one of the following conditions.

(i) $a \in \{0, 1, -1\}$.

(ii) $H = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $\left(a, \dfrac{k}{(k,a)}\right) = 1$.

If $a = 0$, then by Theorem 2.2, $H = \mathbb{Z}$, so $F_{H,a} = F_{\mathbb{Z},a}$ and $(1, a) = 1$. If $H = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$, then $F_{H,a} = F_{k\mathbb{Z},a} \in \text{SMHom}(\mathbb{Z}, +)$, so $(k, a) = 1$ by Thorem 2.3.

Hence the proof is complete. □

Corollary 2.11. $|Reg(SMHom(\mathbb{Z},+))| = |SMHom(\mathbb{Z},+) \setminus Reg(SMHom(\mathbb{Z},+))|$
$= |SMHom(\mathbb{Z},+)| = \aleph_0$

Proof. Since $\{F_{k\mathbb{Z},1} \mid k \in \mathbb{Z}^+\} \subseteq \text{Reg}(\text{SMHom}(\mathbb{Z}, +))$ by Theorem 2.10, it follows that

$$\aleph_0 = |\{F_{k\mathbb{Z},1} \mid k \in \mathbb{Z}^+\}| \leq |\text{Reg}(\text{SMHom}(\mathbb{Z}, +))|. \tag{1}$$

Also, by Theorem 2.2 and Theorem 2.10, $\{F_{\mathbb{Z}_0^+,a} \mid a \in \mathbb{Z} \setminus \{1, -1\}\} \subseteq \text{SMHom}(\mathbb{Z}, +) \setminus \text{Reg}(\text{SMHom}(\mathbb{Z}, +))$. But $F_{\mathbb{Z}_0^+,a}(1) = a + \mathbb{Z}_0^+$ and $a = \min(a + \mathbb{Z}_0^+)$ for all $a \in \mathbb{Z}$, so we have $F_{\mathbb{Z}_0^+,a} \neq F_{\mathbb{Z}_0^+,b}$ for all distinct $a, b \in \mathbb{Z}$. Therefore

$$\aleph_0 = |\{F_{\mathbb{Z}_0^+,a} \mid a \in \mathbb{Z} \setminus \{1, -1\}\}| \leq |\text{SMHom}(\mathbb{Z}, +) \setminus \text{Reg}(\text{SMHom}(\mathbb{Z}, +))|. \tag{2}$$

Hence from Theorem 2.4, (1) and (2), we have

$$\begin{aligned} |\text{Reg}(\text{SMHom}(\mathbb{Z}, +))| &= |\text{SMHom}(\mathbb{Z}, +) \setminus \text{Reg}(\text{SMHom}(\mathbb{Z}, +))| \\ &= |\text{SMHom}(\mathbb{Z}, +)| = \aleph_0. \end{aligned}$$

□

CHAPTER III

REGULAR ELEMENTS OF SEMIGROUPS OF MULTI-VALUED HOMOMORPHISMS OF ($\mathbb{Z}_n$,+)

The regular elements of the semigroup $\mathrm{MHom}(\mathbb{Z}_n, +)$ are characterized in this chapter. Then this characterization is applied to characterize the regularity of the semigroup $\mathrm{MHom}(\mathbb{Z}_n, +)$ in terms of n. Moreover, it is shown that the semigroup $\mathrm{SMHom}(\mathbb{Z}_n, +)$ is always regular.

If $k, a \in \mathbb{Z}$, define the multi-valued function $I_{k,a}$ from $\mathbb{Z}_n$ into itself by

$$I_{k,a}(\overline{x}) = \overline{ax} + k\mathbb{Z}_n \quad \text{for all } x \in \mathbb{Z}.$$

The following known results will be used.

Theorem 3.1 ([6]). *$MHom(\mathbb{Z}_n, +) = \{I_{k,a} \mid k, a \in \mathbb{Z}\}$.*

Theorem 3.2 ([6]). *The following statements hold.*

(i) *If $k, l \in \mathbb{Z}^+$, $k \mid n$, $l \mid n$, $a \in \{0, 1, \ldots, k-1\}$, $b \in \{0, 1, \ldots, l-1\}$ and $I_{k,a} = I_{l,b}$, then $k = l$ and $a = b$.*

(ii) *$MHom(\mathbb{Z}_n, +) = \{\ I_{k,a} \mid k \in \mathbb{Z}^+, k \mid n$ and $a \in \{0, 1, \ldots, k-1\}\}$.*

(iii) *$|MHom(\mathbb{Z}_n, +)| = \sum\limits_{\substack{k \in \mathbb{Z}^+ \\ k \mid n}} k$.*

Note that in Theorem 3.2, (iii) is directly obtained from (i) and (ii).

Theorem 3.3 ([4]). *$SMHom(\mathbb{Z}_n, +) = \{I_{k,a} \mid k, a \in \mathbb{Z}$ and $(n, k, a) = 1\}$.*

To characterize the regular elements of $\mathrm{MHom}(\mathbb{Z}_n, +)$, the following three lemmas are needed.

