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SOME PROPERTIES OF HYPERMODULES OVER KRASNER HYPERRINGS

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เราเรียกระบบ $(R,+,\cdot)$ ว่า *คราสเนอร์ไฮเพอร์ริง* ถ้า (i) $(R,+)$ เป็นคาโนนิคัลไฮเพอร์กรุป (ii) $(R,\cdot)$ เป็นกึ่งกรุปที่มี 0 เป็นศูนย์ โดยที่ 0 เป็นเอกลักษณ์แบบสเกลาร์ของ $(R,+)$ และ (iii) $x\cdot(y+z)=x\cdot y+x\cdot z$ และ $(y+z)\cdot x=y\cdot x+z\cdot x$ สำหรับทุก $x,y,z\in R$ *ไฮเพอร์มอดูลบนคราสเนอร์ไฮเพอร์ริง* R คือ คาโนนิคัลไฮเพอร์กรุป M ที่มีฟังก์ชัน $(r,m)\mapsto rm$ จาก $R\times M$ ไป M ซึ่งทุก $r,r_1,r_2\in R$ และ $m,m_1,m_2\in M$ (i) $r(m_1+m_2)=rm_1+rm_2$ (ii) $(r_1+r_2)m=r_1m+r_2m$ (iii) $(r_1\cdot r_2)m=r_1(r_2m)$ และ (iv) $0_R m=0_M$

ในงานวิจัยนี้ เราขยายสมบัติพื้นฐานที่หลากหลายของมอดูลบนริงไปสู่สมบัติของไฮเพอร์มอดูลบนคราสเนอร์ไฮเพอร์ริง และได้ให้ตัวอย่างที่เป็นรูปธรรมของของไฮเพอร์มอดูลบนคราสเนอร์ไฮเพอร์ริง โดยการพิจารณาจากช่วงทั้งหมดซึ่งมี 0 บนระบบจำนวนจริงที่เป็นกึ่งกรุปกับการคูณแบบปกติ กับไฮเพอร์โอเปอเรชันบางอย่าง มากไปกว่านั้นเราได้นิยามโปรเจคทีฟไฮเพอร์มอดูลซึ่งเป็นบทนิยามที่ขนานไปกับโปรเจคทีฟมอดูลในทฤษฎีมอดูลพร้อมกับศึกษาสมบัติบางประการที่สัมพันธ์กัน

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A system $(R,+,\cdot)$ is said to be a *Krasner hyperring* if (i) $(R,+)$ is a canonical hypergroup, (ii) $(R,\cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(R,+)$ and (iii) $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$ for all $x,y,z\in R$. A *hypermodule over a Krasner hyperring* R is a canonical hypergroup M, for which there is a function $(r,m)\to rm$ from $R\times M$ into M such that for all $r,r_1,r_2\in R$ and $m,m_1,m_2\in M$, (i) $r(m_1+m_2)=rm_1+rm_2$, (ii) $(r_1+r_2)m=r_1m+r_2m$, (iii) $(r_1\cdot r_2)m=r_1(r_2m)$ and (iv) $0_R m=0_M$.

In this research, various elementary properties of modules over rings are generalized to properties of hypermodules over Krasner hyperrings and some concrete examples of hypermodules over Krasner hyperrings are given by considering among the collection of all multiplicative interval semigroups joining 0 on the system of real numbers and some hyperoperations. Moreover, we give a definition of projective hypermodule which is parallel to the definition of projective module in module theory and study some related properties.

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Last but not least, my sincere gratitude and appreciation go to my beloved parents for their support and consistent encouragement throughout my study.

CONTENTS

LIST OF SYMBOLS

$\mathbb{R}$	the set of real numbers
$\mathbb{Z}_n$	the set of integers modulo a natural number n
$\|H\|$	the cardinality of a set H
$\mathcal{P}(H)$	the power set of a set H
$\mathcal{P}^*(H)$	the power set of H not containing $\varnothing$
$\max A$	the maximum of a set A
$\min A$	the minimum of a set A
$\ker(f)$	the kernel of a homomorphism f
$\mathrm{im}(f)$	the image of a function f
$f(X)$	the image of X under a function f
$f^{-1}(X)$	the inverse image of X under a function f
id_M	the identity map on a set M

CHAPTER I

INTRODUCTION

1.1 Motivation

The theory of hyperstructures (also called multialgebras) started with the communication of F. Marty in 1934 at the 8^{th} Congress of Scandinavian Mathematicians. Marty introduced the notion of hypergroups and since then many researchers have worked and developed on this topic. The concept of hyperrings was introduced by M. Krasner. Later, J. Mittas and D. Stratigopoulos, two students of Krasner, earned their theses by studying the structure of hyperrings.

P. Corsini gathered the fundamental concepts in his book "*Prolegomena of hypergroup theory*" and its applications in "*Application of hyperstructure theory*". The structure of hypermodules over hyperrings is defined analogously to one of modules over rings. It has been known that there are many different types of hyperrings, for examples, a Krasner hyperring (or simple hyperring), a feeble hyperring, a multiplicative hyperring, a D-hyperring and a V-S-hyperring. As a result, it is not surprised that a hypermodule over a hyperring is defined in various ways.

The purpose of this thesis is to investigate some properties of hypermodules over Krasner hyperrings that are parallel to those of modules over rings. Moreover, we give some examples of hypermodules which are considered from the collection of all multiplicative interval semigroups of $\mathbb{R}$ joining 0.

In addition, CH. G. Massouros [4] gave a definition of free hypermodules over

Krasner hyperrings and delved into their properties. He accomplished one of the pleasant results stating that a basis of a free hypermodule M is linearly independent and generates M. This leads us to the only remaining objective, namely, studying projective hypermodules. The definition of a projective hypermodule is given along with its properties.

This thesis contains 4 chapters. In Chapter I, we motivate our work and introduce some definitions and examples which are required in the following chapters.

We give, in Chapter II, a definition of hypermodules over Krasner hyperrings and study some elementary properties. Moreover, homomorphisms between hypermodules over Krasner hyperrings and direct sums of hypermodules over Krasner hyperrings are illustrated.

In Chapter III, we explore some examples of canonical hypergroups and Krasner hyperrings in order to construct hypermodules over Krasner hyperrings. In this work, we focus on the collection of all interval subsemigroups of $\mathbb{R}$ under usual multiplication joining the real number 0.

In Chapter IV, a definition and investigation of some properties of free hypermodules and projective hypermodules are presented.

1.2 Preliminaries

In this section, we introduce some definitions of hyperstructures inspired by P. Cosini. Many examples of hyperstructures also are given.

For a set H, let $\mathcal{P}(H)$ denote the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \smallsetminus \{\varnothing\}$.

Definition 1.2.1. [5] A *hyperoperation* on a nonempty set H is a mapping of $H \times H$ into $\mathcal{P}^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets X and Y of H, let

$$X \circ Y = \bigcup_{\substack{x \in X \\ y \in Y}} (x \circ y),$$

and let $X \circ y = X \circ \{y\}$ and $y \circ X = \{y\} \circ X$ for all $y \in H$.

A hypergroupoid $(H, \circ)$ is said to be *commutative* if

$$x \circ y = y \circ x \quad \text{for all } x, y \in H.$$

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that

$$(x \circ y) \circ z = x \circ (y \circ z) \quad \text{for all } x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ such that

$$x \circ H = H \circ x = H \quad \text{for all } x \in H.$$

Definition 1.2.2. [5] Let $(H, \circ)$ be a hypergroupoid.

An element e of H is called an *identity* of H if

$$x \in (x \circ e) \cap (e \circ x) \quad \text{for all } x \in H.$$

An element e of H is called a *scalar identity* of H if

$$x \circ e = e \circ x = \{x\} \quad \text{for all } x \in H.$$

In general, an identity of a hypergroupoid may not be unique see Example 1.2.4. However, a scalar identity is unique since if x and y are scalar identities of a hypergroupoid $(H, \circ)$, then $\{x\} = x \circ y = \{y\}$ so that $x = y$.

Definition 1.2.3. [5] Let $(H, \circ)$ be a semihypergroup. An element x of H is said to be an *inverse* of an element y of H if there exists an identity e of H such that

$$e \in (x \circ y) \cap (y \circ x),$$

that is, $(x \circ y) \cap (y \circ x)$ contains at least one identity of H.

Example 1.2.4. [6] Let H be a nonempty set. Define

$$x \circ y = H \quad \text{for all } x, y \in H.$$

Then $(H, \circ)$ is a commutative hypergroup with the following properties.

i) Every element of H is an identity of H. Consequently H has a scalar identity if and only if $|H| = 1$.

ii) Any pairs of elements of H are inverses of each other.

This hypergroup $(H, \circ)$ is usually called the *total hypergroup.*

Definition 1.2.5. [5] A hypergroup $(H, \circ)$ is called a *canonical hypergroup* if

i) $(H, \circ)$ is commutative,

ii) $(H, \circ)$ has a scalar identity,

iii) every element x of H has a unique inverse, denoted by x^{-1}, in H and

iv) $x \in y \circ z$ implies $z \in y^{-1} \circ x$ for $x, y, z \in H$.

Note that if $(H, \circ)$ is a canonical hypergroup, then $x \in y \circ z$ also implies $z \in x \circ y^{-1}$ for $x, y, z \in H$.

Definition 1.2.6. Let $(H, \circ)$ be a canonical hypergroup. For a nonempty subset X of H, let

$$X^{-1} = \{x^{-1} \mid x \in X\}.$$

Proposition 1.2.7. *Let $(H, \circ)$ be a canonical hypergroup. Then $(x^{-1})^{-1} = x$ and $(x \circ y)^{-1} = x^{-1} \circ y^{-1}$ for all $x, y \in H$.*

Proof. This is obvious. □

Example 1.2.8. Let H be a nonempty set of cardinality at least 2. Choose an element in H and denote by 0. Define a hyperoperation $\circ$ on H by

$$a \circ b = \begin{cases} \{a\}, & \text{if } b = 0, \\ \{b\}, & \text{if } a = 0, \\ H, & \text{if } a = b \neq 0, \\ \{a, b\}, & \text{if } a \neq b,\ a \neq 0 \text{ and } b \neq 0. \end{cases}$$

Then $(H, \circ)$ is a canonical hypergroup with 0 as a scalar identity and a as the inverse of $a \in H$.

Proof. It is obvious that $(H, \circ)$ is commutative. Now, we show that $a \circ H = H$ for all $a \in H$. This is clear if $a = 0$. Let $a \in H \smallsetminus \{0\}$. Then $a \circ H = (a \circ a) \cup \big(a \circ (H \smallsetminus \{a\})\big) = H \cup \big(a \circ (H \smallsetminus \{a\})\big) = H$. Thus $H \circ a = a \circ H = H$ for all $a \in H$.

To show that $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in H$, let $a, b, c \in H$.

Case 1 : $a = 0$ or $b = 0$ or $c = 0$. Without loss of generality, assume that $c = 0$. Then $(a \circ b) \circ c = (a \circ b) \circ 0 = a \circ b$ and $a \circ (b \circ c) = a \circ (b \circ 0) = a \circ \{b\} = a \circ b$.

Case 2 : $a, b, c \neq 0$.

Subcase 2.1 : $a = b = c$. Then $(a \circ b) \circ c = (a \circ a) \circ a = a \circ (a \circ a) = a \circ (b \circ c)$.

Subcase 2.2 : Only two elements of a, b and c are equal. Without loss of generality, let $a = b$. Then $(a \circ b) \circ c = (a \circ a) \circ c = H \circ c = H$ and $a \circ (b \circ c) = a \circ (a \circ c) = a \circ \{a, c\} = H$.

Subcase 2.3 : a, b, c are all distinct. Then $(a \circ b) \circ c = \{a, b\} \circ c = \{a, b, c\}$ and $a \circ (b \circ c) = a \circ \{b, c\} = \{a, b, c\}$.

Thus $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in H$.

Hence $(H, \circ)$ is a commutative hypergroup.

Next, we prove that $(H, \circ)$ is a canonical hypergroup. It is obvious that 0 is a scalar identity and a is the unique inverse of a for all $a \in H$. It remains to show that for each $a, b, c \in H$ if $a \in b \circ c$, then $c \in a \circ b^{-1}$. Let $a, b, c \in H$ be such that $a \in b \circ c$.

Case 1 : $b = 0$. Since $a \in b \circ c = 0 \circ c = \{c\}$, we have $a = c$. Then $c \in a \circ 0 = a \circ b^{-1}$.

Case 2 : $c = 0$. Since $a \in b \circ c = b \circ 0 = \{b\}$, we have $a = b$, i.e., $a = b^{-1}$. Then either $a \circ b^{-1} = \{0\}$ or $a \circ b^{-1} = H$. Thus $c = 0 \in a \circ b^{-1}$.

Case 3 : $b \neq c$ and $b, c \neq 0$. Then $a \in \{b, c\}$. If $a = b$, then $a = b^{-1}$ so that $c \in H = a \circ a = a \circ b^{-1}$. If $a = c$, then $a \neq b$, so $c \in \{a, b\} = a \circ b = a \circ b^{-1}$.

Case 4: $b = c$ and $b, c \neq 0$. Then $a \in H$. If $a = b$, then $c \in H = a \circ a = a \circ b = a \circ b^{-1}$. If $a \neq b$, then $c = b \in a \circ b = a \circ b^{-1}$.

Hence $(H, \circ)$ is a canonical hypergroup with 0 as a scalar identity and a as the inverse of $a \in H$. $\square$

Definition 1.2.9. [5] Let $(H, \circ)$ be a canonical hypergroup. A nonempty subset H' of H is called a *canonical subhypergroup* of $(H, \circ)$ if

i) $x \circ y \subseteq H'$ for all $x, y \in H'$,

ii) $e \in H'$ where e is the scalar identity of H and

iii) $x^{-1} \in H'$ for every $x \in H'$ (where x^{-1} is the inverse of x in H).

Remark 1.2.10. Let H' be a canonical subhypergroup of a canonical hypergroup $(H, \circ)$. It is easy to see that $(H', \circ)$ is a canonical hypergroup such that the scalar identity of H is a scalar identity of H' and the inverse of x in H' is the same as the inverse of x in H for each $x \in H'$.

The following proposition gives a practical method for verifying whether a nonempty subset of a given canonical hypergroup is its canonical subhypergroup.

Proposition 1.2.11. *Let* $(H, \circ)$ *be a canonical hypergroup and* H' *a nonempty subset of* H. *Then* H' *is a canonical subhypergroup of* $(H, \circ)$ *if and only if* $x \circ y^{-1} \subseteq H'$ *for all* $x, y \in H'$.

Proof. First, assume that H' is a canonical subhypergroup of $(H, \circ)$. If $x, y \in H'$, then $y^{-1} \in H'$ so that $x \circ y^{-1} \subseteq H'$.

Conversely, suppose that $x \circ y^{-1} \subseteq H'$ for all $x, y \in H'$. Let $x, y \in H'$. Then $e \in x \circ x^{-1} \subseteq H'$. Since $\{x^{-1}\} = e \circ x^{-1} \subseteq H'$, we have $x^{-1} \in H'$. Hence $x^{-1} \in H'$ for each $x \in H'$. Consequently, $x \circ y = x \circ (y^{-1})^{-1} \subseteq H'$.

This proves that H' is a canonical subhypergroup of $(H, \circ)$. □

For the rest of this chapter, a Krasner hyperring is defined and various examples are given.

Definition 1.2.12. [5] A system $(R, \oplus, \circ)$ is called a *(Krasner) hyperring* if

i) $(R, \oplus)$ is a canonical hypergroup,

ii) $(R, \circ)$ is a semigroup with zero 0 where 0 is the scalar identity of $(R, \oplus)$ and

iii) $x \circ (y \oplus z) = x \circ y \oplus x \circ z$ and $(y \oplus z) \circ x = y \circ x \oplus z \circ x$ for all $x, y, z \in R$.

The hyperoperation $\oplus$ and the operation $\circ$ of a hyperring $(R, \oplus, \circ)$ are called the *addition* and the *multiplication* of R, respectively. Moreover, the scalar identity 0 of $(R, \oplus)$ is called the *zero* of R.

Let $(R, \oplus, \circ)$ be a Krasner hyperring. If $(R, \circ)$ is a monoid with identity 1_R, then we call $(R, \oplus, \circ)$ a Krasner hyperring with identity 1_R.

Remark 1.2.13. Let $\oplus$ be a hyperoperation on $\{0\}$. Then $(\{0\}, \oplus)$ is a canonical hypergroup and $(\{0\}, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on $\mathbb{R}$.

Example 1.2.14. [5] Define a hyperoperation $\oplus$ on $\mathbb{Z}_3$ as follows:

$\oplus$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	$\mathbb{Z}_3$
2	$\{2\}$	$\mathbb{Z}_3$	$\{2\}$

Then $(\mathbb{Z}_3, \oplus, \cdot)$ is a Krasner hyperring with zero 0 in $\mathbb{Z}_3$ where $\cdot$ is the usual multiplication on $\mathbb{Z}_3$.

Next example shows how to construct a Krasner hyperring from a group.

Example 1.2.15. [6] Let $(G, \cdot)$ be a group. For $x, y \in G^0$ where $G^0 = G \cup \{0\}$ and 0 is a new symbol not containing in G and $0 \cdot a = 0 = a \cdot 0$ for all $a \in G^0$, define

$$x \oplus y = \begin{cases} \{x\}, & \text{if } y = 0, \\ \{y\}, & \text{if } x = 0, \\ G^0 \smallsetminus \{x\}, & \text{if } x = y \neq 0, \\ \{x, y\}, & \text{if } x \neq y,\ x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Then $(G^0, \oplus, \cdot)$ is a Krasner hyperring.

Examples 1.2.16–1.2.18 are examples of Krasner hyperrings constructed from real intervals.

Example 1.2.16. [5] Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $R = [0, a]$ or $[0, a)$.

Define a hyperoperation $\oplus$ on R by

$$x \oplus y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y, \\ [0, x], & \text{if } x = y. \end{cases}$$

Then $(R, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on $\mathbb{R}$.

Example 1.2.17. [5] Let $a \in \mathbb{R}$ be such that $a \geq 1$ and $R = [a, \infty) \cup \{0\}$ or $(a, \infty) \cup \{0\}$. Define a hyperoperation $\oplus$ on R by

$$\begin{aligned} &x \oplus 0 = 0 \oplus x = \{x\} && \text{for all } x \in R, \\ &x \oplus x = [x, \infty) \cup \{0\} && \text{for all } x \in R \setminus \{0\} \text{ and} \\ &x \oplus y = \{\min\{x, y\}\} && \text{for all } x, y \in R \setminus \{0\} \text{ with } x \neq y. \end{aligned}$$

Then $(R, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on $\mathbb{R}$.

Example 1.2.18. [5] Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $R = [-a, a]$ or $(-a, a)$. Define a hyperoperation $\oplus$ on R by

$$\begin{aligned} x \oplus x &= \{x\} && \text{for all } x \in R, \\ x \oplus (-x) &= [-|x|, |x|\,] && \text{for all } x \in R \text{ and} \\ x \oplus y = y \oplus x &= \{x\} && \text{for all } x \in R \text{ with } |y| < |x|. \end{aligned}$$

Then $(R, \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication on $\mathbb{R}$.

Examples 1.2.16–1.2.18 will play major roles in Chapter III. We define three multi-valued functions $\oplus_{\max}, \oplus_{\min}$ and $\oplus_{\text{abs}}$ of $\mathbb{R} \times \mathbb{R}$ into $\mathcal{P}(\mathbb{R})$ analogously to hyperoperations defined in these examples in order to form hypermodules over Krasner hyperrings.

CHAPTER II

ELEMENTARY PROPERTIES OF HYPERMODULES OVER KRASNER HYPERRINGS

We investigate elementary properties of hypermodules over Krasner hyperrings that are parallel to those of modules over rings. We demonstrate these in three sections. In the first section, hypermodules, subhypermodules and quotient hypermodules are defined and their examples are given. In Section 2.2, we look up some properties regarding homomorphisms and isomorphism theorems. In the last section, the direct sum of subhypermodules are studied.

2.1 Hypermodules over Krasner Hyperrings

We first introduce a definition and give some examples of hypermodules over Krasner hyperrings.

Definition 2.1.1. [2] Let $(R, \oplus, \circ)$ be a Krasner hyperring, one say that $(M, +, \cdot)$ is a *left R-hypermodule* (or M is a *left R-hypermodule* or M is a hypermodule over R) if

i) $(M, +)$ is a canonical hypergroup,

ii) $\cdot$ is a (left) scalar single-valued operation, that is, a function which associates with any pair $(a, x) \in R \times M$ an element $a \cdot x \in M$ such that for all $x, y \in M$ and all $a, b \in R$, the following conditions hold:

(a) $a \cdot (x + y) = a \cdot x + a \cdot y$,

(b) $(a \oplus b) \cdot x = a \cdot x + b \cdot x$,

(c) $(a \circ b) \cdot x = a \cdot (b \cdot x)$,

(d) $0_R \cdot x = 0_M$ where 0_R and 0_M are the zero of R and the scalar identity of M, respectively.

If R is endowed with an identity 1_R, then M is called *unitary* if $1_R \cdot x = x$ for all $x \in M$.

A *right R-hypermodule* is defined in a similar fashion. Unless stated otherwise, all R-hypermodules in this thesis will be left R-hypermodules.

Let $(M, +, \cdot)$ be an R-hypermodule. For nonempty subsets S of R and N of M, let

$$S \cdot N = \{s \cdot n | s \in S \text{ and } n \in N\},$$

$s \cdot N = \{s\} \cdot N$ and $S \cdot n = S \cdot \{n\}$ for all $s \in S$ and $n \in N$. If there is no ambiguity, then $S \cdot N$, $s \cdot N$, $S \cdot n$ and $s \cdot n$ are denoted by SN, sN, Sn and sn, respectively.

We give some examples of hypermodules over a Krasner hyperrings.

Example 2.1.2. Let R be a Krasner hyperring. Then $\{0\}$ and R are R-hypermodules.

