การออกแบบตัวควบคุมคงทนสำหรับเพนดูลัมผกผันสองท่อนแบบหมุนโดยใช้อสมการเมตริกซ์เชิงเส้น

นายหวง ฮา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิศวกรรมศาสตรมหาบัณฑิต
สาขาวิชาวิศวกรรมไฟฟ้า ภาควิชาวิศวกรรมไฟฟ้า
คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2550

ROBUST CONTROLLER DESIGN FOR A ROTARY DOUBLE INVERTED PENDULUM USING LINEAR MATRIX INEQUALITIES

Mr. Ha Hoang

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Engineering Program in Electrical Engineering
Department of Electrical Engineering
Faculty of Engineering
Chulalongkorn University
Academic Year 2007

Thesis Title	ROBUST CONTROLLER DESIGN FOR A ROTARY DOUBLE INVERTED PENDULUM USING LINEAR MATRIX INEQUALITIES
By	Mr. Ha Hoang
Field of Study	Electrical Engineering
Thesis Advisor	Assistant Professor Manop Wongsaisuwan, D. Eng.

Accepted by the Faculty of Engineering, Chulalongkorn University in Partial Fulfillment of Requirements for Master's Degree

.... Dean of the Faculty of Engineering
(Professor Direk Lavansiri, Ph.D.)

THESIS COMMITTEE

.... Chairman
(Associate Professor David Banjerdpongchai, Ph.D.)

.... Thesis Advisor
(Assistant Professor Manop Wongsaisuwan, D. Eng.)

.... Member
(Naebboon Hoonchareon, Ph.D.)

หวง ฮา: การออกแบบตัวควบคุมคงทนสำหรับเพนดูลัมผกผันสองท่อนแบบหมุนโดยใช้อสมการเมตริกซ์เชิงเส้น (ROBUST CONTROLLER DESIGN FOR A ROTARY DOUBLE INVERTED PENDULUM USING LINEAR MATRIX INEQUALITIES), อ. ที่ปรึกษา: ผศ.ดร. มานพ วงศ์สายสุวรรณ, 67 หน้า.

ชุดทดลองเพนดูลัมผกผันสองท่อนแบบหมุนได้ถูกเลือกให้ใช้เป็นปัญหาทดสอบ (benchmark) สำหรับการสัมมนาเฉพาะสาขาในปีค.ศ. 2006 ที่กรุงเทพฯ ประเทศไทยวัตถุประสงค์หลักของงานวิจัยนี้คือเพื่อออกแบบตัวควบคุมคงทนสำหรับเพนดูลัมผกผันสองท่อนแบบหมุนโดยใช้อสมการเมตริกซ์เชิงเส้น โดยต้องทำให้ระบบมีเสถียรภาพที่ตำแหน่งในแนวตั้งตรงซึ่งเดิมไม่เสถียร แม้ว่าจะมีผลของความไม่แน่นอนเชิงพารามิเตอร์และสัญญาณรบกวน นอกจากนี้ยังจะได้พิจารณาถึงการหาค่าต่ำสุดของฟังก์ชันจุดประสงค์ กล่าวคือผลของการรบกวนที่มีต่อระบบต้องได้รับการลดทอนลง เนื่องจากปัญหาการออกแบบตัวควบคุมจำนวนมากสามารถแสดงได้ในรูปแบบของปัญหาเชิงคอนเวกซ์ เช่น ปัญหาการหาค่าต่ำสุดโดยวิธีอสมการเชิงเส้นและการทดสอบเสถียรภาพคงทนของระบบวงปิด ดังนั้นเราจึงใช้วิธีอสมการเมตริกซ์เชิงเส้นซึ่งเป็นเครื่องมือที่มีประสิทธิภาพในการศึกษาการออกแบบตัวควบคุม วิทยานิพนธ์นี้เกี่ยวข้องกับการออกแบบตัวควบคุมป้อนกลับสถานะแบบคงทน H_∞ และตัวควบคุมป้อนกลับสัญญาณขาออกแบบคงทน H_∞ เพื่อวัตถุประสงค์ข้างต้น

ภาควิชา ......วิศวกรรมไฟฟ้า...... ลายมือชื่อนิสิต ....
สาขาวิชา ......วิศวกรรมไฟฟ้า...... ลายมือชื่ออาจารย์ที่ปรึกษา ....
ปีการศึกษา ......2550......

##4870725721: MAJOR ELECTRICAL ENGINEERING

KEY WORD: ROTARY DOUBLE INVERTED PENDULUM / ROBUST STABILITY / STATE FEEDBACK CONTROLLER / OUTPUT FEEDBACK CONTROLLER / LINEAR MATRIX INEQUALITIES / H_∞ CONTROL WITH REGIONAL POLE CONSTRAINTS.

HA HOANG: ROBUST CONTROLLER DESIGN FOR A ROTARY DOUBLE INVERTED PENDULUM USING LINEAR MATRIX INEQUALITIES, THESIS ADVISOR: ASSISTANT. PROFESSOR. MANOP WONGSAISUWAN, D. Eng., 67+xi pp.,

The AUN/SEED-Net benchmark problem rendered at the field wise seminar in Bangkok, Thailand in 2006 is a rotary double inverted pendulum (RDIP). The primary objective of this research is to a design robust controller for the RDIP using Linear Matrix Inequalities (LMIs). The system is required to be stable at the upright unstable position regardless of some parametric uncertainties and disturbances. In addition, the minimization of a given criterion function must be considered. The effect of the disturbances on the system needs to be attenuated. The LMIs tool is a very powerful one in the study of design controllers. Many design problems can be cast as convex problems such as an LMI minimization problem and a robust stability test of closed-loop systems. This thesis is concerned with the design of a robust H_∞ state feedback controller and a robust H_∞ output feedback controller.

Department ...... Electrical Engineering

Field of study ...... Electrical Engineering

Academic year 2007

Student's signature ....

Advisor's signature ....

Acknowledgments

I would first like to thank Assist. Prof. Manop Wongsaisuwan, my principle advisor, for his many useful suggestions, constructive remarks and guidance along the way. I am profoundly grateful for his kind support, encouragement. He is always willing to make his free time for discussions and giving comments during this research despite his bustling schedule. It has been very rewarding and stimulating to work together with him.

I am also thankful to Assoc. Prof. David Banjerdpongchai, for his helpful comments, advice, insight. I am grateful to Assoc. Prof. David Banjerdpongchai, Assist. Prof. Manop Wongsaisuwan, Assist. Prof. Suchin Arusawatwong, Assoc. Prof. Watharapong Khovidhungij, for providing me essential acknowledge. Their courses on the field of control systems formed the seed for this research.

I would like to express my profound gratitude to Assoc. Prof. David Banjerdpongchai, Dr. Naebboon Hoonchareon for thorough reading my manuscript. Many thanks to all graduate students in control systems research laboratory, Chulalongkorn University for their friendliness and for useful discussions. I am grateful to AUN/Seed-Net for sponsoring me with a scholarship to do postgraduate studies at Chulalongkorn University.

Finally, I would like to show my deep gratitude towards my family for their patience, love and having given chance to study in Thailand.

Contents

List of Figures

List of Notations

Symbols

$\mathbb{R}$	The set of real number
$\mathbb{R}^n(\mathbb{R}^{m\times n})$	The set of real, ordered n- tuples (real, $m \times n$ matrices)
$0^n(0^{m\times n})$	The zero element of $\mathbb{R}^n(\mathbb{R}^{m\times n})$
I_n	The identity matrix of size n
$\overline{\sigma}(M)$	Largest singular value of M
$\mathcal{N}_M$	The null space of M
$M > 0(M \geq 0)$	The symmetric matrix M is positive (semi-) definite
$M < 0(M \leq 0)$	The symmetric matrix M is negative (semi-) definite
$\mathrm{diag}(A_1, \ldots, A_n)$	A block-diagonal matrix with $A_1, \ldots, A_n$ along the diagonal
$\Vert . \Vert_2$	$\mathcal{L}_2$ of l_2 norm of a signal of sequence
$\mathcal{H}_\infty$	Hardy space of essentially bounded functions on the imaginary axis, with analytic continuation into right half-plane
J	The performance index
F	State feedback gain matrix
P	Lyapunov matrix
γ	Upper bound on J

List of Acronyms

Co	Convex hull
LFT	Linear Fractional Transformation
LMI	Linear Matrix Inequality
LTI	Linear Time Invariant
LPV	Linear Parameter Varying
LQR	Linear Quadratic Regulator
MIMO	Multi-Input Multi-Output
RDIP	Rotary Double Inverted Pendulum
SVD	Singular Value Decomposition

CHAPTER I

INTRODUCTION

In this chapter some issues are discussed regarding the control problem of inverted pendulums. The philosophy and technical difficulties as well as the control problem associated with a rotary double inverted pendulum (RDIP) are illustrated in section 1.1. Section 1.2 covers literature review on the control of general inverted pendulums. The objectives of this thesis are addressed in section 1.3 and its scope is also outlined in section 1.4. Finally, the distribution of the thesis are stated briefly in section 1.5.

1.1 Motivation

An inverted pendulum is a popular experiment for control research and education. The system not only intrigues students by performing an exciting balancing act, but also captures some of the more challenging characteristics inherent in many real-world systems: nonlinearity, open loop instability, non minimum-phase behavior, model uncertainty, etc. Because of the relative simplicity of the system and its practical relevance, the inverted pendulum has also been used extensively as an experimental test to demonstrate a new control algorithm. In addition, dynamic model of the pendulum is nonlinear, hence it is now used to illustrate many of the ideas emerging in the field of nonlinear control. Especially, the RDIP consists of the two inverted pendulums, hence it is very highly nonlinear. It will not be able to stabilize the system when the two pendulums have the same length and material. In addition to the difficulty of the naturally nonlinear characteristic of the system , we also need to deal with some sets of uncertainties and the minimization of a given function. The uncertainties here are the parameter uncertainties and disturbances.

1.2 Literature Review

In this section, the main aim is to review the literature on control schemes for the some kinds of the inverted pendulum. The basic model is an inverted pendulum mounted on the cart. It is also a trivial control problem of the inverted pendulum. This kind of inverted pendulum has several types such as single pendulum and double pendulums as well as triple pendulums in which the control problem can be sorted into two problems which are stabilizing and swing-up. Firstly, we get to know to the single pendulum. In this model, the inverted pendulum is mounted on the cart. There were two controllers introduced by Mori [1] The first one is a feedforward controller that is to swing-up the pendulum from its pending position to upright position. The second one is linear feedback controller with a full state feedback observer based on LQR to keep the inverted pendulum at the upright position. Besides the two types of the above controllers, Ishida [2] had succeed in applying neural network control with the back propagation algorithm. The neural network controller provides the appropriate force to balance the inverted pendulum. In their method, there are two neural network controllers in order to deal simultaneously with identification and controlling the inverted pendulum. Renou and Saydy [3] used approximate linearization controller for the inverted pendulum. In this algorithm, a linear transformation and a state feedback control law are found by solving a quadratic linearlization problem. We now consider the double inverted pendulum which are consists of two pendulum. This system had been study in Henmi [4] for the swing-up control of the pendulum. The control of the swing up and stabilizing was designed in three steps: (1) to swing-up the first pendulum using energy control method, (2) to swing-up the second pendulum using energy control method while stabilizing the first pendulum using slide mode control method, (3) stabilizing the both pendulums at upright position using sliding mode control. Another self-tuning controller for stabilizing the double inverted pendulum was proposed in Fujinaka [5] by using combination of two controllers together which are PID and neural networks controller. The gain of PID controller is adjusted by the neural networks controller. In addition, many approaches for swinging and catching of an inverted pendulum have been proposed in the literature; from Furuta et al. [6] with minimum time controller, which is unfortunately not very robust, to Astrom and Furuta [7] with energy control strategy, which controls of energy of the inverted pendulum toward a value equal to the steady-state upright position, and Yi et al. [8] with a fuzzy controller based on single input rule modules. There have been several fuzzy-model-based approaches concerning the stability of such nonlinear systems. Yurkovich and Widjaja [9] fully analyzed the control-engineering design procedures for an implementation of fuzzy-system concepts, and extend the linear quadratic fuzzy-based controller design to adapt to the changing system parameters in balancing control for the rotational inverted pendulum. Wang et al. [10] also presented a design methodology for stabilization of a class of nonlinear systems based on Takagi-Sugeno fuzzy model and PDC control design, where stability analysis and control-design problems are reduced to linear matrix inequality problems. Both solutions have been successfully applied in simulations on a cart inverted pendulum model.

While controlling a real inverted pendulum, we are coped with several limitations and constrains that were not considered in those approaches. Instead of rolling the disc velocity (acceleration) directly, in particular case, the disc velocity is driven over dc-motor voltage, so we must consider the disc velocity limitations instead of the disc-acceleration limitations.

In the literature, there are several solutions for swing up and stabilization of an inverted pendulum with a restricted travel. Wei et al. [11] presented a nonlinear control strategy by decomposing the control law into a sequence of steps. Chung and Hauser [12] proposed a nonlinear state controller that controls the position and the swinging energy of the pendulum at the same time. Zhao and Spong [13] applied a hybrid-control strategy, which globally asymptotically stabilizes the system for all initial conditions.

Finally, we will mention to a rotary double inverted pendulum consisting of an inverted pendulum mounted on a rotating disc. This system has been developed by K. Furuta from Tokyo Institute of Technology, therefore it has been known as Furuta pendulum. There are some control algorithms proposed for swing-up and stabilizing control problem. Grossimon and Barbieri [14] proposed a sliding mode control to stabilize the system. For this method, the position of the tip of the rotating disc and the inclination of the angle pendulum are formulated by a function of the angle of the rotating disc. The simulation was done for two cases of the inclination angle of the pendulum: $45°$ and $0°$. The first case is not stable but the latter one. Sugie [15] and Nair [16] proposed a nonlinear controller to solve stabilization of the Furuta pendulum at the upright unstable position. This method is based on approximate linearization by transforming the nonlinear system into Brunowsky canonical form. The feedback gain matrix of the linearized system is then computed by solving LQR.

In this research, a robust H_∞ state feedback and a robust H_∞ output feedback controller are proposed. The controllers with minimum cost are designed via LMIs. The advantage of LMIs is that the design of control systems can be cast or recast as convex problems that involve LMIs such as an LMI minimization problem and a robust stability test of the closed-loop system. The problem of design controller is to stabilize the pendulum at the upper steady-state position regardless of some parametric uncertainties and input disturbances. For this purpose, exact mathematical model of the real inverted pendulum has been derived and linearized at the upright steady-state position. All experiments have been done in simulations on the nonlinear model of the inverted pendulum.

1.3 Objectives

The primary objective of this thesis is to design a robust controller for a rotary double inverted pendulum (RDIP) using linear matrix inequalities (LMIs) regardless of some parametric uncertainties and input disturbances. In addition, the minimization of value of a given criterion function must be considered. The advantage of LMIs is that the design problem can be cast as convex problems such as an LMI minimization problem and a robust stability test of the closed-loop system. A state feedback H_∞ and an output feedback H_∞ controller will be proposed to solve the problem.

1.4 Scope of Thesis

1. This thesis deals with the AUN/SEED-Net benchmark problem rendered at the field wise seminar in Bangkok, Thailand 2006. .
2. The design problem is the stabilization of a rotary double inverted pendulum regardless of some uncertain parameters and disturbance inputs.
3. In addition to the stabilization problem, a given performance index is minimized.
4. Some comparisons between the proposed controllers and LQR, LPV controllers.

1.5 Methodology

1. The computational tool used in this thesis is the YALMIP package with SeDuMi solver.
2. The design problem is formulated into a convex optimization problem involving linear matrix inequalities (LMIs).
3. A polytopic approach is used to find the controllers.
4. A regional pole constraint is considered to solve the problem.

1.6 Contributions

The expected contributions from this thesis are:

1. Design two robust controllers for a rotary double inverted pendulum
2. The method can be applied to a certain inverted pendulum.
3. A useful algorithm for designing controllers for uncertain systems.

CHAPTER II

BASIC KNOWLEDGE

In this chapter basic knowledge on control systems are briefly reviewed. The section 2.1 gives introduction to linear matrix inequalities (LMIs). It turns out that LMIs is a very powerful tool in control systems. Model of uncertain systems are studied in section 2.2. Finally, nominal stabilities and nominal performances are also reviewed in section 2.3.

2.1 Introduction to linear matrix inequalities

In this section, we will briefly discuss about LMIs in control system. As we will see, many problems in control systems can be formulated (or reformulated) using LMIs. LMIs entail a sign definiteness constraint on a matrix that depends linearly on its variable space. LMI is an expression of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{n} x_i F_i < 0 \tag{2.1}$$

where

- $x = (x_1, ..., x_n)$ is a vector of n real numbers called the decision variables.
- $F_0, ..., F_n$ are real symmetric matrices, i.e., $F_i = F_i^T$, for $i = 0, ..., n$.