Lemma 3.4. *If $r, s, t \in \mathbb{Z}$, $r \neq 0$ and $t \neq 0$ are such that $r \mid \left(s, \dfrac{t}{(s,t)}\right)$, then $r^2 \mid t$.*

Proof. From the assumption, $r \mid s$ and $r \mid \dfrac{t}{(s,t)}$. Then $r(s,t) \mid t$. Hence $r \mid s$ and $r \mid t$ which implies that $r \mid (s,t)$, and thus $r^2 \mid r(s,t)$. But $r(s,t) \mid t$, so $r^2 \mid t$. □

Lemma 3.5. *For $k, l, a, b \in \mathbb{Z}$,*

$$I_{k,a}I_{l,b} = \begin{cases} I_{(k,al),ab} & \text{if } k \neq 0 \text{ or } al \neq 0, \\ I_{0,ab} & \text{if } k = al = 0. \end{cases}$$

Proof. For $x \in \mathbb{Z}$,

$$\begin{aligned} I_{k,a}I_{l,b}(\overline{x}) &= I_{k,a}(\overline{bx} + l\mathbb{Z}_n) \\ &= \overline{a}(\overline{bx} + l\mathbb{Z}_n) + k\mathbb{Z}_n \\ &= \overline{abx} + al\mathbb{Z}_n + k\mathbb{Z}_n \\ &= \begin{cases} \overline{abx} + (k,al)\mathbb{Z}_n = I_{(k,al),ab}(\overline{x}) & \text{if } k \neq 0 \text{ or } al \neq 0, \\ \overline{abx} + 0\mathbb{Z}_n = I_{0,ab}(\overline{x}) & \text{if } k = al = 0, \end{cases} \end{aligned}$$

so the lemma is proved. □

Lemma 3.6. *If $k, l, a, b \in \mathbb{Z}$ are such that $I_{k,a} = I_{l,b}$, then $k\mathbb{Z}_n = l\mathbb{Z}_n$ and $(n,k) \mid (a-b)$.*

Proof. We have that $k\mathbb{Z}_n = I_{k,a}(\overline{0}) = I_{l,b}(\overline{0}) = l\mathbb{Z}_n$. Then $I_{k,a} = I_{k,b}$, so $\overline{a} + k\mathbb{Z}_n = I_{k,a}(\overline{1}) = I_{k,b}(\overline{1}) = \overline{b} + k\mathbb{Z}_n$. Hence $\overline{a-b} = \overline{kt}$ for some $t \in \mathbb{Z}$, thus $n \mid (a-b-kt)$. Since $(n,k) \mid n$ and $(n,k) \mid kt$, it follows that $(n,k) \mid (a-b)$. □

Theorem 3.7. *For $k, a \in \mathbb{Z}$, $I_{k,a}$ is a regular element of the semigroup $MHom(\mathbb{Z}_n, +)$ if and only if a and $\dfrac{(n,k)}{(n,k,a)}$ are relatively prime.*

Proof. First, assume that $I_{k,a}$ is a regular element of $\mathrm{MHom}(\mathbb{Z}_n, +)$. Then there are $l, b \in \mathbb{Z}$ such that $I_{k,a} = I_{k,a}I_{l,b}I_{k,a}$. By Lemma 3.5, $I_{k,a}I_{l,b}I_{k,a} = I_{s,a^2b}$ for some

$s \in \mathbb{Z}$, and so by Lemma 3.6, $(n,k) \mid (a^2b-a)$. This implies that $\dfrac{(n,k)}{(n,k,a)} \mid (ab-1)$. Therefore $ab + \dfrac{(n,k)}{(n,k,a)}t = 1$ for some $t \in \mathbb{Z}$. Consequently, a and $\dfrac{(n,k)}{(n,k,a)}$ are relatively prime.

Conversely, assume that a and $\dfrac{(n,k)}{(n,k,a)}$ are relatively prime. Then there are $b, c \in \mathbb{Z}$ such that $ab + \dfrac{(n,k)}{(n,k,a)}c = 1$. It follows that for every $x \in \mathbb{Z}$,

$$\begin{aligned}
\overline{(a^2b-a)x} &= \overline{(ab-1)ax} \\
&= \overline{\left(\frac{(n,k)}{(n,k,a)}(-c)ax\right)} \\
&= (n,k)\overline{\left(\frac{a}{(n,k,a)}(-c)x\right)} \\
&\in (n,k)\mathbb{Z}_n = k\mathbb{Z}_n.
\end{aligned}$$

Consequently, $\overline{a^2bx} + k\mathbb{Z}_n = \overline{ax} + k\mathbb{Z}_n$ for every $x \in \mathbb{Z}$. By Lemma 3.5,

$$I_{k,a}I_{k,b}I_{k,a} = \begin{cases} I_{(k,a(k,bk)),a^2b} = I_{k,a^2b} & \text{if } k \neq 0, \\ I_{0,a^2b} = I_{k,a^2b} & \text{if } k = 0. \end{cases}$$

Thus for every $x \in \mathbb{Z}$, $I_{k,a}I_{k,b}I_{k,a}(\overline{x}) = \overline{a^2bx} + k\mathbb{Z}_n = \overline{ax} + k\mathbb{Z}_n = I_{k,a}(\overline{x})$, so $I_{k,a}I_{k,b}I_{k,a} = I_{k,a}$. Hence $I_{k,a}$ is a regular element of $\mathrm{MHom}(\mathbb{Z}_n,+)$, as desired. $\square$

Corollary 3.8. *Let QF be the set of all square-free positive integers. Then the following statements hold.*

(i) $Reg(MHom(\mathbb{Z}_n,+))$

$= \{\ I_{k,a} \mid k \in \mathbb{Z}^+, k \mid n, a \in \{0,1,\ldots,k-1\}$ *and* $\left(a, \dfrac{k}{(k,a)}\right) = 1\}$