Example 2.1.3. Let $a, b \in \mathbb{R}$ be such that $a \geq 1$ and $0 < b \leq 1$, $R = [a, \infty) \cup \{0\}$ and $M = [0, b]$. We recall from Example 1.2.16 and Example 1.2.17 that $(R, \oplus, \circ)$ is a Krasner hyperring and $(M, +)$ is a canonical hypergroup where $\circ$ is the usual multiplication on $\mathbb{R}$, $\oplus$ and $+$ are defined as follows

$$
\begin{aligned}
r \oplus 0 = 0 \oplus r &= \{r\} && \text{for all } r \in R,\\
r \oplus r &= [r, \infty) \cup \{0\} && \text{for all } r \in R \setminus \{0\} \text{ and}\\
r \oplus s &= \{\min\{r, s\}\} && \text{for all } r, s \in R \setminus \{0\} \text{ with } r \neq s,
\end{aligned}
$$

and

$$
x + y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y,\\ [0, x], & \text{if } x = y. \end{cases}
$$

Define a scalar single-valued operation $\cdot : R \times M \to M$ by, for all $c \in R$ and $x \in M$

$$
c \cdot x = \begin{cases} 0 & \text{if } c = 0,\\ \dfrac{x}{c}, & \text{if } c \neq 0. \end{cases}
$$

Then $(M, +, \cdot)$ is an R-hypermodule. The proof will be given later in Proposition 3.2.25.

Example 2.1.4. Let M, M' be R-hypermodules and $L = \{f \mid f : M \to M'\}$. Define a hyperoperation $\oplus$ on L by, for each $f, g \in L$,

$$
f \oplus g = \{h : M \to M' \mid h(x) \in f(x) + g(x) \text{ for all } x \in M\}
$$

and $* : R \times L \to L$ by, for each $r \in R$ and $f \in L$,

$$
(r * f)(x) = r(f(x)) \text{ for all } x \in M.
$$

Then $(L, \oplus, *)$ is an R-hypermodule.

Proof. It is easy to show that $(L, \oplus)$ is a canonical hypergroup where the zero function 0_L is a scalar identity and $-f$ is the inverse of f in L and $0_R * f = 0_L$.

For each $a, b \in R$ and $f \in L$, since $\big((a \circ b) * f\big)(x) = (a \circ b)(f(x)) = a(b(f(x))) = a((b*f)(x)) = (a*(b*f))(x)$ for all $x \in M$, we obtain that $(a \circ b)*f = a*(b*f)$.

Next, we show that $r*(f \oplus g) = r*f \oplus r*g$ for all $r \in R$ and $f, g \in L$. First, let $h \in f \oplus g$. Then $h(x) \in f(x)+g(x)$ for all $x \in M$. Thus $(r*h)(x) = r(h(x)) \in r(f(x)+g(x)) = r(f(x)) + r(g(x)) = (r*f)(x) + (r*g)(x)$ so that $(r*h)(x) \in (r*f)(x) + (r*g)(x)$ for all $x \in M$. Hence $r*h \in r*f \oplus r*g$. On the other hand, let $h \in r*f \oplus r*g$. Then $h(x) \in (r*f)(x) + (r*g)(x) = r(f(x)) + r(g(x)) = r(f(x)+g(x))$ for all $x \in M$. So $h(x) \in r(f(x)+g(x))$ for all $x \in M$. Then for each $x \in M$ there exists $l_x \in f(x)+g(x) \subseteq M'$ such that $h(x) = r(l_x)$. Define $l : M \to M'$ by $l(x) = l_x$ for all $x \in M$. Then for each $x \in M$, $l(x) \in f(x)+g(x)$ and $h(x) = r(l(x)) = (r*l)(x)$, i.e., $l \in f \oplus g$ and $h = r*l$. Hence $h \in r*(f \oplus g)$. Therefore $r*(f \oplus g) = r*f \oplus r*g$.

Finally, we show that $(a+b)*f = a*f \oplus b*f$. First, let $r \in a+b$. Then $(r*f)(x) = r(f(x)) \in (a+b)(f(x)) = a(f(x)) + b(f(x)) = (a*f)(x) + (b*f)(x)$ for all $x \in M$. Hence $r*f \in a*f \oplus b*f$. Next, let $h \in a*f \oplus b*f$. Then $h(x) \in a(f(x)) + b(f(x)) = (a+b)(f(x)) = ((a+b)*f)(x)$ for all $x \in M$. Hence $h \in (a+b)*f$. Therefore $(a+b)*f = a*f \oplus b*f$.

As a result $(L, \oplus, *)$ is an R-hypermodule. □

From now on, we use $+$, $\oplus$ for hyperoperations on an R-hypermodule M and a Krasner hyperring R, respectively. Besides we denote scalar identities of M and R and the inverse of $m \in M$ and $r \in R$ by 0_M, 0_R, $-m$ and $-r$, respectively. If there is no ambiguity, then 0_M and 0_R are denoted by 0.

As in a module over ring, the following proposition for a hypermodule over a Krasner hyperring is obtained.

Proposition 2.1.5. *Let M be an R-hypermodule. Then*

i) $r0_M = 0_M$ *for all* $r \in R$,

ii) $r(-m) = -(rm) = (-r)m$ *for all* $r \in R$ *and* $m \in M$*.*

Proof. *i)* Clearly $r0_M = r(0_R 0_M) = (r0_R)0_M = (0_R)0_M = 0_M$ for all $r \in R$.

ii) Let $m \in M$ and $r \in R$. By *i)*, we have $0_M = r0_M \in r(m + (-m)) = rm + r(-m)$. Then $0_M \in rm + r(-m)$ so that $-(rm) = r(-m)$. Also $0_M = 0_R m \in (r \oplus (-r))m = rm + (-r)m$. Then $0_M \in rm + (-r)m$ so that $-(rm) = (-r)m$. Therefore $r(-m) = -(rm) = (-r)m$. □

The concept of subhypermodules of an R-hypermodule have been studied.

Definition 2.1.6. Let M be an R-hypermodule. A nonempty subset N of M is a *subhypermodule* of M if N is a canonical subhypergroup of M and $rN \subseteq N$ for all $r \in R$

Proposition 2.1.7. *Let* M *be an* R*-hypermodule. A nonempty subset* N *is a subhypermodule of* M *if and only if* $x - y \subseteq N$ *and* $rx \in N$ *for all* $r \in R$ *and* $x, y \in N$*.*

Proof. This follows from Proposition 1.2.11 and Definition 2.1.6. □

For a collection of subhypermodules of an R-hypermodule M, the largest subhypermodule of M contained in these subhypermodules and the smallest sub hypermodule of M containing these subhypermodules alway exist.

Proposition 2.1.8. *Let* M *be an* R*-hypermodule and* N_λ *a subhypermodule of* M *for each* $\lambda \in \Lambda$*. Then* $\bigcap_{\lambda \in \Lambda} N_\lambda$ *is the largest subhypermodule of* M *contained in all* N_λ*.*

Proof. It is obvious that $\bigcap_{\lambda \in \Lambda} N_\lambda$ is contained in all N_λ. Let $x, y \in \bigcap_{\lambda \in \Lambda} N_\lambda$ and $r \in R$. Then $x, y \in N_\lambda$ for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, since N_λ is a subhypermodule of M, it follows that $x - y \subseteq N_\lambda$ and $rx \in N_\lambda$. Hence $\bigcap_{\lambda \in \Lambda} N_\lambda$ is again a subhypermodule of M.

Next, assume that L is an R- hypermodule contained in all N_λ. To show that $L \subseteq \bigcap_{\lambda\in\Lambda} N_\lambda$, let $x \in L$. Then $x \in N_\lambda$ for all $\lambda \in \Lambda$ so that $x \in \bigcap_{\lambda\in\Lambda} N_\lambda$.

Therefore $\bigcap_{\lambda\in\Lambda} N_\lambda$ is the largest subhypermodule of M contained in all N_λ. $\square$

Definition 2.1.9. Let M be an R-hypermodule and N_λ a subhypermodule of M for each $\lambda \in \Lambda$. Define

$$\sum_{\lambda\in\Lambda} N_\lambda = \big\{x \in M \;\big|\; \exists r \in \mathbb{N}\ \exists\ \lambda_1, \dots, \lambda_r \in \Lambda,\ x \in N_{\lambda_1} + \cdots + N_{\lambda_r}\big\}.$$

We call $\sum_{\lambda\in\Lambda} N_\lambda$ the *sum of subhypermodules* N_λ.

Remark 2.1.10. Let M be an R-hypermodule and N_λ a subhypermodule of M for each $\lambda \in \Lambda$. Then

$$\sum_{\lambda\in\Lambda} N_\lambda = \bigcup_{\substack{\lambda_1,\dots,\lambda_r\in\Lambda \\ r\in\mathbb{N}}} (N_{\lambda_1} + \cdots + N_{\lambda_r}).$$

Proposition 2.1.11. *Let M be an R-hypermodule and N_λ a subhypermodule of M for each $\lambda \in \Lambda$. Then $\sum_{\lambda\in\Lambda} N_\lambda$ is the smallest subhypermodule of M containing all N_λ.*

Proof. It is obvious that $\sum_{\lambda\in\Lambda} N_\lambda$ contains all N_λ. Let $x, y \in \sum_{\lambda\in\Lambda} N_\lambda$ and $r \in R$. Then there exist $r_1, r_2 \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_{r_1}$, $\mu_1, \dots \mu_{r_2} \in \Lambda$ such that $x \in N_{\lambda_1} + \cdots + N_{\lambda_{r_1}}$ and $y \in N_{\mu_1} + \cdots + N_{\mu_{r_2}}$. Since each N_λ is a subhypermodule of M, we have $-y \in N_{\mu_1} + \cdots + N_{\mu_{r_2}}$ and $rx \in rN_{\lambda_1} + \cdots + rN_{\lambda_{r_1}} \subseteq N_{\lambda_1} + \cdots + N_{\lambda_{r_1}} \subseteq \sum_{\lambda\in\Lambda} N_\lambda$. Thus $x - y \subseteq N_{\lambda_1} + \cdots + N_{\lambda_{r_1}} + N_{\mu_1} + \cdots + N_{\mu_{r_2}} \subseteq \sum_{\lambda\in\Lambda} N_\lambda$. Hence $\sum_{\lambda\in\Lambda} N_\lambda$ is a subhypermodule of M containing all N_λ.

Assume that K is a subhypermodule of M containing all N_λ. To show that $\sum_{\lambda\in\Lambda} N_\lambda \subseteq K$, let $x \in \sum_{\lambda\in\Lambda} N_\lambda$. Then there exists $r \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_r \in \Lambda$

such that $x \in N_{\lambda_1} + \cdots + N_{\lambda_r}$. Then $x \in K$ since K is a subhypermodule of M containing all N_λ. Therefore $\sum_{\lambda \in \Lambda} N_\lambda$ is the smallest subhypermodule of M containing all N_λ. □

In the case where the index set Λ is finite, say $\Lambda = \{1, 2, 3, \ldots, n\}$, we often write $\sum_{\lambda \in \Lambda} N_\lambda$ as $\sum_{i=1}^{n} N_i$ or $N_1 + N_2 + \cdots + N_n$. We see that

$$N_1 + N_2 + \cdots + N_n = \{x \in M \mid \exists n_i \in N_i, x \in n_1 + n_2 + \cdots + n_n\}$$

and call the *sum of subhypermodules* $N_1, \ldots, N_n$. With this notation we obtain the following consequences.

Proposition 2.1.12. ***Modularity Condition***

Let M be an R-hypermodule. If K, H and L are subhypermodules of M and $K \subseteq H$, then $H \cap (K + L) = K + (H \cap L)$.

Proof. First, let $a \in H \cap (K + L)$. Then $a \in H$ and there exists $k \in K$ and $l \in L$ such that $a \in k + l$. Then $l \in a - k \subseteq H$, i.e., $l \in H$ so that $l \in H \cap L$. As a result, $a \in k + l \subseteq K + (H \cap L)$. Hence $H \cap (K + L) \subseteq K + (H \cap L)$.

On the other hand, since $K + (H \cap L) \subseteq H + H = H$ and $K + (H \cap L) \subseteq K + L$, we have $K + (H \cap L) \subseteq H \cap (K + L)$. □

Corollary 2.1.13. *Let M be an R-hypermodule and K, H and L are subhypermodules of M. If $K \subseteq H$, $K + L = H + L$ and $K \cap L = H \cap L$, then $K = H$.*

Proof. Assume that $K \subseteq H$, $K + L = H + L$ and $K \cap L = H \cap L$. By the Modularity Condition, we have $H \cap (K + L) = K + (H \cap L)$. Then $H \subseteq H \cap (H + L) = H \cap (K + L) = K + (H \cap L) = K + (K \cap L) \subseteq K$. Hence $K = H$. □

This section is ended by constructing a quotient hypermodule over a Krasner hyperring.

Proposition 2.1.14. *Let M be an R-hypermodule and N a subhypermodule of M. Define the relation ρ on M by*

$$x\rho y \quad \Leftrightarrow \quad x+N=y+N \quad \text{for all } x,y\in M.$$

Then ρ is an equivalence relation on M.

Proof. This is obvious. □

Definition 2.1.15. Let M be an R-hypermodule, N a subhypermodule of M and ρ the equivalence relation defined in Proposition 2.1.14. Denote the set of all equivalence classes by M/N, i.e.,

$$M/N=\big\{[x]_\rho|x\in M\big\}=\{x+N|x\in M\}$$

Moreover, $N=0+N$.

Proposition 2.1.16. *Let M be an R-hypermodule and N a subhypermodule of M. Then $x\in y+N$ if and only if $x+N=y+N$ for all $x,y\in M$.*

Proof. This follows from the fact that $\{x+N|x\in M\}$ forms a partition of M. □

Theorem 2.1.17. *Let M be an R-hypermodule and N a subhypermodule of M. Define the hyperoperation $\diamond$ on M/N by*

$$(m_1+N)\diamond(m_2+N)=\{v+N|\ v\in m_1+m_2\} \quad \text{for all } m_1,m_2\in M.$$

Then $(M/N,\diamond)$ is a canonical hypergroup.

Proof. First, we show that $\diamond$ is well-defined. Let $m_1 + N = n_1 + N$ and $m_2 + N = n_2 + N$ where $m_1, m_2, n_1, n_2 \in M$. To show that $A := \{v + N | v \in m_1 + m_2\} = \{w + N | w \in n_1 + n_2\} := B$, let $v \in m_1 + m_2$. Then $v \in m_1 + m_2 \subseteq (n_1 + N) + (n_2 + N) = (n_1 + n_2) + N$. So there exists $w \in n_1 + n_2$ such that $v \in w + N$, i.e., $v + N = w + N$. Hence $A \subseteq B$. The proof of the reverse inclusion is similar. Consequently, $\diamond$ is well-defined.

Next, we show that $(M/N, \diamond)$ is a hypergroup. Let $m_1, m_2, m_3 \in M$. Then

$$\begin{aligned}
\big((m_1 + N) \diamond (m_2 + N)\big) \diamond (m_3 + N) &= \{v + N | v \in m_1 + m_2\} \diamond (m_3 + N) \\
&= \bigcup_{v \in m_1 + m_2} (v + N) \diamond (m_3 + N) \\
&= \bigcup_{v \in m_1 + m_2} \{w + N | w \in v + m_3\} \\
&= \{w + N | w \in (m_1 + m_2) + m_3\} \\
&= \{w + N | w \in m_1 + (m_2 + m_3)\} \\
&= \bigcup_{v \in m_2 + m_3} \{w + N | w \in m_1 + v\} \\
&= \bigcup_{v \in m_2 + m_3} (m_1 + N) \diamond (v + N) \\
&= (m_1 + N) \diamond \{v + N | v \in m_2 + m_3\} \\
&= (m_1 + N) \diamond \big((m_2 + N) \diamond (m_3 + N)\big).
\end{aligned}$$

Thus $(M/N, \diamond)$ is associative. In order to show that $(m_1 + N) \diamond (M/N) = M/N$, let $m \in M$. Since M is a hypergroup, $M = m_1 + M$ so that there exists $n \in M$ such that $m \in m_1 + n$. Then $m + N \in (m_1 + N) \diamond (n + N) \subseteq (m_1 + N) \diamond M/N$.

Now, we prove that $(M/N, \diamond)$ is canonical. It is clear that $(M/N, \diamond)$ is commutative because $(M, +)$ is commutative. We see that N is a scalar identity of $(M/N, \diamond)$ as follows. To show that $-m + N$ is the unique inverse of $m + N$ for each $m \in M$, let $m \in M$. Then $(m+N) \diamond (-m+N) = \{v+N | v \in m+(-m)\}$. Thus $N \in (m+N) \diamond (-m+N)$. Hence $-m+N$ is an inverse of $m+N$. For the uniqueness of an inverse of $m+N$, we let $n \in M$ be such that $N \in (m+N) \diamond (n+N)$. There exists $t \in m+n$ $t+N = N$. Then $t \in N$ and $n \in n+N \subseteq (-m+t)+N = -m+N$. Hence $n+N = -m+N$.

Finally, assume that $m_1 + N \in (m_2+N) \diamond (m_3+N)$ where $m_1, m_2, m_3 \in M$. There exists $t \in m_2 + m_3$ such that $m_1 + N = t + N$. Then $t \in m_1 + u$ for some $u \in N$. Since $t \in m_2 + m_3$, we obtain that $m_3 \in t - m_2 \subseteq m_1 + u - m_2 = (m_1 - m_2) + u$. There exists $s \in m_1 - m_2$ such that $m_3 \in s + u$ so that $m_3 \in s + N$, i.e., $m_3 + N = s + N$. Hence $m_3 + N \in (m_1+N) \diamond (-m_3+N) = (m_1+N) \diamond -(m_3+U)$. $\square$

Theorem 2.1.18. *Let M be an R-hypermodule and N a subhypermodule of M. Define the scalar single-valued operation $\cdot : R \times M/N \to M/N$ by*

$$r \cdot (m+N) = rm + N \quad \textit{for all } m \in M \textit{ and } r \in R.$$

Then $(M/N, \diamond, \cdot)$ is an R-hypermodule.

Proof. First, we show that $\cdot$ is well-defined. Let $r \in R$ and $m_1, n_1 \in M$ be such that $m_1 + N = n_1 + N$. We show that $rm_1 + N = rn_1 + N$. There exists $u_1 \in N$ such that $m_1 \in n_1 + u_1$ since $m_1 + N = n_1 + N$. Then $rm_1 \in r(n_1 + u_1) = rn_1 + ru_1 \subseteq rn_1 + N$. Hence $rm_1 \in rn_1 + N$, i.e., $rm_1 + N = rn_1 + N$. Thus $\cdot$ is well-defined.

To show that $(M/N, \diamond, \cdot)$ is an R-hypermodule, let $r_1, r_2 \in R$ and $m_1, m_2 \in M$.

We see that

$$\begin{aligned}
r\cdot\big((m_1+N)+(m_2+N)\big) &= r\cdot\{t+N|t\in m_1+m_2\}\\
&= \{rt+N|t\in m_1+m_2\}\\
&= \{s+N|s\in rm_1+rm_2\}\\
&= (rm_1+N)\diamond(rm_2+N)\\
&= r\cdot(m_1+N)\diamond r\cdot(m_2+N),\\
(r_1+r_2)\cdot(m_1+N) &= \{rm_1+N|r\in r_1+r_2\}\\
&= \{m+N|m\in r_1m_1+r_2m_1\}\\
&= (r_1m_1+N)\diamond(r_2m_1+N)\\
&= r_1\cdot(m_1+N)\diamond r_2\cdot(m_1+N),\\
r\cdot\big(s\cdot(m_1+N)\big) &= r\circ(sm_1+N)\\
&= r(sm_1)+N\\
&= (rs)m_1+N\\
&= (rs)\cdot(m_1+N),
\end{aligned}$$

finally,

$$0_R\cdot(m_1+N)=0_Rm_1+N=0_M+N=N.$$

Therefore $(M/N,\diamond,\cdot)$ is an R-hypermodule. □

From the previous theorem, we are able to give a definition of quotient hypermodules.

Definition 2.1.19. Let M be an R-hypermodule and N a subhypermodule of M. The hypermodule $(M/N,\diamond,\cdot)$ is called the *quotient hypermodule* of M by N.

2.2 Homomorphisms and Isomorphism Theorems

In this section, we are interested in exploring homomorphisms, and isomorphism theorems of hypermodules over Krasner hyperrings.

Definition 2.2.1. [2] Let M and M' be R-hypermodules. A function $f : M \to M'$ is called a *(hypermodule) homomorphism* if

i) $f(x+y) = f(x) + f(y)$ for all $x, y \in M$ and

ii) $f(rx) = rf(x)$ for all $r \in R$ and $x \in M$.

The followings are simple examples of hypermodule homomorphisms

Example 2.2.2. Let M be an R-hypermodule. The identity function, id_M, on M is obvious a homomorphism.

Example 2.2.3. Let M be an R-hypermodule and N a subhypermodule of M. The *canonical map* $p_N : M \to M/N$ defined by

$$p_N(m) = m + N \text{ for all } m \in M,$$

is a surjective homomorphism.

Next proposition shows elementary properties of hypermodule homomorphisms.

Proposition 2.2.4. *Let M and N be R-hypermodules. If $f : M \to N$ be a homomorphism. then $f(0_M) = 0_N$ and $f(-m) = -f(m)$ for all $m \in M$.*

Proof. We see that $f(0_M) = f(0_R 0_M) = 0_R f(0_M) = 0_N$. Consequently, for each $m \in M$, we obtain that $0_N = f(0_M) \in f\big(m + (-m)\big) = f(m) + f(-m)$ so that $f(-m) = -f(m)$ as desired. □

Proposition 2.2.5. *Let M, N and U be R-hypermodules and $f : M \to N$ and $g : N \to U$ homomorphisms. Then the composite function $g \circ f : M \to U$ is a homomorphism.*

Proof. Let $m, m' \in M$. Then $(g \circ f)(m+m') = g\big(f(m+m')\big) = g\big(f(m)+f(m')\big) = g\big(f(m)\big)+g\big(f(m')\big) = (g \circ f)(m)+(g \circ f)(m')$. Hence $g \circ f$ is a homomorphism. $\square$

For a given homomorphism of hypermodules, its kernel and image are defined in the usual way. The property that, if $f : M \to N$ is a hypermodule homomorphism, then $\ker(f)$ and $\operatorname{im}(f)$ are subhypermodules of M and N, respectively, are obtained unsurprisingly.

Definition 2.2.6. Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. We define the *kernel* and the *image* of f, denoted by $\ker(f)$ and $\operatorname{im}(f)$, respectively, by

$$\ker(f) = \{m \in M \mid f(m) = 0\} \text{ and}$$
$$\operatorname{im}(f) = \{f(m) \mid m \in M\}.$$

Proposition 2.2.7. *Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. Then $f(X)$ is a subhypermodule of N for every subhypermodule X of M, and $f^{-1}(Y)$ is a subhypermodule of M for every subhypermodule Y of N.*

Proof. First, let X be a subhypermodule of M. Since $X \neq \varnothing$, we let $x_1, x_2 \in X$ and $r \in R$. Then it is clear that $f(x_1) - f(x_2) = f(x_1 - x_2) \subseteq f(X)$ and $rf(x_1) = f(rx_1) \in f(X)$. Thus $f(X)$ is a subhypermodule of N.