Requiring the matrix $F(x)$ to be positive definite is a convex constraint on the variable space x. In the most control applications, LMIs arise as functions of matrix variables rather than scalar valued decision variables. This means that in the inequalities of the form (2.1) where $\mathcal{X} = \mathbb{R}^{n_1 \times n_2}$ is the set of real matrices of dimension $n_1 \times n_2$.

Remark A non-strict LMI is a linear matrix inequality where $(<)$ in (2.1) is replaced by $(\leq)$. In this case, the non-strict LMI includes an implicit equality constraint, and allows the matrix $F(x)$ to be singular. Thereby, infeasibility of the strict LMI may incorrectly suggest that the non-strict LMI is infeasible.

A system of LMIs is a finite set of LMIs

$$F_1(x) < 0, ..., F_k(x) < 0. \tag{2.2}$$

The set x satisfying (2.2) can be found using a block diagonal LMI whose diagonal blocks are the individual LMIs

$$F(x) := \begin{bmatrix} F_1(x) & & & \\ & F_2(x) & & \\ & & \ddots & \\ & & & F_k(x) \end{bmatrix} < 0.$$

Hence, multiple LMIs constraints can always be considered as a single LMI constraint. Nonlinear matrix inequalities in Schur complement form define convex constraints on the variable x and can be converted to LMI. In particular, the set of nonlinear inequalities

$$R(x) > 0, \qquad Q(x) - S(x)^T R(x)^{-1} S(x) > 0, \tag{2.3}$$

where $Q(x)$ and $R(x)$ are symmetric, i.e., $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x, are equivalent to the LMI

$$\begin{bmatrix} Q(x) & S(x)^T \\ S(x) & R(x) \end{bmatrix} > 0 \tag{2.4}$$

In some cases, we have constraint of the form

$$Tr[S(x)^T P(x)^{-1} S(x)] < 1, \quad P(x) > 0, \tag{2.5}$$

where $P(x)$ is a symmetric matrix, i.e., $P(x) = P(x)^T$, and $S(x)$ depend affinely on x. A slack variable, $X = X^T$, is introduced to solve such problem. The constraint of Eqn. (2.5) can be rewritten as:

$$TrX < 1, \quad \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0 \tag{2.6}$$

2.1.1 Some simple applications of linear matrix inequalities

a Stability

Consider the linear autonomous system

$$\dot{x} = Ax \tag{2.7}$$

The exponential stability of the system (2.7) is equivalent to the feasibility of the LMI

$$\begin{bmatrix} -X & 0 \\ 0 & A^T X + XA \end{bmatrix} < 0$$

b μ-analysis

Determine a diagonal matrix such that $\|DMD^{-1}\| < 1$ where M is a given matrix.

$$\begin{aligned}\|DMD^{-1}\| < 1 &\Longleftrightarrow D^{-T}M^TD^TDMD^{-1} < I\\ &\Longleftrightarrow M^TD^TDM < D^TD\\ &\Longleftrightarrow M^TXM - X < 0\end{aligned}$$

c Singular value minimization

Let $F : \mathcal{X} \to \mathbb{S}$ be an affine function and consider the problem to minimize $f(x) := \sigma(F(x))$ over x. It can be clearly seen that

$$\begin{aligned}f(x) < \gamma \Longleftrightarrow \lambda_{max}(F^T(x)F(x)) < \gamma^2 &\Longleftrightarrow \tfrac{1}{\gamma}F^T(x)F(x) - \gamma I < 0\\ &\Longleftrightarrow \begin{bmatrix}\gamma I & F(x)\\ F^T(x) & \gamma I\end{bmatrix} > 0\end{aligned}$$

d Evaluation of quadratic cost

Consider the linear autonomous system (2.7) with initial value of state variable $x(0) = x_0$ and a criterion function $J := \int_0^\infty x^T(t)Qx(t)dt$ where $Q = Q^T \geq 0$. Assume that the system is asymptotically stable. For any matrix $X = X^T$ which is solution of LMI: $A^TX + XA + Q \leq 0$, we can differentiate $x^T(t)Xx(t)$ along solution of Eqn (2.7) to yield

$$\frac{d}{dt}[x^T(t)Xx(t)] = x^T(t)[A^TX + XA]x(t) \leq -x^T(t)Qx(t)$$

Integrating the latter inequalities from $t = 0$ till ∞ yields the upper bound

$$J = \int_0^\infty x^T(t)Qx(t)dt \leq x_0^TXx_0$$

Minimizing the function $f(X) := x_0^TXx_0$ over all $X = X^T$ satisfying $X > 0$ and $A^TX + XA + Q \leq 0$ leads to the smallest upper bound of J. Clearly, this is an optimization problem with an LMI constraint.

2.2 Models of uncertain systems

In control systems, systems are often analyzed using models that are approximated from the actual system dynamics. The difference between a real-life system and its model is due to uncertainty in the identification of the system parameters. Small parameter variations may have a major effect on the dynamics of a system, and these uncertainties make worse performance or cause system unstable. Hence, it is very important to analyze parametric uncertainties of dynamical system. Robust control design strives to guarantee stability and performance for such uncertain systems. We now consider robust stability and robust performance of an uncertain system.

2.2.1 Parametric uncertainties

Let $\delta = (\delta_1, ..., \delta_p)$ be a vector of uncertain parameters in a given dynamical system. Then, there are two cases of parametric uncertainties

- **time-invariant parametric uncertainties:** the vector δ is fixed but element of an uncertainty set $\Delta \subseteq \mathbb{R}^p$ is unknown.
- **time-varying parametric uncertainties:** the vector δ is an unknown time varying function $\delta : \mathbb{R} \to \mathbb{R}^k$ whose values $\delta(t)$ belong to an uncertainty set $\Delta \subseteq \mathbb{R}^p$.

In the fist case, the physical parameters of the system are fixed but approximately known up to some level of accuracy. In the second case, parametric uncertainties, coefficients, or other physical quantities are time-dependent.

a Affine parameter dependent systems

Consider a uncertain system

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{2.8}$$

These matrices $A(\delta)$, $B(\delta)$, $C(\delta)$ and $D(\delta)$ can be expressed as

$$\begin{aligned} A(\delta) &= A_0 + \delta_1 A_1 + ... + \delta_p A_p \\ B(\delta) &= B_0 + \delta_1 B_1 + ... + \delta_p B_p \\ C(\delta) &= C_0 + \delta_1 C_1 + ... + \delta_p C_p \\ D(\delta) &= D_0 + \delta_1 D_1 + ... + \delta_p D_p \end{aligned}$$

These above matrices are rewritten as

$$S(\delta) = S_0 + \delta_1 S_1 + ... + \delta_p S_p$$

where

$$S(\delta) = \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix}$$

We now consider control system models of the form

$$\frac{d}{dt}x(t) = f(x, w, u, t), \quad z(t) = g(x, w, u, t), \quad y(t) = h(x, w, u, t) \tag{2.9}$$

where $x(t) \in \mathbf{R}^{n \times n}$,$w(t) \in \mathbf{R}^r$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^q$ and $z(t) \in \mathbf{R}^p$. The function x is called the "state" of the system, while w and u are "inputs", z and y are "outputs". w consists of exogenous inputs, i.e., inputs that we have no control over, such as noises. Reference input u consists of control inputs. We may set $u(t)$ to any value we wish. The output z is of interest: this may consist, for instant, of components of x of even those of u. y consists of outputs that can be measured. In order to accommodate uncertainties, it is assumed that f, g and h are not known exactly, but only known to satisfy some certain properties. Robust control analysis problems consist of the study of the solutions of equations (2.8). Robust design problems contain designing control laws $u(t) = \mathcal{K}(y,t)$, so that with the control law in place desired answers are obtained of the analysis questions.

b Linear fractional representation of uncertain systems

We now focus on a special instance of system, consisting of an interconnection of a linear time-invariant system and an "uncertainty" or "perturbation" in the feedback loop. The model is described by

$$\frac{d}{dt}x(t) = Ax(t)+B_{xp}p(t)+B_{xu}u(t)+B_{xw}w(t), \quad q(t) = C_q x(t)+D_{qp}p(t)+D_{qu}u(t)+D_{qw}w(t),$$

$$y(t) = C_y x(t)+D_{yp}p(t)+D_{yu}u(t)+D_{yw}w(t), \quad z(t) = C_z x(t)+D_{zp}p(t)+D_{zu}u(t)+D_{zw}w(t),$$

$$p(t) = \Delta(q,t),$$

where $p \in \mathbf{R}^m, q \in \mathbf{R}^m, A, B_{xp}, B_{xu}, B_{xw}, C_q, C_y, C_z, D_{yp}, D_{yu}, D_{qp}, D_{qu}, D_{qw}, D_{zp}, D_{zu}$ and D_{zw} are real matrices of appropriate size. $\Delta : \mathbf{L}_2^m[0,\infty) \to \mathbf{L}_2^m[0,\infty)$ is in general a nonlinear operation representing the "uncertainty" in modeling, often Δ contains the origin, i.e., Δ=0. The linear time-invariant system in this case is called the "nominal model". The above model is also known as the "Linear Fractional Representation" of the uncertain system.

c Polytopic systems

Polytopic systems form a special class of **LFR** systems. For these systems, there exists an extensive body of work on analysis and synthesis using quadratic Lyapunov functions. These systems are described by

$$\frac{d}{dt}x(t) = Ax(t) + B_u u(t) + B_w w(t), \quad y(t) = C_y x(t) + D_{yu}u(t) + D_{yw}w(t),$$

$$z(t) = C_z x(t) + D_{zu}u(t) + D_{zw}w(t), \quad \Sigma(t) = \begin{bmatrix} A(t) & B_u(t) & B_w(t) \\ C_y(t) & D_{yu}(t) & D_{yw}(t) \\ C_z(t) & D_{zu}(t) & D_{zw}(t) \end{bmatrix} \in \mathbf{\Xi} \quad (2.10)$$

where

$$\mathbf{\Xi} = \mathbf{Co}\left\{ \begin{bmatrix} A_1 & B_{u,1} & B_{w,1} \\ C_{y,1} & D_{yu,1} & D_{yw,1} \\ C_{z,1} & D_{zu,1} & D_{zw,1} \end{bmatrix}, \ldots, \begin{bmatrix} A_L & B_{u,L} & B_{w,L} \\ C_{y,L} & D_{yu,L} & D_{yw,L} \\ C_{z,L} & D_{zu,L} & D_{zw,L} \end{bmatrix} \right\} \quad (2.11)$$

where **Co** denotes the convex hull. (The matrices $\begin{bmatrix} A_i & B_{u,i} & B_{w,i} \\ C_{y,i} & D_{yu,i} & D_{yw,i} \\ C_{z,i} & D_{zu,i} & D_{zw,i} \end{bmatrix}$, i= 1,..., L are given.)

2.3 Nominal stability and nominal performance

The study of Lyapunov stability concerns the asymptotic behavior of the state of the dynamical system around an equilibrium. The main contributions of Lyapunov are the concept of stability, asymptotic stability, and method of verification of these concepts in terms of the existence of functions, called Lyapunov functions. Fortunately, the problem of finding Lyapunov functions can be solved by testing a feasibility of LMIs.

2.3.1 Nominal stability of linear system

Consider the linear autonomous system

$$\dot{x} = Ax \tag{2.12}$$

where $A : \mathbb{R}_n \to \mathbb{R}_n$ is a linear map obtained as the linearization of $f : \mathcal{X} \to \mathcal{X}$ around an equilibrium point $x^* \in \mathcal{X}$ of the following system.

$$\dot{x}(t) = f(x(t), t) \tag{2.13}$$

Clearly, for $x^* \in \mathcal{X}$, we can write

$$f(x) = f(x^*) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x^*)[x - x^*] + ...$$

f is assumed to be differentiable at least once. The linearization of f around x^* is defied by the system (2.12) with A defined by the real $n \times n$ matrix.
All elements in ker A are equilibrium points, but we consider the stability of (2.13) at the equilibrium point $x^* = 0$. Then, the following positive definite function $V : \mathcal{X} \to \mathbb{R}$ defined by

$$V(x) = x^T X x$$

severs as a quadratic Lyapunov function.

Theorem 2.1. *Let the system (2.12) be a linearizaion of (2.13) at the equilibrium point x^*. The following statements are equivalent*
(a) The origin is an asymptotically stable equilibrium for (2.12).
(b) The origin is a globally asymptotically stable equilibrium for (2.12).
(c) All eigenvalues $\lambda(A)$ of A have stricly negative real part.
(d) The linear matrix inequalities

$$A^T X + XA < 0, \quad X > 0$$

are feasible.

2.3.2 Nominal performance and LMIs

In this section, we will examine some performance criteria for dynamical systems. Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{2.14}$$

where $x(t) \in \mathcal{X} = \mathbb{R}^{n\times n}$ is the state, $w(t) \in \mathcal{W} = \mathbb{R}^m$ the input and $z(t) \in \mathcal{Z} = \mathbb{R}^p$ the output. Let $T(s) = C(Is - A)^{-1}B + D$ denote the corresponding transfer function. The system is assumed to be asymptotically stable. The input w is considered as an input variable (a 'disturbance') whose effect on the output z should be minimized. This effect can be depicted by some ways. For instant, with a given input w, the quotient $\|z\|_2/\|w\|_2$ shows the relationship between the input w and the output z. The wort case gain of the system is usually considered

$$\|T\|_\infty := \sup_{0<\|w\|<\infty} \frac{\|z\|_2}{\|w\|_2}$$

The H_∞ of the transfer function $T(s)$ is also defined as by

$$\|T\|_\infty = \sup_{\omega} |T(j\omega)|$$

The H_∞ norm can be interpreted as the maximal power of y, $y = T(s)u$, given a power of u of 1 Watt, or, alternatively, as the maximal energy of y, for all possible signals u having the energy 1 Joule.

2.4 Conclusions

In this chapter we have reviewed basic knowledge that is very helpful for solving control problems. It can be seen that many problem in control systems can be cast or recast into LMIs formulation such as stability problem, μ analysis, so on. It has been a very powerful tool in the field of control systems. In order to design controller of an actual system, it is required to approximate the system to get its mathematical model. For an uncertain system, we need to describe it in terms of a certain dynamical system such as LFR or polytopic sytems.

CHAPTER III

ROTARY DOUBLE INVERTED PENDULUM

In this chapter some issues pertaining to the rotary double inverted pendulum (RDIP) are introduced. Section 3.1 describes the physical structure of the RDIP. The statements of the control problem of the RDIP are stated in section 3.2. In order to deal with controller design next chapter, the nonlinear model of the RDIP is analyzed in the section 3.3. The nonlinear model is then linearized at an upright unstable position . The linearized model will be used to design controller in next chapter.

3.1 Introduction

The (RDIP) shown in Fig 3.1 consists of two rigid pendulums. They are mounted on perpendicularly rotating disc which is connected to a DC motor. The two pendulum can only move on the vertical plane and the rotating disc can only move on the horizontal plane. The pendulums are controlled to be at the inverted position by rotating the disc. In fact, it is impossible to control the two pendulums when they have the same length and material as well as homogeneity. The result obtained in this section is cited from [17]. The dynamic models are based on Euler-Lagrange equations derived by specifying a Lagrangian, difference between kinetic and potential energy of the RDIP. The nonlinear dynamic is linearized around a point of operation which is position of both pendulums up.

The schematic of the RDIP is dawn in Fig 3.2. The system variables and their nominal values are described in Table 3.1.

τ: The external torque applied to the disc (N.m)

α: The angular displacement of the rotating disc (rad)

β_1: The 1^{st} pendulum angle with respect to the vertical axis (rad)

β_2: The 2^{nd} pendulum angle with respect to the vertical axis (rad)

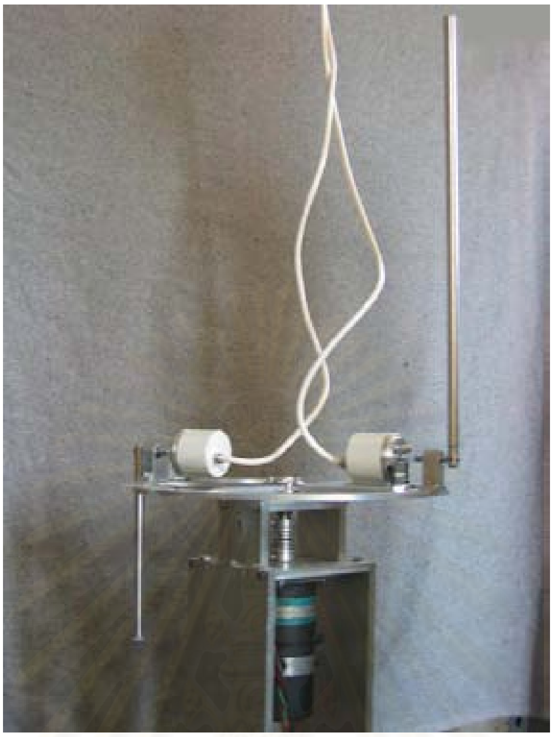

Figure 3.1: Physical System.