$= \{\ I_{k,a} \mid k \in QF, k \mid n$ *and* $a \in \{0,1,\ldots,k-1\}\}$

$\cup \{\ I_{k,a} \mid k \in \mathbb{Z}^+ \smallsetminus QF, k \mid n, a \in \{0,1,\ldots,k-1\}$ *and* $\left(a, \dfrac{k}{(k,a)}\right) = 1\}$

(ii) $|Reg(MHom(\mathbb{Z}_n,+))|$

$$= \sum_{\substack{k\in QF\\ k|n}} k + \sum_{\substack{k\in\mathbb{Z}^+\setminus QF\\ k|n}} |\{a\in\{0,1,\ldots,k-1\} \mid \left(a, \frac{k}{(k,a)}\right) = 1\}|$$

Proof. (i) The first equality follows from Theorem 3.2(ii) and Theorem 3.7 and the second equality is obtained from Lemma 3.4.

(ii) is obtained from (i) and Theorem 3.2(i). □

Theorem 3.9. *The semigroup $MHom(\mathbb{Z}_n,+)$ is regular if and only if n is square-free.*

Proof. From Theorem 3.1 and Theorem 3.7, we have respectively that

$$\mathrm{MHom}(\mathbb{Z}_n,+) = \{I_{k,a} \mid k,a\in\mathbb{Z}\}$$

and

$$\mathrm{Reg}(\mathrm{MHom}(\mathbb{Z}_n,+)) = \{I_{k,a} \mid k,a\in\mathbb{Z} \text{ and } \left(a, \frac{(n,k)}{(n,k,a)}\right) = 1\}.$$

First, assume that n is not square-free. Then there exists an integer $r>1$ such that $r^2|n$. Then

$$\left(r, \frac{(n,n)}{(n,n,r)}\right) = \left(r, \frac{n}{r}\right) = r > 1,$$

which implies by Theorem 3.7 that $I_{n,r}\in \mathrm{MHom}(\mathbb{Z}_n,+)\setminus \mathrm{Reg}(\mathrm{MHom}(\mathbb{Z}_n,+))$. This proves that if $\mathrm{MHom}(\mathbb{Z}_n,+)$ is a regular semigroup, then n is square-free.

For the converse, assume that n is square-free. Then k is square-free for every $k\in\mathbb{Z}^+$ with $k\mid n$. Therefore we deduce from Corollary 3.8(i) that

$$\mathrm{Reg}(\mathrm{MHom}(\mathbb{Z}_n,+)) = \{I_{k,a} \mid k\in\mathbb{Z}^+, k\mid n \text{ and } a\in\{0,1,\ldots,k-1\}\}.$$

By Theorem 3.2(ii), we have $\mathrm{Reg}(\mathrm{MHom}(\mathbb{Z}_n,+)) = \mathrm{MHom}(\mathbb{Z}_n,+)$. Hence $\mathrm{MHom}(\mathbb{Z}_n,+)$ is a regular semigroup. □

The following corollary is obtained directly from Theorem 3.2(iii) and Theorem 3.9.

Corollary 3.10. *For any prime p, $MHom(\mathbb{Z}_p, +)$ is a regular semigroup of order $1 + p$.*

Example 3.11. By Theorem 3.2(iii) and Theorem 3.9, $\mathrm{MHom}(\mathbb{Z}_6, +)$ is a regular semigroup of order $1 + 2 + 3 + 6 = 12$.

By Corollary 3.8(ii),

$$\begin{aligned} |\mathrm{Reg}(MHom(\mathbb{Z}_{20}, +))| &= (1 + 2 + 5 + 10) + |\{a \in \{0, 1, 2, 3\} \mid \left(a, \frac{4}{(4, a)}\right) = 1\}| \\ &\quad + |\{a \in \{0, 1, \ldots, 19\} \mid \left(a, \frac{20}{(20, a)}\right) = 1\}| \\ &= 18 + (3 + 15) \\ &= 36 \end{aligned}$$

since for $a \in \{0, 1, 2, 3\}$,

$$\left(a, \frac{4}{(4, a)}\right) = 1 \Leftrightarrow a \in \{0, 1, 3\},$$

and for $a \in \{0, 1, \ldots, 19\}$,

$$\left(a, \frac{20}{(20, a)}\right) = 1 \Leftrightarrow a \in \{0, 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19\}.$$

By Theorem 3.2(iii),

$$\begin{aligned} |MHom(\mathbb{Z}_{20}, +) \backslash \mathrm{Reg}(MHom(\mathbb{Z}_{20}, +))| &= (1 + 2 + 4 + 5 + 10 + 20) - 36 \\ &= 42 - 36 = 6. \end{aligned}$$

Theorem 3.12. *For every $n \in \mathbb{N}$, $SMHom(\mathbb{Z}_n, +)$ is a regular semigroup.*

Proof. Let $k, a \in \mathbb{Z}$ be such that $I_{k,a} \in \mathrm{SMHom}(\mathbb{Z}_n, +)$. By Theorem 3.3, $(n, k, a) = 1$. Then $((n, k), a) = 1$, so there are $b, c \in \mathbb{Z}$ such that

$$ab + (n, k)c = 1, \tag{1}$$

Hence for every $x \in \mathbb{Z}$,

$$\begin{aligned}\overline{(a^2b-a)x} &= \overline{(ab-1)ax} \\ &= -\overline{(n,k)cax} \qquad\qquad \text{from (1)} \\ &\in (n,k)\mathbb{Z}_n = k\mathbb{Z}_n,\end{aligned}$$

which implies that

$$\text{for every } x \in \mathbb{Z},\quad \overline{a^2bx} + k\mathbb{Z}_n = \overline{ax} + k\mathbb{Z}_n. \tag{2}$$