Next, let Y be a subhypermodule of N. From Proposition 2.2.4, $0 \in f^{-1}(Y)$ so that $f^{-1}(Y) \neq \varnothing$. Let $x_1, x_2 \in f^{-1}(Y)$ and $r \in R$. Then $f(x_1), f(x_2) \in Y$ so that $f(x_1 - x_2) = f(x_1) - f(x_2) \in Y$. Hence $x_1 - x_2 \subseteq f^{-1}(Y)$. Moreover,

$f(rx_1) = rf(x_1) \in Y$ so that $rx_1 \in f^{-1}(Y)$. Hence $f^{-1}(Y)$ is a subhypermodule of M. □

Corollary 2.2.8. *Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. Then* $\ker(f)$ *and* $\operatorname{im}(f)$ *are subhypermodules of M and N, respectively.*

Proof. The results follow from Proposition 2.2.7 since $\ker(f) = f^{-1}(\{0\})$ and $\operatorname{im}(f) = f(M)$. □

Proposition 2.2.9. *Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. Then f is injective if and only if* $\ker(f) = \{0\}$.

Proof. First, the injectivity of f and the fact that $f(0) = 0$ imply $\ker(f) = \{0\}$. Next, we assume that $\ker(f) = \{0\}$. Let $x, y \in M$ be such that $f(x) = f(y)$. Then $0 \in f(x) - f(y) = f(x - y)$. Thus there exists $z \in x - y$ such that $f(z) = 0$, i.e., $z \in \ker(f)$. Hence $z = 0$. This shows that $0 \in x - y$ and then $x \in y + 0 = \{y\}$. Thus $x = y$. As a result, f is injective. □

Proposition 2.2.10. *Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. If X is a subhypermodule of M and Y is a subhypermodule of N then*

i) $f\big[X \cap f^{-1}(Y)\big] = f(X) \cap Y$ *(this property, in fact, holds even if f is just a function)*

ii) $f^{-1}\big[Y + f(X)\big] = f^{-1}(Y) + X$.

Proof. Let X and Y be subhypermodules of M and N, respectively. We prove only the property *ii*).

ii) First, let $x \in f^{-1}\big[Y + f(X)\big]$. Then $f(x) \in Y + f(X)$ so that $f(x) \in y_1 + f(x_2)$ for some $y_1 \in Y$ and $x_2 \in X$. Thus $y_1 \in f(x) - f(x_2) = f(x - x_2)$. There exists $x_1 \in x - x_2$ such that $f(x_1) = y_1 \in Y$, i.e., $x_1 \in f^{-1}(Y)$. Moreover,

$x \in x_1 + x_2$ and then $x \in f^{-1}(Y) + X$. Hence $f^{-1}\big[Y + f(X)\big] \subseteq f^{-1}(Y) + X$. Next, let $x \in f^{-1}(Y) + X$. Then there exist $x_1 \in f^{-1}(Y)$ and $x_2 \in X$ such that $x \in x_1 + x_2$. Thus $f(x) \in f(x_1 + x_2) = f(x_1) + f(x_2) \in Y + f(X)$. Hence $x \in f^{-1}[Y + f(X)]$. This shows that $f^{-1}(Y) + X \subseteq f^{-1}\big[Y + f(X)\big]$.

Therefore $ii)$ follows. □

Corollary 2.2.11. *Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. If X is a subhypermodule of M and Y is a subhypermodule of N then*

i) $f\big[f^{-1}(Y)\big] = Y \cap \mathrm{im}(f)$ *(this property, in fact, holds even if f is just a function)*

ii) $f^{-1}\big[f(X)\big] = X + \ker(f)$.

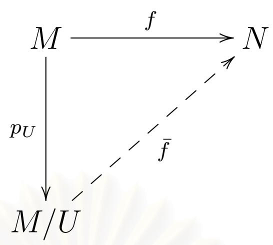
Proof. This follows immediately from Proposition 2.2.10 and the fact that $\mathrm{im}(f) = f(M)$ and $\ker(f) = f^{-1}\big(\{0\}\big)$. □

We give a definition of an isomorphism of R-hypermodule. Then the main theorems for isomorphism are proved.

Definition 2.2.12. A hypermodule homomorphism is called an *isomorphism* if it is also a bijection. If there exists an isomorphism between R-hypermodules M_1 and M_2, we say that M_1 and M_2 are *isomorphic* and denote $M_1 \cong M_2$.

Theorem 2.2.13. ***Factorization (Homomorphism) Theorem***

Let M and N be R-hypermodules and $f : M \to N$ a homomorphism. If U is a subhypermodule of M with $U \subseteq \ker(f)$, then there exists a unique homomorphism $\bar{f} : M/U \to N$ with $f = \bar{f} \circ p_U$, the composite function, i.e., the following diagram *commutes.*



Moreover, $\mathrm{im}(\bar{f}) = \mathrm{im}(f)$ *and* $\ker(\bar{f}) = \ker(f)/U$.

Proof. Note that M/U and $\ker(f)/U$ are R-hypermodules because U is a subhypermodule of M and $\ker(f)$. Define $\bar{f} : M/U \to N$ by

$$\bar{f}(m+U) = f(m) \text{ for all } m \in M.$$

First, we show that $\bar{f}$ is well-defined. Let $x, y \in M$ be such that $x+U = y+U$. Then there is $u \in U$ with $x \in y+u$ so that $f(x) \in f(y+u)$. Since f is a homomorphism and $U \subseteq \ker(f)$, it follows that $f(x) \in f(y+u) = f(y)+f(u) = f(y)+0 = \{f(y)\}$. Hence $f(x) = f(y)$. Consequently, $\bar{f}$ is well-defined.

It is clear that $f = \bar{f} \circ p_U$ and $\bar{f}$ is a homomorphism.

Next, we show the uniqueness of $\bar{f}$. Let $h_1, h_2 : M/U \to N$ be homomorphisms such that $f = h_1 \circ p_U$ and $f = h_2 \circ p_U$. For each $x+U \in M/U$ where $x \in M$,

$$h_1(x+U) = h_1(p_U(x)) = (h_1 \circ p_U)(x) = f(x) = (h_2 \circ p_U)(x) = h_2(p_U(x)) = h_2(x+U).$$

This shows that $h_1 = h_2$. Hence $\bar{f}$ is the unique homomorphism such that $f = \bar{f} \circ p_U$.

Moreover, we see that

$$\mathrm{im}(\bar{f}) = \{\bar{f}(x+U) | x+U \in M/U\} = \{f(x) | x \in M\} = \mathrm{im}(f)$$

and

$$\begin{aligned}\ker(\bar{f}) &= \{x+U \in M/U | \bar{f}(x+U) = 0\} \\ &= \{x+U \in M/U | f(x) = 0\} \\ &= \{x+U \in M/U | x \in \ker(f)\} \\ &= \ker(f)/U.\end{aligned}$$

Hence $\operatorname{im}(\bar{f}) = \operatorname{im}(f)$ and $\ker(\bar{f}) = \ker(f)/U$ as desired. □

Theorem 2.2.14. ***The First Isomorphism Theorem***

Let M and N be R-hypermodules and $f : M \to N$ a surjective homomorphism. Then $M/\ker(f) \cong N$.

Proof. Apply the Factorization Theorem by setting $U = \ker(f)$, then there exists a homomorphism $\bar{f} : M/\ker(f) \to N$ such that $\operatorname{im}(\bar{f}) = \operatorname{im}(f) = N$ and $\ker(\bar{f}) = \ker(f)/\ker(f) = \{0\}$. Thus $\bar{f}$ is a bijection. Hence $M/\ker(f) \cong N$. □

Theorem 2.2.15. ***The Second Isomorphism Theorem***

Let M be an R-hypermodule, N and U subhypemodules of M such that $N \subseteq U \subseteq M$. Then $(M/N)\big/(U/N) \cong M/U$.

Proof. Clearly, M/N and U/N are R-hypermodules. Define $f : M/N \to M/U$ by

$$f(m+N) = m+U \text{ for all } m \in M.$$

It is easy to show that f is well-defined and surjective. To show that f is a homomorphism, let $m_1 + N, m_2 + N \in M/N$. Then

$$\begin{aligned} f\big((m_1+N) \diamond (m_2+N)\big) &= \{f(v+N) | v \in m_1 + m_2\} \\ &= \{v + U | v \in m_1 + m_2\} \\ &= (m_1 + U) \diamond (m_2 + U) \\ &= f(m_1 + N) \diamond f(m_2 + N). \end{aligned}$$

Hence f is a homomorphism.

Next, we show that $\ker(f) = U/N$. It is clear that $U/N \subseteq \ker(f)$. Thus, let $m + N \in \ker(f)$. Then $m + U = f(m + N) = U$, i.e., $m \in U$ so that $m + N \in U/N$. Hence $\ker(f) = U/N$.

Thus $(M/N)\big/(U/N) \cong M/U$ by the First Isomorphism Theorem. □

Theorem 2.2.16. ***The Third Isomorphism Theorem***

Let M be an R-hypermodule, N and U subhypemodules of M. Then $(N+U)/U \cong N/N \cap U$.

Proof. Note that $(N+U)/U$ and $N/N\cap U$ are R-hypermodules because U, $N \cap U$ are subhypermodules of $N + U$ and N, respectively. Define $f : N \to (N + U)/U$ by

$$f(n) = n + U \text{ for all } n \in N.$$

We show that f is a surjective homomorphism whose kernel is $N \cap U$.

It is clear that $n + U \in (N + U)/U$ for all $n \in N$ since $0 \in U$. Moreover, f is surjective obviously. To show that f is a homomorphism, let $n_1, n_2 \in N$. Then

$$\begin{aligned} f(n_1 + n_2) &= \{v + U | v \in n_1 + n_2\} \\ &= (n_1 + U) \diamond (n_2 + U) \\ &= f(n_1) \diamond f(n_2). \end{aligned}$$

Hence f is a homomorphism. Next, we show that $\ker(f) = N \cap U$. It is clear that $N \cap U \subseteq \ker(f)$. Now, let $x \in \ker(f)$. Then $x \in N$ and $x + U = f(x) = U$, i.e., $x \in U$ so that $x \in N \cap U$. Hence $\ker(f) = N \cap U$.

Thus $(N + U)/U \cong N/N \cap U$ by the First Isomorphism Theorem. □

Theorem 2.2.17. ***The Butterfly of Zazzenhaus***

Let M be an R-hypermodule, N, U, N' and U' subhypemodules of M such that $N \subseteq U$ and $N' \subseteq U'$. Then

$$\frac{N + (U \cap U')}{N + (U \cap N')} \cong \frac{U \cap U'}{(N \cap U') + (N' \cap U)} \cong \frac{N' + (U \cap U')}{N' + (N \cap U')}.$$

Proof. Let $S = U \cap U'$ and $T = N + (U \cap N')$. Then we claim that

i) $S + T = N + (U \cap U')$ and

ii) $S \cap T = (N \cap U') + (N' \cap U)$.

First, $S + T = (U \cap U') + (N + (U \cap N')) = N + (U \cap U')$ since $U \cap N' \subseteq U \cap U'$.

Next, to show that $S \cap T \subseteq (N \cap U') + (N' \cap U)$, let $s \in S \cap T$. Then $s \in U \cap U'$ and $s \in N + (U \cap N')$. There exists $n \in N$ and $n' \in U \cap N'$ such that $s \in n + n'$. Since $U \cap N' \subseteq U \cap U'$, we have $n \in s - n' \subseteq U'$. Thus $n \in N \cap U'$. Hence $s \in n + n' \subseteq (N \cap U') + (N' \cap U)$. Conversely, since $N \cap U', N' \cap U \subseteq S \cap T$, we obtain that $(N \cap U') + (N' \cap U) \subseteq S \cap T$. Hence the second claim is proved.

By the Third Isomorphism Theorem, $(S+T)/T \cong S/S\cap T$, i.e.,

$$\frac{N+(U\cap U')}{N+(U\cap N')} \cong \frac{U\cap U'}{(N\cap U')+(N'\cap U)}.$$

Similarly,

$$\frac{N'+(U\cap U')}{N'+(N\cap U')} \cong \frac{U\cap U'}{(N\cap U')+(N'\cap U)}.$$

Thus

$$\frac{N+(U\cap U')}{N+(U\cap N')} \cong \frac{U\cap U'}{(N\cap U')+(N'\cap U)} \cong \frac{N'+(U\cap U')}{N'+(N\cap U')}.$$

□

2.3 Direct Sums

This final section devotes to studying elementary properties of direct sums of R-hypermodules.

Definition 2.3.1. Let N and P be subhypermodules of an R-hypermodule M. If $M = N+P$ and $N\cap P = \{0\}$, then M is called the *(internal) direct sum* of N and P. This is written as $M = N\bigoplus P$.

Definition 2.3.2. A subhypermodule N of M is called a *direct summand* of M if there is a subhypermodule P of M such that $M = N\bigoplus P$.

Proposition 2.3.3. *Let $M = N\bigoplus P$. Then every $m\in M$ there exist unique $n\in N$ and $p\in P$ such that $m\in n+p$ with $n\in N$ and $p\in P$.*

Proof. Let $m\in M$. Since $M = N\bigoplus P$, there exist $n\in N$ and $p\in P$ with $m\in n+p$. Now, we show the uniqueness of n and p. Let $n_1, n_2\in N$ and $p_1, p_2\in P$ be such that $m\in n_1+p_1$ and $m\in n_2+p_2$. Then $n_1\in m-p_1$ so that $n_1\in n_2+p_2-p_1 = n_2+(p_2-p_1)$. Thus there exists $x\in p_2-p_1\subseteq P$ such that

$n_1 \in n_2 + x$. Hence $x \in n_1 - n_2 \subseteq N$. This shows that $x \in N \cap P = \{0\}$, i.e., $x = 0$. Consequently, $n_1 \in n_2 + 0$ and $0 \in p_2 - p_1$. Thus $n_1 = n_2$ and $p_1 = p_2$.

This finishes the proof. □

Example 2.3.4. Let R-hypermodule M be the direct sum of subhypermodules P and Q. Define the maps $\pi : P \bigoplus Q \to P$ and $\iota : P \to P \bigoplus Q$ by

$$\pi(x) = p \ \text{ for all } \ x \in p + q, \text{ and}$$
$$\iota(p) = p \ \text{ for all } \ p \in P.$$

Then π is a surjective homomorphism and ι is an injective homomorphism.

It is easy to show that ι is an injective homomorphism and π is surjective. To show that π is a homomorphism, let $x_1, x_2 \in P \bigoplus Q$. Then $x_1 \in p_1 + q_1$ and $x_2 \in p_2 + q_2$ for some $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. Thus $p_1 \in x_1 - q_1$ and $p_2 \in x_2 - q_2$. We obtain that

$$x_1 + x_2 \subseteq (p_1 + q_1) + (p_2 + q_2) = (p_1 + p_2) + (q_1 + q_2)$$
$$p_1 + p_2 \subseteq (x_1 - q_1) + (x_2 - q_2) = (x_1 + x_2) - (q_1 + q_2).$$

First, let $a \in \pi(x_1 + x_2)$. There exists $x \in x_1 + x_2$ such that $a = \pi(x)$. Thus $x \in p + q$ for some $p \in p_1 + p_2$ and $q \in q_1 + q_2$. Hence $a = \pi(x) = p \in p_1 + p_2 = \pi(x_1) + \pi(x_2)$.

Conversely, let $a \in \pi(x_1) + \pi(x_2)$, i.e., $a \in p_1 + p_2$. Then there exist $x \in x_1 + x_2$ and $q \in q_1 + q_2$ such that $a \in x - q$. Then $x \in a + q \subseteq P \bigoplus Q$. Hence $a = \pi(x)$.

Therefore π is a surjective homomorphism.

Definition 2.3.5. Let P and Q be subhypermodules of an R-hypermodule. The surjective homomorphism $\pi : P \bigoplus Q \to P$ and the injective homomorphism

$\iota : P \to P \bigoplus Q$ defined in the previous example are called the *projection* map and the *inclusion map*, respectively.

Proposition 2.3.6. *Let M and N be R-hypermodules $f : M \to N$ and $g : N \to M$ be homomorphisms such that the composition between f and g is the identity map on N, i.e., $f \circ g = id_N$ where id_N is the identity map on N. Then $M = \ker(f) \bigoplus \operatorname{im}(g)$.*

Proof. First, we show that $\ker(f) \cap \operatorname{im}(g) = \{0\}$. Let $x \in \ker(f) \cap \operatorname{im}(g)$. Then $f(x) = 0$ and $x = g(n)$ for some $n \in N$. Then

$$0 = f(x) = f(g(n)) = (f \circ g)(n) = id_N(n) = n.$$

Hence $n = 0$ and $x = g(n) = g(0) = 0$. As a result, $\ker(f) \cap \operatorname{im}(g) = \{0\}$.

Next, we show that $M = \ker(f) + \operatorname{im}(g)$. It is enough to show only that $M \subseteq \ker(f) + \operatorname{im}(g)$. Let $m \in M$. Then $0 \in f\big(m - (g \circ f)(m)\big)$ because

$$f\big(m-(g\circ f)(m)\big) = f(m)-f(g\circ f)(m)) = f(m)-(f\circ g)(f(m)) = f(m)-f(m).$$

Then there exists $v \in m - (g \circ f)(m)$ such that $f(v) = 0$, i.e., $v \in \ker(f)$. Thus $m \in v + (g \circ f)(m)$. In fact, $v \in \ker(f)$ and $(g \circ f)(m) \in \operatorname{im}(g)$, so we can conclude that $m \in \ker(f) + \operatorname{im}(g)$. Therefore $M = \ker(f) + \operatorname{im}(g)$. □

CHAPTER III

EXAMPLES OF HYPERMODULES

The goal of this chapter is to investigate some examples of hypermodules over Krasner hyperrings by considering among the collection of all multiplicative interval semigroups of $\mathbb{R}$ joining the real number 0 which are motivated by [3].

There are two sections in this chapter. In Section 3.1, we construct certain canonical hypergroups and Krasner hyperrings. In Section 3.2, we apply the results from the previous section in order to explore certain examples of hypermodules over Krasner hyperrings which is our main purpose

We would like to recall the characterization of interval semigroups of $\mathbb{R}$ under usual multiplication.

Proposition A [3] *Let I be a real interval. Then I is a subsemigroup of $\mathbb{R}$ under usual multiplication if and only if I is one of the following forms :*

i) $\mathbb{R}$, *ii)* $\{0\}$, *iii)* $\{1\}$, *iv)* $(0,\infty)$, *v)* $[0,\infty)$,

vi) (a,∞) *where* $a \geq 1$, *vii)* $[a,\infty)$ *where* $a \geq 1$,

viii) $(0,b)$ *where* $0 < b \leq 1$, *ix)* $(0,b]$ *where* $0 < b \leq 1$,

x) $[0,b)$ *where* $0 < b \leq 1$, *xi)* $[0,b]$ *where* $0 < b \leq 1$,

xii) (a,b) *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

xiii) $(a,b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

xiv) $[a,b)$ *where* $-1 \leq a < 0 < a^2 < b \leq 1$,

xv) $[a,b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$.

We are interested in interval semigroups I^0 of $\mathbb{R}$ under usual multiplication joining the real number 0 because 0 will be needed as the scalar identity of canonical hypergroups. In another word, for a multiplicative interval subsemigroup I of $\mathbb{R}$, let

$$I^0 = \begin{cases} I & \text{if } 0 \in I, \\ I \cup \{0\} & \text{if } 0 \notin I. \end{cases}$$

Proposition B *Let I be a real interval. Then I is a subsemigroup of $\mathbb{R}$ containing the real number 0 under usual multiplication if and only if I is one of the following forms :*

i) $\mathbb{R}$, *ii)* $\{0\}$, *iii)* $\{0,1\}$, *iv)* $[0,\infty)$,

v) $(a,\infty) \cup \{0\}$ *where* $a \geq 1$, *vi)* $[a,\infty) \cup \{0\}$ *where* $a \geq 1$,

vii) $[0,b)$ *where* $0 < b \leq 1$, *viii)* $[0,b]$ *where* $0 < b \leq 1$,

ix) (a,b) *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

x) $(a,b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

xi) $[a,b)$ *where* $-1 \leq a < 0 < a^2 < b \leq 1$,

xii) $[a,b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$.

We denote the collection of all multiplicative interval subsemigroups of $\mathbb{R}$ joining the real number 0 induced from all multiplicative interval subsemigroups of $\mathbb{R}$ by $\mathcal{I}^0$.

Next, we would like to construct canonical hypergroups and then Krasner hyperrings from $\mathcal{I}^0$.

3.1 Hyperoperations $\oplus_{\max}$, $\oplus_{\min}$ and $\oplus_{\mathbf{abs}}$

3.1.1 Hyperoperation $\oplus_{\max}$

Define a multi-valued function $\oplus_{\max} : \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$x \oplus_{\max} y = \begin{cases} \{\max\{x, y\}\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y. \end{cases}$$

Note that if $x < 0$, then $x \oplus_{\max} x = \varnothing$.

We investigate a nonempty set $H \in \mathcal{I}^0$ such that $\oplus_{\max}$ is a hyperoperation on H.

Lemma 3.1.1. *Let $H \in \mathcal{I}^0$ be one of the following forms :*

i) $\mathbb{R}$,

ii) (a, b) *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

iii) $(a, b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

iv) $[a, b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

v) $[a, b)$ *where* $-1 \leq a < 0 < a^2 < b \leq 1$.

Then $\oplus_{\max}$ is not a hyperoperation on H.

Proof. This is obvious because there is a negative $x \in H$ such that $x \oplus_{\max} x = \varnothing$. □

Lemma 3.1.2. *Let $H \in \mathcal{I}^0$ be one of the following forms :*

i) $\{0, 1\}$,

ii) $(a, \infty) \cup \{0\}$ *where* $a \geq 1$, *iii)* $[a,\infty)\cup\{0\}$ *where* $a\geq 1$.

Then $\oplus_{\max}$ *is not a hyperoperation on* H.