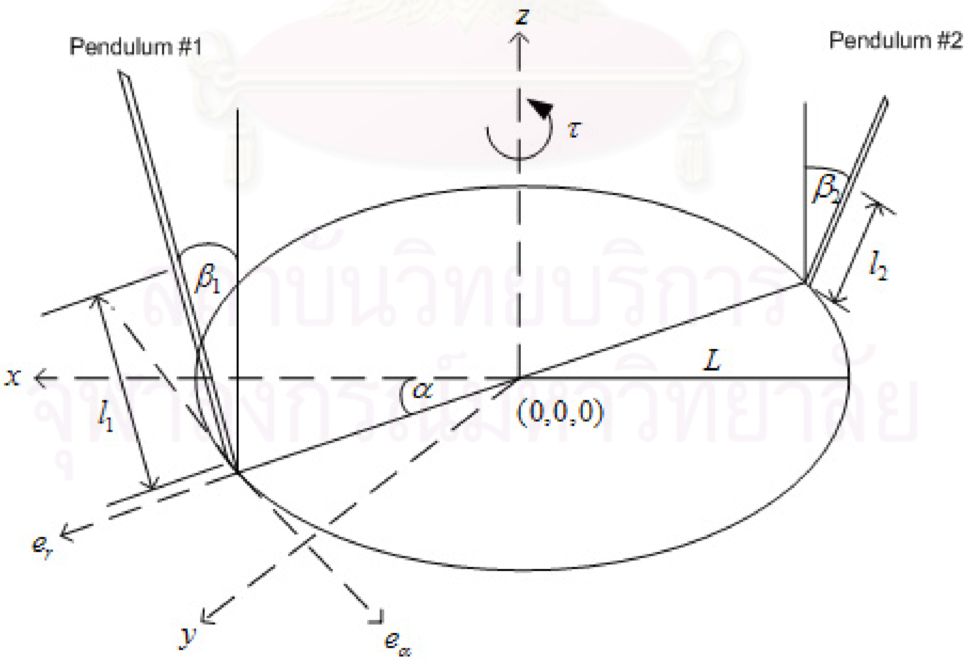

Figure 3.2: Rotary Double Inverted Pendulum.

Notation	Value	Unit	Remark
J_0	0.06	$kg - m^2$	Inertia of the rotating disc
J_1	0.008	$kg - m^2$	Inertia of the 1^{st} pendulum
J_2	0.002	$kg - m^2$	Inertia of the 2^{nd} pendulum
c_0	0.004	$N - m.$s	Viscous coef. of rotating disc
c_1	0.003	$N - m.s$	Viscous coef. of the 1^{st} pendulum
c_2	0.009	$N - m.s$	Viscous coef. of the 2^{nd} pendulum
m_1	0.25	kg	Mass of the 1^{st} pendulum
m_2	0.13	kg	Mass of the 2^{nd} pendulum
l_1	0.24	m	The displacement from the joint to the c.m of the 1^{st} pendulum
l_2	0.13	m	The displacement from the joint to the c.m of the 2^{nd} pendulum
L	0.172	m	The radius of the rotating disc
g	9.8	m/s	The gravity constant
K_m	0.374	$N.m/A$	Torque constant
K_b	0.374	$Volt/rad$	Back emf. constant
R	8.26	Ω	Resistant in motor circuit

Table 3.1: System parameters.

3.2 Statement of the benchmark problem

The aim of the design controller is robust stabilization of the RDIP. The RDIP to be controlled is an uncertain plant. There are three uncertain parameters in the plant as follows

1. Length of the long pendulum,

2. Length of the short pendulum,

3. Inertia of the rotational arm.

Other parameters are fixed as nominal values. In addition to the stabilization problem, we are interested in minimizing performance measure when the RDIP is injected by disturbances. The system is set to upright position. The disturbance inputs, $w_1(t)$ and $w_2(t)$, injected to two passive joints of long and short pendula are defined as

$$w_1(t) = a_1(t)\sin(\omega_{11}t + \phi_{11}) + b_1(t)\sin(\omega_{12}t + \phi_{12})$$

$$w_2(t) = a_2(t)\sin(\omega_{21}t + \phi_{21}) + b_2(t)\sin(\omega_{22}t + \phi_{22})$$

where
a_1, a_2, b_1 and b_2 are in the set $[0.0, 10^{-3}][Nm]$

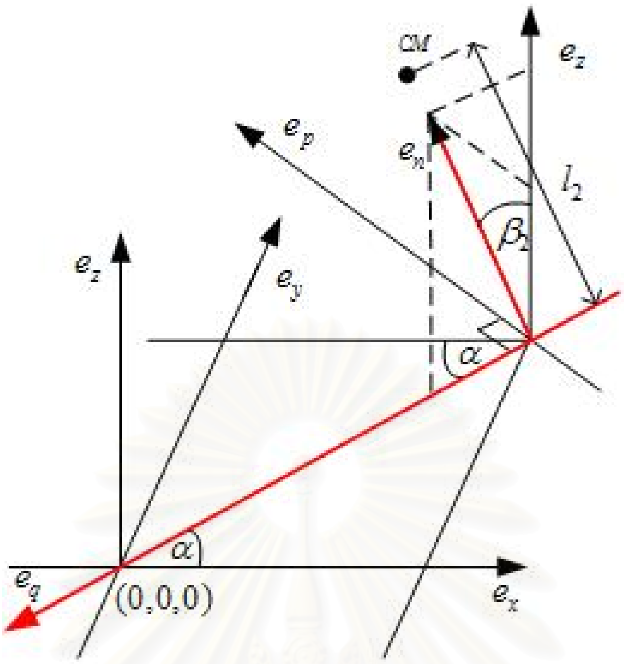

Figure 3.3: Unit vectors in the cartesian and polar coordinate.

$\omega_{11}, \omega_{12}, \omega_{21}$ and ω_{22} are in the set $[0.5, 10.0]$ [Hz]
$\phi_{11}, \phi_{12}, \phi_{21}$ and ϕ_{22} are in the set $[0.0, \pi]$ [rad]

The performance index is calculated as:

$$J = \frac{\|Qx\|_2}{\|w\|_2}, \quad Q = \text{diag}(10, 10, 10, 1, 1, 1) \tag{3.1}$$

where x stands for the state of the system.

The time duration of the calculation is defined from 0 to 10 (sec).

3.3 Nonlinear Dynamic Model

We now compute kinetic energy, potential energy, and loss energy for the system.

Kinetic Energy

The kinetic energy of the system is calculated as

$$K = \frac{1}{2}(J_0\dot{\alpha}^2 + J_1\dot{\beta}_1^2 + J_2\dot{\beta}_2^2 + m_1 v_1^2 + m_2 v_2^2) \tag{3.2}$$

where v_1, v_2 are the velocities of the center mass of the 1^{st} pendulum and the 2^{nd} pendulum, respectively. The cartesian coordinate is required to compute each velocity of the center mass of the pendulum. As shown in Fig 3.3, $\{e_x, e_y, e_z\}$ denotes the set of unit vectors of cartesian coordinate and $\{e_p, e_q, e_z\}$ denotes the set of unit vectors of cartesian coordinate in angle of the rotating disc and pendulum, respectively.

The unit vector of which direction along the line connecting the two pendulums is calculated

$$e_q = e_x \cos\alpha + e_y \sin\alpha \tag{3.3}$$

The unit vector e_p is similarly calculated

$$e_p = -e_x \sin\alpha + e_y \cos\alpha \tag{3.4}$$

In the coordinate $\{e_p, e_z\}$ for the pendulum, an unit vector e_n along direction of pendulum is computed

$$e_n = -e_z \cos\beta_i - e_p \sin\beta_i \tag{3.5}$$

The vector of the center mass of the pendulum is then found in coordinate pqz

$$\begin{aligned} v_{CM} &= l_i e_n + e_q L && (3.6) \\ &= e_z l_i \cos\beta_i - e_p l_i \sin\beta_i + L e_q && (3.7) \end{aligned}$$

We substitute (3.3) and (3.4) into (3.7). It is easy to find

$$v_{CM} = (l_i \sin\beta_i \sin\alpha + L\cos\alpha)e_x + (L\sin\alpha - l_i \sin\beta_i \cos\alpha)e_y + (l_i \cos\beta_i)e_z \tag{3.8}$$

The velocity of each pendulum is computed in the coordinate xyz. For more details, see [17].

$$\begin{aligned} v_i^x &= l_i \sin\beta_i \cos\alpha\dot{\alpha} + \sin\alpha(l_i \cos\beta_i \dot{\beta}_i - L\dot{\alpha}) \\ v_i^y &= l_i \sin\beta_i \sin\alpha\dot{\alpha} - \cos\alpha(l_i \cos\beta_i \dot{\beta}_i - L\dot{\alpha}) \\ v_i^z &= -l_i \sin\beta_i \sin\alpha\dot{\alpha} \end{aligned}$$

Substituting the velocities calculated above into (3.2) yields

$$\begin{aligned} K &= \frac{1}{2}(J_0\dot{\alpha}^2 + J_1\dot{\beta}_1^2 + J_2\dot{\beta}_2^2 + m_1(l_1 \sin\beta_1\dot{\alpha})^2 + m_1(L\dot{\alpha})^2 + m_1(l_1\dot{\beta}_1)^2 \\ &+ m_2(l_2 \sin\beta_2\dot{\alpha})^2 + m_2(L\dot{\alpha})^2 + m_2(l_2\dot{\beta}_2)^2) && (3.9) \\ &- m_1 l_1 L \cos\beta_1 \dot{\beta}_1 \dot{\alpha} - m_2 l_2 L \cos\beta_2 \dot{\beta}_2 \dot{\alpha} \end{aligned}$$

Potential Energy

The potential energy of the system is

$$P = m_1 g l_1 \cos\beta_1 + m_2 g l_2 \cos\beta_2 \tag{3.10}$$

Loss energy

The loss energy of the system resulting from the frictional force is

$$W = \frac{1}{2}(c_0)\dot{\alpha}^2 + c_1\dot{\beta}_1^2 + c_2\dot{\beta}_2^2 \tag{3.11}$$

Lagrangian

From the kinetic energy and potential energy, the Lagrangian is given by

$$L = K - P \tag{3.12}$$

From (3.9) and (3.10), we get

$$\begin{aligned} L &= \frac{1}{2}(J_0\dot{\alpha}^2 + J_1\dot{\beta}_1^2 + J_2\dot{\beta}_2^2 + m_1(l_1\sin\beta_1\dot{\alpha})^2 + m_1(L\dot{\alpha})^2 + m_1(l_1\dot{\beta}_1)^2 \\ &+ m_2(l_2\sin\beta_2\dot{\alpha})^2 + m_2(L\dot{\alpha})^2 + m_2(l_2\dot{\beta}_2)^2) - m_1l_1L\cos\beta_1\dot{\beta}_1\dot{\alpha} \\ &- m_2l_2L\cos\beta_2\dot{\beta}_2\dot{\alpha} - m_1gl_1\cos\beta_1 - m_2gl_2\cos\beta_2 \end{aligned} \tag{3.13}$$

State equations can be generated using Lagrange's Equation

$$\frac{d}{dt}\frac{L}{\partial \dot{q}_i} - \frac{L}{\partial q_i} + \frac{W}{\partial \dot{q}_i} = F_i \tag{3.14}$$

where F_i, q_i are the generalized forces, the generalized coordinates, respectively. Here, $q_i \in \{\alpha, \beta_1, \beta_2\}$

The dynamic equation of the system is computed as follows:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{3.15}$$

It is trivial to verify that

$$\begin{aligned} A_{11} &= J_0 + m_1l_1^2\sin^2\beta_1 + m_1L^2 + m_2l_2^2\sin^2\beta_2 + m_2L^2 \\ A_{12} &= -m_1l_1L\cos\beta_1 \\ A_{13} &= -m_2l_2L\cos\beta_2 \\ A_{21} &= -m_1l_1L\cos\beta_1 \\ A_{22} &= J_1 + m_1l_1^2 \\ A_{23} &= 0 \\ A_{31} &= -m_2l_2L\cos\beta_2 \\ A_{32} &= 0 \\ A_{33} &= J_2 + m_2l_2^2 \\ a_1 &= m_1l_1^2\dot{\beta}_1\dot{\alpha}\sin(2\beta_1) + m_1l_1L\dot{\beta}_1^2\sin\beta_1 + c_0\dot{\alpha} + m_2l_2^2\dot{\beta}_2\dot{\alpha}\sin(2\beta_2) + m_2l_2L\dot{\beta}_2^2\sin\beta_2 \\ a_2 &= -m_1l_1^2\dot{\alpha}^2\sin\beta_1\cos\beta_1 - m_1gl_1\sin\beta_1 + c_1\dot{\beta}_1 \\ a_3 &= -m_2l_2^2\dot{\alpha}^2\sin\beta_2\cos\beta_2 - m_2gl_2\sin\beta_2 + c_2\dot{\beta}_2 \end{aligned}$$

where τ is the torque applied to the rotating disc. In fact, the control input of the system is the voltage of DC motor, therefore, the torque is expressed in terms of control input. The inductor in the motor circuit is assumed to be zero. As a results, the τ is

$$\tau = \frac{K_m V}{R} - \frac{K_m K_b \dot{\alpha}}{R}$$

3.4 Linearized Dynamic Model

The system is linearized at the equilibrium point $\beta_1^* = \beta_2^* = 0°$. It can be seen that the if the value x is very small then $\sin x = x$ and $\sin^2 x = 0$.

Let $x = [\alpha \quad \beta_1 \quad \beta_2 \quad \dot{\alpha} \quad \dot{\beta}_1 \quad \dot{\beta}_2]^T$ be a state variable, and $u = V$ be an input. The equation (3.15) can be explicitly rewritten as

$$\begin{aligned}
(J_0 + m_1 L^2 + m_2 L^2)\ddot{\alpha} - m_1 l_1 L \ddot{\beta}_1 - m_2 l_2 L \ddot{\beta}_2 &= -(c_0 + \frac{K_m K_b}{R})\dot{\alpha} \\
-m_1 l_1 L \ddot{\alpha} + (J_1 + m_1 l_1^2)\ddot{\beta}_1 &= m_1 g l_1 \beta_1 - c_1 \dot{\beta}_1 + \frac{K_m}{R} u \\
-m_2 l_2 L \ddot{\alpha} + (J_2 + m_2 l_2^2)\ddot{\beta}_2 &= m_2 g l_2 \beta_2 - c_2 \dot{\beta}_2
\end{aligned}$$

It is easy to obtain the state space equation of the RDIP as

$$\begin{aligned}
E\dot{x} &= Fx + Gu \\
\dot{x} &= E^{-1}Fx + E^{-1}Gu \\
&=: Ax + Bu
\end{aligned}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_0 + m_1 L^2 + m_2 L^2 & -m_1 l_1 L & -m_2 l_2 L \\ 0 & 0 & 0 & -m_1 l_1 L & J_1 + m_1 l_1^2 & 0 \\ 0 & 0 & 0 & -m_2 l_2 L & 0 & J_2 + m_2 l_2^2 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -(c_0 + \frac{K_m K_b}{R}) & 0 & 0 \\ 0 & m_1 g l_1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & m_2 g l_2 & 0 & 0 & -c_2 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & \frac{K_m}{R} & 0 & 0 \end{bmatrix}^T$$

3.5 Summary and Discussion

3.5.1 Summary

In this chapter, we have stated formulation problem for RDIP. The system is an uncertain system having three uncertain parameters which are the length of the long and short pendulum as well as the inertia of the rotating disc. In addition to its uncertain parameters, the system is consider under two disturbance inputs injected to the passive joint of long and short pendulum. We also constructed a nonlinear dynamic model of the system based on Lagrangian equation. The system is then linearized at the upright unstable equilibrium position.

3.5.2 Discussion

By looking at the state equation of the RDIP, it turns out that our system is sixth order and is very highly nonlinear model. The RDIP was only linearized at the operating point which is the upright position, because the controller design problem is the stabilization of the pendulums at that point. We can linearize the nonlinear model at another operating point which is $\beta_1^* = \beta_2^* = 180°$. In this case, the state variables $,x,$ should be changed to $x = [\alpha \quad \beta_1 - \pi \quad \beta_2 - pi \quad \dot{\alpha} \quad \dot{\beta}_1 \quad \dot{\beta}_2]^T$. For more details, see [17].