By Lemma 3.5,

$$I_{k,a}I_{k,b}I_{k,a} = \begin{cases} I_{(k,a(k,bk)),a^2b} = I_{k,a^2b} & \text{if } k \neq 0, \\ I_{0,a^2b} = I_{k,a^2b} & \text{if } k = 0. \end{cases} \tag{3}$$

Then from (2) and (3), we have

$$\begin{aligned}\text{for every } x \in \mathbb{Z},\ (I_{k,a}I_{k,b}I_{k,a})(\overline{x}) &= \overline{a^2bx} + k\mathbb{Z}_n \\ &= \overline{ax} + k\mathbb{Z}_n \\ &= I_{k,a}(\overline{x}).\end{aligned}$$

Hence $I_{k,a} = I_{k,a}I_{k,b}I_{k,a}$. From (1), $(n,k,b) = ((n,k),b) = 1$. Thus $I_{k,b} \in \mathrm{SMHom}(\mathbb{Z}_n,+)$ by Theorem 3.3.

This proves that $\mathrm{SMHom}(\mathbb{Z}_n,+)$ is a regular semigroup, as desired. $\square$

CHAPTER IV

MULTI-VALUED HOMOMORPHISMS OF CERTAIN SEMIGROUPS

In this chapter, we are concerned with the following semigroups: left zero semigroups, right zero semigroups, zero semigroups and Kronecker semigroups. We characterize the multi-valued homomorphisms of these semigroups.

Recall that $\mathrm{MF}(S)$ and $\mathrm{MHom}(S)$ denote the set of all multi-valued functions of S and the set of all multi-valued homomorphisms of S, respectively.

Theorem 4.1. (i) *If S is a left zero semigroup, then $MHom(S) = MF(S)$, that is, every multi-valued function of S is a multi-valued homomorphism.*
(ii) *If S is a right zero semigroup, then $MHom(S) = MF(S)$.*

Proof. (i) Since $xy = x$ for all $x, y \in S$, it follows that $AB = A$ for all nonempty subsets A and B of S. Then for $f \in \mathrm{MF}(S)$,

$$f(xy) = f(x) = f(x)f(y) \quad \text{for all } x, y \in S.$$

Therefore we deduce that $\mathrm{MHom}(S) = \mathrm{MF}(S)$.

(ii) Since $xy = y$ for all $x, y \in S$, we have similarly that for every $f \in \mathrm{MF}(S)$,

$$f(xy) = f(y) = f(x)f(y) \quad \text{for all } x, y \in S.$$

Hence $\mathrm{MHom}(S) = \mathrm{MF}(S)$. □

Theorem 4.2. *Let S be a zero semigroup. Then*

$$MHom(S) = \{f \in MF(S) \mid f(0) = \{0\}\}.$$

Proof. Since $xy = 0$ for all $x, y \in S$, it follows that $AB = \{0\}$ for all nonempty subsets A and B of S. If $f \in \text{MHom}(S)$, then

$$f(0) = f(00) = f(0)f(0) = \{0\}.$$

Hence $\text{MHom}(S) \subseteq \{f \in \text{MF}(S) \mid f(0) = \{0\}\}$. For the reverse inclusion, let $f \in \text{MF}(S)$ be such that $f(0) = \{0\}$. If $x, y \in S$, then

$$f(xy) = f(0) = \{0\} = f(x)f(y),$$

so $f \in \text{MHom}(S)$. Therefore $\{f \in \text{MF}(S) \mid f(0) = \{0\}\} \subseteq \text{MHom}(S)$, and hence the theorem is proved. $\square$

Let S be a Kronecker semigroup. Since $xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$ we have that for all nonempty subsets A and B of S,

$$|A| > 1 \text{ or } |B| > 1 \Rightarrow AB = (A \cap B) \cup \{0\}, \tag{1}$$

$$|A| = |B| = 1 \quad \Rightarrow AB = \begin{cases} A & \text{if } A = B, \\ \{0\} & \text{if } A \neq B, \end{cases} \tag{2}$$

$$0 \in A \text{ or } 0 \in B \quad \Rightarrow AB = (A \cap B) \cup \{0\}. \tag{3}$$

For $f \in \text{MF}(S)$, let

$$Z(f) = \{x \in S \mid 0 \in f(x)\}.$$

To characterize the elements of $\text{MHom}(S)$ where S is a Kronecker semigroup, the following lemmas are needed.