Proof. If $H=\{0,1\}$, then $1\oplus_{\max}1=[0,1]\nsubseteq\{0,1\}$, so that $\oplus_{\max}$ is not a hyperoperation on H.

For other cases, we consider $(a+1)\oplus_{\max}(a+1)=[0,a+1]\nsubseteq(a,\infty)\cup\{0\}$ and not a subset of $[a,\infty)\cup\{0\}$. Hence, $\oplus_{\max}$ is not a hyperoperation on H. □

Theorem 3.1.3. *Let* $H\in\mathcal{I}^0$. *Then* $\oplus_{\max}$ *is a hyperoperation on* H *if and only if* H *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,b)$ *where* $0<b\leq 1$,

iv) $[0,b]$ *where* $0<b\leq 1$.

Proof. First, assume that H is not one of the above forms. By Lemma 3.1.1 and Lemma 3.1.2, we obtain that $\oplus_{\max}$ is not a hyperoperation on H. Next, if H is one of the above forms, then it is clear that $\oplus_{\max}$ is a hyperoperation on H. □

Next, we characterize when $(M,\oplus_{\max})$ is a canonical hypergroup where $M\in\mathcal{I}^0$.

Theorem 3.1.4. *Let* $M\in\mathcal{I}^0$. *Then* $(M,\oplus_{\max})$ *is a canonical hypergroup if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,a)$ *where* $0<a\leq 1$,

iv) $[0, a]$ *where* $0 < a \leq 1$.

Proof. First, assume that M is not one of the above forms. By Theorem 3.1.3, $\oplus_{\max}$ is not a hyperoperation on M. Hence $(M, \oplus_{\max})$ is not a canonical hypergroup.

Conversely, by Theorem 3.1.3, Remark 1.2.13 and Example 1.2.16, we have $\big(\{0\}, \oplus_{\max}\big)$, $\big([0, a), \oplus_{\max}\big)$ and $\big([0, a], \oplus_{\max}\big)$ are canonical hypergroups where $0 < a \leq 1$. For the final case, it is straightforward to show that $([0, \infty), \oplus_{\max})$ is a commutative hypergroup and 0 is its scalar identity. Next, we show that every nonnegative real numbers has a unique inverse in $[0, \infty)$. Let $x \in [0, \infty)$. Then x is an inverse of x because $0 \in x \oplus_{\max} x$. And it is clear that this inverse is unique. Finally, we show that $x \in y \oplus_{\max} z$ implies $z \in y \oplus_{\max} x$ for $x, y, z \in [0, \infty)$. Let $x, y, z \in [0, \infty)$ be such that $x \in y \oplus_{\max} z$.

Case 1. $y = z$. Then $x \in [0, y]$. If $x = y$, then $z \in [0, z] = z \oplus_{\max} z = y \oplus_{\max} x$. If $x \neq y$, then $x < y$ so that $y \oplus_{\max} x = \{y\} = \{z\}$, i.e., $z \in y \oplus_{\max} x$.

Case 2. $y < z$. Then $x \in y \oplus_{\max} z = \{z\}$ so that $x = z$. Since $z \in z \oplus_{\max} y$, we have $z \in x \oplus_{\max} y = y \oplus_{\max} x$.

Case 3. $z < y$. Then $x \in y \oplus_{\max} z = \{y\}$ so that $x = y$. Hence $y \oplus_{\max} x = [0, y]$. Since $z < y$, we have $z \in y \oplus_{\max} x$.

Therefore $\big([0, \infty), \oplus_{\max}\big)$ is a canonical hypergroup. □

From now on, let $\cdot$ be the usual multiplication on $\mathbb{R}$.

We investigate when $(R, \oplus_{\max}, \cdot)$ is a Krasner hyperring where $R \in \mathcal{I}^0$.

Theorem 3.1.5. *Let* $R \in \mathcal{I}^0$. *Then* $(R, \oplus_{\max}, \cdot)$ *is a Krasner hyperring if and only if* R *is one of the following forms :*

i) $\{0\}$,

ii) $[0, \infty)$,

iii) $[0, a)$ *where* $0 < a \leq 1$,

iv) $[0, a]$ *where* $0 < a \leq 1$.

Proof. If $(R, \oplus_{\max}, \cdot)$ is a Krasner hyperring, then the result follows from Theorem 3.1.3.

Conversely, it follows from Theorem 3.1.3, Remark 1.2.13 and Example 1.2.16, that $\big(\{0\}, \oplus_{\max}, \cdot\big)$, $\big([0, a), \oplus_{\max}, \cdot\big)$ and $\big([0, a], \oplus_{\max}, \cdot\big)$ are Krasner hyperrings where $0 < a \leq 1$. The other case, $([0, \infty), \oplus_{\max})$ is a canonical hypergroup by Theorem 3.1.4 and $([0, \infty), \cdot)$ is a semigroup with zero 0. Finally, we show that $x \cdot (y \oplus_{\max} z) = x \cdot y \oplus_{\max} x \cdot z$ for all $x, y, z \in [0, \infty)$. Let $x, y, z \in [0, \infty)$.

Case 1. $y = z$. Then $xy = xz$. So $x \cdot (y \oplus_{\max} z) = x \cdot [0, y] = [0, xy]$ and $x \cdot y \oplus_{\max} x \cdot z = xy \oplus_{\max} xz = [0, xy]$. Hence $x \cdot (y \oplus_{\max} z) = x \cdot y \oplus_{\max} x \cdot z$.

Case 2. $y \neq z$. Without loss of generality, assume that $y > z$. If $x = 0$, then $xy = xz = 0$ so $x \cdot (y \oplus_{\max} z) = x \cdot \{y\} = \{xy\} = \{0\}$ and $x \cdot y \oplus_{\max} x \cdot z = xy \oplus_{\max} xz = \{0\}$. If $x \neq 0$, then $xy > xz$ so $x \cdot (y \oplus_{\max} z) = x \cdot \{y\} = \{xy\}$ and $x \cdot y \oplus_{\max} x \cdot z = xy \oplus_{\max} xz = \{xy\}$. Hence $x \cdot (y \oplus_{\max} z) = x \cdot y \oplus_{\max} x \cdot z$.

From both cases, $x \cdot (y \oplus_{\max} z) = x \cdot y \oplus_{\max} x \cdot z$ for all $x, y, z \in [0, \infty)$. Since $\big([0, \infty), \cdot\big)$ is commutative, $(y \oplus_{\max} z) \cdot x = y \cdot x \oplus_{\max} z \cdot x$ for all $x, y, z \in [0, \infty)$. Therefore, $\big([0, \infty), \oplus_{\max}, \cdot\big)$ is a Krasner hyperring such that 0 is the zero and the additive inverse of $x \in M$ is x itself. □

3.1.2 Hyperoperation $\oplus_{\min}$

Define a multi-valued function $\oplus_{\min} : \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$
\begin{aligned}
&x \oplus_{\min} 0 = 0 \oplus_{\min} x = \{x\} && \text{for all } x \in \mathbb{R}, \\
&x \oplus_{\min} x = [x, \infty) \cup \{0\} && \text{for all } x \in \mathbb{R} \smallsetminus \{0\} \text{ and} \\
&x \oplus_{\min} y = \big\{\min\{x, y\}\big\} && \text{for all } x, y \in \mathbb{R} \smallsetminus \{0\} \text{ with } x \neq y.
\end{aligned}
$$

We investigate when $\oplus_{\min}$ is a hyperoperation on H where $H \in \mathcal{I}^0$.

Lemma 3.1.6. *Let H be one of the following forms :*

i) $\{0, 1\}$,

ii) $(0, a]$ *where* $0 < a \leq 1$,

iii) $[0, a]$ *where* $0 < a \leq 1$,

iv) (a, b) *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

v) $(a, b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

vi) $[a, b]$ *where* $-1 \leq a < 0 < a^2 \leq b \leq 1$,

vii) $[a, b)$ *where* $-1 \leq a < 0 < a^2 < b \leq 1$.

Then $\oplus_{\min}$ is not a hyperoperation on H.

Proof. Let $x \in H \smallsetminus \{0\}$. Consider $x \oplus_{\min} x = [x, \infty) \cup \{0\} \nsubseteq H$. Hence $\oplus_{\min}$ is not a hyperoperation on H. □

Theorem 3.1.7. *Let $H \in \mathcal{I}^0$. Then $\oplus_{\min}$ is a hyperoperation on H if and only if H is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $[0,\infty)$,

iv) $(a,\infty)\cup\{0\}$ *where* $a\geq 1$,

v) $[a,\infty)\cup\{0\}$ *where* $a\geq 1$.

Proof. First, assume that H is not one of the above forms. We obtain that $\oplus_{\min}$ is not a hyperoperation on H by Lemma 3.1.6. Next, it is obvious that $\oplus_{\min}$ is a hyperoperation on H if H is one of the above forms. □

Now, we examine when $(M,\oplus_{\min})$ is a canonical hypergroup where $M\in\mathcal{I}^0$.

Theorem 3.1.8. *Let* $M\in\mathcal{I}^0$. *Then* $(M,\oplus_{\min})$ *is a canonical hypergroup if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $[0,\infty)$,

iv) $(a,\infty)\cup\{0\}$ *where* $a\geq 1$,

v) $[a,\infty)\cup\{0\}$ *where* $a\geq 1$.

Proof. First, if M is not one of the above forms, then $(M,\oplus_{\min})$ is not a canonical hypergroup which is a result of Theorem 3.1.7.

Conversely, Theorem 3.1.7, Remark 1.2.13 and Example 1.2.17 show that $(\{0\},\oplus_{\min})$, $((a,\infty)\cup\{0\},\oplus_{\min})$ and $([a,\infty)\cup\{0\},\oplus_{\min})$ are canonical hypergroups where $a\geq 1$. For the other cases, since $[0,\infty)\subseteq\mathbb{R}$, it suffices to show only that $(\mathbb{R},\oplus_{\min})$ is a canonical hypergroup.

It is easy to see that $(\mathbb{R}, \oplus_{\min})$ is a commutative hypergroup, has a scalar identity 0 and the inverse of $x \in R$ is x itself. Next, we show that for $x, y, z \in M$, $x \in y \oplus_{\min} z$ implies $z \in y \oplus_{\min} x$. Let $x, y, z \in M$. Assume that $x \in y \oplus_{\min} z$.

Case 1. $y = 0$ and $z = 0$. Then $y \oplus_{\min} z = 0 \oplus_{\min} 0 = \{0\}$. So $x = 0$. We have $y \oplus_{\min} x = 0 \oplus_{\min} 0 = \{0\}$. Hence $z \in y \oplus_{\min} x$.

Case 2. $y = z$ and $y, z \neq 0$. Then $y \oplus_{\min} z = y \oplus_{\min} y = [y, \infty) \cup \{0\}$. Thus $x \geq y$ or $x = 0$. If $x = y$, then $y \oplus_{\min} x = y \oplus_{\min} y = [y, \infty) \cup \{0\}$ so $z \in y \oplus_{\min} x$. If $x > y$ or $x = 0$, then $y \oplus_{\min} x = \{y\}$ so that $z \in y \oplus_{\min} x$.

Case 3. $y \neq 0$ and $z = 0$. Then $y \oplus_{\min} z = y \oplus_{\min} 0 = \{y\}$ and then $x = y$. We have $y \oplus_{\min} x = y \oplus_{\min} y = [y, \infty) \cup \{0\}$. Hence $z \in y \oplus_{\min} x$.

Case 4. $y = 0$ and $z \neq 0$. Then $y \oplus_{\min} z = 0 \oplus_{\min} z = \{z\}$ and then $x = z$. We have $y \oplus_{\min} x = 0 \oplus_{\min} z = \{z\}$. Hence $z \in y \oplus_{\min} x$.

Case 5. $y < z$. and $y, z \neq 0$. Then $y \oplus_{\min} z = \{y\}$, so $x = y$. Thus $y \oplus_{\min} x = y \oplus_{\min} y = [y, \infty) \cup \{0\}$. Hence $z \in y \oplus_{\min} x$.

Case 6. $y > z$. and $y, z \neq 0$. Then $y \oplus_{\min} z = \{z\}$, so $x = z$. Thus $y \oplus_{\min} x = y \oplus_{\min} z = \{z\}$. Hence $z \in y \oplus_{\min} x$.

By all cases, we obtain that for $x, y, z \in M$, $x \in y \oplus_{\min} z$ implies $z \in y \oplus_{\min} x$. Hence $(\mathbb{R}, \oplus_{\min})$ is a canonical hypergroup. $\square$

We consider when $(R, \oplus_{\min}, \cdot)$ is a Krasner hyperring where $R \in \mathcal{I}^0$.

Lemma 3.1.9. *$(\mathbb{R}, \oplus_{\min}, \cdot)$ is not a Krasner hyperring.*

Proof. Consider $(-2) \cdot (2 \oplus_{\min} 2) = (-2) \cdot \big([2, \infty) \cup \{0\}\big) = (-\infty, -4] \cup \{0\}$ and $\big((-2) \cdot 2\big) \oplus_{\min} \big((-2) \cdot 2\big) = (-4) \oplus_{\min} (-4) = [-4, \infty)$. Then $(-2) \cdot \big(2 \oplus_{\min} 2\big) \neq ((-2) \cdot 2) \oplus_{\min} ((-2) \cdot 2)$. Hence $(\mathbb{R}, \oplus_{\min}, \cdot)$ is not a Krasner hyperring. $\square$

Theorem 3.1.10. *Let $R \in \mathcal{I}^0$. Then $(R, \oplus_{\min}, \cdot)$ is a Krasner hyperring if and only if R is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $(a,\infty)\cup\{0\}$ *where* $a\geq 1$,

iv) $[a,\infty)\cup\{0\}$ *where* $a\geq 1$.

Proof. Assume that R is not one of the above forms. By Theorem 3.1.8 and Lemma 3.1.9, $(R,\oplus_{\min},\cdot)$ is not a Krasner hyperring.

Conversely, Theorem 3.1.8, Remark 1.2.13 and Example 1.2.17 show that $(\{0\},\oplus_{\min},\cdot)$, $((a,\infty)\cup\{0\},\oplus_{\min},\cdot)$ and $([a,\infty)\cup\{0\},\oplus_{\min},\cdot)$ are Krasner hyperrings where $a\geq 1$. For the other case, it is obvious that $([0,\infty),\cdot)$ is a semigroup with zero 0 where 0 is a scalar identity. We conclude from Theorem 3.1.8 that $([0,\infty),\oplus_{\min})$ is a canonical hypergroup. It remains to show only that $x\cdot(y\oplus_{\min}z)=x\cdot y\oplus_{\min}x\cdot z$ for all $x,y,z\in[0,\infty)$. This is clear when $x=0$. Now we let $x,y,z\in[0,\infty)$ and $x\neq 0$.

Case 1. $y=0$ and $z=0$. Then $x\cdot y=0$ and $x\cdot z=0$. Thus $x\cdot(y\oplus_{\min}z)=x\cdot(0\oplus_{\min}0)=x\cdot\{0\}=\{0\}$ and $(x\cdot y)\oplus_{\min}(x\cdot z)=0\oplus_{\min}0=\{0\}$. Hence $x\cdot(y\oplus_{\min}z)=(x\cdot y)\oplus_{\min}(x\cdot z)$.

Case 2. $y=z$ and $y,z\neq 0$. Then $x\cdot y=x\cdot z$ and $x\cdot y,x\cdot z\neq 0$. Thus $x\cdot(y\oplus_{\min}z)=x\cdot(y\oplus_{\min}y)=x\cdot\big([y,\infty)\cup\{0\}\big)=[x\cdot y,\infty)\cup\{0\}$ and $(x\cdot y)\oplus_{\min}(x\cdot z)=(x\cdot y)\oplus_{\min}(x\cdot y)=[x\cdot y,\infty)\cup\{0\}$. Hence $x\cdot(y\oplus_{\min}z)=(x\cdot y)\oplus_{\min}(x\cdot z)$.

Case 3. $y\neq 0$ and $z=0$. Then $x\cdot y\neq 0$ and $x\cdot z=0$. It follows that $x\cdot(y\oplus_{\min}z)=x\cdot(y\oplus_{\min}0)=x\cdot\{y\}=\{x\cdot y\}$ and $(x\cdot y)\oplus_{\min}(x\cdot z)=(x\cdot y)\oplus_{\min}0=\{x\cdot y\}$. Hence $x\cdot(y\oplus_{\min}z)=(x\cdot y)\oplus_{\min}(x\cdot z)$.

Case 4. $y=0$ and $z\neq 0$. The proof is similar to the proof of Case 3.

Case 5. $y<z$ and $y,z\neq 0$. Then $x\cdot y<x\cdot z$ and $x\cdot y,x\cdot z\neq 0$. This leads

to $x \cdot (y \oplus_{\min} z) = x \cdot \{y\} = \{x \cdot y\}$ and $(x \cdot y) \oplus_{\min} (x \cdot z) = \{x \cdot y\}$. Hence $x \cdot (y \oplus_{\min} z) = (x \cdot y) \oplus_{\min} (x \cdot z)$.

Case 6. $y > z$ and $y, z \neq 0$. The proof is similar to the proof of Case 5.

We obtain from any cases that $x \cdot (y \oplus_{\min} z) = (x \cdot y) \oplus_{\min} (x \cdot z)$. Note that $(y \oplus_{\min} z) \cdot x = (y \cdot x) \oplus_{\min} (z \cdot x)$ because of the commutativity of $\big([0, \infty), \cdot\big)$. Hence $([0, \infty), \oplus_{\min}, \cdot)$ is a Krasner hyperring. $\square$

3.1.3 Hyperoperation $\oplus_{\text{abs}}$

Define a multi-valued function $\oplus_{\text{abs}} : \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$
\begin{aligned}
&x \oplus_{\text{abs}} x = \{x\} && \text{for all } x \in \mathbb{R},\\
&x \oplus_{\text{abs}} y = y \oplus_{\text{abs}} x = \{x\} && \text{for all } x, y \in \mathbb{R} \text{ with } |y| < |x| \text{ and}\\
&x \oplus_{\text{abs}} (-x) = [-|x|, |x|\,] && \text{for all } x \in \mathbb{R}.
\end{aligned}
$$

Proposition 3.1.11. *The multi-valued function $\oplus_{\text{abs}}$ is a hyperoperation on H for all $H \in \mathcal{I}^0$.*

Proof. This is obvious. $\square$

We verify when $(M, \oplus_{\text{abs}})$ is a canonical hypergroup where $M \in \mathcal{I}^0$.

Proposition 3.1.12. *Let $M \in \mathcal{I}^0 \smallsetminus \big\{\{0\}\big\}$. If $(M, \oplus_{\text{abs}})$ is a canonical hypergroup, then M must contain a negative real number.*

Proof. Assume that $(M, \oplus_{\text{abs}})$ is a canonical hypergroup with $M \neq \{0\}$. There exists $m \in M$ such that $m \neq 0$. If $m < 0$, then we are done. Let $m > 0$. Since $-m$ is the inverse of m and $(M, \oplus_{\text{abs}})$ is a canonical hypergroup, we have $-m \in M$. Therefore the proof is complete. $\square$

Applying Proposition 3.1.12, we obtain the following corollary.

Corollary 3.1.13. *Let M be one of the following forms :*

i) $\{0,1\}$,

ii) $[0,\infty)$,

iii) $(a,\infty)\cup\{0\}$ *where* $a\geq 1$,

iv) $[a,\infty)\cup\{0\}$ *where* $a\geq 1$,

v) $[0,b)$ *where* $0<b\leq 1$,

vi) $[0,b]$ *where* $0<b\leq 1$.

Then $(M,\oplus_{\text{abs}})$ *is not a canonical hypergroup.*

Lemma 3.1.14. *Let M be one of the following forms :*

i) $(a,b]$ *where* $-1\leq a<0<a^2\leq b\leq 1$,

ii) $[a,b)$ *where* $-1\leq a<0<a^2<b\leq 1$.

Then $(M,\oplus_{\text{abs}})$ *is not a canonical hypergroup.*

Proof. Let $a,b\in M$ be such that $-1\leq a<0<a^2\leq b\leq 1$. First, we show that $\big((a,b],\oplus_{\text{abs}}\big)$ is not a canonical hypergroup.

Case 1. $b\geq -a$. Then an inverse of b does not exist. Hence $\big((a,b],\oplus_{\text{abs}}\big)$ is not a canonical hypergroup.

Case 2. $b<-a$. Then an inverse of $\dfrac{-b+a}{2}$ does not exist. Hence $\big((a,b],\oplus_{\text{abs}}\big)$ is not a canonical hypergroup.

We can prove similarly for the case $M=[a,b)$ where $-1\leq a<0<a^2<b\leq 1$.

□

Proposition 3.1.15. *Let M be one of the following forms :*

i) (a,b) *where* $-1 \le a < 0 < a^2 \le b \le 1$,

ii) $[a,b]$ *where* $-1 \le a < 0 < a^2 \le b \le 1$.

Then $(M, \oplus_{\text{abs}})$ *is a canonical hypergroup if and only if* $a = -b$.

Proof. First, we consider $M = (a,b)$ where $-1 \le a < 0 < a^2 \le b \le 1$. Suppose that $a \ne -b$. If $-b > a$, then an inverse of $\dfrac{a+(-b)}{2}$ does not exist. And if $-b < a$, then an inverse of $\dfrac{b+(-a)}{2}$ does not exist. Then $(M, \oplus_{\text{abs}})$ is not a canonical hypergroup.

Conversely, the result holds by Example 1.2.18. □

Theorem 3.1.16. *Let* $M \in \mathcal{I}^0$. *Then* $(M, \oplus_{\text{abs}})$ *is a canonical hypergroup if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $(-a,a)$ *where* $0 < a \le 1$,

iv) $[-a,a]$ *where* $0 < a \le 1$.

Proof. Assume that M is not one of the above forms. Corollary 3.1.13, Lemma 3.1.14 and Proposition 3.1.15 show that $(M, \oplus_{\text{abs}})$ is not a canonical hypergroup.

Conversely, by Proposition 3.1.11, Remark 1.2.13 and Example 1.2.18, we obtain that $\big(\{0\}, \oplus_{\text{abs}}\big)$, $\big((-a,a), \oplus_{\text{abs}}\big)$ and $\big([-a,a], \oplus_{\text{abs}}\big)$ are canonical hypergroups where $0 < a \le 1$. For the remaining case, it is easy to show that $(\mathbb{R}, \oplus_{\text{abs}})$ is a commutative hypergroup, has a scalar identity 0 and the inverse of $x \in \mathbb{R}$ is $-x$. Next, we show that for $x, y, z \in \mathbb{R}$, $x \in y \oplus_{\text{abs}} z$ implies $z \in -y \oplus_{\text{abs}} x$. Let $x, y, z \in \mathbb{R}$. Assume that $x \in y \oplus_{\text{abs}} z$.