CHAPTER IV

CONTROLLER DESIGN

This chapter presents the method of the controller design for the RDIP. Section 4.1 discusses about a robust state feedback controller using LMIs. A regional pole constraints is considered in order to improve the performance of the system. It is noted that the regional pole constraints are also cast in terms of some formulations of LMIs. A robust output feedback controller is mentioned in section 4.2. In this section, we will design an output controller of the 6 order.

4.1 H_∞ control

The LQR, Kalman filter and LQG problems can be posed as 2-norm optimization problems. However, these problems can be alternatively posed using the system H_∞-norm as a cost function. It is clear that H_∞-norm is the worst case gain of the system, hence, it provides a good match to engineering specifications.

It is very important to understand that the terms H_∞-norm and H_∞ are not terms which can impart a lot of engineering specifications. When one mentions about H_∞, that means a design method which is used to minimize the peak of a certain transfer function. The H_∞ norm of a stable scalar transfer function $F(s)$ is the peak value of $|F(j\omega)|$ as a function of frequency

$$\|F(s)\|_\infty \triangleq \sup_\omega |F(j\omega)|$$

The symbol ∞ comes from the fact that the maximum magnitude over frequency might be rewritten as

$$\sup_\omega |F(j\omega)| = \lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |F(j\omega)|^p \right)^{1/p}$$

H_∞ is the set of transfer functions with bounded ∞ norm. In other words, it is the set of stable and proper transfer functions.
An H_∞ controller minimized the worst case gain of the system. The problem can be thought of as an interesting matter: the designer will seek a controller that minimizes the gain in case of worst case input that maximizes the gain.

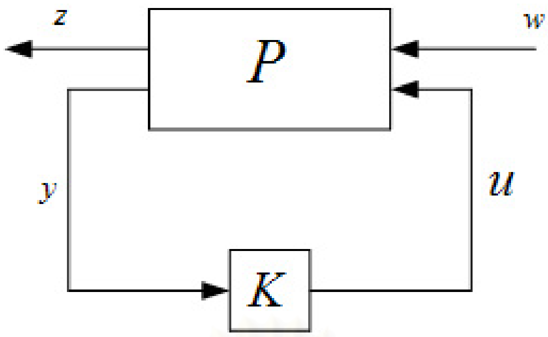

Figure 4.1: Block diagram.

4.2 LMI-based H_∞ controller design

In this section, we will design H_∞ state feedback controller with regional pole constraints. The advantage of this controller design is that the transient response of the closed-loop system can be improved. It can locate the closed-loop system poles into a suitable subregion of the left half plane. One way of simultaneously tuning the H_∞ performance and transient behavior is to combine the H_∞ and pole placement objectives. The disturbance inputs, $w_1(t)$ and $w_2(t)$, are injected to two passive joints of long and short pendulums. Our goal is to design a robust controller to achieve closed-loop stability and to attenuate to the effect of the disturbances on the peak value of the regulated signal z. On the other hand, the performance index

$$J = \frac{\|Qx\|_2}{\|\omega\|_2} = \frac{\sqrt{\int_0^\infty (x^T Q^T Qx)dt}}{\sqrt{\int_0^\infty (w^T w)dt}} \tag{4.1}$$

must be minimized.

The state space of the nominal plant is described as follows :

$$\begin{aligned} \dot{x} &= Ax + B_w w + Bu, \\ z &= C_z x, \\ y &= Cx. \end{aligned} \tag{4.2}$$

where $x \in \mathbb{R}^{6\times 6}$ is the state, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}^{3\times 3}$ is the measured output. $w \in \mathbb{R}^{2\times 2}$ is the disturbance input, $z \in \mathbb{R}^{6\times 6}$ is the output to be regulated. The matrix A, B of which elements were defined in chapter 3. According to the control problem, the RDIP has three outputs which are the angle of the two pendulums and the angle of the rotating disc.

The output matrix is then defined as

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The effect of disturbance inputs on the system needs to be attenuated. That means H_∞ of the transfer function from disturbance inputs w_1 and w_2 to the output z must be minimized.

$$\begin{aligned} \|H_{wz}\|_\infty &= \sup_{0<\|w\|<\infty} \frac{\|z\|_2}{\|w\|_2} \\ &= \sup_{0<\|w\|<\infty} \frac{\|C_z x\|_2}{\|\omega\|_2} \end{aligned}$$

By comparison with (4.1), the matrix C_z is defined by

$$C_z = Q$$

The signal w is considered as an additional input injected to the short and long pendulums. As a results, B_w is computed as

$$B_w = E^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

4.2.1 LMI formulation of pole-placement objectives

A concept of an LMI region as a convenient LMI-based representation of general of general stability regions. We now recall how the seeking of pole clustering in specific regions of the left-half complex plane is characterized as a formulation of LMIs. For instance, consider a second-order system with poles $\lambda = -\zeta\omega_n \pm j\omega_d$. It is clear that the step response of this system is characterized in terms of the undamped natural frequency $\omega_n = |\lambda|$, the damping ratio ζ, and the damped natural frequency $\omega - d$. Some regions such as α- stability regions $Re(s) \le -\alpha$, vertical strips, disks, conic sectors, etc. are very interesting. One combination of these regions is $S(\alpha, r, \theta)$ of complex numbers $x + jy$ is

$$x < -\alpha < 0, \quad |x + jy| < r, \quad \tan\theta x < -|y|$$

If the closed-poles of a certain system lie on this region then it can be ensured a minimum decay rate α, a minimum damping ratio $\zeta = \cos\theta$, and a maximum undamped natural frequency $\omega_d = r\sin\theta$. In other words, the maximum overshoot, the frequency of oscillatory modes, the delay time, the rise time, and the settling time can be improved as well. We now look at Lyapunov-based characterizations of pole constraints in stability subregions of the complex plane.

Lyapunov Conditions for regional pole constraints

Let $\mathcal{D}$ be a subregion of the left-half of the complex plane. A system $\dot{x} = Ax$ is called $\mathcal{D}$ stable if its poles lie in the region $\mathcal{D}$, the matrix A is then called $\mathcal{D}$ stable. For example when the region $\mathcal{D}$ is the left-half plane, the Lyapunov condition is stated as: A is stable if and only if there exists a symmetric matrix X satisfying

$$AX + XA^T < 0, \quad X > 0$$

Based on this knowledge, we consider an alternative LMI-based representation of $\mathcal{D}$ stability regions.

Definition 4.1 *LMI stability region*

For a symmetric matrix $P \in \mathbf{R}^{m\times m}$ and a matrix $Q \in \mathbf{R}^{m\times m}$, the set of complex numbers

$$\mathcal{D} = \{z \in \mathbf{C} : \quad f_{\mathcal{D}}(z) < 0\}$$

where

$$f_{\mathcal{D}}(z) = P + Qz + Q^T\overline{z}$$

is called LMI region.

On the other hand, an LMI region is a subset of the complex plane that is represented by an LMI in z and $\overline{z}$. Therefore, LMI regions are convex and symmetric with respect to the real axis for any $z \in \mathcal{D}$, $f_{\mathcal{D}}(\overline{z}) = \overline{f_{\mathcal{D}}(z)} < 0$. Below are a few examples of LMI regions:

- Half-plane $Re(z) < -\alpha : f_{\mathcal{D}}(z) = z + \overline{z} + 2\alpha < 0$

- Disk centered at $(-q, 0)$ with radius r:
  The matrices $P, Q \in \mathbb{R}^{2\times 2}$ exist as

$$P = \begin{bmatrix} -r & q \\ q & -r \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  Therefore, the characteristic function $f_{\mathcal{D}}(z)$ takes

$$f_{\mathcal{D}}(z) = \begin{bmatrix} -r & q+z \\ q+\overline{z} & -r \end{bmatrix} < 0$$

- Conic sector with apex at the origin and inner angle 2θ:
  The matrices $P, Q \in \mathbb{R}^{2\times 2}$ exist as

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

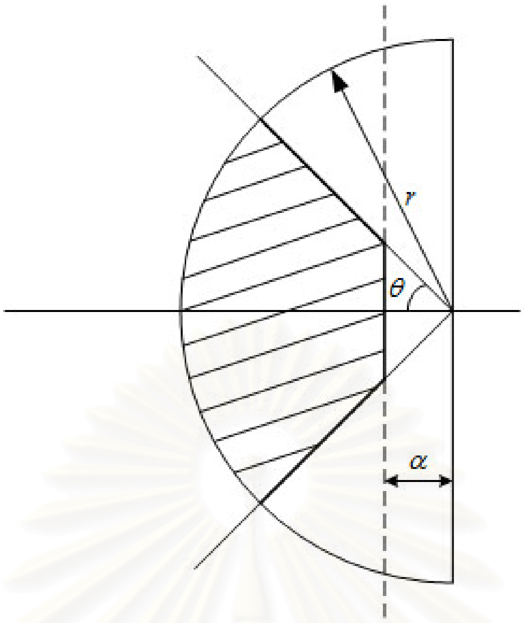

Figure 4.2: Region $S(\alpha, r, \theta)$.

$$Q = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

Hence

$$f_{\mathcal{D}}(z) = \begin{bmatrix} \sin(\theta)(z+\overline{z}) & \cos(\theta)(z-\overline{z}) \\ \cos(\theta)(\overline{z}-z) & \sin(\theta)(z+\overline{z}) \end{bmatrix} < 0$$

Consider a state feedback controller $u = Kx$, the closed-loop system becomes

$$\begin{aligned} \dot{x} &= (A+BK)x + B_w w \\ z &= C_z x \\ y &= Cx \end{aligned} \tag{4.3}$$

The H_∞ norm of the system admits an interpretation in terms of LMI known as bounded real lemma.

Lemma 4.1 (Bounded Real Lemma). *Suppose that the system described by (4.3) is controllable and has transfer function T, let* $\gamma > 0$*. Then the following statements are equivalent*

(a) $\|T\|_\infty < \gamma$*,*

(b) For all w *there holds that*

$$\sup_{0<\|w\|<\infty} \frac{\|z\|_2}{\|w\|_2} < \gamma$$

subject to initial condition $x(0) = 0$*,*

(c)There exists a solution $P = P^T$ *to the LMI*

$$(A+BK)^T P + P(A+BK) + C_z^T C_z + \frac{1}{\gamma^2} P B_w B_w^T P < 0$$

Suppose that there exists a quadratic function $V(\xi) = \xi^T P \xi, \quad P > 0$, and $\gamma \geq 0$ such that for all t,

$$\frac{d}{dt}V(x) + z^T z - \gamma^2 w^T w < 0 \tag{4.4}$$

then H_∞ of the system is less than γ. We now integrate (4.4) from 0 to T, with the condition $x(0) = 0$ to obtain

$$V(x) + \int_0^T (z^T z - \gamma^2 w^T w) dt < 0$$

The inequality (4.4) is equivalent to

$$((A+BK)x + B_w w)^T P x + x^T P((A+BK)x + B_w w) + x^T C_z^T C_z x - \gamma^2 w^T w$$

$$= x^T (A+BK)^T P x + w^T B_w^T P x + x^T P B_w w + x^T P(A+BK)x + x^T C_z^T C_z x - \gamma^2 w^T w < 0$$

or

$$\begin{bmatrix} x \\ w \end{bmatrix} \begin{bmatrix} (A+BK)^T P + P(A+BK) + C_z^T C_z & PB_w \\ B_w^T P & -\gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}^T < 0 \tag{4.5}$$

By using Schur complement, we have

$$(A+BK)^T P + P(A+BK) + C_z^T C_z + \frac{1}{\gamma^2} P B_w B_w^T P < 0 \tag{4.6}$$

Let $\frac{1}{\gamma^2} Y^{-1} = P$, the above LMI becomes

$$(A+BK)Y + Y(A+BK)^T + \frac{1}{\gamma^2} Y C_z^T C_z Y + B_w B_w^T < 0$$

Let $L = KY$, we get

$$\begin{bmatrix} AY + YA^T + BL + L^T B^T + B_w B_w^T & Y C_z^T \\ C_z Y & -\gamma^2 I \end{bmatrix} < 0$$

The closed-loop poles of $(A + BK)$ is constrained to lie on the region $S(\alpha, r, \theta)$. The LMI formulations are as the following LMIs: if there exist a symmetric matrix $Y > 0$ and a scalar γ such that

$$(A+BK)Y + Y(A+BK)^T + 2\alpha Y < 0$$

$$\begin{bmatrix} -rY & (A+BK)Y \\ Y(A+BK)^T & -rY \end{bmatrix} < 0$$

and

$$\begin{bmatrix} \sin(\theta)((A+BK)Y + Y(A+BK)^T) & \cos(\theta)((A+BK)Y - Y(A+BK)^T) \\ \cos(\theta)(Y(A+BK)^T - (A+BK)Y) & \sin(\theta)((A+BK)Y - Y(A+BK)^T) \end{bmatrix} < 0$$

In order to design a robust state feedback controller, we will take into account the system described by a polytope of linear affine systems. In general, a polytope description of uncertainties results in a less conservative controller design than other characterization of uncertainty. Note that, with the increasing of uncertain parameters, the number of vertices increases exponentially and the design time increases exponentially as well.

Let the system be represented by the state realization with uncertainties

$$\begin{aligned} \dot{x} &= A(\theta)x + B_w w + B(\theta)u \\ z &= C_z x \\ y &= Cx \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} A(\theta) &= A_0 + A_1\theta_1 + ... + A_p\theta_p \\ B(\theta) &= B_0 + B_1\theta_1 + ... + B_p\theta_p \end{aligned} \tag{4.8}$$

The system represented by (4.7) is a polytope of linear systems. The system is described by a list of its vertices as follows

$$\{(A_{v1}, B_{v1}), ..., (A_{vN}, B_{vN})\} \tag{4.9}$$

where $N = 8$, is the number of vertices.

Consider the polytope of the system, and with $L = KY$, the above LMIs are equivalent to

$$\begin{bmatrix} A_{vi}Y + YA_{vi}^T + B_{vi}L + L^T B_{vi}^T + B_w B_w^T & YC_z^T \\ C_z Y & -\gamma^2 I \end{bmatrix} < 0 \tag{4.10}$$

$$A_{vi}Y + YA_{vi}^T + B_{vi}L + (B_{vi}L)^T + 2\alpha Y < 0 \tag{4.11}$$

$$\begin{bmatrix} -rY & A_{vi}Y + B_{vi}L \\ YA_{vi}^T + (B_{vi}L)^T & -rY \end{bmatrix} < 0 \tag{4.12}$$

$$\begin{bmatrix} \sin(\theta)(A_{vi}Y + B_{vi}L + YA_{vi}^T + (B_{vi}L)^T) & \cos(\theta)(A_{vi}Y + B_{vi}L - YA_{vi}^T - (B_{vi}L)^T) \\ \cos(\theta)(YA_{vi}^T + (B_{vi}L)^T - A_{vi}Y - B_{vi}L) & \sin(\theta)(A_{vi}Y + B_{vi}L + YA_{vi}^T + (B_{vi}L)^T) \end{bmatrix} < 0 \tag{4.13}$$

Algorithm 4.1. *Consider the system described by (4.7). The robust H_∞ control with regional pole constraints can be characterized as follows*

$$\min_{L,Y} \gamma$$

s.t LMIs (4.10)-(4.13)

The state feedback gain matrix can be computed as $K = LY^{-1}$ which leads to $\|T_{zw}\|_\infty^2 \leqslant \gamma^2$.

4.3 Robust output feedback H_∞ controller

In this section, we will design a robust H_∞ output feedback controller. It was shown in [18] that the existence of H_∞ is equivalent to the feasibility of a system of three LMIs whose unknown X, Y are two symmetric matrices of size equal to the plant order. The explicit characterizations of solutions to the Bounded Real Lemma were also derived in [19] [20]. In comparison, the formulas given here are simpler and suitable for numerically stable implementations.

We assume that only partial state information is available through the output y. Our output feedback law is generated by a strictly proper full order linear controller.