Lemma 4.3. *Let S be a Kronecker semigroup and $f \in MHom(S)$. Then for every $x \in S \smallsetminus Z(f)$, $|f(x)| = 1$.*

Proof. If $x \in S$ is such that $|f(x)| > 1$, then by (1),

$$0 \in f(x)f(x) = f(xx) = f(x),$$

so $x \in Z(f)$. Hence for every $x \in S \smallsetminus Z(f)$, $|f(x)| = 1$. $\square$

Lemma 4.4. *Let S be a Kronecker semigroup and $f \in MHom(S)$. If $0 \notin Z(f)$, then $|f(0)| = 1$ and $f(x) = f(0)$ for all $x \in S$.*

Proof. Since $0 \notin Z(f)$, by Lemma 4.3, $|f(0)| = 1$. But $0 \notin f(0)$, so $f(0) = \{a\}$ for some $a \in S \smallsetminus \{0\}$. Hence for every $x \in S$,

$$\{a\} = f(0) = f(0x) = f(0)f(x) = \{a\}f(x),$$

Since $a \neq 0$, from (1) and (2), we have $f(x) = \{a\}$ for all $x \in S$. □

Lemma 4.5. *Let S be a Kronecker semigroup and $f \in MHom(S)$. If $0 \in Z(f)$, then for all distinct $x, y \in S \smallsetminus Z(f)$, $f(x) \neq f(y)$.*

Proof. Let $x, y \in S \smallsetminus Z(f)$ be distinct. Then $xy = 0$, so

$$0 \in f(0) = f(xy) = f(x)f(y).$$

By Lemma 4.3, $|f(x)| = |f(y)| = 1$. We also have that $f(x) \neq \{0\} \neq f(y)$. It follows from (2) that $f(x) \neq f(y)$. □

Lemma 4.6. *Let S be a Kronecker semigroup and $f \in MHom(S)$. If $|Z(f)| > 1$, then $0 \in Z(f)$ and*

$$f(x) \cap f(y) = f(0) \quad \textit{for all distinct} \;\; x, y \in Z(f).$$

Proof. Let $x, y \in Z(f)$ be distinct. Then $xy = 0$, $0 \in f(x)$ and $0 \in f(y)$. From (3),

$$0 \in f(x) \cap f(y) = f(x)f(y) = f(xy) = f(0),$$

and thus $0 \in Z(f)$. □

Lemma 4.7. *Let S be a Kronecker semigroup and $f \in MHom(S)$. If $0 \in Z(f)$ and $|S \smallsetminus Z(f)| > 1$, then $f(0) = \{0\}$ and $f(Z(f)) \cap f(S \smallsetminus Z(f)) = \varnothing$.*

Proof. Let $x, y \in S \setminus Z(f)$ be distinct. Then $xy = 0$. By Lemma 4.3 and Lemma 4.5, $|f(x)| = |f(y)| = 1$ and $f(x) \neq f(y)$. Therefore we have

$$f(0) = f(xy) = f(x)f(y) = \{0\}.$$

Suppose that $f(Z(f)) \cap f(S \setminus Z(f)) \neq \varnothing$. Let $s \in f(Z(f)) \cap f(S \setminus Z(f))$. Then $s \in f(t) \cap f(u)$ for some $t \in Z(f)$ and $u \in S \setminus Z(f)$. Since $u \in S \setminus Z(f)$ and $s \in f(u)$, we have that $s \neq 0$. But

$$s = ss \in f(t)f(u) = f(tu) = f(0) = \{0\},$$

so we have a contradiction. Therefore $f(Z(f)) \cap f(S \setminus Z(f)) = \varnothing$, as desired. □

Lemma 4.8. *Let S be a Kronecker semigroup and $f \in MHom(S)$. If $0 \in Z(f)$ and $S \setminus Z(f) = \{a\}$, then*

$$f(0) = \begin{cases} f(a) \cup \{0\} & \text{if } f(a) \subseteq f(Z(f)), \\ \{0\} & \text{if } f(a) \nsubseteq f(Z(f)). \end{cases}$$

Proof. Since $a \notin Z(f)$, $0 \notin f(a)$. By Lemma 4.3, $|f(a)| = 1$.

Case 1: $f(a) \subseteq f(Z(f))$. Then $f(a) \subseteq f(x)$ for some $x \in Z(f)$, so $0 \in f(x) \setminus f(a)$. Hence by (1)

$$f(0) = f(ax) = f(a)f(x) = f(a) \cup \{0\}.$$

Case 2: $f(a) \nsubseteq f(Z(f))$. Since $0 \in Z(f)$, $f(a) \nsubseteq f(0)$. But $|f(a)| = 1$, so we have from (3) that

$$f(0) = f(a0) = f(a)f(0) = \{0\}.$$

Therefore the lemma is proved. □

Theorem 4.9. *Let S be a Kronecker semigroup and $f \in MF(S)$. Then $f \in MHom(S)$ if and only if one of the following two conditions holds.*

(i) $0 \notin Z(f)$, $|f(0)| = 1$ *and* $f(x) = f(0)$ *for all* $x \in S$.

(ii) $0 \in Z(f)$ *and*

(a) $|f(x)| = 1$ *for every* $x \in S \smallsetminus Z(f)$,

(b) $f(x) \neq f(y)$ *for all distinct* $x, y \in S \smallsetminus Z(f)$,

(c) $f(x) \cap f(y) = f(0)$ *for all distinct* $x, y \in Z(f)$,

(d) $|S \smallsetminus Z(f)| > 1 \Rightarrow f(0) = \{0\}$ *and* $f(Z(f)) \cap f(S \smallsetminus Z(f)) = \varnothing$ *and*

(e) $S \smallsetminus Z(f) = \{a\} \Rightarrow f(0) = f(a) \cup \{0\}$ *if* $f(a) \subseteq f(Z(f))$ *and*

$f(0) = \{0\}$ *if* $f(a) \nsubseteq f(Z(f))$.

Proof. Assume that $f \in \mathrm{MHom}(S)$. If $0 \notin Z(f)$, then (i) holds by Lemma 4.4. Next, assume that $0 \in Z(f)$. We have that (a), (b), (c), (d) and (e) hold by Lemma 4.3, Lemma 4.5, Lemma 4.6, Lemma 4.7 and Lemma 4.8, respectively.