Case 1. $y = z$. Then $y \oplus_{\text{abs}} z = y \oplus_{\text{abs}} y = \{y\}$ so $x = y$. We have

$-y \oplus_{\text{abs}} x = -y \oplus_{\text{abs}} y = [-|y|, |y|]$. Hence $z \in -y \oplus_{\text{abs}} x$.

Case 2. $|y| < |z|$. Then $y \oplus_{\text{abs}} z = \{z\}$ so $x = z$. We have $-y \oplus_{\text{abs}} x = -y \oplus_{\text{abs}} z = \{z\}$. Hence $z \in -y \oplus_{\text{abs}} x$.

Case 3. $|z| < |y|$. Then $y \oplus_{\text{abs}} z = \{y\}$ so $x = y$. We obtain that $-y \oplus_{\text{abs}} x = -y \oplus_{\text{abs}} y = [-|y|, |y|]$. Hence $z \in y \oplus_{\text{abs}} x$.

Case 4. $y = -z$. Then $y \oplus_{\text{abs}} z = y \oplus_{\text{abs}} -y = [-|y|, |y|]$. Thus $-|y| \leq x \leq |y|$. If $-|y| < x < |y|$, then $-y \oplus_{\text{abs}} x = \{-y\}$ so that $z \in -y \oplus_{\text{abs}} x$. If $x = y$ or $-y$, then $-y \oplus_{\text{abs}} x = [-|y|, |y|]$ or $\{-y\}$, again, $z \in -y \oplus_{\text{abs}} x$.

For any cases, we obtain that for $x, y, z \in \mathbb{R}$, $x \in y \oplus_{\text{abs}} z$ implies $z \in -y \oplus_{\min} x$. Hence $(\mathbb{R}, \oplus_{\text{abs}})$ is a canonical hypergroup. □

We characterize when $(R, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring where $R \in \mathcal{I}^0$.

Theorem 3.1.17. *Let $R \in \mathcal{I}^0$. Then $(R, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring if and only if R is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $(-a, a)$ *where* $0 < a \leq 1$,

iv) $[-a, a]$ *where* $0 < a \leq 1$.

Proof. If M is not one of the above forms, then $(M, \oplus_{\text{abs}})$ is not a Krasner hyperring which is a result of Theorem 3.1.16.

Conversely, by Proposition 3.1.11, Remark 1.2.13 and Example 1.2.18, we conclude that $\big(\{0\}, \oplus_{\text{abs}}, \cdot\big)$, $\big((-a, a), \oplus_{\text{abs}}, \cdot\big)$ and $\big([-a, a], \oplus_{\text{abs}}, \cdot\big)$ are canonical hypergroups where $0 < a \leq 1$. The remaining case, it is obvious that $(\mathbb{R}, \cdot)$ is a semigroup with zero 0 where 0 is a scalar identity of $(\mathbb{R}, \oplus_{\text{abs}})$. Theorem 3.1.16 shows that $(\mathbb{R}, \oplus_{\text{abs}})$ is a canonical hypergroup. Next, we show that $x \cdot (y \oplus_{\text{abs}} z) =$

$x \cdot y \oplus_{\text{abs}} x \cdot z$ for all $x, y, z \in \mathbb{R}$. This is clear when $x = 0$. Let $x, y, z \in \mathbb{R}$ and $x \neq 0$.

Case 1. $y = z$. Then $x \cdot y = x \cdot z$. Thus $x \cdot (y \oplus_{\text{abs}} z) = x \cdot (y \oplus_{\text{abs}} y) = x \cdot \{y\} = \{x \cdot y\}$ and $(x \cdot y) \oplus_{\text{abs}} (x \cdot z) = (x \cdot y) \oplus_{\text{abs}} (x \cdot y) = \{x \cdot y\}$. Hence $x \cdot (y \oplus_{\text{abs}} z) = (x \cdot y) \oplus_{\text{abs}} (x \cdot z)$.

Case 2. $|y| < |z|$. Then $|x \cdot y| < |x \cdot z|$. Thus $x \cdot (y \oplus_{\text{abs}} z) = x \cdot \{z\} = \{x \cdot z\}$ and $(x \cdot y) \oplus_{\text{abs}} (x \cdot z) = \{x \cdot z\}$. Hence $x \cdot (y \oplus_{\text{abs}} z) = (x \cdot y) \oplus_{\text{abs}} (x \cdot z)$.

Case 3. $|z| < |y|$. The proof is similar to the proof of Case 2.

Case 4. $y = -z$. Then $x \cdot y = -(x \cdot z)$. So $x \cdot (y \oplus_{\text{abs}} z) = x \cdot (y \oplus_{\text{abs}} -y) = x \cdot ([-|y|, |y|]) = [-|x \cdot y|, |x \cdot y|]$ and $(x \cdot y) \oplus_{\text{abs}} (x \cdot z) = (x \cdot y) \oplus_{\text{abs}} -(x \cdot y) = [-|x \cdot y|, |x \cdot y|]$. Hence $x \cdot (y \oplus_{\text{abs}} z) = (x \cdot y) \oplus_{\text{abs}} (x \cdot z)$.

We obtain from all cases that $x \cdot (y \oplus_{\text{abs}} z) = (x \cdot y) \oplus_{\text{abs}} (x \cdot z)$. Since $(\mathbb{R}, \cdot)$ is commutative, we have $(y \oplus_{\text{abs}} z) \cdot x = (y \cdot x) \oplus_{\text{abs}} (z \cdot x)$. Hence $(\mathbb{R}, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring. □

3.2 Hypermodules over Krasner Hyperrings

We apply the results from the previous section to costruct hypermodules over Krasner hyperrings. We focus R-hypermodules M in two aspects. One hand, hyperoperations on M and R are the same. On the other hand, the difference of hyperoperations on M and R are considered.

Throughout this section, for each $R, M \in \mathcal{I}^0$, let $\circ, * : R \times M \to \mathbb{R}$ be the functions defined by

$$r \circ m = r \cdot m$$

and

$$r * m = \begin{cases} 0 & \text{if } r = 0, \\ \dfrac{1}{r} \cdot m & \text{if } r \neq 0. \end{cases}$$

where $\cdot$ is the usual multiplication on $\mathbb{R}$.

We provide a proposition for investigating some hypermodules over Krasner hyperrings.

Proposition 3.2.1. *Let $(M, \oplus)$ be a canonical hypergroup and R a Krasner hyperring. If $M = \{0\}$ or $R = \{0\}$ and there exists a function $\bullet : R \times M \to M$ such that $0 \bullet m = 0$ for all $m \in M$, then $(M, \oplus, \bullet)$ is an R-hypermodule.*

Proof. The proof is trivial. □

Applying Proposition 3.2.1, we obtain the immediate corollaries.

Corollary 3.2.2. *Let R be a Krasner hyperring such that $R \in \mathcal{I}^0$ and $\oplus$ is a hyperoperation on $\{0\}$. Then $(\{0\}, \oplus, \circ)$ and $(\{0\}, \oplus, *)$ are R-hypermodules.*

Corollary 3.2.3. *Let $(M, \oplus)$ be a canonical hypergroup such that $M \in \mathcal{I}^0$. If $R = (\{0\}, +, \cdot)$ is a Krasner hyperring, then $(M, \oplus, \circ)$ and $(M, \oplus, *)$ are R-hypermodules.*

3.2.1 Hypermodules over Krasner Hyperrings Induced by the Same Hyperoperations

We study the existence of R-hypermodules M where hyperoperations on M and R are the same among $\oplus_{\max}$, $\oplus_{\min}$ and $\oplus_{\text{abs}}$.

Proposition 3.2.4. *Let $M, R \in \mathcal{I}^0$ be such that $M \subseteq R$ and $(R, \oplus, \cdot)$ a Krasner hyperring. Then $(M, \oplus, \circ)$ is an R-hypermodule if and only if $(M, \oplus)$ is a canonical hypergroup and $R \circ M \subseteq M$.*

Proof. Let $(M, \oplus, \circ)$ be an R-hypermodule. It is clear that $(M, \oplus)$ is a canonical hypergroup and $R \circ M \subseteq M$.

Conversely, assume that $(M, \oplus)$ is a canonical hypergroup and $R \circ M \subseteq M$. Note that $\circ : R \times M \to M$. Let $a, b \in R$ and $x, y \in M$. Since $M \subseteq R$ and R is a Krasner hyperring, it follows that

1. $a \circ (x \oplus y) = a \cdot (x \oplus y) = a \cdot x \oplus a \cdot y = a \circ x \oplus a \circ y$
2. $(a \oplus b) \circ x = (a \oplus b) \cdot x = a \cdot x \oplus b \cdot x = a \circ x \oplus b \circ x$
3. $(a \cdot b) \circ x = a \cdot b \cdot x = a \cdot (b \cdot x) = a \circ (b \circ x)$
4. $0 \circ x = 0 \cdot x = 0$.

This shows that $(M, \oplus, \circ)$ is an R-hypermodule. □

Proposition 3.2.5. *Let $M, R \in \mathcal{I}^0$ be such that $R \subseteq M$ and $(R, \oplus, \cdot)$ a Krasner hyperring. If $(M, \oplus, \cdot)$ is a Krasner hyperring, then $(M, \oplus, \circ)$ is an R-hypermodule.*

Proof. The proof is similar to the proof of Proposition 3.2.4. □

Now, we study on a hyperoperation $\oplus_{\max}$. We obtain the following two results by applying Corollary 3.2.3 and Theorem 3.1.4.

Proposition 3.2.6. *Let $R = \big(\{0\}, \oplus_{\max}, \cdot\big)$ and $M \in \mathcal{I}^0$. Then $(M, \oplus_{\max}, \circ)$ is an R-hypermodule if and only if M is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,a)$ *where* $0 < a \leq 1$,

iv) $[0,a]$ *where* $0 < a \leq 1$.

Proposition 3.2.7. *Let* $R = \big(\{0\}, \oplus_{\max}, \cdot\big)$ *and* $M \in \mathcal{I}^0$. *Then* $(M, \oplus_{\max}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,a)$ *where* $0 < a \leq 1$,

iv) $[0,a]$ *where* $0 < a \leq 1$.

Proposition 3.2.8. *Let* $R = \big([0,\infty), \oplus_{\max}, \cdot\big)$. *Then* $(M, \oplus_{\max}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$.

Proof. First, assume that $(M, \oplus_{\max}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\max})$ is a canonical hypergroup. By Theorem 3.1.4, M must be one of $\{0\}$, $[0,\infty)$, $[0,a)$ and $[0,a]$ where $0 < a \leq 1$. If M is $[0,a)$ or $[0,a]$ where $0 < a \leq 1$, then $R \circ M \nsubseteq M$. Hence M is either $\{0\}$ or $[0,\infty)$.

Conversely, suppose that M is one of $\{0\}$ and $[0,\infty)$. By Theorem 3.1.4, $(M, \oplus_{\max})$ is a canonical hypergroup. Since $M \subseteq R$ and $R \circ M \subseteq M$, $(M, \oplus_{\max}, \circ)$ is an R-hypermodule from Proposition 3.2.4. $\square$

Corollary 3.2.9. *Let* $R = \big([0,a), \oplus_{\max}, \cdot\big)$ *or* $\big([0,a], \oplus_{\max}, \cdot\big)$ *where* $0 < a \leq 1$. *Then* $(M, \oplus_{\max}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,b)$ *where* $0<b\le 1$,

iv) $[0,b]$ *where* $0<b\le 1$.

Proof. First, we consider $R = \big([0,a),\oplus_{\max},\cdot\big)$ where $0<a\le 1$. Assume that $(M,\oplus_{\max},\circ)$ is an R-hypermodule. Then $(M,\oplus_{\max})$ is a canonical hypergroup. By Theorem 3.1.4, M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0<b\le 1$.

Conversely, suppose that M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0<b\le 1$. Then $(M,\oplus_{\max},\cdot)$ is a Krasner hyperring by Theorem 3.1.5.

Case 1. $M=\{0\}$. Then $(M,\oplus_{\max})$ is a canonical hypergroup. Since $M\subseteq R$ and $R\circ M\subseteq M$, we conclude that $(M,\oplus_{\max},\circ)$ is an R-hypermodule from Proposition 3.2.4.

Case 2. $M=[0,\infty)$. Since $R\subseteq M$, we have $(M,\oplus_{\max},\circ)$ is an R-hypermodule from Proposition 3.2.5.

Case 3. $M=[0,b)$ where $0<b\le 1$. If $b<a$, then $M\subseteq R$ and $R\circ M\subseteq M$ so that $(M,\oplus_{\max},\circ)$ is an R-hypermodule by Proposition 3.2.4. Otherwise, we have $R\subseteq M$ and by Proposition 3.2.5, $(M,\oplus_{\max},\circ)$ is an R-hypermodule.

Case 4. $M=[0,b]$. The proof is similar to the proof of Case 3.

We can proof similarly for the case $R=([0,a],\oplus_{\max},\cdot)$. □

Proposition 3.2.10. *Let* $R,M\in\mathcal{I}^0$ *and* $(R,\oplus_{\max},\cdot)$ *a Krasner hyperring such that* $R\neq\{0\}$. *Then* $(M,\oplus_{\max},*)$ *is an* R*-hypermodule if and only if* $M=\{0\}$.

Proof. First, assume that $(M,\oplus_{\max},*)$ is an R-hypermodule. Then $(M,\oplus_{\max})$ is a canonical hypergroup. Since $(R,\oplus_{\max},\cdot)$ is a Krasner hyperring and $(M,\oplus_{\max})$ is a canonical hypergroup, M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0<$

$b \leq 1$ and R is one of $[0, \infty)$, $[0, a)$ and $[0, a]$ where $0 < a \leq 1$. Now we claim that there exist $a, b \in R$ and $x \in M$ such that $(a \oplus_{\max} b) * x \neq a * x \oplus_{\max} b * x$ when R is one of $[0, \infty)$, $[0, a)$ and $[0, a]$ and M is one of $[0, \infty)$ or $[0, b)$ or $[0, b]$. It suffices to show only the case that $R = [0, a)$ and $M = [0, b)$. We see that

$$(\frac{a}{2} \oplus_{\max} \frac{a}{2}) * \frac{b}{2} = [0, \frac{a}{2}] * \frac{b}{2} = [\frac{b}{a}, \infty) \cup \{0\}$$

while

$$\frac{a}{2} * \frac{b}{2} \oplus_{\max} \frac{a}{2} * \frac{b}{2} = \frac{b}{a} \oplus_{\max} \frac{b}{a} = [0, \frac{b}{a}]$$

Thus $(\frac{a}{2} \oplus_{\max} \frac{a}{2}) * \frac{b}{2} \neq \frac{a}{2} * \frac{b}{2} \oplus_{\max} \frac{a}{2} * \frac{b}{2}$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

Conversely, since $\oplus_{\max}$ is a hyperoperation on $\{0\}$, it follows that $\big(\{0\}, \oplus_{\max}, *\big)$ is an R-hypermodule by Corollary 3.2.2. □

Next, we explore the hyperoperation $\oplus_{\min}$. Applying Corollary 3.2.3 and Theorem 3.1.8, we obtain the next two results.

Proposition 3.2.11. *Let* $R = \big(\{0\}, \oplus_{\min}, \cdot\big)$. *Then* $(M, \oplus_{\min}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $[0, \infty)$,

iv) $(a, \infty) \cup \{0\}$ *where* $a \geq 1$,

v) $[a, \infty) \cup \{0\}$ *where* $a \geq 1$.

Proposition 3.2.12. *Let* $R = \big(\{0\}, \oplus_{\min}, \cdot\big)$*. Then* $(M, \oplus_{\min}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $[0, \infty)$,

iv) $(a, \infty) \cup \{0\}$ *where* $a \geq 1$,

v) $[a, \infty) \cup \{0\}$ *where* $a \geq 1$.

Proposition 3.2.13. *Let* $R = \big([0, \infty), \oplus_{\min}, \cdot\big)$*. Then* $(M, \oplus_{\min}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0, \infty)$.

Proof. Suppose that $(M, \oplus_{\min}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. By Theorem 3.1.8, M is one of $\mathbb{R}$, $\{0\}$, $[0, \infty)$ and $(a, \infty) \cup \{0\}$ or $[a, \infty) \cup \{0\}$ where $a \geq 1$. If $M = \mathbb{R}$, we see that

$$(2 \oplus_{\min} 2) \circ (-2) = ([2, \infty) \cup \{0\}) \circ (-2) = (-\infty, -4] \cup \{0\}$$

while

$$2 \circ (-2) \oplus_{\min} 2 \circ (-2) = -4 \oplus_{\min} -4 = [-4, \infty).$$

Thus $(2 \oplus_{\min} 2) \circ (-2) \neq 2 \circ (-2) \oplus_{\min} 2 \circ (-2)$, hence, M is not an R-hypermodule. If $M = (a, \infty) \cup \{0\}$ or $[a, \infty) \cup \{0\}$, then $R \circ M \nsubseteq M$. Thus M is $\{0\}$ or $[0, \infty)$.

Next, assume that M is one of $\{0\}$ and $[0,\infty)$. By Theorem 3.1.8, $(M,\oplus_{\min})$ is a canonical hypergroup. Since $M\subseteq R$ and $R\circ M\subseteq M$, $(M,\oplus_{\min},\circ)$ is an R-hypermodule by Proposition 3.2.4. $\square$

Proposition 3.2.14. *Let* $R=\Big((a,\infty)\cup\{0\},\oplus_{\min},\cdot\Big)$ *or* $\Big([a,\infty)\cup\{0\},\oplus_{\min},\cdot\Big)$. *Then* $(M,\oplus_{\min},\circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $(b,\infty)\cup\{0\}$ *where* $b\geq 1$,

iv) $[b,\infty)\cup\{0\}$ *where* $b\geq 1$.

Proof. Assume that $R=\Big((a,\infty)\cup\{0\},\oplus_{\min},\cdot\Big)$.

First, suppose that $(M,\oplus_{\min},\circ)$ is an R-hypermodule. Then $(M,\oplus_{\min})$ is a canonical hypergroup. By Theorem 3.1.8, M is one of $\mathbb{R}$, $\{0\}$, $[0,\infty)$ and $(b,\infty)\cup\{0\}$ or $[b,\infty)\cup\{0\}$ where $a\geq 1$. If $M=\mathbb{R}$, we see that

$$(a+1\oplus_{\min}a+1)\circ(-2)=\big([a+1,\infty)\cup\{0\}\big)\circ(-2)=(-\infty,-2(a+1)]\cup\{0\}$$

while

$$(a+1)\circ(-2)\oplus_{\min}(a+1)\circ(-2)=-2(a+1)\oplus_{\min}-2(a+1)=[-2(a+1),\infty),$$

so $(a+1\oplus_{\min}a+1)\circ(-2)\neq a+1\circ(-2)\oplus_{\min}a+1\circ(-2)$. Hence M is not an R-hypermodule.

Conversely, assume that M is one of $\{0\}$, $[0,\infty)$, $(b,\infty)\cup\{0\}$ andr $[b,\infty)\cup\{0\}$ where $b\geq 1$. Then $(M,\oplus_{\min},\cdot)$ is a Krasner hyperring by Theorem 3.1.10.

Case 1. $M = \{0\}$. Then $(M, \oplus_{\min})$ is a canonical hypergroup. Since $M \subseteq R$ and $R \circ M \subseteq M$, we obtain that $(M, \oplus_{\min}, \circ)$ is an R-hypermodule from Proposition 3.2.4.

Case 2. $M = [0, \infty)$. Since $R \subseteq M$, we have $(M, \oplus_{\min}, \circ)$ is an R-hypermodule by Proposition 3.2.5.

Case 3. $(b, \infty) \cup \{0\}$. If $b \geq a$, then $M \subseteq R$ and $R \circ M \subseteq M$. Then $(M, \oplus_{\min}, \circ)$ is an R-hypermodule by Proposition 3.2.4. Otherwise, we have $R \subseteq M$ and by Proposition 3.2.5, $(M, \oplus_{\min}, \circ)$ is an R-hypermodule.

Case 4. $[b, \infty) \cup \{0\}$. The proof is similar to the proof of Case 3.

The proof of the case $R = \big([a, \infty) \cup \{0\}, \oplus_{\min}, \cdot\big)$ is similar. □

Proposition 3.2.15. *Let $R, M \in \mathcal{I}^0$ and $(R, \oplus_{\min}, \cdot)$ a Krasner hyperring be such that $R \neq \{0\}$. Then $(M, \oplus_{\min}, *)$ is an R-hypermodule if and only if $M = \{0\}$.*

Proof. First, assume that $(M, \oplus_{\min}, *)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. Since $(R, \oplus_{\min}, \cdot)$ is a Krasner hyperring and $(M, \oplus_{\min})$ is a canonical hypergroup, M is one of $\mathbb{R}$, $\{0\}$, $[0, \infty)$ and $(b, \infty) \cup \{0\}$ or $[b, \infty) \cup \{0\}$ where $b \geq 1$ and R is one of $[0, \infty)$, $(a, \infty) \cup \{0\}$ and $[a, \infty) \cup \{0\}$ where $a \geq 1$. Now we show that there exist $a, b \in R$ and $x \in M$ such that $(a \oplus_{\min} b) * x \neq a * x \oplus_{\min} b * x$ when R is one of $[0, \infty)$, $(a, \infty) \cup \{0\}$ and $[a, \infty) \cup \{0\}$ and M is one of $\mathbb{R}$, $[0, \infty)$ and $(b, \infty) \cup \{0\}$, $[b, \infty) \cup \{0\}$. It suffices to show only the case that $R = (a, \infty) \cup \{0\}$ and $M = (b, \infty) \cup \{0\}$. We see that

$$(2a \oplus_{\min} 2a) * 2b = (2a, \infty) \cup \{0\} * 2b = [0, \frac{b}{a}]$$

while

$$(2a * 2b) \oplus_{\min} (2a * 2b) = \frac{b}{a} \oplus_{\min} \frac{b}{a} = [\frac{b}{a}, \infty) \cup \{0\}.$$

This shows that $(2a \oplus_{\min} 2a) * 2b \neq (2a * 2b) \oplus_{\min} (2a * 2b)$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

Conversely, since $\oplus_{\min}$ is a hyperoperation on $\{0\}$, it follows that $(\{0\}, \oplus_{\min}, *)$ is an R-hypermodule by Corollary 3.2.2. $\square$

Finally, the hyperoperation $\oplus_{\text{abs}}$ is taken into account. Applying Corollary 3.2.3 and Theorem 3.1.16, the following two results are obtained.