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c u_c \\ y_c &= C_c x_c + D_c u_c \end{aligned} \tag{4.14}$$

where $x_c \in \mathbb{R}^{6\times 6}$ is the controller state, $u_c \in \mathbb{R}$ is the controller input, $y_c \in \mathbb{R}^{3\times 1}$ is the controller output. The method of controller design is first derived in the nominal case and then extended to uncertain systems described by a polytope of models. A convenient way to proceed is to find a realization of the closed-loop transfer function from w to z: $\mathcal{T}(G, C) = D_{cl} + C_{cl}(sI - A_{cl})^{-1} B_{cl}$ where

$$A_{cl} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B_w \\ 0 \end{bmatrix},$$

$$C_{cl} = \begin{pmatrix} C_z & 0 \end{pmatrix}, \quad D_{cl} = 0. \tag{4.15}$$

Consider the LMI (4.6), we also get

$$A_{cl}^T P + P A_{cl} + C_{cl}^T C_{cl} + \frac{1}{\gamma^2} P B_{cl} B_{cl}^T P < 0 \tag{4.16}$$

and let $\gamma X_{cl} = P$, we get

$$X_{cl} A_{cl} + A_{cl}^T X_{cl} + \frac{1}{\gamma} C_{cl}^T C_{cl} + \frac{1}{\gamma} X_{cl} B_{cl} B_{cl}^T X_{cl} < 0 \tag{4.17}$$

Applying Schur complement yields

$$\begin{bmatrix} A_{cli}^T X_{cl} + X_{cl} A_{cli} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} < 0 \tag{4.18}$$

for some symmetric matrices $X_{cl} > 0$ of dimension (6×6). The unknown matrices are X_{cl} and the controller parameters inside $A_{cl}, B_{cl}, C_{cl}, D_{cl}$. Internal stability and the H_∞-norm constraint are equivalent to above feasibility of the matrix inequality. It is noted that (4.18) is not an LMI. This is Bilinear matrix inequality (BMI) because it consists of some products of two variables. It is impossible to solve BMI by any solver. Therefore, we need to transform an BMI into a certain LMI.

Let X_{cl} be partitioned as follows

$$X_{cl} = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \quad X_{cl}^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix} \tag{4.19}$$

with $X, Y, M, N \in \mathbb{R}^{6\times 6}$.

Let $\mathcal{N}_{12}$ and $\mathcal{N}_{21}$ denote orthonormal bases of the null spaces of $(B^T, 0)$ and $(C, 0)$, respectively. Substituting the matrix X_{cl} into the LMI (4.17) and consider each term of this LMI

$$\mathcal{N}_{21}^T A_{cl}^T X_{cl} \mathcal{N}_{21} = \mathcal{N}_{21}^T \begin{bmatrix} A^T + C^T D_c^T B^T)Y + C^T B_c^T N^T & * \\ C_c^T B^T Y + A_c^T N^T & * \end{bmatrix} \mathcal{N}_{21} \tag{4.20}$$

$$= \mathcal{N}_{21}^T \begin{bmatrix} A^T Y & 0 \\ 0 & 0 \end{bmatrix} \mathcal{N}_{21} \tag{4.21}$$

$$\mathcal{N}_{21} C_{cl}^T C_{cl} \mathcal{N}_{21} = \mathcal{N}_{21}^T \begin{bmatrix} C_z^T C_z & 0 \\ 0 & 0 \end{bmatrix} \mathcal{N}_{21} \tag{4.22}$$

The other terms are also reduced, and the LMI (4.17) becomes

$$\mathcal{N}_{21}^T \begin{bmatrix} A^T Y + YA + \frac{1}{\gamma} C_z^T C_z + \frac{1}{\gamma} Y B_w B_w^T Y & 0 \\ 0 & 0 \end{bmatrix} \mathcal{N}_{21} < 0 \tag{4.23}$$

It is trivial to get

$$\mathcal{N}_{12}^T \begin{bmatrix} AX + XA^T + \frac{1}{\gamma} B_w B_w^T + \frac{1}{\gamma} X C_z^T C_w X & 0 \\ 0 & 0 \end{bmatrix} \mathcal{N}_{12} < 0 \tag{4.24}$$

Hence, the controller parameters will be thrown out (4.18) to achieve an LMI associated with X and Y only.

We now extend to uncertain systems described by a polytopic state-space model. Such polytopic models may result from convex interpolation of a set of model (A_i, B_i) identified in different operating points.

Algorithm 4.2. *Consider the system described by (4.2), the LMIs (4.23) and (4.24). The optimal H_∞ problem is solvable if and only if there exist two symmetric matrices $X, Y \in \mathbb{R}^{6\times 6}$ satisfying the following system of LMIs*

$$\min_{X,Y} \gamma$$

$$\begin{bmatrix} \mathcal{N}_{12i} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_i X + X A_i^T & X C_z^T & B_w \\ C_z X & -\gamma I & 0 \\ B_w^T & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_{12i} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{4.25}$$

$$\begin{bmatrix} \mathcal{N}_{21} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_i^T Y + Y A_i & Y B_w & C_z^T \\ B_w^T Y & -\gamma I & 0 \\ C_z & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_{21} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{4.26}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{4.27}$$

where $\mathcal{N}_{12i}$ denotes orthonormal bases of the null spaces of $(B_i^T, 0)$. The last LMI is to make sure that the matrix $X_{cl} > 0$. The pairs (X, Y) are called feasible for the LMI systems (4.25)-(4.27) and computing feasible pairs is a convex optimization problem. Let $k :=$ Rank$(I - XY)$ and compute via SVD two full column rank matrices $M, N \in \mathbb{R}^{6\times k}$ such that

$$MN^T = I - XY \tag{4.28}$$

The matrix X_{cl} can be computed as an unique solution of the linear equation

$$X_{cl} \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{4.29}$$

Once X_{cl} is determined, (4.18) becomes an LMI with respect to the controller parameters A_c, B_c, C_c, D_c only. Therefore, the controller parameters could be computed by solving the LMI (4.18). Generally, this option is appropriate for most cases, however it can procedure a numerical problem. A particular solution can be computed more efficiently using elementary linear algebra. In order to construct controllers parameter, we will rewrite the LMI (4.18). By using a Schur complement argument, the LMI (4.18) is equivalent to the two LMIs

$$\Delta_{cl} = \begin{bmatrix} \gamma I & -D_{cl}^T \\ -D_{cl} & \gamma I \end{bmatrix} > 0 \tag{4.30}$$

$$A_{cli}^T X_{cl} + X_{cl} A_{cli} + \begin{bmatrix} B_{cl}^T X_{cl} \\ C_{cl} \end{bmatrix}^T \Delta_{cl}^{-1} \begin{bmatrix} B_{cl}^T X_{cl} \\ C_{cl} \end{bmatrix} < 0 \tag{4.31}$$

We set up two new variables R, S such that $X_{cl} R = S$ where

$$R = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, \quad S = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{4.32}$$

Thus, $X_{cl} = SR^{-1}$. Substituting this term into (4.31), and multiplying both side of the LMI (4.31) with the matrix R^T, R yield

$$R^T A_{cli}^T S + S^T A_{cli} R + \begin{bmatrix} B_{cl}^T S \\ C_{cl} R \end{bmatrix}^T \Delta_{cl}^{-1} \begin{bmatrix} B_{cl}^T S \\ C_{cl} R \end{bmatrix} < 0 \tag{4.33}$$

For simplicity, some shorthands are introduced

$$\mathcal{A}_i = A_i + B_i D_c C,$$

$$K_B = B_c^T N^T + D_c^T B_i^T Y, \quad K_C = C_c M^T + D_c C X. \tag{4.34}$$

We have

$$S^T A_{cli} R = \begin{bmatrix} I & 0 \\ Y & N \end{bmatrix} \begin{bmatrix} A_i + B_i D_c C & B_i C_c \\ B_c C & A_c \end{bmatrix} \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} \tag{4.35}$$

$$= \begin{bmatrix} A_i X + B_i K_C & \mathcal{A}_i \\ N A_c M^T + Y \mathcal{A}_i X + Y B_i C_c M^T + N B_c C X & Y A_i + K_B^T C \end{bmatrix} \tag{4.36}$$

$$R^T A_{cli}^T S = \begin{bmatrix} X A_i^T + K_C^T B_i^T & \mathcal{A}_i^T \\ M A_c^T N + X \mathcal{A}_i^T Y + M C_c^T B_i^T Y + X C^T B_c^T Y & A_i^T Y + C^T K_B \end{bmatrix} \tag{4.37}$$

and

$$\begin{bmatrix} B_{cl}^T S \\ C_{cl} R \end{bmatrix}^T \Delta_{cl}^{-1} \begin{bmatrix} B_{cl}^T S \\ C_{cl} R \end{bmatrix} = [\ldots]^T \Delta_{cl}^{-1} \begin{bmatrix} B_w^T & C_z X \\ B_w^T Y & C_z^T \end{bmatrix} \tag{4.38}$$

The BMI (4.33) becomes

$$\begin{bmatrix} \Delta_{Xi} & \Delta_{21i}^T \\ \Delta_{21i} & \Delta_{Yi} \end{bmatrix} < 0 \tag{4.39}$$

where

$$\Delta_{Xi} = A_i X + X A_i^T + B_i K_C + K_C^T B_i^T + [\ldots]^T \Delta_{cl}^{-1} \begin{bmatrix} B_w^T \\ C_z X \end{bmatrix} \tag{4.40}$$

$$\Delta_{Yi} = A_i^T Y + Y A_i + C^T K_B + K_B^T C + [\ldots]^T \Delta_{cl}^{-1} \begin{bmatrix} B_w^T Y \\ C_z^T \end{bmatrix} \tag{4.41}$$

$$\Delta_{21i} = N A_c M^T + \mathcal{A}_i^T + Y \mathcal{A}_i X + Y B_i C_c M^T + Y N B_c C X + \begin{bmatrix} Y B_w & C_z^T \end{bmatrix} \Delta_{cl}^{-1} \begin{bmatrix} B_w^T \\ C_z X \end{bmatrix} \tag{4.42}$$

It can be seen that the products of variable X_{cl} and state matrices of the closed-loop system are pulled out the BMI (4.31). We have transform the BMI (4.31) into the LMI (4.39). The controller parameters can be computed by solving the LMI (4.39).

4.4 Summary and Discussion

4.4.1 Summary

In this chapter, we have presented a robust state feedback H_∞ controller and a robust output feedback H_∞ controller. In order to design the controllers, an LMI regional pole constrains has been used to improved the transient response of the system and to make the uncertain system more stable. As given in Algorithm 4.1, the robust state feedback H_∞ controller was found by solving four LMIs system over the two variable L and Y. In case of the robust output feedback H_∞ controller, we also need to solve a minimization problem as shown in Algorithm 4.2. Unlike state feed back controller, in this case after solving three LMIs system we need to reconstruct the controller. It is noted that in both cases a polytopic system of eight vertices was considered. For the first case, we need to solve a system of thirty-two LMIs. In next chapter, we will show the results of the controller designs.

4.4.2 Discussion

The computation of adequate X, Y and the controller reconstruction reduce to solving LMIs, and hence to convex optimization programs. However, its solutions parameterize the set of H_∞ controllers and bear important connections with the controller order. H_∞ control is a natural for applications where the specifications are given in terms of frequency dependent bounds on the output. Requiring the input to output gain to remain below prescribed levels is typical of engineering design specifications. The existence conditions for a optimal H_∞ controller are useful when performing trade-offs between competing control objectives. In addition, H_∞ control can be used as an alternative to LQG optimal problem. Both approaches are reasonable for a wide range of problems.

CHAPTER V

SIMULATION RESULTS

This chapter presents the result of the controller design for the DRIP. In section 5.1, the result of the robust state feedback controller design based on a regional pole constraints is shown. Specifically, we will inject the two disturbance inputs and seek the worst-case gain while varying the uncertain parameters of the system. Section 5.2 shows the result of the robust output feedback controller design. With the same steps in previous case, the responses of the system corresponding to the disturbance inputs are displayed. Some comparison between the two controllers are then shown.

5.1 Robust state feedback H_∞ controller

The feasibility problem was solved for (Y, L) and the state feedback matrix was obtained as $K = LY^{-1}$. In order to find the state feedback, the minimization problem in Algorithm 4.2 was solved using YALMIP package with solver SeDuMi. The problem was solved with regional pole constraints in the region of $S(0.5, 10000, 1.04)$. For the state feedback controller with regional pole constraints, the state feedback gain is found

$$K = [-84 \quad 43971 \quad -38633 \quad -130 \quad 8633 \quad -6285]$$

The poles of the closed-loop system are

$$-343.1307 \quad -3.5960 \pm j2.0394 \quad -2.3478 \quad -1.5377 \pm j1.4401$$

It is clear that the poles of the closed-loop system are satisfied the region $S(0.5, 10000, 1.04)$. The H_∞ performance of the resulting closed loop systems is found $\gamma = 72.4$.
We now show the simulation results of pendulum angle near the operating point $\beta_1^* = 0$ and $\beta_2^* = 0$. Firstly, the two long and The system is set to the upright position to calculate the performance when the disturbances are injected to passive joints of long and short pendulums. We will consider the effects of uncertainties and two sets of the disturbances specified above on the system. The simulation is also tested by using a virtual plant. The worst value of performance index can be found by varying a set of three uncertain parameters and two disturbances. This step can be done more efficiently by using an interface control panel in Figure 5.1. It is noted that the cost function must be calculated from 0 to 10 (sec). Therefore, the value obtained by simulation is less than the H_∞-norm obtained