For the converse, assume that f satisfies (i) or (ii). To show that $f \in \mathrm{MHom}(S)$, let $u, v \in S$.

Case 1: f satisfies (i), that is, $|f(0)| = 1$ and $f(x) = f(0)$ for all $x \in S$. Then $f(0)f(0) = f(0)$ and $f(u) = f(0) = f(v)$, so

$$f(uv) = f(0) = f(0)f(0) = f(u)f(v).$$

Case 2: f satisfies (ii).

Subcase 2.1: $u, v \in S \smallsetminus Z(f)$. By (a), $|f(u)| = |f(v)| = 1$. If $u = v$, then $f(u) = f(v)$, so

$$f(uv) = f(u) = f(u)f(v).$$

Assume that $u \neq v$. Hence

$$\begin{aligned} f(uv) &= f(0) \\ &= \{0\} \qquad \text{by (d)} \\ &= f(u)f(v) \qquad \text{by (b).} \end{aligned}$$

Subcase 2.2: $u, v \in Z(f)$. Then $0 \in f(u) \cap f(v)$. If $u = v$, then $f(uv) = f(u)$ and by (3), $f(u)f(v) = f(u)f(u) = f(u)$. Thus $f(uv) = f(u)f(v)$. Assume that $u \neq v$. Then $f(uv) = f(0)$ and

$$f(u)f(v) = f(u) \cap f(v) \quad \text{by (2)}$$
$$= f(0) \quad \text{by (c).}$$

Hence $f(uv) = f(u)f(v)$.

Subcase 2.3: $u \in Z(f)$, $v \in S \setminus Z(f)$ and $|S \setminus Z(f)| > 1$. Then

$$\begin{aligned} f(uv) &= f(0) \\ &= \{0\} && \text{by (d)} \\ &= f(u)f(v) && \text{since } f(Z(f)) \cap f(S \setminus Z(f)) = \varnothing. \end{aligned}$$

Subcase 2.4: $u \in Z(f)$, $S \setminus Z(f) = \{v\}$ and $f(v) \subseteq f(Z(f))$. By (a), $|f(v)| = 1$. Then $f(v) \subseteq f(w)$ for some $w \in Z(f)$. If $w = u$, then

$$\begin{aligned} f(uv) &= f(0) \\ &= f(v) \cup \{0\} && \text{by (e)} \\ &= f(u)f(v) && \text{by (3) and the facts that} \\ & && f(v) \subseteq f(w) = f(u) \\ & && \text{and } 0 \in f(u). \end{aligned}$$

Next, assume that $w \neq u$. Then

$$\begin{aligned} f(uv) &= f(0) \\ &= f(v) \cup \{0\} && \text{by (e),} \end{aligned}$$

and

$$f(0) = f(w) \cap f(u) \quad \text{by (c),}$$

which imply that $f(v) \subseteq f(u)$. Hence

$$\begin{aligned} f(u)f(v) &= (f(u) \cap f(v)) \cup \{0\} && \text{by (3) and the fact} \\ & && \text{that } 0 \in f(u) \\ &= f(v) \cup \{0\} && \text{since } f(v) \subseteq f(u). \end{aligned}$$

Consequently, $f(uv) = f(u)f(v)$.

Subcase 2.5: $u \in Z(f)$, $S \smallsetminus Z(f) = \{v\}$ and $f(v) \nsubseteq f(Z(f))$. Since $|f(v)| = 1$ by (a), it follows that $f(v) \cap f(u) = \varnothing$. Hence $f(u)f(v) = \{0\}$, by (3), so

$$f(uv) = f(0) = \{0\} = f(u)f(v).$$

Hence the theorem is proved. □

Two direct remarkable consequences of Theorem 4.9 are the following corollaries.

Corollary 4.10. *Let S be a Kronecker semigroup, $f \in MF(S)$ and $0 \notin f(0)$. Then $f \in MHom(S)$ if and only if there is an element $a \in S \smallsetminus \{0\}$ such that $f(x) = f(0) = \{a\}$ for all $x \in S$.*

Corollary 4.11. *Let S be a Kronecker semigroup and $\varphi : S \smallsetminus \{0\} \to S \smallsetminus \{0\}$ a bijection. If $f(0) = \{0\}$ and $f(x) = \{\varphi(x)\}$ for all $x \in S \smallsetminus \{0\}$, then $f \in MHom(S)$.*

Example 4.12. Let $(\mathbb{Z}, *)$ be a Kronecker semigroup with zero 0, that is,

$$x * y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Define $f, g \in \mathrm{MF}(\mathbb{Z})$ by

$$f(x) = \begin{cases} 2\mathbb{Z} & \text{if } x \text{ is even}, \\ \{x\} & \text{if } x \text{ is odd}, \end{cases}$$

$$g(x) = \begin{cases} \{0, x\} & \text{if } x \text{ is even}, \\ \{x\} & \text{if } x \text{ is odd}. \end{cases}$$

Then $Z(f) = 2\mathbb{Z} = Z(g)$, $\mathbb{Z} \smallsetminus Z(f) = 2\mathbb{Z}+1 = \mathbb{Z} \smallsetminus Z(g)$, $f(0) = 2\mathbb{Z}$, $f(x) \cap f(y) = f(0)$ and $g(x) \cap g(y) = g(0)$ for all $x, y \in 2\mathbb{Z}$. Moreover, $f(2\mathbb{Z}) \cap f(2\mathbb{Z}+1) = \varnothing$ and $g(2\mathbb{Z}) \cap g(2\mathbb{Z}+1) = \varnothing$. However, $f(0) = 2\mathbb{Z}$ and $g(0) = \{0\}$. It follows from Theorem 4.9 that $f \notin \mathrm{MHom}(\mathbb{Z}, *)$ but $g \in \mathrm{MHom}(\mathbb{Z}, *)$.