Proposition 3.2.16. *Let* $R = \big(\{0\}, \oplus_{\text{abs}}, \cdot\big)$. *Then* $(M, \oplus_{\text{abs}}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $(-a, a)$ *where* $0 < a \leq 1$,

iv) $[-a, a]$ *where* $0 < a \leq 1$.

Proposition 3.2.17. *Let* $R = \big(\{0\}, \oplus_{\text{abs}}, \cdot\big)$. *Then* $(M, \oplus_{\text{abs}}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $(-a, a)$ *where* $0 < a \leq 1$,

iv) $[-a, a]$ *where* $0 < a \leq 1$.

Proposition 3.2.18. *Let* $R = \big(\mathbb{R}, \oplus_{\text{abs}}, \cdot\big)$. *Then* $(M, \oplus_{\text{abs}}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$.

Proof. Assume that $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. By Theorem 3.1.16, M is one of $\mathbb{R}$, $\{0\}$, $(-a, a)$ and $[-a, a]$ where $0 < a \leq 1$. If $M = (-a, a)$ or $[-a, a]$, then $R \circ M \nsubseteq M$. Hence M is $\mathbb{R}$ or $\{0\}$.

Conversely, suppose that M is $\mathbb{R}$ or $\{0\}$. By Theorem 3.1.16, $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. Since $M \subseteq R$ and $R \circ M \subseteq M$, $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule by Proposition 3.2.4. □

Proposition 3.2.19. *Let* $R = \big((-a, a), \oplus_{\text{abs}}, \cdot\big)$ *or* $\big([-a, a], \oplus_{\text{abs}}, \cdot\big)$ *where* $0 < a \leq 1$. *Then* $(M, \oplus_{\text{abs}}, \circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $(-b, b)$ *where* $0 < b \leq 1$,

iv) $[-b, b]$ *where* $0 < b \leq 1$.

Proof. Assume that $R = \big((-a, a), \oplus_{\text{abs}}, \cdot\big)$ where $0 < a \leq 1$.

First, assume that $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. By Theorem 3.1.16, M is one of $\mathbb{R}$, $\{0\}$, $(-b, b)$ and $[-b, b]$ where $0 < b \leq 1$.

Conversely, suppose M is one of $\mathbb{R}$ or $\{0\}$ or $(-b, b)$ or $[-b, b]$ where $0 < b \leq 1$.

Case 1. $M = \{0\}$. Then $(M, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring by Theorem 3.1.16. Thus $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. Since $M \subseteq R$ and $R \circ M \subseteq M$, Proposition 3.2.4 shows that $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule.

Case 2. $M = \mathbb{R}$. Since $R \subseteq M$, Proposition 3.2.5 shows that $(M, \oplus_{\text{abs}}, \circ)$ is an

R-hypermodule.

Case 3. $M = (-b, b)$. If $b < a$, then $M \subseteq R$ and $R \circ M \subseteq M$. Then $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule by Proposition 3.2.4. Otherwise, we have $R \subseteq M$ and by Proposition 3.2.5, $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule.

Case 4. $M = [-b, b]$. The proof is similar to the proof of Case 3.

The proof of the case $R = \big([0, a], \oplus_{\max}, \cdot\big)$ where $0 < a \leq 1$ is obtained similarly. □

Proposition 3.2.20. *Let $R, M \in \mathcal{I}^0$ and $(R, \oplus_{\text{abs}}, \cdot)$ a Krasner hyperring be such that $R \neq \{0\}$. Then $(M, \oplus_{\text{abs}}, *)$ is an R-hypermodule if and only if $M = \{0\}$.*

Proof. Assume that $(M, \oplus_{\text{abs}}, *)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. Since $(R, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring and $(M, \oplus_{\text{abs}})$ is a canonical hypergroup, M is one of $\mathbb{R}$, $\{0\}$, $(-b, b)$ and $[-b, b]$ where $0 < b \leq 1$ and R is one of $\mathbb{R}$, $(-a, a)$ and $[-a, a]$ where $0 < a \leq 1$. Now we show that there exist $a, b \in R$ and $x \in M$ such that $(a \oplus_{\text{abs}} b) * x \neq a * x \oplus_{\text{abs}} b * x$ when R is one of $\mathbb{R}$, $(-a, a)$ and $[-a, a]$ and M is one of $\mathbb{R}$, $(-b, b)$ and $[-b, b]$. It suffices to show only the case that $R = (-a, a)$ and $M = (-b, b)$. Note that

$$(\frac{a}{2} \oplus_{\text{abs}} -\frac{a}{2}) * \frac{b}{2} = [-\frac{a}{2}, \frac{a}{2}] * \frac{b}{2} = (-\infty, -\frac{b}{a}] \cup [\frac{b}{a}, \infty) \cup \{0\}$$

while

$$(\frac{a}{2} * \frac{b}{2}) \oplus_{\text{abs}} (-\frac{a}{2} * \frac{b}{2}) = \frac{b}{a} \oplus_{\text{abs}} -\frac{b}{a} = [-\frac{b}{a}, \frac{b}{a}].$$

Thus $(\frac{a}{2} \oplus_{\text{abs}} -\frac{a}{2}) * \frac{b}{2} \neq (\frac{a}{2} * \frac{b}{2}) \oplus_{\text{abs}} (-\frac{a}{2} * \frac{b}{2})$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

Conversely, since $\oplus_{\max}$ is a hyperoperation on $\{0\}$, Corollary 3.2.2 shows that

$\big(\{0\}, \oplus_{\rm abs}, *\big)$ is an R-hypermodule. $\square$

3.2.2 Hypermodules over Krasner Hyperrings Induced by the Different Hyperoperations

We consider how to construct R-hypermodules M where hyperoperations on M and R are different among $\oplus_{\max}$, $\oplus_{\min}$ and $\oplus_{\rm abs}$.

We first study the case that Krasner hyperring R and canonical hypergroup M are equipped with $\oplus_{\min}$ and $\oplus_{\max}$, respectively. Next, we consider the case that hyperoperations in the previous case interchange their places.

Let $(R, \oplus_{\min}, \cdot)$ be a Krasner hyperring such that $R \in \mathcal{I}^0$. We examine, where $M \in \mathcal{I}^0$, when $(M, \oplus_{\max}, \circ)$ and $(M, \oplus_{\max}, *)$ are R-hypermodules. From the assumption, we see that R is one of $\{0\}$, $[0,\infty)$, $(a,\infty)\cup\{0\}$ and $[a,\infty)\cup\{0\}$ where $a \geq 1$ and M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0 < b \leq 1$.

Applying Corollary 3.2.3 and Theorem 3.1.4, we obtain the first two propositions.

Proposition 3.2.21. *Let $R = \big(\{0\}, \oplus_{\min}, \cdot\big)$ and $M \in \mathcal{I}^0$. Then $(M, \oplus_{\max}, \circ)$ is an R-hypermodule if and only if M is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,a)$ *where* $0 < a \leq 1$,

iv) $[0,a]$ *where* $0 < a \leq 1$.

Proposition 3.2.22. *Let $R = \big(\{0\}, \oplus_{\min}, \cdot\big)$ and $M \in \mathcal{I}^0$. Then $(M, \oplus_{\max}, *)$ is an R-hypermodule if and only if M is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$,

iii) $[0,a)$ *where* $0 < a \leq 1$,

iv) $[0,a]$ *where* $0 < a \leq 1$.

Corollary 3.2.23. *Let* $R, M \in \mathcal{I}^0$ *and* $(R, \oplus_{\min}, \cdot)$ *a Krasner hyperring be such that* $R \neq \{0\}$. *Then* $(M, \oplus_{\max}, \circ)$ *is an* R*-hypermodule if and only if* $M = \{0\}$.

Proof. First, assume that $(M, \oplus_{\max}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\max})$ is a canonical hypergroup. Since $(R, \oplus_{\min}, \cdot)$ is a Krasner hyperring and $(M, \oplus_{\max})$ is a canonical hypergroup, M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0 < b \leq 1$ and R is one of $[0,\infty)$, $(a,\infty) \cup \{0\}$ and $[a,\infty) \cup \{0\}$ where $a \geq 1$. Now we show that there exist $a, b \in R$ and $x \in M$ such that $(a \oplus_{\min} b) \circ x \neq a \circ x \oplus_{\max} b \circ x$ when R is one of $[0,\infty)$, $(a,\infty) \cup \{0\}$ and $[a,\infty) \cup \{0\}$ and M is one of $[0,\infty)$, $[0,b)$ and $[0,b]$. It suffices to show only the case that $R = (a,\infty) \cup \{0\}$ and $M = [0,b)$. We see that

$$(2a \oplus_{\min} 2a) \circ \frac{b}{2} = ([2a,\infty) \cup \{0\}) \circ 2b = [ab,\infty) \cup \{0\}$$

while

$$(2a \circ \frac{b}{2}) \oplus_{\max} (2a \circ \frac{b}{2}) = ab \oplus_{\max} ab = [0, ab],$$

so $(2a \oplus_{\min} 2a) \circ \frac{b}{2} \neq (2a \circ \frac{b}{2}) \oplus_{\max} (2a \circ \frac{b}{2})$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

Conversely, since $\oplus_{\max}$ is a hyperoperation on $\{0\}$, $\big(\{0\}, \oplus_{\max}, \circ\big)$ is an R-hypermodule by Corollary 3.2.1. $\square$

Proposition 3.2.24. *Let* $R = \big([0,\infty), \oplus_{\min}, \cdot\big)$ *and* $M \in \mathcal{I}^0$. *Then* $(M, \oplus_{\max}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0,\infty)$.

Proof. First, assume that $(M,\oplus_{\max},*)$ is an R-hypermodule. Then $(M,\oplus_{\max})$ is a canonical hypergroup. By Theorem 3.1.4, M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0<b\leq 1$. If $M=[0,b)$ or $[0,b]$, then $R*M\nsubseteq M$. Hence M is $\{0\}$ or $[0,\infty)$.

Conversely, suppose that M is $\{0\}$ or $[0,\infty)$. If $M=\{0\}$, then $(M,\oplus_{\max},*)$ is an R-hypermodule by Corollary 3.2.2. Let $M=[0,\infty)$. Theorem 3.1.4 shows that $\big([0,\infty),\oplus_{\max}\big)$ is a canonical hypergroup. Let $x,y\in M$ and $a,b\in R$. First, we claim that $a*(x\oplus_{\max}y)=a*x\oplus_{\max}a*y$. It is clear if $a=0$. Assume that $a\neq 0$

Case 1. $x=y$. Then $\frac{x}{a}=\frac{y}{a}$. So $a*(x\oplus_{\max}y)=a*([0,x])=[0,\frac{x}{a}]$ and $a*x\oplus_{\max}a*y=\frac{x}{a}\oplus_{\max}\frac{y}{a}=\frac{x}{a}\oplus_{\max}\frac{x}{a}=[0,\frac{x}{a}]$. Hence $a*(x\oplus_{\max}y)=a*x\oplus_{\max}a*y$.

Case 2. $x>y$. Then $\frac{x}{a}>\frac{y}{a}$. So $a*(x\oplus_{\max}y)=a*\{x\}=\{\frac{x}{a}\}$ and $a*x\oplus_{\max}a*y=\frac{x}{a}\oplus_{\max}\frac{y}{a}=\{\frac{x}{a}\}$. Hence $a*(x\oplus_{\max}y)=a*x\oplus_{\max}a*y$.

Case 3. $x<y$. The proof is similar to the proof of Case 2.

Hence $a*(x\oplus_{\max}y)=a*x\oplus_{\max}a*y$.

Second, we show that $(a\oplus_{\min}b)*x=a*x\oplus_{\max}b*x$.

Case 1. $a=0$ and $b=0$. Then $a*x=0$ and $b*x=0$. So $(a\oplus_{\min}b)*x=(0\oplus_{\min}0)*x=\{0\}*x=\{0\}$ and $a*x\oplus_{\max}b*x=0\oplus_{\max}0=\{0\}$. Hence $(a\oplus_{\min}b)*x=a*x\oplus_{\max}b*x$.

Case 2. $a\neq 0$ and $b=0$. Then $b*x=0$. So $(a\oplus_{\min}b)*x=(a\oplus_{\min}0)*x=\{a\}*x=\{\frac{x}{a}\}$ and $a*x\oplus_{\max}b*x=\frac{x}{a}\oplus_{\max}0=\{\frac{x}{a}\}$. Hence $(a\oplus_{\min}b)*x=a*x\oplus_{\max}b*x$.

Case 3. $a=0$ and $b\neq 0$. The proof is similar to the proof of Case 2.

Case 4. $a=b$ and $a,b\neq 0$. Then $\frac{x}{a}=\frac{x}{b}$ and $\frac{x}{a},\frac{x}{b}\neq 0$. So $(a\oplus_{\min}b)*x=(a\oplus_{\min}a)*x=([a,\infty)\cup\{0\})*x=[0,\frac{x}{a}]$ and $a*x\oplus_{\max}b*x=\frac{x}{a}\oplus_{\max}\frac{x}{b}=$

$\frac{x}{a} \oplus_{\max} \frac{x}{a} = [0, \frac{x}{a}]$. Hence $(a \oplus_{\min} b) * x = a * x \oplus_{\max} b * x$.

Case 5. $a < b$ and $a, b \neq 0$. Then $\frac{x}{a} > \frac{x}{b}$ and $\frac{x}{a}, \frac{x}{b} \neq 0$. So $(a \oplus_{\min} b) * x = \{a\} * x = \{\frac{x}{a}\}$ and $a*x \oplus_{\max} b*x = \frac{x}{a} \oplus_{\max} \frac{x}{b} = \{\frac{x}{a}\}$.Hence $(a \oplus_{\min} b)*x = a*x \oplus_{\max} b*x$.

Case 6. $a > b$ and $a, b \neq 0$. The proof is similar to the proof of Case 5.

Hence $(a \oplus_{\min} b) * x = a * x \oplus_{\max} b * x$.

Finally, we show that $(a \cdot b) * x = a * (b * x)$. It is obvious if $a = 0$ or $b = 0$. Assume that $a \neq 0$ and $b \neq 0$. Then $(a \cdot b) * x = \frac{x}{a \cdot b}$ and $a * (b * x) = a * \frac{x}{b} = \frac{x}{a \cdot b}$. Since $0_R \cdot x = 0_M$, $(M, \oplus_{\max}, *)$ is an R-hypermodule. □

Proposition 3.2.25. *Let* $R = \big((a, \infty) \cup \{0\}, \oplus_{\min}, \cdot\big)$ *or* $\big([a, \infty) \cup \{0\}, \oplus_{\min}, \cdot\big)$ *where* $a \geq 1$ *and* $M \in \mathcal{I}^0$. *Then* $(M, \oplus_{\max}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0, \infty)$,

iii) $[0, b)$ *where* $0 < b \leq 1$,

iv) $[0, b]$ *where* $0 < b \leq 1$.

Proof. Assume that $(M, \oplus_{\max}, *)$ is an R-hypermodule. Then $(M, \oplus_{\max})$ is a canonical hypergroup. We obtain from Theorem 3.1.4 that M is one of $\{0\}$, $[0, \infty)$, $[0, b)$ and $[0, b]$ where $0 < b \leq 1$.

Conversely, suppose that M is one of $\{0\}$, $[0, \infty)$, $[0, b)$ and $[0, b]$ where $0 < b \leq 1$. Again, Theorem 3.1.4 shows that $(M, \oplus_{\max})$ is a canonical hypergroup. Since $M, R \subseteq [0, \infty)$, the proof is obtained similarly from the proof of Proposition 3.2.24. □

Let $(R, \oplus_{\max}, \cdot)$ be a Krasner hyperring such that $R \in \mathcal{I}^0$. We consider when $(M, \oplus_{\min}, \circ)$ or $(M, \oplus_{\min}, *)$ is an R-hypermodule when $M \in \mathcal{I}^0$. So, R is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0<b\leq 1$ and M is one of $\mathbb{R}$, $\{0\}$, $[0,\infty)$ and $(a,\infty)\cup\{0\}$ or $[a,\infty)\cup\{0\}$ where $a\geq 1$.

Applying Corollary 3.2.3 and Theorem 3.1.8, we obtain the following two statements.

Proposition 3.2.26. *Let* $R=\big(\{0\},\oplus_{\max},\cdot\big)$. *Then* $(M,\oplus_{\min},\circ)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $[0,\infty)$,

iv) $(a,\infty)\cup\{0\}$ *where* $a\geq 1$,

v) $[a,\infty)\cup\{0\}$ *where* $a\geq 1$.

Proposition 3.2.27. *Let* $R=\big(\{0\},\oplus_{\max},\cdot\big)$. *Then* $(M,\oplus_{\min},*)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\mathbb{R}$,

ii) $\{0\}$,

iii) $[0,\infty)$,

iv) $(a,\infty)\cup\{0\}$ *where* $a\geq 1$,

v) $[a,\infty)\cup\{0\}$ *where* $a\geq 1$.

Corollary 3.2.28. *Let* $R,M\in\mathcal{I}^0$ *and* $(R,\oplus_{\max},\cdot)$ *a Krasner hyperring be such that* $R\neq\{0\}$. *Then* $(M,\oplus_{\min},\circ)$ *is an* R*-hypermodule if and only if* $M=\{0\}$.

Proof. First, assume that $(M, \oplus_{\min}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. Since $(R, \oplus_{\max}, \cdot)$ is a Krasner hyperring and $(M, \oplus_{\min})$ is a canonical hypergroup, M is one of $\{0\}$, $[0, \infty)$, $(b, \infty) \cup \{0\}$ and $[b, \infty) \cup \{0\}$ where $b \geq 1$ and R is one of $[0, \infty)$, $[0, a)$ and $[0, a]$ where $0 < a \leq 1$. Now we show that there exist $a, b \in R$ and $x \in M$ such that $(a \oplus_{\max} b) \circ x \neq a \circ x \oplus_{\min} b \circ x$ when R is one of $[0, \infty)$, $[0, a)$ and $[0, a]$ and M is one of $[0, \infty)$, $(b, \infty) \cup \{0\}$ and $[b, \infty) \cup \{0\}$. It suffices to show only the case that $R = [0, a)$ and $M = (b, \infty) \cup \{0\}$. We notice that

$$(\frac{a}{2} \oplus_{\max} \frac{a}{2}) \circ 2b = [0, \frac{a}{2}] \circ 2b = [0, ab]$$

while

$$(\frac{a}{2} \circ 2b) \oplus_{\min} (\frac{a}{2} \circ 2b) = ab \oplus_{\min} ab = [ab, \infty) \cup \{0\},$$

so $(\frac{a}{2} \oplus_{\max} \frac{a}{2}) \circ 2b \neq (\frac{a}{2} \circ 2b) \oplus_{\min} (\frac{a}{2} \circ 2b)$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

Conversely, since $\oplus_{\min}$ is a hyperoperation on $\{0\}$, $\big(\{0\}, \oplus_{\min}, *\big)$ is an R-hypermodule by Corollary 3.2.2. □

Corollary 3.2.29. *Let* $R = \big([0, \infty), \oplus_{\max}, \cdot\big)$. *Then* $(M, \oplus_{\min}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0, \infty)$.

Proof. First, assume that $(M, \oplus_{\min}, *)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. It follows from Theorem 3.1.8 that M is one of $\mathbb{R}$, $\{0\}$,

$[0,\infty)$ and $(a,\infty)\cup\{0\}$ or $[a,\infty)\cup\{0\}$ where $a \geq 1$. If $M=(a,\infty)\cup\{0\}$ or $[a,\infty)\cup\{0\}$, then $R*M \nsubseteq M$. If $M=\mathbb{R}$, we see that

$$(2 \oplus_{\max} 2)*(-2) = [0,2]*(-2) = (-\infty,-1]\cup\{0\}$$

while

$$(2*-2) \oplus_{\min} (2*-2) = (-1) \oplus_{\min} (-1) = [-1,\infty)\cup\{0\}.$$

Then $(2\oplus_{\max}2)*(-2) \neq (2*-2)\oplus_{\min}(2*-2)$. Hence M is not an R-hypermodule. Therefore M is $\{0\}$ or $[0,\infty)$.

Conversely, suppose that M is $\{0\}$ or $[0,\infty)$. If $M=\{0\}$, then $(M,\oplus_{\max},*)$ is an R-hypermodule from Corollary 3.2.2. Let $M=[0,\infty)$. Then $(M,\oplus_{\min})$ is a canonical hypergroup. Let $x,y\in M$ and $a,b\in R$. First, we show that $a*(x\oplus_{\min}y) = a*x\oplus_{\min}a*y$. It is clear if $a=0$. Assume that $a\neq 0$

Case 1. $x=0$ and $y=0$. Then $a*x=0$ and $a*y=0$. So $a*(x\oplus_{\min}y) = a*(0\oplus_{\min}0) = a*\{0\} = \{0\}$ and $a*x\oplus_{\min}a*y = 0\oplus_{\min}0 = \{0\}$. Hence $a*(x\oplus_{\min}y) = a*x\oplus_{\min}a*y$.

Case 2. $x\neq 0$ and $y=0$. Then $a*y=0$. So $a*(x\oplus_{\min}y) = a*(x\oplus_{\min}0) = a*\{x\} = \{\frac{x}{a}\}$ and $a*x\oplus_{\min}a*y = \frac{x}{a}\oplus_{\min}0 = \{\frac{x}{a}\}$. Hence $a*(x\oplus_{\min}y) = a*x\oplus_{\min}a*y$.

Case 3. $x<y$. The proof is similar to the proof of Case 2.

Case 4. $x=y$ and $x,y\neq 0$. Then $\frac{x}{a}=\frac{y}{a}$ and $\frac{x}{a},\frac{y}{a}\neq 0$. So $a*(x\oplus_{\min}y) = a*(x\oplus_{\min}x) = a*([x,\infty)\cup\{0\}) = [\frac{x}{a},\infty)\cup\{0\}$ and $a*x\oplus_{\min}a*y = \frac{x}{a}\oplus_{\min}\frac{y}{a} = \frac{x}{a}\oplus_{\min}\frac{x}{a} = [\frac{x}{a},\infty)\cup\{0\}$. Hence $a*(x\oplus_{\min}y) = a*x\oplus_{\min}a*y$.