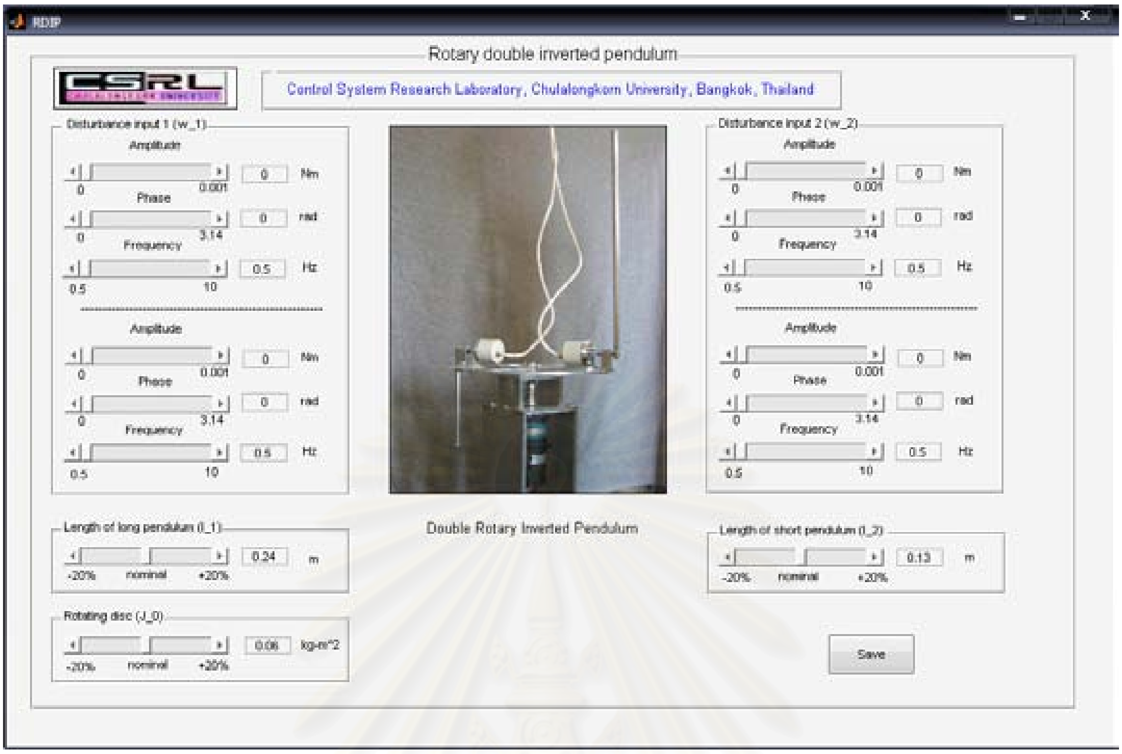

Figure 5.1: The control panel

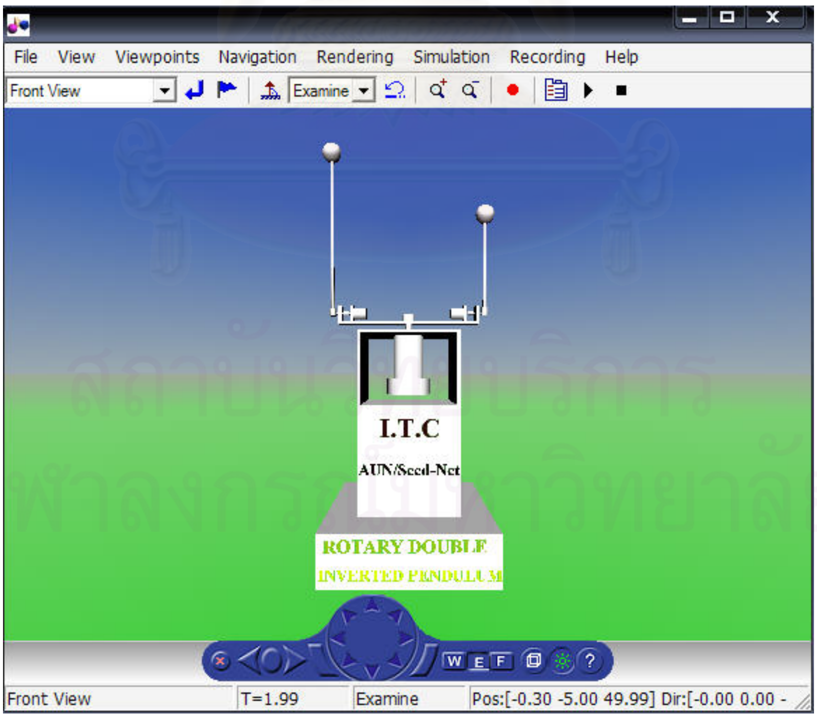

Figure 5.2: The virtual plant

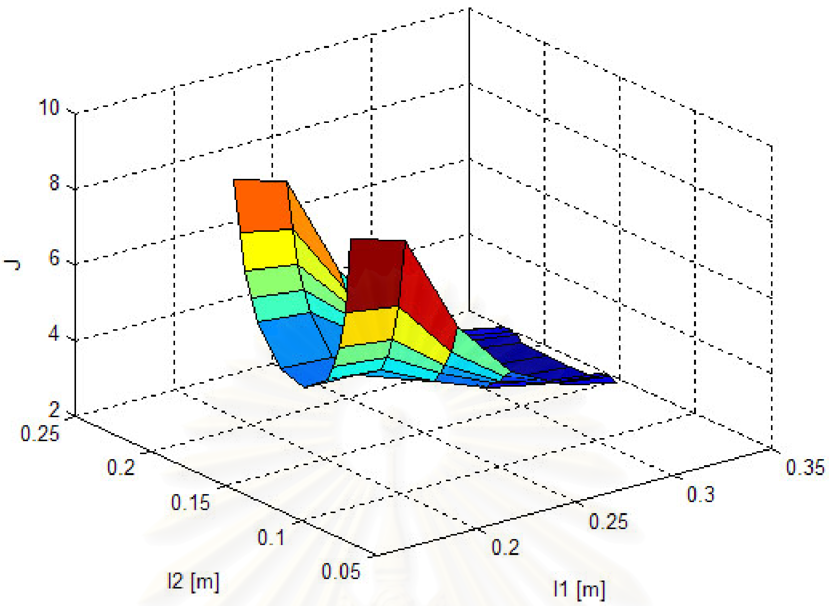

Figure 5.3: The performance index with uncertainties l_1, l_2.

by solving Algorithm 4.1. In the first step, we examine the performance index when the uncertain parameters which are length of the short and the short pendulums are being varied. By observing Fig 5.3, it is transparent that the worst value of performance index occurs when l_1 and l_2 are minimum. In the second step, we fix the length of the short pendulum at the minimum values. The inertia of rotating disc and the length of the long pendulum are changed 25 percent around their nominal values. The performance index corresponding to the uncertain parameters J_0, l_1 is shown in Fig 5.4. The worst-case value occurs when the inertia of rotating disc is maximum, and the length of the long pendulum is minimum. The greater the value of J_0 is the slower the position of the rotating disc changes. As a results, the stabilization of two pendulums will be more affected by the disturbance inputs.

In this third step, we fix the length of the long pendulum at the minimum values and vary the length of the short pendulum 20 percent around its nominal value. It turns out that, the maximum value occurs when the length of the short pendulum is at the minimum value and the inertia of the rotating disc is maximum.

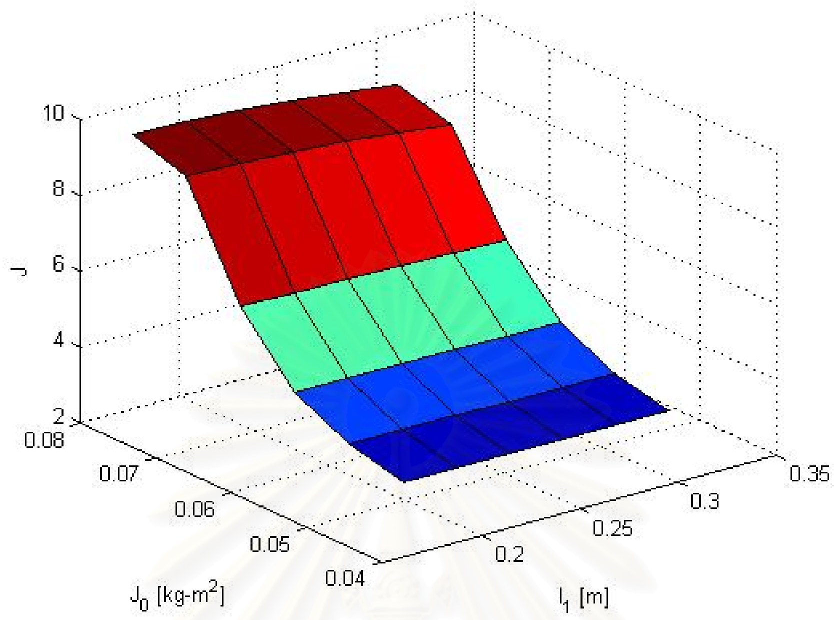

Figure 5.4: The performance index with uncertainties J_0, l_1.

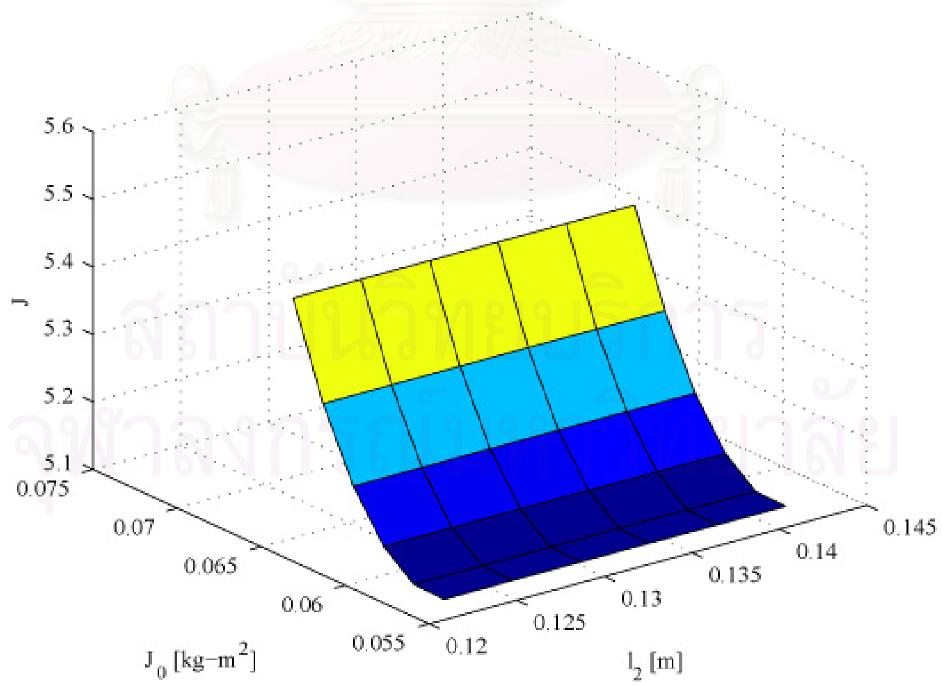

Figure 5.5: The performance index with uncertainties J_0, l_2.

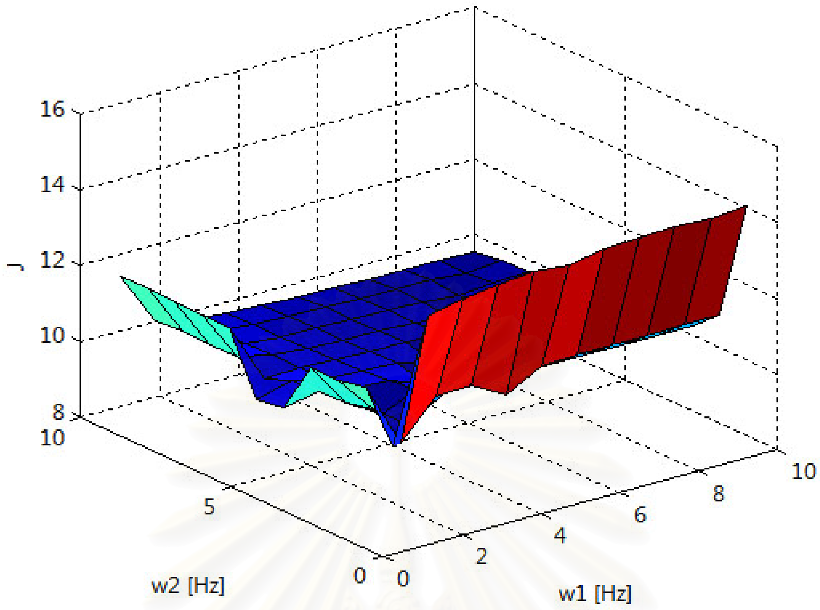

Figure 5.6: The performance index with disturbances ω_1, ω_2.

We now will take into account the effect of disturbance inputs on the system. The length of each pendulum is fixed at the minimum value and the inertia of the rotating disc is fixed at the maximum value. From Fig 5.6, the worst-case inputs is found

$$w_1(t) = 0.001\sin(20\pi t + \pi/2) + 0.001\sin(20\pi t + \pi/2)$$

$$w_2(t) = 0.001\sin(\pi t + \pi/2) + 0.001\sin(\pi t + \pi/2)$$

Consequently, the worst value of the performance index is 14.2. The response of the system corresponding to the disturbance inputs is shown in Fig 5.7. We observe that the amplitude of the pendulums is very small, 0.03 [deg]. Therefore, the system is stable while the disturbances are working on it.

Consider the initial values $\beta_1 = 6(deg)$ and $\beta_2 = 6(deg)$. The closed-system is stable as shown in Fig 5.8~5.10.

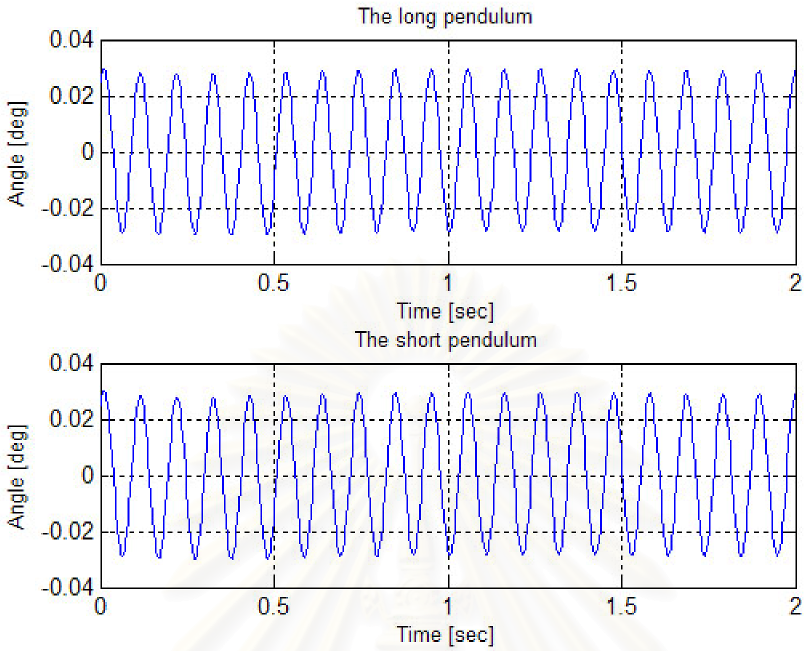

Figure 5.7: The angle of disc α.

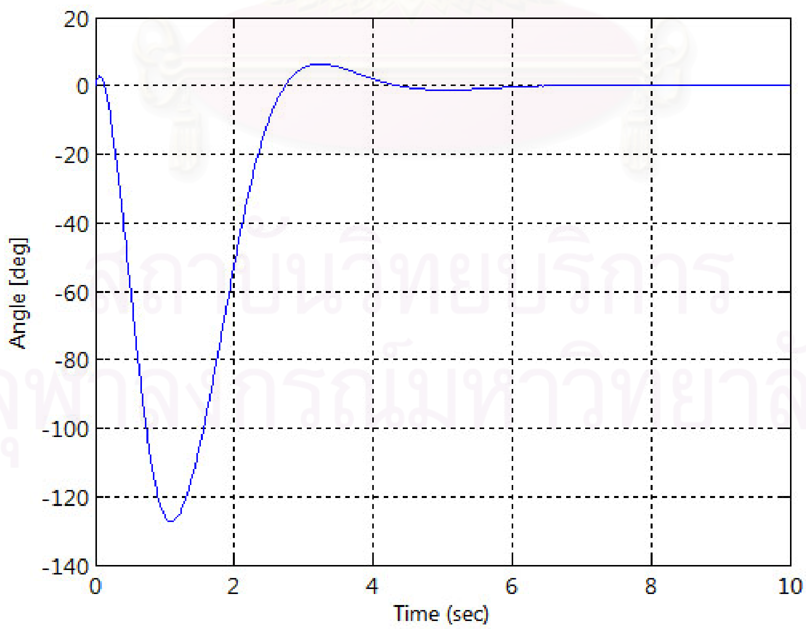

Figure 5.8: The angle of disc α.

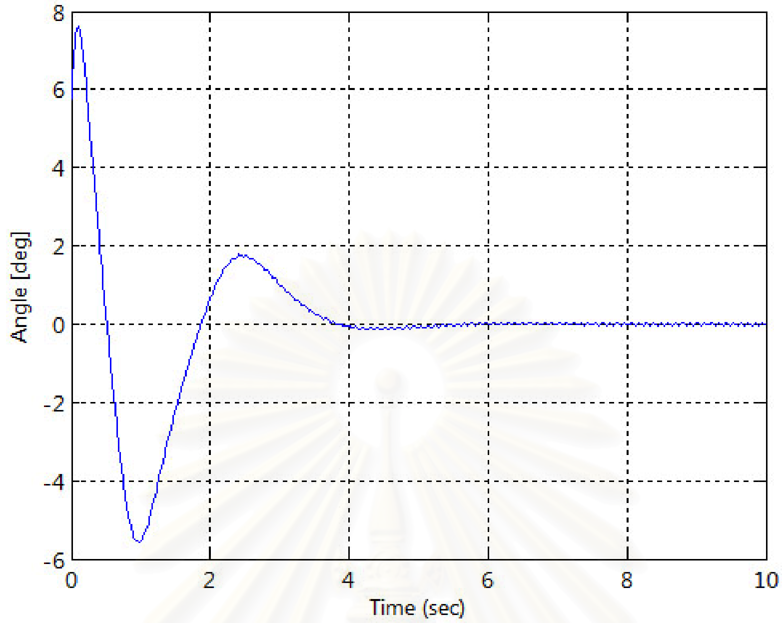

Figure 5.9: The angle of the long pendulum β_1.

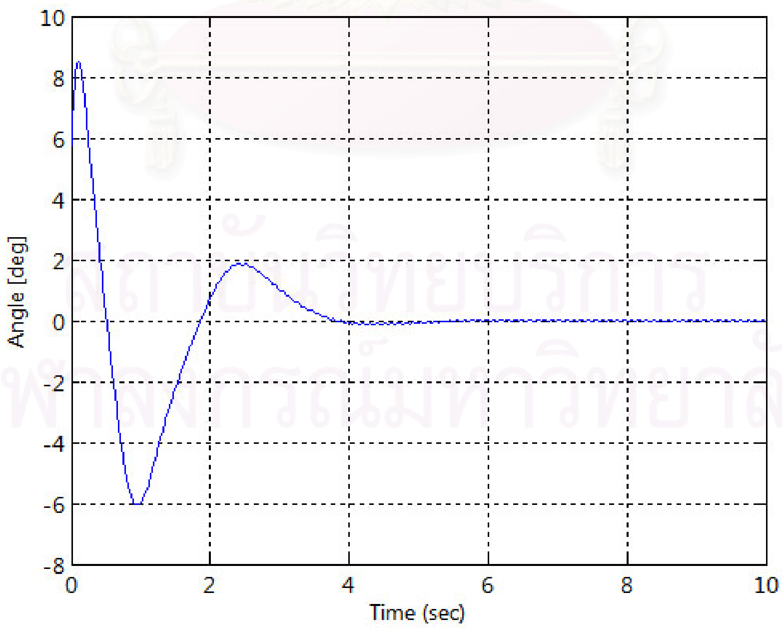

Figure 5.10: The angle of the short pendulum β_2.