CHAPTER V

SOME REGULAR ELEMENTS OF THE SEMIGROUP OF MULTI-VALUED FUNCTIONS OF A SET

In this chapter, some sufficient conditions for the regularity of the elements of $\mathrm{MF}(X)$ are given where X is a nonempty set.

We know that $T(X)$, the full transformation semigroup on X, is a regular subsemigroup of $\mathrm{MF}(X)$ for every nonempty set X. We first give some examples of regular elements of $\mathrm{MF}(X)$ in $\mathrm{MF}(X) \smallsetminus T(X)$ and some nonregular elements of $\mathrm{MF}(X)$.

Example 5.1. (i) If $f \in \mathrm{MF}(\{1,2,3\})$ is defined by

$$f(1) = \{1,2\},\ f(2) = \{2,3\},\ f(3) = \{3,1\}, \tag{1}$$

then f is not regular in $\mathrm{MF}(\{1,2,3\})$. To show this, suppose that $f = fgf$ for some $g \in \mathrm{MF}(\{1,2,3\})$. Then

$$\{1,2\} = f(1) = (fgf)(1) = f(g(\{1,2\})) = f(g(1) \cup g(2)), \tag{2}$$

$$\{2,3\} = f(2) = (fgf)(2) = f(g(\{2,3\})) = f(g(2) \cup g(3)). \tag{3}$$

Therefore (1) and (2) imply $g(1) = g(2) = \{1\}$ and (1) and (3) imply $g(2) = g(3) = \{2\}$. This is a contradiction.

(ii) Let $f \in \mathrm{MF}(\{1,2,3,4\})$ be defined by

$$f(1) = \{1,2,3\},\ f(2) = \{1,3\} \text{ and } f(3) = \{2\} = f(4).$$

Define $g \in \mathrm{MF}(\{1,2,3\})$ by

$$g(1) = \{2\},\ g(2) = \{3\},\ g(3) = \{2\},\ g(4) = \{4\}.$$

Then we have

$$(fgf)(1) = fg(\{1,2,3\}) = f(\{2,3\}) = \{1,2,3\} = f(1),$$
$$(fgf)(2) = fg(\{1,3\}) = f(2),$$
$$(fgf)(3) = fg(2) = f(3),$$
$$(fgf)(4) = fg(2) = f(3) = f(4).$$

Hence $f = fgf$, so f is a regular element of $\mathrm{MF}(\{1,2,3\})$.

Notice that f in Example 5.1(ii) satisfies the fact that

$$\mathrm{ran}f = \{1,2,3\},$$
$$\bigcap_{1\in f(t)} f(t) = f(1) \cap f(2) = f(2),$$
$$\bigcap_{2\in f(t)} f(t) = f(1) \cap f(3) \cap f(4) = f(3) = f(4),$$
$$\bigcap_{3\in f(t)} f(t) = f(1) \cap f(2) = f(2).$$

Hence f has the property that

$$\text{for every } x \in \mathrm{ran}f, \quad \bigcap_{x\in f(t)} f(t) = f(x') \text{ for some } x' \in X. \tag{I}$$

Observe that f in Example 5.1(i) does not satisfy (I).

The following theorem shows that the property (I) of $f \in \mathrm{MF}(X)$ is sufficient for f to be regular in $\mathrm{MF}(X)$ where X is any nonempty set.

Theorem 5.2. *Let X be a nonempty set and $f \in MF(X)$. If for every $x \in \mathrm{ran}f$, $\bigcap_{\substack{t\in X \\ x\in f(t)}} f(t) = f(x')$ for some $x' \in X$, then f is a regular element of the semigroup $MF(X)$.*

Proof. Assume that

$$\text{for every } x \in \mathrm{ran}f, \text{ there is an element } x' \in X \text{ such that}$$
$$\bigcap_{\substack{t\in X \\ x\in f(t)}} f(t) = f(x'). \tag{1}$$

Define $g \in \mathrm{MF}(X)$ by

$$g(x) = \begin{cases} \{x'\} & \text{if } x \in \mathrm{ran} f, \\ \{x\} & \text{if } X \smallsetminus \mathrm{ran} f. \end{cases} \tag{2}$$

To show that $fgf = f$, that is, $(fgf)(y) = f(y)$ for all $y \in X$, let $y \in X$ be given. If $x \in f(y)$, then $x \in \mathrm{ran} f$, so

$$\begin{aligned} x \in \bigcap_{\substack{t \in X \\ x \in f(t)}} f(t) &= f(x') && \text{from (1)} \\ &= fg(x) && \text{from (2)} \\ &\subseteq (fg)(f(y)) && \text{since } x \in f(y) \\ &= (fgf)(y). \end{aligned}$$