Case 5. $x<y$ and $x,y\neq 0$. Then $\frac{x}{a}<\frac{y}{a}$ and $\frac{x}{a},\frac{y}{a}\neq 0$. So $a*(x\oplus_{\min}y) = a*\{x\} = \{\frac{x}{a}\}$ and $a*x\oplus_{\min}a*y = \frac{x}{a}\oplus_{\min}\frac{y}{a} = \{\frac{x}{a}\}$. Hence $a*(x\oplus_{\min}y) =$

$a * x \oplus_{\min} a * y$.

Case 6. $a > b$ and $a, b \neq 0$. The proof is similar to the proof of Case 5.

Hence $a * (x \oplus_{\min} y) = a * x \oplus_{\min} a * y$.

Second, we show that $(a \oplus_{\max} b) * x = a * x \oplus_{\min} b * x$.

Case 1. $a = b$.Then $a * x = b * x$. So $(a \oplus_{\max} b) * x = (a \oplus_{\max} a) * x = ([0, a]) * x = [\frac{x}{a}, \infty) \cup \{0\}$ and $a * x \oplus_{\min} b * x = a * x \oplus_{\min} a * x = \frac{x}{a} \oplus_{\min} \frac{x}{a} = [\frac{x}{a}, \infty) \cup \{0\}$. Hence $(a \oplus_{\max} b) * x = a * x \oplus_{\min} b * x$.

Case 2. $a > b$. Then $\frac{x}{a} < \frac{x}{b}$. So $(a \oplus_{\max} b) * x = \{a\} * x = \{\frac{x}{a}\}$ and $a * x \oplus_{\min} b * x = \frac{x}{a} \oplus_{\min} \frac{x}{b} = \{\frac{x}{a}\}$. Hence $(a \oplus_{\max} b) * x = a * x \oplus_{\min} b * x$.

Case 3. $x < y$. The proof is similar to the proof of Case 2.

Hence $(a \oplus_{\max} b) * x = a * x \oplus_{\min} b * x$.

Finally, we show that $(a \cdot b) * x = a * (b * x)$. It is obvious if $a = 0$ or $b = 0$. Assume that $a \neq 0$ and $b \neq 0$. Then $(a \cdot b) * x = \frac{x}{a \cdot b}$ and $a * (b * x) = a * \frac{x}{b} = \frac{x}{a \cdot b}$. Since $0_R \cdot x = 0_M$, $(M, \oplus_{\min}, *)$ is an R-hypermodule. □

Proposition 3.2.30. *Let* $R = \big([0, b), \oplus_{\max}, \cdot\big)$ *or* $\big([0, b], \oplus_{\max}, \cdot\big)$ *where* $0 < b \leq 1$. *Then* $(M, \oplus_{\min}, *)$ *is an* R*-hypermodule if and only if* M *is one of the following forms :*

i) $\{0\}$,

ii) $[0, \infty)$,

iii) $(a, \infty) \cup \{0\}$ *where* $a \geq 1$,

iv) $[a, \infty) \cup \{0\}$ *where* $a \geq 1$.

Proof. Let $R = \big([0, b), \oplus_{\max}, \cdot\big)$ where $0 < b \leq 1$. Assume that $(M, \oplus_{\min}, *)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. We can see from Theorem 3.1.8 that M is one of $\mathbb{R}$, $\{0\}$, $[0, \infty)$ and $(a, \infty) \cup \{0\}$ or $[a, \infty) \cup \{0\}$

where $a \geq 1$. If $M = \mathbb{R}$, we see that

$$(\frac{b}{2} \oplus_{\max} \frac{b}{2}) * (-2) = [0, \frac{b}{2}] * (-2) = (-\infty, \frac{-4}{b}] \cup \{0\}$$

while

$$(\frac{b}{2} * -2) \oplus_{\min} (\frac{b}{2} * -2) = \frac{-4}{b} \oplus_{\min} \frac{-4}{b} = [\frac{-4}{b}, \infty) \cup \{0\},$$

so $(\frac{b}{2} \oplus_{\max} \frac{b}{2}) * (-2) \neq (\frac{b}{2} * -2) \oplus_{\min} (\frac{b}{2} * -2)$. Hence M is not an R-hypermodule. Therefore M is one of $\{0\}$, $[0, \infty)$, $(a, \infty) \cup \{0\}$ and $[a, \infty) \cup \{0\}$ where $a \geq 1$.

Conversely, suppose that M is one of $\{0\}$ or $[0, \infty)$ or $(a, \infty) \cup \{0\}$ or $[a, \infty) \cup \{0\}$ where $a \geq 1$. By Theorem 3.1.8, $(M, \oplus_{\min})$ is a canonical hypergroup. Since $M, R \subseteq [0, \infty)$, the proof is obtained similarly from the proof of Proposition 3.2.29.

□

Now we focus the hyperoperation $\oplus_{\text{abs}}$ on R and the hyperoperation $\oplus_{\max}$ on M.

Proposition 3.2.31. *Let $R, M \in \mathcal{I}^0$ and $(R, \oplus_{\text{abs}}, \cdot)$ a Krasner hyperring such that $R \neq \{0\}$. Then the followings are equivalent :*

i) $M = \{0\}$.

ii) $(M, \oplus_{\max}, \circ)$ *is an R-hypermodule.*

iii) $(M, \oplus_{\max}, *)$ *is an R-hypermodule.*

Proof. *i)* ⇒ *ii)* Let $M = \{0\}$. Since $\oplus_{\max}$ is a hyperoperation on M, $(M, \oplus_{\max}, \circ)$ is an R-hypermodule from Corollary 3.2.2.

ii) ⇒ *i)* Suppose that $(M, \oplus_{\max}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\max})$ is a canonical hypergroup. By Theorem 3.1.4, M is one of $\{0\}$, $[0, \infty)$, $[0, b)$

and $[0,b]$ where $0 < b \leq 1$. Since $(R, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring such that $R \neq \{0\}$, R is one of $\mathbb{R}$, $(-a,a)$ and $[-a,a]$ where $0 < a \leq 1$. If M is $[0,\infty)$ or $[0,b)$ or $[0,b]$, it is seen that

$$(-\frac{a}{2}) \circ \frac{b}{2} = -\frac{ab}{4} \notin M.$$

Then $(M, \oplus_{\max}, \circ)$ is not an R-hypermodule. Hence $M = \{0\}$.

i) $\Rightarrow$ *iii)* Let $M = \{0\}$. Since $\oplus_{\max}$ is a hyperoperation on M, $(M, \oplus_{\max}, *)$ is an R-hypermodule by Corollary 3.2.2.

iii) $\Rightarrow$ *i)* Suppose that $(M, \oplus_{\max}, *)$ is an R-hypermodule. Then $(M, \oplus_{\max})$ is a canonical hypergroup. By Theorem 3.1.4, M is one of $\{0\}$, $[0,\infty)$, $[0,b)$ and $[0,b]$ where $0 < b \leq 1$. Since $(R, \oplus_{\text{abs}}, \cdot)$ is a Krasner hyperring such that $R \neq \{0\}$, R is one of $\mathbb{R}$, $(-a,a)$ and $[-a,a]$ where $0 < a \leq 1$. If M is one of $[0,\infty)$, $[0,b)$ and $[0,b]$, then

$$(-\frac{a}{2}) * \frac{b}{2} = -\frac{a}{b} \notin M.$$

Then $(M, \oplus_{\max}, *)$ is not an R-hypermodule. Hence $M = \{0\}$. □

Proposition 3.2.32. *Let $R, M \in \mathcal{I}^0$ and $(R, \oplus_{\max}, \cdot)$ be a Krasner hyperring such that $R \neq \{0\}$. Then the followings are equivalent :*

i) $M = \{0\}$.

ii) $(M, \oplus_{\text{abs}}, \circ)$ *is an R-hypermodule.*

iii) $(M, \oplus_{\text{abs}}, *)$ *is an R-hypermodule.*

Proof. *i)* $\Rightarrow$ *ii)* Let $M = \{0\}$. Since $\oplus_{\text{abs}}$ is a hyperoperation on M, $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule by Corollary 3.2.2.

$ii) \Rightarrow i)$ Suppose that $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. By Theorem 3.1.16, M is one of $\{0\}$, $\mathbb{R}$, $(-b, b)$ and $[-b, b]$ where $0 < b \leq 1$. Since $(R, \oplus_{\max}, \cdot)$ is a Krasner hyperring such that $R \neq \{0\}$, R is one of $[0, \infty)$, $[0, a)$ and $[0, a]$ where $0 < a \leq 1$. If M is one of $\mathbb{R}$, $(-b, b)$ and $[-b, b]$, then

$$(\frac{a}{2} \oplus_{\max} \frac{a}{2}) \circ \frac{b}{2} = [0, \frac{a}{2}] \circ \frac{b}{2} = [0, \frac{ab}{4}]$$

while

$$(\frac{a}{2} \circ \frac{b}{2}) \oplus_{\text{abs}} (\frac{a}{2} \circ \frac{b}{2}) = \frac{ab}{4} \oplus_{\text{abs}} \frac{ab}{4} = \frac{ab}{4}.$$

So $(\frac{a}{2} \oplus_{\max} \frac{a}{2}) \circ \frac{b}{2} \neq (\frac{a}{2} \circ \frac{b}{2}) \oplus_{\text{abs}} (\frac{a}{2} \circ \frac{b}{2})$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

$i) \Rightarrow iii)$ Let $M = \{0\}$. Since $\oplus_{\text{abs}}$ is a hyperoperation on M, $(M, \oplus_{\text{abs}}, *)$ is an R-hypermodule by Corollary 3.2.2.

$iii) \Rightarrow i)$ Suppose that $(M, \oplus_{\text{abs}}, *)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. By Theorem 3.1.16, M is one of $\{0\}$, $\mathbb{R}$, $(-b, b)$ and $[-b, b]$ where $0 < b \leq 1$. Since $(R, \oplus_{\max}, \cdot)$ is a Krasner hyperring such that $R \neq \{0\}$, R is one of $[0, \infty)$, $[0, a)$ and $[0, a]$ where $0 < a \leq 1$. If M is one of $\mathbb{R}$, $(-b, b)$ and $[-b, b]$, we obtain that

$$(\frac{a}{2} \oplus_{\max} \frac{a}{2}) * \frac{b}{2} = [0, \frac{a}{2}] * \frac{b}{2} = [\frac{b}{a}, \infty] \cup \{0\}$$

while

$$(\frac{a}{2} * \frac{b}{2}) \oplus_{\text{abs}} (\frac{a}{2} * \frac{b}{2}) = \frac{b}{a} \oplus_{\text{abs}} \frac{b}{a} = \frac{b}{a}.$$

So $(\frac{a}{2} \oplus_{\max} \frac{a}{2}) * \frac{b}{2} \neq (\frac{a}{2} * \frac{b}{2}) \oplus_{\text{abs}} (\frac{a}{2} * \frac{b}{2})$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$. $\square$

Proposition 3.2.33. *Let $R, M \in \mathcal{I}^0$ and $(R, \oplus_{\text{abs}}, \cdot)$ a Krasner hyperring such that $R \neq \{0\}$. Then the followings are equivalent :*

i) $M = \{0\}$.

ii) $(M, \oplus_{\min}, \circ)$ *is an R-hypermodule.*

iii) $(M, \oplus_{\min}, *)$ *is an R-hypermodule.*

Proof. $i) \Rightarrow ii)$ Let $M = \{0\}$. Since $\oplus_{\min}$ is a hyperoperation on M, $(M, \oplus_{\min}, \circ)$ is an R-hypermodule by Corollary 3.2.2.

$ii) \Rightarrow i)$ Suppose that $(M, \oplus_{\min}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. By Theorem 3.1.8 M is one of $\mathbb{R}$, $\{0\}$, $[0, \infty)$, $(b, \infty) \cup \{0\}$ and $[b, \infty) \cup \{0\}$ where $b \geq 1$. Since $(R, \oplus_{\text{abs}}, \cdot)$ a Krasner hyperring such that $R \neq \{0\}$, R is one of $\mathbb{R}$, $(-a, a)$ and $[-a, a]$ where $0 < a \leq 1$. If M is one of $\mathbb{R}$, $[0, \infty)$, $(b, \infty) \cup \{0\}$ and $[b, \infty) \cup \{0\}$, we see that

$$(\frac{a}{2} \oplus_{\text{abs}} \frac{a}{2}) \circ 2b = \frac{a}{2} \circ 2b = ab$$

while

$$(\frac{a}{2} \circ 2b) \oplus_{\min} (\frac{a}{2} \circ 2b) = ab \oplus_{\min} ab = [ab, \infty) \cup \{0\}.$$

So $(\frac{a}{2} \oplus_{\text{abs}} \frac{a}{2}) \circ 2b \neq (\frac{a}{2} \circ 2b) \oplus_{\min} (\frac{a}{2} \circ 2b)$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

$i) \Rightarrow iii)$ Let $M = \{0\}$. Since $\oplus_{\min}$ is a hyperoperation on M, $(M, \oplus_{\min}, *)$ is an R-hypermodule by Corollary 3.2.2.

iii) $\Rightarrow$ *i)* Suppose that $(M, \oplus_{\min}, *)$ is an R-hypermodule. Then $(M, \oplus_{\min})$ is a canonical hypergroup. By Theorem 3.1.8 M is one of $\mathbb{R}$, $\{0\}$, $[0, \infty)$, $(b, \infty) \cup \{0\}$ and $[b, \infty) \cup \{0\}$ where $b \geq 1$. Since $(R, \oplus_{\text{abs}}, \cdot)$ a Krasner hyperring such that $R \neq \{0\}$, R is one of $\mathbb{R}$, $(-a, a)$ and $[-a, a]$ where $0 < a \leq 1$. If M is one of $\mathbb{R}$, $[0, \infty)$, $(b, \infty) \cup \{0\}$ and $[b, \infty) \cup \{0\}$, then

$$(\frac{a}{2} \oplus_{\text{abs}} \frac{a}{2}) * 2b = \frac{a}{2} * 2b = \frac{4b}{a}$$

while

$$(\frac{a}{2} *) \oplus_{\min} (\frac{a}{2} * 2b) = \frac{4b}{a} \oplus_{\min} \frac{4b}{a} = [\frac{4b}{a}, \infty) \cup \{0\}.$$

So $(\frac{a}{2} \oplus_{\text{abs}} \frac{a}{2}) * 2b \neq (\frac{a}{2} * 2b) \oplus_{\min} (\frac{a}{2} * 2b)$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$. $\square$

Proposition 3.2.34. *Let $R, M \in \mathcal{I}^0$ and $(R, \oplus_{\min}, \cdot)$ be a Krasner hyperring such that $R \neq \{0\}$. Then the followings are equivalent :*

i) $M = \{0\}$.

ii) $(M, \oplus_{\text{abs}}, \circ)$ *is an R-hypermodule.*

iii) $(M, \oplus_{\text{abs}}, *)$ *is an R-hypermodule.*

Proof. *i)* $\Rightarrow$ *ii)* Let $M = \{0\}$. Since $\oplus_{\text{abs}}$ is a hyperoperation on M, $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule by Corollary 3.2.2.

ii) $\Rightarrow$ *i)* Suppose that $(M, \oplus_{\text{abs}}, \circ)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. By Theorem 3.1.16, M is one of $\{0\}$, $\mathbb{R}$, $(-b, b)$ and $[-b, b]$ where $0 < b \leq 1$. Since $(R, \oplus_{\min}, \cdot)$ is a Krasner hyperring such that $R \neq \{0\}$, R is one of $[0, \infty)$, $(a, \infty) \cup \{0\}$ and $[a, \infty) \cup \{0\}$ where $a \geq 1$. If M

is one of $\mathbb{R}$, $(-b, b)$ and $[-b, b]$, we see that

$$(2a \oplus_{\max} 2a) \circ \frac{b}{2} = ([2a, \infty) \cup \{0\}) \circ \frac{b}{2} = [ab, \infty) \cup \{0\}$$

while

$$(2a \circ \frac{b}{2}) \oplus_{\text{abs}} (2a \circ \frac{b}{2}) = ab \oplus_{\text{abs}} ab = ab.$$

So $(2a \oplus_{\max} 2a) \circ \frac{b}{2} \neq (2a \circ \frac{b}{2}) \oplus_{\text{abs}} (2a \circ \frac{b}{2})$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$.

i) $\Rightarrow$ *iii)* Let $M = \{0\}$. Since $\oplus_{\text{abs}}$ is a hyperoperation on M, $(M, \oplus_{\text{abs}}, *)$ is an R-hypermodule by Corollary 3.2.2.

iii) $\Rightarrow$ *i)* Suppose that $(M, \oplus_{\text{abs}}, *)$ is an R-hypermodule. Then $(M, \oplus_{\text{abs}})$ is a canonical hypergroup. By Theorem 3.1.16, M is one of $\{0\}$, $\mathbb{R}$, $(-b, b)$ and $[-b, b]$ where $0 < b \leq 1$. Since $(R, \oplus_{\min}, \cdot)$ is a Krasner hyperring such that $R \neq \{0\}$, R is one of $[0, \infty)$, $(a, \infty) \cup \{0\}$ and $[a, \infty) \cup \{0\}$ where $a \geq 1$. If M is one of $\mathbb{R}$, $(-b, b)$ and $[-b, b]$, then

$$(2a \oplus_{\max} 2a) * \frac{b}{2} = ([2a, \infty) \cup \{0\}) * \frac{b}{2} = [0, \frac{b}{4a}]$$

while

$$(2a * \frac{b}{2}) \oplus_{\text{abs}} (2a * \frac{b}{2}) = \frac{b}{4a} \oplus_{\text{abs}} \frac{b}{4a} = \frac{b}{4a}.$$

So $(2a \oplus_{\max} 2a) * \frac{b}{2} \neq (2a * \frac{b}{2}) \oplus_{\text{abs}} (2a * \frac{b}{2})$. Hence M is not an R-hypermodule. Therefore $M = \{0\}$. $\square$

So far, for fixed single-valued operations $\circ$ and $*$, we are able to characterize when $(M, \oplus_M, \circ)$ and $(M, \oplus_M, *)$ are $(R, \oplus_R, \cdot)$-hypermodules where $\oplus_M, \oplus_R \in \{\oplus_{\max}, \oplus_{\min}, \oplus_{\text{abs}}\}$ and $\cdot$ is the usual multiplication on $\mathbb{R}$. We obtain the following facts.

1. If $R = \{0\}$, then $(M, \oplus_M, \circ)$ and $(M, \oplus_M, *)$ are $(R, \oplus_R, \cdot)$-hypermodules.
2. If $R \neq \{0\}$, then
   (a) there exists $M \neq \{0\}$ in $\mathcal{I}^0$ such that $(M, \oplus_{\max}, \circ)$ and $(M, \oplus_{\max}, *)$ are $(R, \oplus_{\min}, \cdot)$-hypermodules,
   (b) there exists $M \neq \{0\}$ in $\mathcal{I}^0$ such that $(M, \oplus_{\min}, \circ)$ and $(M, \oplus_{\min}, *)$ are $(R, \oplus_{\max}, \cdot)$-hypermodules,
   (c) $M = \{0\}$ is the only case such that $(M, \oplus_{\max}, \circ)$, $(M, \oplus_{\max}, *)$, $(M, \oplus_{\min}, \circ)$, $(M, \oplus_{\min}, *)$ are $(R, \oplus_{\text{abs}}, \cdot)$-hypermodules, $(M, \oplus_{\text{abs}}, \circ)$, $(M, \oplus_{\text{abs}}, *)$ are $(R, \oplus_{\max}, \cdot)$-hypermodules and $(M, \oplus_{\text{abs}}, \circ)$, $(M, \oplus_{\text{abs}}, *)$ are $(R, \oplus_{\min}, \cdot)$-hypermodules.

This brings us to look for an appropriate single-valued operation $\bullet$ such that $(M, \oplus_M, \bullet)$ is an $(R, \oplus_R, \cdot)$-hypermodule with $M \in \mathcal{I}^0 \smallsetminus \big\{\{0\}\big\}$ and $\oplus_M$ and $\oplus_R$ satisfy the case (b) above.