5.2 Output feedback H_∞ controller

We consider effect of the disturbances on the system. The initial angles are also set $\beta_1(0) = 0°$ (deg) and $\beta_2(0) = 0°$ (deg). The output feedback controller is found

$$A_c = \begin{bmatrix} -4.3039 & -1.1172 & -2.7539 & -0.3162 & 36.1764 & -494.6792 \\ -2.8702 & -19.5836 & -0.4055 & -3.4490 & -18.2823 & 112.3659 \\ 3.2645 & 0.7971 & -3.9003 & -3 & -17 & 88 \\ 1.5346 & 4.3969 & 0.1010 & -2.5191 & -65.6825 & 752.9302 \\ 43.8694 & -56.1111 & 139.5282 & 25.4766 & -111.8484 & 149.2250 \\ -3512.9 & 68565 & -1520.4 & -1852.6 & 19544 & -748.6134 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1.4811 & 2.8340 & 0.4968 \\ -10.3463 & 1.9112 & -303.8622 \\ 11.0831 & 63.0928 & 15.2727 \\ 6.5628 & 482472 & 49.7058 \\ -5066 & -40916 & -10058 \\ 25100 & -138200 & 1593100 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -3.6169 & -1.1411 & -2.4524 & -0.4004 & 35.9913 & -491.3821 \end{bmatrix}; \quad D_c = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

The output feedback controller can also be written as

$$K = \text{diag}(K_1(s), K_2(s), K_3(s))$$

where

$$\begin{aligned} K_1(s) &= \frac{-12371458.8283(s-371.7)(s+73.3)(s+4.96)(s+0.4135)(s-0.1236)}{(s+490.4)(s+64.25)(s+15.21)(s+3.249)(s^2+1329s+8.437e005)} \\ K_2(s) &= \frac{67767121.7747(s+623.9)(s+77.81)(s+5.283)(s^2+0.5575s+0.986)}{(s+490.4)(s+64.25)(s+15.21)(s+3.249)(s^2+1329s+8.437e005)} \\ K_3(s) &= \frac{-782847598.0097(s+6.28)(s^2+0.5415s+0.9182)(s^2+96.37s+2624)}{(s+490.4)(s+64.25)(s+15.21)(s+3.249)(s^2+1329s+8.437e005)} \end{aligned}$$

The poles of the closed-loop system are

$$\begin{array}{cccc} -548.66 & -101.02 \pm j30.78 & -39.31 \pm j28.66 & -49.96 \\ -0.66 & -1.46 \pm j2.35 & -2.22 & -0.27 \pm j0.93 \end{array}$$

The Fig 5.11$\sim$ 5.13 show the performance index. The worst value of performance index in case of consideration of the disturbances calculated from 0 [sec] to 10 [sec] is $J = 27.5$. This worst value is greater than the wort value obtained in the previous case.

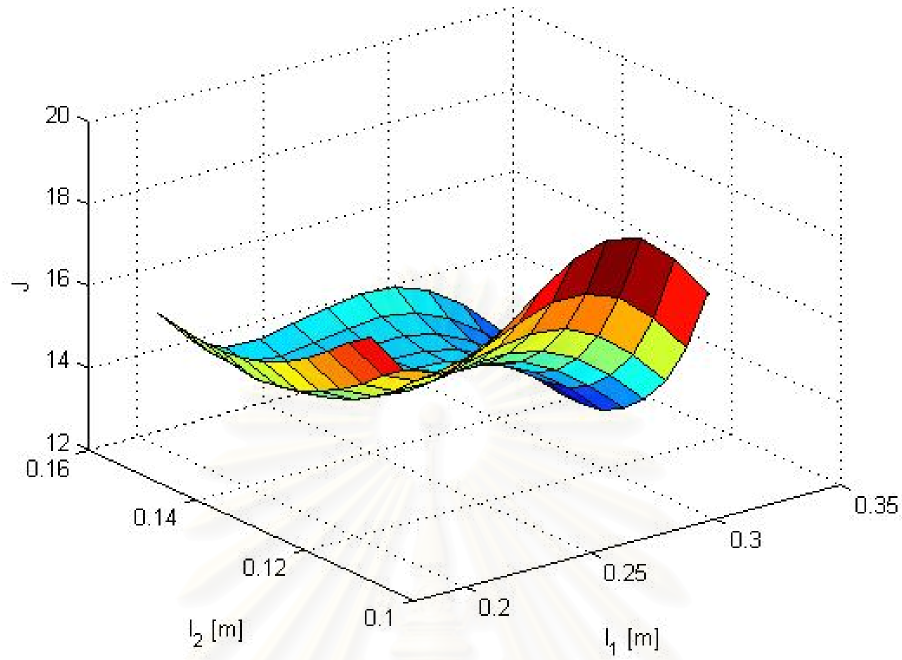

Figure 5.11: The performance index with uncertainties l_1, l_2.

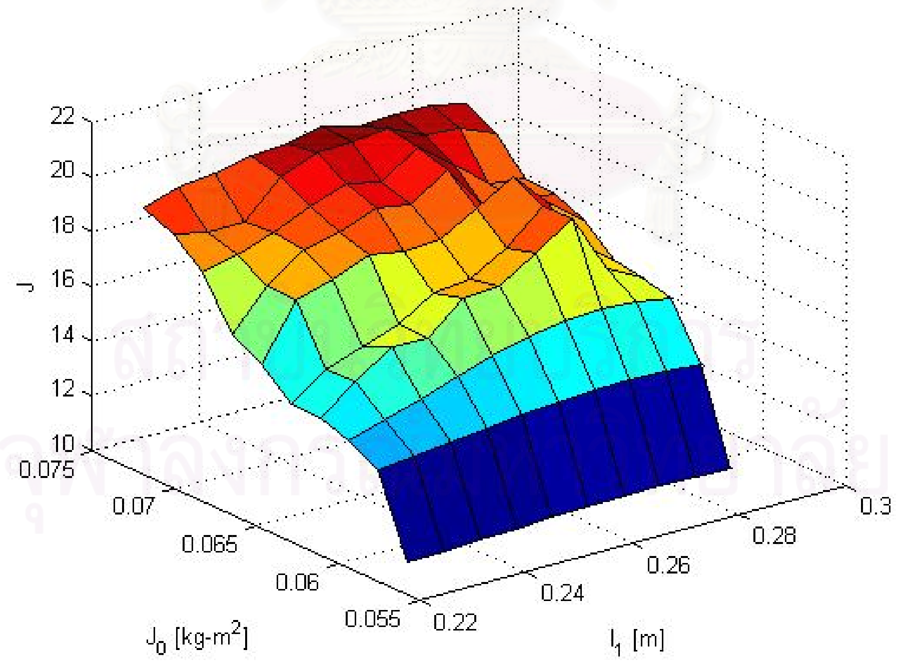

Figure 5.12: The performance index with uncertainties J_0, l_1.

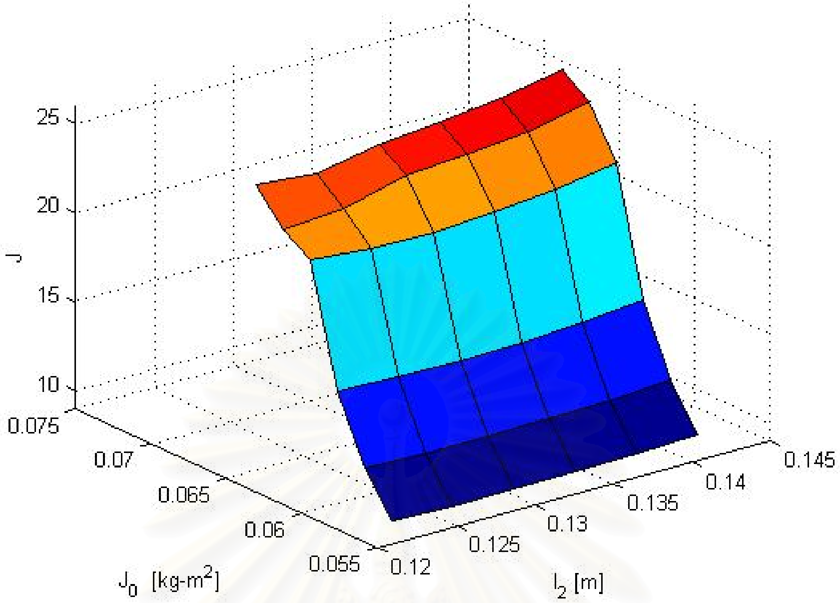

Figure 5.13: The performance index with uncertainties J_0, l_2.

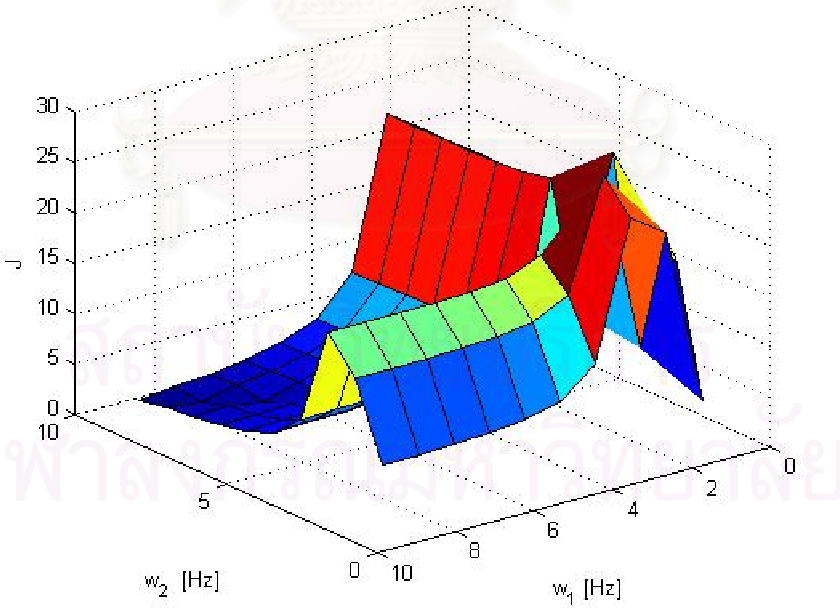

Figure 5.14: The performance index with disturbances ω_1, ω_2.

We set the initial values as $\beta_1(0) = 0°$ and $\beta_2(0) = 0°$, that means the two pendulum are set at the upright unstable position. Responses of the outputs of the nominal plant with respect to the disturbances are shown Fig 5.15 and 5.16. Because of the sinusoidal inputs, hence, the system will oscillate with a small amplitude.

In comparison with the state feedback controller, it is transparent that the amplitude of the outputs in this case is greater. The system is now examined when the initial positions are not at the upright unstable position. As shown in Fig 5.17, the angle of the short and the long pendulums are $\beta_1(0) = 6°$ and $\beta_2(0) = 6°$. The peak value of the angle of the short pendulum is $11.5°$ and greater than that in case of robust state feedback controller. The same remark is stated for the long pendulum.

In short, the robust H_∞ state feedback controller gives the better results than the robust H_∞ output feedback controller does. It is reasonable because in the state feedback controller we know all information of the system. For the output feedback controller, only the angles of pendulum and disc are known.

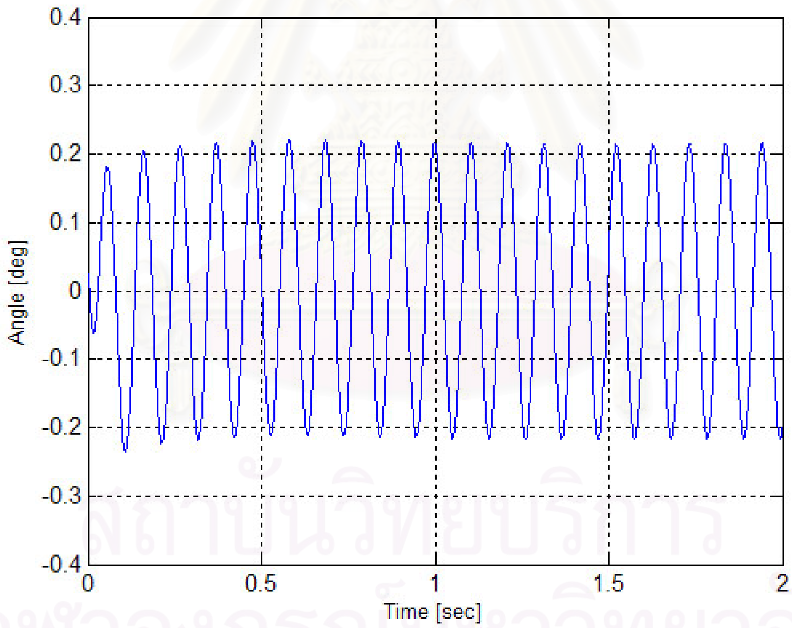

Figure 5.15: The angle of long pendulum β_1.

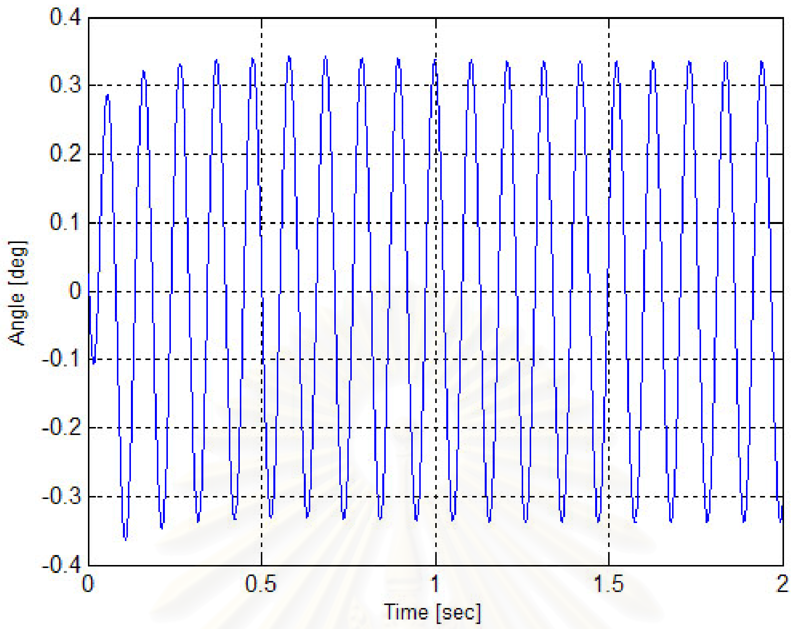

Figure 5.16: The angle of short pendulum β_2.

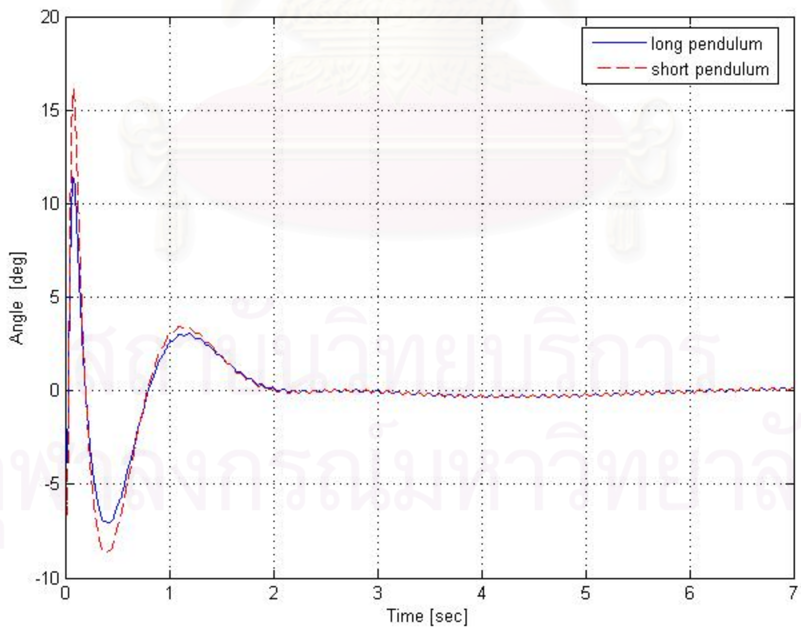

Figure 5.17: The angle of long pendulum and short pendulum.

Some comparisons between the proposed controllers and LQR controller are shown in Fig 5.18∼ 5.21. Firstly, we consider the nominal plant and set the initial angle of the pendulums at $6°$. In this case, the LQR controller gives a result as good as the proposed controller. The LQR controller causes a higher peak value in comparison with the proposed controllers but it is acceptable. Therefore, the LQR controller gives a result as good as the proposed controller.