This shows that $f(y) \subseteq (fgf)(y)$. Since

$$\begin{aligned} (fgf)(y) &= (fg)(f(y)) \\ &= f\Big(\bigcup_{t \in f(y)} g(t)\Big) \\ &= \bigcup_{t \in f(y)} f(g(t)) \\ &= \bigcup_{t \in f(y)} f(t') && \text{by (2) and the fact that} \\ & && t \in f(y) \subseteq \mathrm{ran} f \\ &= \bigcup_{t \in f(y)} \Big(\bigcap_{\substack{r \in X \\ t \in f(r)}} f(r)\Big) && \text{from (1)} \\ &\subseteq \bigcup_{t \in f(y)} f(y) \\ &= f(y), \end{aligned}$$

we deduce that $(fgf)(y) = f(y)$. Hence f is a regular element of $\mathrm{MF}(X)$, as desired. □

We have a direct consequence of Theorem 5.2 as follows:

Corollary 5.3. *Let X be a nonempty set and $f \in MF(X)$. If for all $x, y \in \operatorname{ran} f$, either $f(x) \cap f(y) = \varnothing$ or $f(x) = f(y)$, then f is a regular element of $MF(X)$.*

Also, we have

Corollary 5.4. *Let X be a finite nonempty set and $f \in MF(X)$. If for all $x, y \in \operatorname{ran} f$, either* (i) *$f(x) \cap f(y) = \varnothing$ or* (ii) *$f(x) \subseteq f(y)$ or $f(y) \subseteq f(x)$, then f is regular in $MF(X)$.*

Proof. Let $x \in \operatorname{ran} f$ and let $A = \{t \in X \mid x \in f(t)\}$. Since X is finite, A is finite. But $x \in f(t) \cap f(t')$ for all $t, t' \in A$, so by assumption, $f(t) \subseteq f(t')$ or $f(t') \subseteq f(t)$ for all $t, t' \in A$. Hence $\{f(t) \mid t \in A\}$ contains a smallest element under inclusion ($\subseteq$), say $f(t_0)$ where $t_0 \in A$. Hence $\bigcap_{x \in f(t)} f(t) = f(t_0)$. By Theorem 5.2, we deduce that f is a regular element of $\mathrm{MF}(X)$. □

The following example shows that the converse of Theorem 5.2 is not generally true.

Example 5.5. Let $f \in \mathrm{MF}((0, \infty))$ be defined by

$$f(x) = (x, \infty) \quad \text{for all } x \in (0, \infty).$$

Then $\operatorname{ran} f = (0, \infty)$ and for $x \in (0, \infty)$,

$$\begin{aligned}(ff)(x) &= f((x, \infty)) \\ &= \bigcup_{t \in (x, \infty)} f(t) \\ &= \bigcup_{t \in (x, \infty)} (t, \infty) \\ &= (x, \infty) = f(x),\end{aligned}$$

so f is regular in $\mathrm{MF}((0, \infty))$. If $x \in (0, \infty) (= \operatorname{ran} f)$, then

$$\bigcap_{x \in f(t)} f(t) = \bigcap_{x \in (t, \infty)} (t, \infty) = [x, \infty) \neq f(y) \quad \text{for all } y \in X.$$

The following example shows that the finiteness of X cannot be omitted in Corollary 5.4.

Example 5.6. Let $f \in \mathrm{MF}([0,1))$ be defined by

$$f(x) = [0, 1 - \frac{x}{2}) \quad \text{for all } x \in [0,1).$$

Then for all $x, y \in [0,1)$, $f(x) \subseteq f(y)$ or $f(y) \subseteq f(x)$. Note that $0 \in f(x)$ for every $x \in [0,1)$.

Suppose that f is regular in $\mathrm{MF}([0,1))$. Then there exists an element $g \in \mathrm{MF}([0,1))$ such that $fgf = f$. Let $a \in g(0)$. Then there exists an element $b \in [0,1)$ such that $0 \leq a < b < 1$. It follows that $f(b) \subseteq f(a)$ and $f(b) \neq f(a)$. Since $a \in g(0)$ and $0 \in f(b)$, we have

$$f(a) \subseteq f(g(0)) \subseteq fgf(b) = f(b),$$

which is a contradiction. Therefore f is not regular in $\mathrm{MF}([0,1))$.

REFERENCES

[1] Feichtinger, O. More on lower semi-continuity. **Amer. Math. Monthly** 83 (1978), 30.

[2] Higgins, P. M. **Techniques of semigroup theory**. New York: Oxford University Press, 1992.

[3] Howie, J. M. **Fundamentals of semigroup theory**. Oxford: Clarendon Press, 1995.

[4] Nenthien, S., and Lertwichitsilp, P. Surjective multihomomorphisms between cyclic groups. **Thai J. Math.** 4 (2006), 35-42.

[5] Smithson, R. E. A characterization of lower semi-continuity. **Amer. Math. Monthly** 75 (1968), 505.

[6] Triphop, N., Harnchoowong, A., and Kemprasit, Y., Multihomomorphisms between cyclic groups. **Set-valued Mathematics and Applications**, to appear.

[7] Whyburn, G. T. Continuity of multifunctions. **Proc. Nat. Acad. Sciences**, U.S.A. 54 (1965), 1494-1501.

[8] Youngkhong, P., and Savettaraseranee, K., Multihomomorphisms from groups into groups of real numbers. **Thai J. Math.** 4 (2006), 43-48.

VITA

Name	Mr. Watchara Teparos.
Date of Birth	5 September 1982.
Place of Birth	Kalasin, Thailand.
Education	B.Sc. (Mathematics), KhonKaen University, 2004.