Proposition 3.2.35. *Let $(M, \oplus)$ be a canonical hypergroup and $(R, +, \cdot)$ a Krasner hyperrinng. If we define a funtion $\bullet : R \times M \to M$ by $r \bullet m = 0_M$ for all r in R and m in M, then $(M, \oplus, \bullet)$ is an R-hypermodule.*

Proof. This is obvious. □

Proposition 3.2.36. *Let $(M, \oplus_{\max})$ be a canonical hypergroup and $(R, \oplus_{\text{abs}}, \cdot)$ be a Krasner hyperring such that M, $R \in \mathcal{I}^0 \smallsetminus \{\{0\}\}$. Then $(M, \oplus_{\max}, \bullet)$ is an R-hypermodule if and only if the single-valued operation $\bullet : R \times M \to M$ must be uniquely defined by $r \bullet m = 0_M$ for all r in R and m in M.*

Proof. First, assume that $(M, \oplus_{\max}, \bullet)$ is an R-hypermodule. Let $r \in R$ and $m \in M$. Then

$$\begin{aligned}(r \oplus_{\text{abs}} r) \bullet m &= r \bullet m \oplus_{\max} r \bullet m \\ \{r\} \bullet m &= [0, r \bullet m] \\ \{r \bullet m\} &= [0, r \bullet m] \\ r \bullet m &= 0.\end{aligned}$$

Hence $r \bullet m = 0_M$ for all r in R and m in M. Conversely, the result holds from Proposition 3.2.35. □

Proposition 3.2.37. *Let $(M, \oplus_{\text{abs}})$ be a canonical hypergroup and $(R, \oplus_{\max}, \cdot)$ be a Krasner hyperring such that M, $R \in \mathcal{I}^0 \smallsetminus \{\{0\}\}$. Then $(M, \oplus_{\text{abs}}, \bullet)$ is an R-hypermodule if and only if the single-valued operation $\bullet : R \times M \to M$ must be uniquely defined by $r \bullet m = 0_M$ for all r in R and m in M.*

Proof. First, assume that $(M, \oplus_{\text{abs}}, \bullet)$ is an R-hypermodule. Then $0_R \bullet m = 0_M$ for all $m \in M$. Let $r \in R \smallsetminus \{0\}$ and $m \in M$. Then

$$\begin{aligned}(r \oplus_{\max} r) \bullet m &= r \bullet m \oplus_{\text{abs}} r \bullet m \\ [0, r] \bullet m &= \{r \bullet m\}\end{aligned}$$

Since $0 \in [0, r] \bullet m$, we have $r \bullet m = 0_M$ for all r in R and m in M. Conversely, the result holds by Proposition 3.2.35. □

Proposition 3.2.38. *Let $(M, \oplus_{\min})$ be a canonical hypergroup and $(R, \oplus_{\text{abs}}, \cdot)$ a Krasner hyperring such that M, $R \in \mathcal{I}^0 \smallsetminus \{\{0\}\}$. Then $(M, \oplus_{\min}, \bullet)$ is an R-hypermodule if and only if the single-valued operation $\bullet : R \times M \to M$ must be uniquely defined by $r \bullet m = 0_M$ for all r in R and m in M.*

Proof. First, assume that $(M, \oplus_{\min}, \bullet)$ is an R-hypermodule. Let $r \in R$ and $m \in M$. Then

$$\begin{aligned} (r \oplus_{\text{abs}} r) \bullet m &= r \bullet m \oplus_{\min} r \bullet m \\ \{r\} \bullet m &= [r \bullet m, \infty) \cup \{0\} \text{ or } \{0\} \end{aligned}$$

Then $\{r \bullet m\} = \{0\}$. Hence $r \bullet m = 0_M$ for all r in R and m in M. Conversely, the result holds by Proposition 3.2.35. □

Proposition 3.2.39. *Let $(M, \oplus_{\text{abs}})$ be a canonical hypergroup and $(R, \oplus_{\min}, \cdot)$ a Krasner hyperring such that M, $R \in \mathcal{I}^0 \setminus \{\{0\}\}$. Then $(M, \oplus_{\text{abs}}, \bullet)$ is an R-hypermodule if and only if the single-valued operation $\bullet : R \times M \to M$ must be uniquely defined by $r \bullet m = 0_M$ for all r in R and m in M.*

Proof. First, assume that $(M, \oplus_{\text{abs}}, \bullet)$ is an R-hypermodule. Then $0_R \bullet m = 0_M$ for all $m \in M$. Let $r \in R \setminus \{0\}$ and $m \in M$. Then

$$\begin{aligned} (r \oplus_{\min} r) \bullet m &= r \bullet m \oplus_{\text{abs}} r \bullet m \\ ([r, \infty) \cup \{0\} \text{ or } \{0\}) \bullet m &= \{r \bullet m\} \end{aligned}$$

Then $r \bullet m = 0$. Hence $r \bullet m = 0_M$ for all r in R and m in M. Conversely, the result holds by Proposition 3.2.35. □

CHAPTER IV

FREE AND PROJECTIVE HYPERMODULES

In this chapter, we separate into two sections. The first section is based on [4] where we adopt the notion of free hypermodules over Krasner hyperrings. However, we give some certain examples of free hypermodules over Krasner hyperrings at the end. In the last section, we define a projective hypermodule and some properties of projective hypermodules over Krasner hyperrings are studied.

4.1 Free Hypermodules

We give some definitions and propositions regarding free R-hypermodules from [4]. Moreover, examples of free hypermodules are provided at the end of the section.

First, we give a definition of a (hypermodule) weak homomorphism and a multi-valued (hypermodule) weak homomorphism which has a major role in the followings section.

Definition 4.1.1. Let M and M' be R-hypermodules. A *multi-valued (hypermodule) homomorphism from M into M'* is a multi-valued function from M into M', i.e., $f : M \to \mathcal{P}^*(M')$ such that

i) $f(x+y) \subseteq f(x) + f(y)$ for all $x, y \in M$,

ii) $f(rx) = rf(x)$ for all $r \in R$ and $x \in M$.

Definition 4.1.2. [4] Let M be an R-hypermodule and X a nonempty subset of M.

A *linear combination* of X is a sum of the form $r_1x_1 + r_2x_2 + \cdots + r_nx_n$ where $n \in \mathbb{N}$, $r_i \in R$ and $x_i \in X$ for all $i \in \{1, 2, \ldots, n\}$. Note that if $n = 1$, a linear combination of the form r_1x_1 where $r_1 \in R$ and $x_1 \in X$ is $\{r_1x_1\}$.

We say that X *generates* M if every element of M belongs to a linear combination of X, i.e., X generates M if and only if for each $m \in M$ there exist $r_1, r_2, \ldots, r_n \in R$ and $x_1, x_2, \ldots, x_n \in X$ such that $m \in r_1x_1 + r_2x_2 + \cdots + r_nx_n$.

Moreover, X is said to be *linearly dependent* if there exists distinct $x_1, x_2, \ldots, x_n$ in X and $r_1, r_2, \ldots, r_n$ in R, not all of which are 0, such that

$$0 \in r_1x_1 + r_2x_2 + \cdots + r_nx_n.$$

A subset of M which is not linearly dependent is called *linearly independent*, i.e., X is linearly independent if and only if for all distinct $x_1, x_2, \ldots, x_n \in X$ and all $r_1, r_2, \ldots, r_n \in R$ if $0 \in r_1x_1 + r_2x_2 + \cdots + r_nx_n$, then $r_i = 0$ for all $i \in \{1, 2, \ldots, n\}$.

Finally, let $\varnothing$ generate $\{0\}$.

Note that if $X = \varnothing$, then X is linearly independent.

Definition 4.1.3. [4] Let M be a unitary R-hypermodule. We call an R-hypermodule M a *free R-hypermodule* if there exists a subset B of M such that

i) B generates M and

ii) for every function f from B into an R-hypermodule N there exists a multi-valued homomorphism $f^m : M \to \mathcal{P}^*(N)$ such that $f^m(x) = \{f(x)\}$ for all $x \in B$.

The set B is called a *basis* of M.

Proposition 4.1.4. *[4] Let M be a unitary R-hypermodule and $B = \{b_1, b_2, \ldots, b_n\}$ a finite subset of M. Then the followings are equivalent:*

i) B is a basis of M,

ii) B is linearly independent and generates M,

iii) for every $m \in M$ there are uniquely defined elements $r_1, \ldots, r_n \in R$ such that $m \in r_1b_1 + r_2b_2 + \cdots + r_nb_n$.

Obviously, $\{0\}$ is a free hypermodule with the basis $\varnothing$.

Recall that a basis of a free module M over a ring is a maximal linearly independent subset of M and is a minimal spanning subset of M. This also holds for a basis of a free hypermodule.

Proposition 4.1.5. *Let M be a free R-hypermodule. If B is a basis of M, then B is a maximal linearly independent subset of M.*

Proof. Assume that B is a basis of M and there is a linearly independent subset C of M such that $B \subsetneq C$. Let $v \in C \smallsetminus B$. Since B is a basis of M, there exist $r_1, \ldots, r_n \in R$ and $b_1, \ldots, b_n \in B$ such that $v \in r_1b_1 + \cdots + r_nb_n$. Then $0 \in v - v \subseteq r_1b_1 + \cdots + r_nb_n + (-1)v$. This contradicts the fact that C is linearly independent. Hence B is a maximal linearly independent subset of M. □

Proposition 4.1.6. *Let M be a free R-hypermodule. If B is a basis of M, then B is a minimal generating subset of M.*

Proof. Assume that B is a basis of M and there is a generating subset C of M such that $C \subsetneq B$. Then there exists $v \in B \smallsetminus C$. Since C is a generating subset of M, we have $v \in r_1c_1 + \cdots + r_nc_n$ for some $r_1, \ldots, r_n \in R$ and $c_1, \ldots, c_n \in C$. Then $0 \in v - v \subseteq r_1c_1 + \cdots + r_nc_n + (-1)v$. This contradicts the fact that B is linearly independent. Hence B is a minimal generating subset of M. □

We give some examples of free hypermodules.

Example 4.1.7. [4] Let Ω be a nonempty set and $(R, +, \cdot)$ a Krasner hyperring. We consider the set $R^\Omega = \{f | f : \Omega \to R\}$. Denote $E(\Omega)$ as a set of all functions in R^Ω which vanish almost everywhere, i.e.,

$$E(\Omega) = \{f \in R^\Omega | f(x) = 0 \text{ almost all } x \in \Omega\}.$$

Define a hyperoperation $\oplus : E(\Omega) \times E(\Omega) \to \mathcal{P}^*\big(E(\Omega)\big)$ and a single-valued operation $\circ : R \times E(\Omega) \to E(\Omega)$ by

$$f \oplus g = \big\{h \in E(\Omega) \ \big| \ h(x) \in f(x) + g(x) \text{ for all } x \in \Omega\big\}$$

and

$$r \circ f = rf \text{ where } rf : \Omega \to R \text{ defined by } (r \circ f)(x) = rf(x) \ \text{ for all } \ x \in \Omega$$

for all $f, g \in E(\Omega)$ and $r \in R$. Hence $(E(\Omega), \oplus, \circ)$ is a free R-hypermodule.

Moreover, a basis of $(E(\Omega), \oplus, \circ)$ is $\{f_a | a \in \Omega\}$ where

$$f_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases}$$

The previous example shows that we can construct a free R-hypermodule from a nonempty subset Ω. In addition, $E(\Omega)$ has a basis B such that B and Ω have the same cardinalities.

Example 4.1.8. If R is a hyperring with identity 1, then R is clearly a free R-hypermodule with a basis $\{1\}$.

Example 4.1.9. Let $R = \Big([0,1], \oplus_{\max}, \cdot\Big)$ and $M = \Big([0,a], \oplus_{\max}, \circ\Big)$ where $0 < a \leq 1$. Then M is a free R-hypemodule with a basis $\{a\}$.

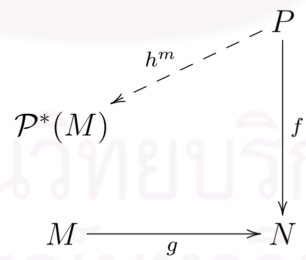
Example 4.1.10. Let $R = \Big([1,\infty) \cup \{0\}, \oplus_{\min}, \cdot\Big)$ and $M = \Big([a,\infty) \cup \{0\}, \oplus_{\min}, \circ\Big)$ where $a \geq 1$. Then M is a free R-hypemodule with a basis $\{a\}$.

Example 4.1.11. Let $R = \Big([-1,1], \oplus_{\text{abs}}, \cdot\Big)$ and $M = \Big([-a,a], \oplus_{\text{abs}}, \circ\Big)$ where $0 < a \leq 1$. Then M is a free R-hypemodule with a basis $\{a\}$.

4.2 Projective Hypermodules

We introduce a definition of a projective hypermodule and investigate some properties that are parallel to those of a projective module.

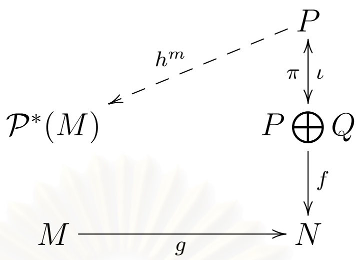
Definition 4.2.1. An R-hypermodule P is *projective* if for any R-hypermodules M and N, a homomorphism $f : P \to N$ and a surjective homomorphism $g : M \to N$, there exists a multi-valued homomorphism $h^m : P \to \mathcal{P}^*(M)$ such that $g\big[h^m[P]\big] \subseteq f[P]$.



The following proposition shows that a direct sum $P \bigoplus Q$ of an R-hypermodule is also projective if at least one of P or Q is projective.

Proposition 4.2.2. *Let R-hypermodule M be the direct sum of subhypermodules P and Q. If P is projective, then $P \bigoplus Q$ is a projective R-hypermodule.*

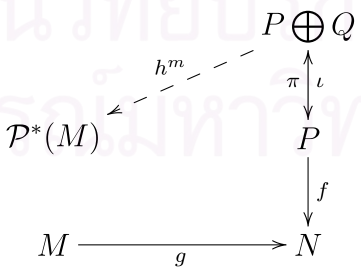
Proof. Let N be an R-hypermodule, $f : P \bigoplus Q \to N$ and $g : M \to N$ homomorphisms such that g is surjective. Consider the following diagram :



Then $f \circ \iota : P \to N$ is a homomorphism. Since P is projective, there exists a multi-valued homomorphism $h^m : P \to \mathcal{P}^*(M)$ such that $g\big[h^m[P]\big] \subseteq (f \circ \iota)[P]$, i.e., $g\big[h^m[P]\big] \subseteq f[P]$. Then $h^m \circ \pi : P \bigoplus Q \to \mathcal{P}^*(M)$ is a multi-valued homomorphism. Thus $g\Big[(h^m \circ \pi)(P \bigoplus Q)\Big] = g\big[h^m[P]\big] \subseteq f[P] \subseteq f[P \bigoplus Q]$. Hence $P \bigoplus Q$ is projective. □

Proposition 4.2.3. *Let P and Q be subhypermodules of an R-hypermodule. If $P \bigoplus Q$ is a projective R-hypermodule, then P and Q are projective.*

Proof. To show that P is a projective R-hypermodule, let M and N be R-hypermodules, $f : P \to N$ and $g : M \to N$ homomorphisms such that g is surjective. Consider the following diagram :



Then $f \circ \pi : P \bigoplus Q \to N$ is a homomorphism. Since $P \bigoplus Q$ is projective, there exists a multi-valued homomorphism $h^m : P \bigoplus Q \to \mathcal{P}^*(M)$ such that $g\big[h^m[P \bigoplus Q]\big] \subseteq (f \circ \pi)[P \bigoplus Q]$, i.e., $g\big[h^m[P \bigoplus Q]\big] \subseteq f[P]$. Then $h^m \circ \iota : P \to \mathcal{P}^*(M)$

is a multi-valued homomorphism. Thus

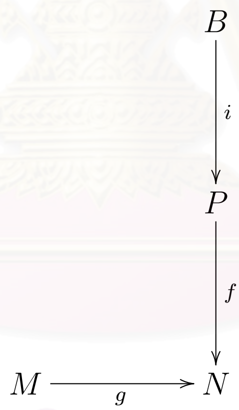
$$g\Big[(h^m \circ \iota)(P)\Big] = g\big[h^m[P]\big] \subseteq g\big[h^m[P \bigoplus Q]\big] \subseteq f[P].$$

Hence P is projective.

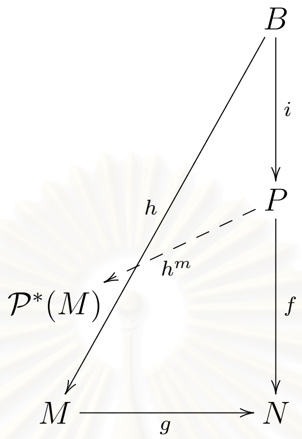
Similarly, Q is projective. □

Proposition 4.2.4. *Every free R-hypermodule is projective.*

Proof. Suppose that P is a free R-hypermodule with a basis B. Let M and N be R-hypermodules, $f : P \to N$ and $g : M \to N$ homomorphisms such that g is surjective. Let $i : B \to P$ be the map defined by $i(x) = x$ for all $x \in B$.



Since g is surjective, for each $b \in B$ there exists $m_b \in M$ such that $g(m_b) = (f \circ i)(b) = f(b)$. Thus, for each $b \in B$, choose once and for all an element $m_b \in M$ such that $f(b) = g(m_b)$. Define a mapping $h : B \to M$ by $h(b) = m_b$. We have $g \circ h = f \circ i$. Since P is free, we can extend h to a multi-valued homomorphism $h^m : P \to \mathcal{P}^*(M)$ such that $h^m(b) = \{h(b)\}$ for all $b \in B$.



To show that $g\big[h^m[P]\big] \subseteq f[P]$, let $p \in P$. Since B generates P, there exist $\alpha_1, \ldots, \alpha_n \in R$ and $b_1, \ldots, b_n \in B$ such that $p \in \alpha_1 b_1 + \cdots + \alpha_n b_n$. Thus

$$\begin{aligned}
g\big[h^m(p)\big] &\subseteq g\big[h^m(\alpha_1 b_1 + \cdots + \alpha_n b_n)\big] \\
&\subseteq g\big[\alpha_1 h^m(b_1) + \cdots + \alpha_n h^m(b_n)\big] \\
&= g\big[\alpha_1\{h(b_1)\} + \cdots + \alpha_n\{h(b_n)\}\big] \\
&= g\big[\{\alpha_1 h(b_1)\} + \cdots + \{\alpha_n h(b_n)\}\big] \\
&= g\big[\alpha_1 h(b_1) + \cdots + \alpha_n h(b_n)\big] \\
&= \alpha_1 g(h(b_1)) + \cdots + \alpha_n g(h(b_n)) \\
&= \alpha_1 f(i(b_1)) + \cdots + \alpha_n f(i(b_n)) \\
&= \alpha_1 f(b_1) + \cdots + \alpha_n f(b_n) \\
&= f(\alpha_1 b_1 + \cdots + \alpha_n b_n) \ \subseteq f[P].
\end{aligned}$$

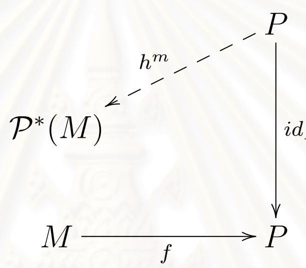
Thus $g\big[h^m(p)\big] \subseteq f[P]$ for all $p \in P$, i.e., $g\big[h^m[P]\big] \subseteq f[P]$. Hence P is projective.

□

We can conclude from the previous proposition that Examples 4.1.6–4.1.10 are examples of projective hypermodules.

Proposition 4.2.5. *Let P be a projective R-hypermodule and M an R-hypermodule. If $f : M \to P$ is a surjective homomorphism, then there exists a multi-valued homomorphism $h^m : P \to \mathcal{P}^*(M)$ such that $f\big[h^m[P]\big] \subseteq P$.*

Proof. Assume that $f : M \to P$ is a surjective homomorphism. Let $id_P : P \to P$ be the identity function on P.



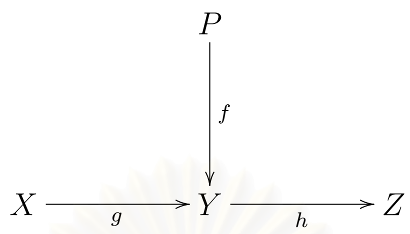
There exists a multi-valued homomorphism $h^m : P \to \mathcal{P}^*(M)$ such that $f\big[h^m[P]\big] \subseteq id_P[P] = P$ since P is projective. □

Proposition 4.2.6. *Let P be an R-hypermodule. If P is a direct summand of a free hypermodule, then P is projective.*

Proof. Assume that P is a direct summand of a free R-hypermodule F. Then there exists a hypermodule Q such that $F = P \bigoplus Q$. Since F is a free R-hypermodule, Proposition 4.2.4 shows that F is projective. By Proposition 4.2.3, P is projective. □

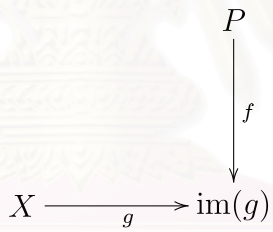
Proposition 4.2.7. *Let P be a projective R-hypermodule. Suppose that X, Y and Z*

are R-hypermodules and the diagram

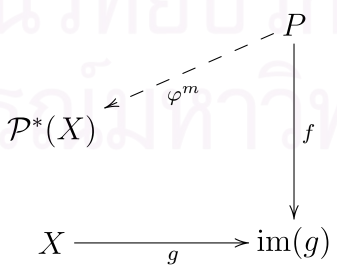


is such that f, g and h are homomorphisms, $\ker(h) = \operatorname{im}(g)$ *and* $h \circ f = 0$*. Then there exists a multi-valued homomorphism* $\varphi^m : P \to \mathcal{P}^*(X)$ *such that* $g\big[\varphi^m[P]\big] \subseteq f[P]$*.*

Proof. Since $h \circ f = 0$, we have $\operatorname{im}(f) \subseteq \ker(h) = \operatorname{im}(g)$. Then we can consider the given diagram as

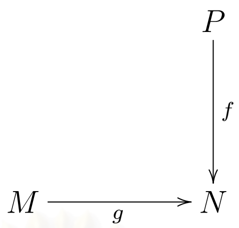


Applying the projectivity of P to the diagram, there exists a multi-valued homomorphism $\varphi^m : P \to \mathcal{P}^*(X)$ such that $g\big[\varphi^m[P]\big] \subseteq f[P]$.



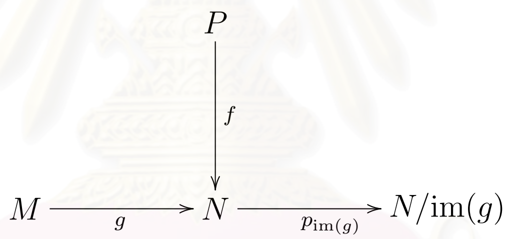
□

Proposition 4.2.8. *Let P, M and N be R-hypermodules. Suppose that P is* *projective and the diagram*

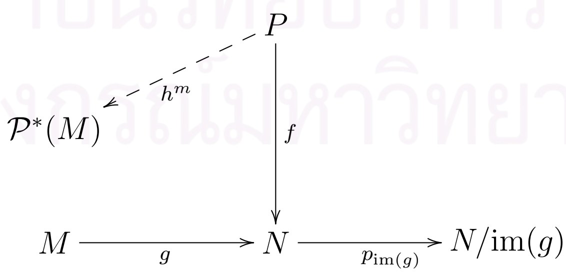


is such that f *and* g *are homomorphisms. If* $\mathrm{im}(f) \subseteq \mathrm{im}(g)$*, then there exists a multi-valued homomorphism* $h^m : P \to \mathcal{P}^*(M)$ *such that* $g\big[h^m[P]\big] \subseteq f[P]$*. The converse holds if* $g\big[h^m[P]\big] = f[P]$*.*

Proof. First, assume that $\mathrm{im}(f) \subseteq \mathrm{im}(g)$. Recall that the canonical map $p_{\mathrm{im}(g)}$ is a surjective homomorphism with $\ker(p_{\mathrm{im}(g)}) = \mathrm{im}(g)$. Consider the diagram



Then $p_{\mathrm{im}(g)} \circ f = 0$ because $im(f) \subseteq im(g)$. By Proposition 4.2.7, there exists a multi-valued homomorphism $h^m : P \to \mathcal{P}^*(M)$ such that $g\big[h^m[P]\big] \subseteq f[P]$.



Conversely, assume that $g\big[h^m[P]\big] = f[P]$. Then $\mathrm{im}(f) = g\big[h^m[P]\big] \subseteq g[M] = \mathrm{im}(g)$. Hence $\mathrm{im}(f) \subseteq \mathrm{im}(g)$. □

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VITA

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