We investigate the stability of the system when the uncertain parameters change 15 percent around their nominal value. From Fig 5.20 and 5.21, it can be seen that the LQR controller presents a bad response. The both pendulums oscillate with a large amplitude. If we change 20 percent of the uncertain parameters then the LQR controller can not stabilize the system.

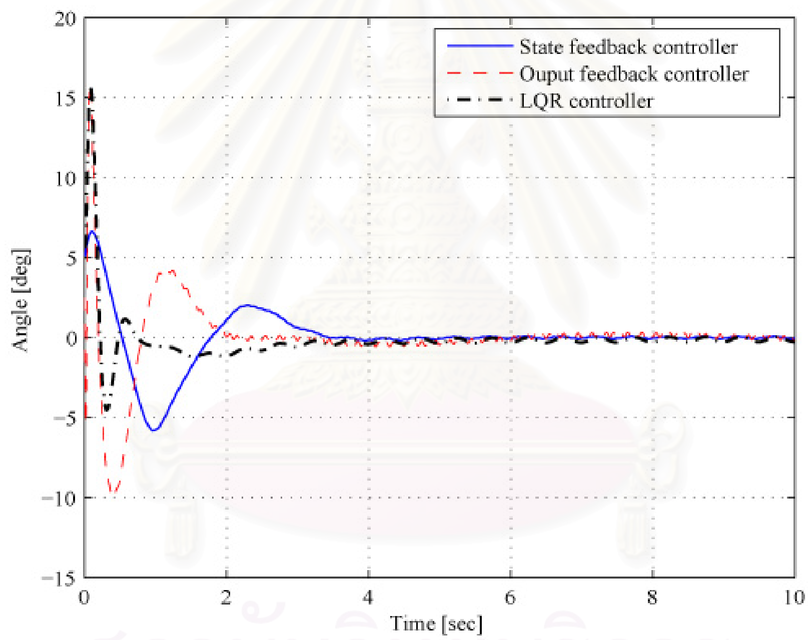

Figure 5.18: The angle of long pendulum β_1.

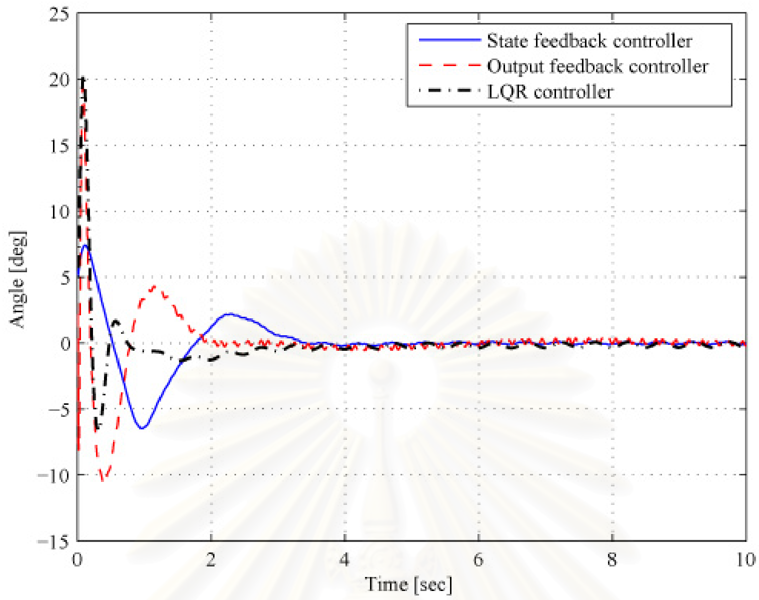

Figure 5.19: The angle of short pendulum β_1.

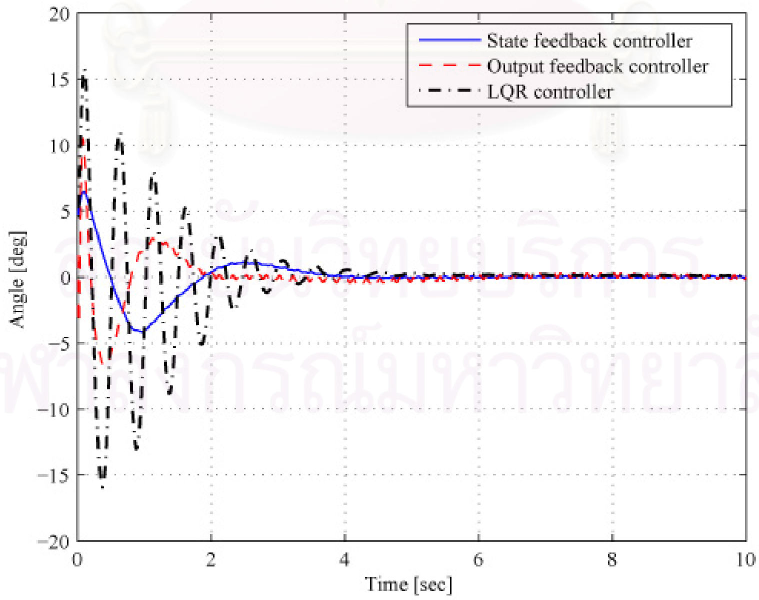

Figure 5.20: The angle of long pendulum with 15 percent uncertain parameter.

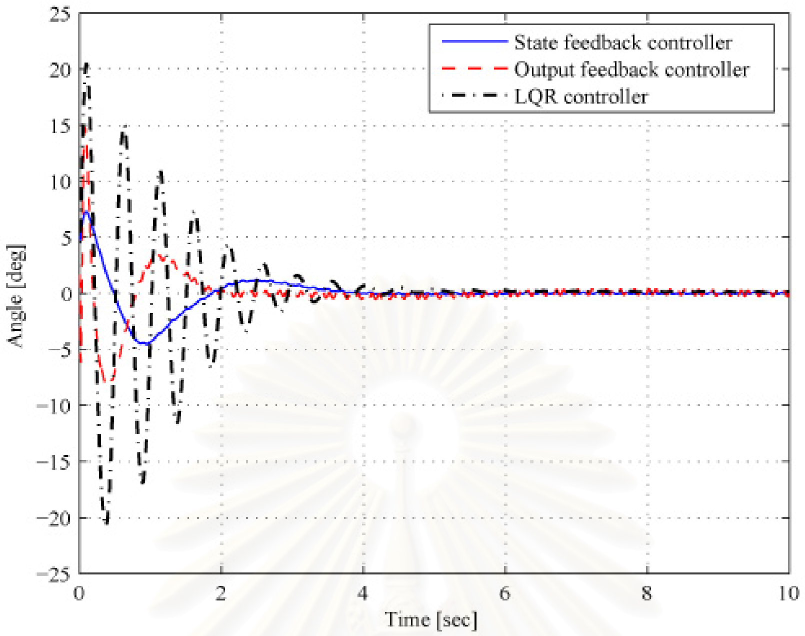

Figure 5.21: The angle of short pendulum with 15 percent uncertain parameter.

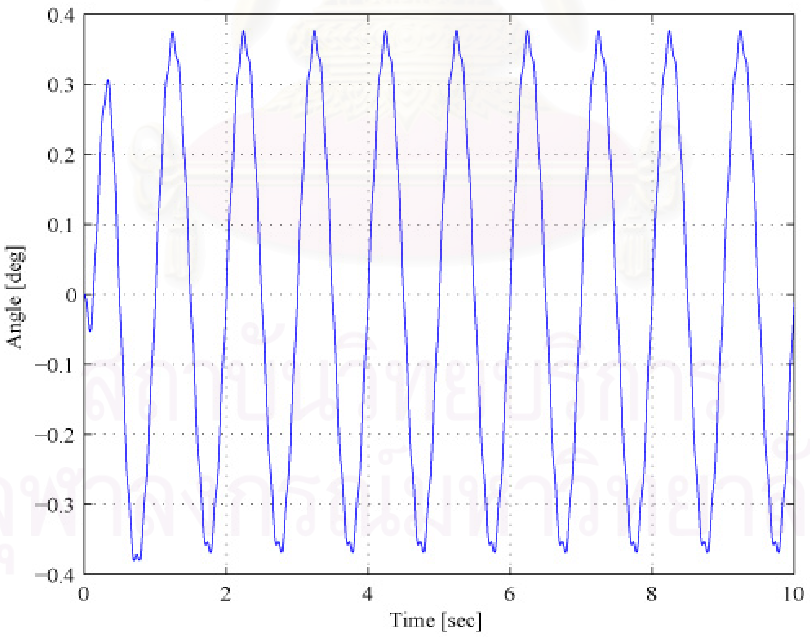

Figure 5.22: The angle of long pendulum with respect to disturbance inputs for LQR controller.

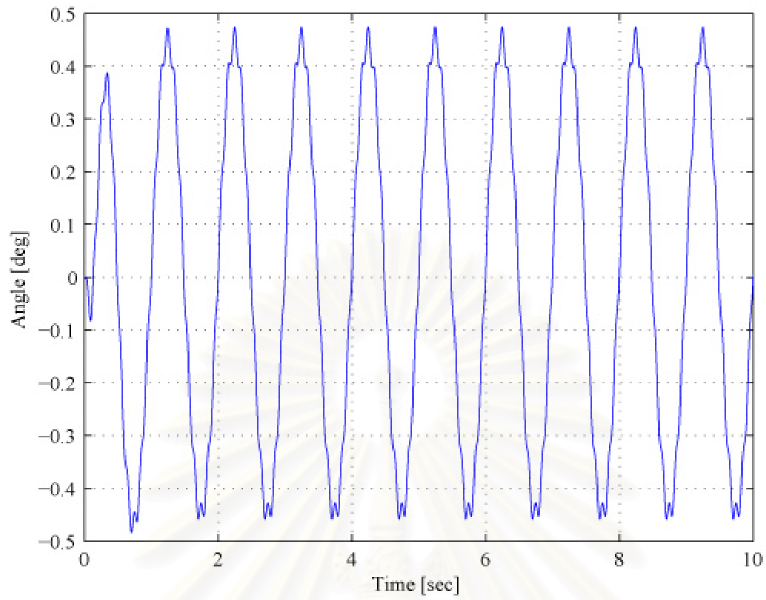

Figure 5.23: The angle of short pendulum with respect to disturbance inputs for LQR controller.

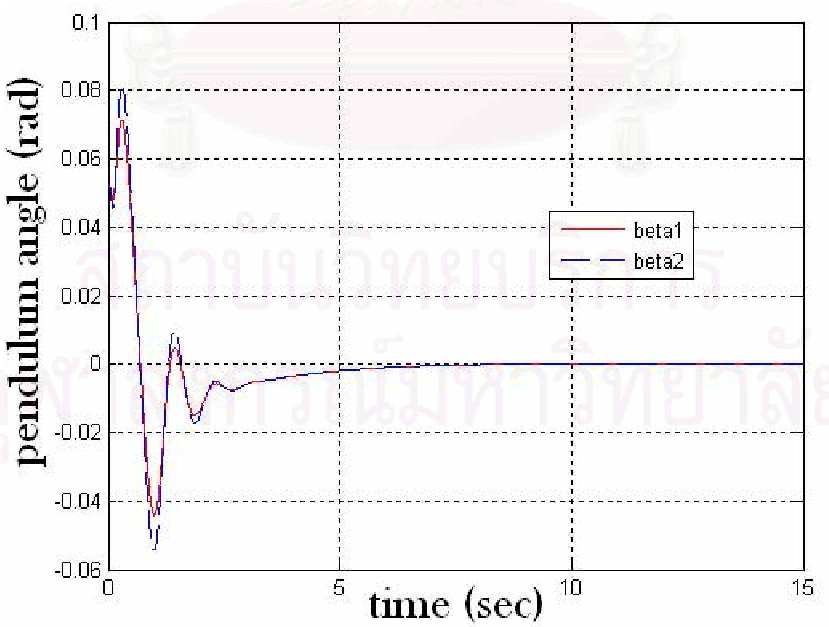

Figure 5.24: The angle of pendulum for LPV controller.

The response of the system with respect to the disturbance inputs for the LQR controller is shown in Fig 5.22 and 5.23. The amplitude of the angle is greater than that in the proposed controllers. We continue comparing the proposed controllers and an LPV controller. This

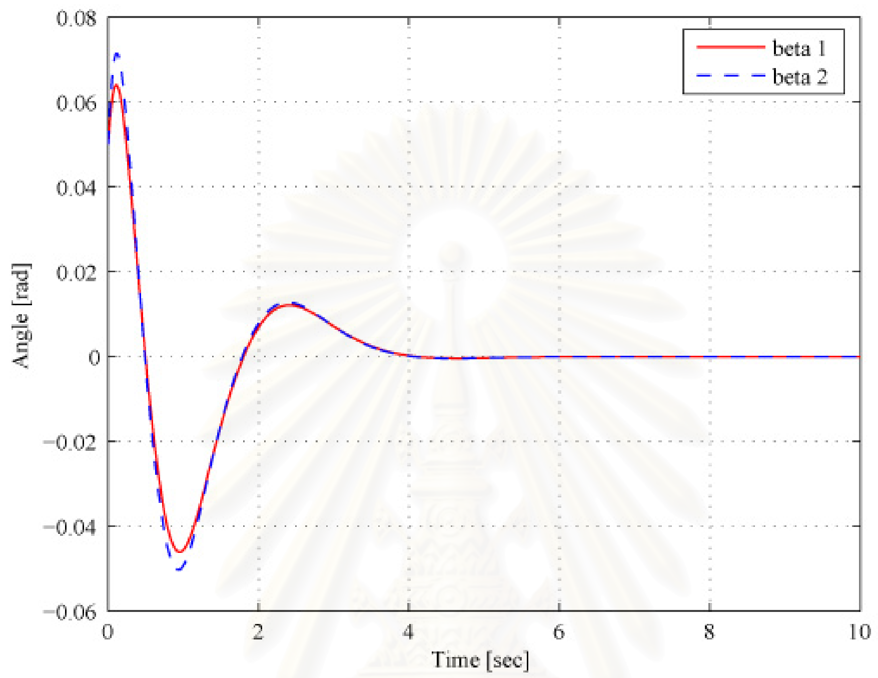

Figure 5.25: The angle of pendulum with initial 3°.

LPV controller was presented by Aribowo and Nazaruddin [21]. The results show that the proposed controller are quite good in comparison with the LPV controller.

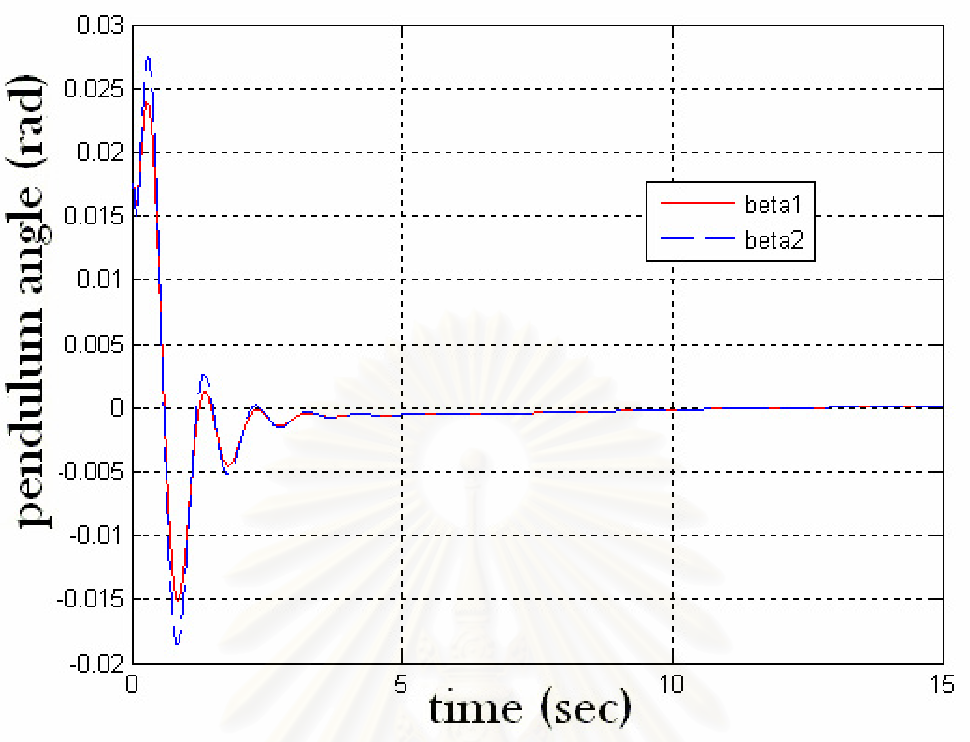

Figure 5.26: The angle of pendulum for LPV controller.

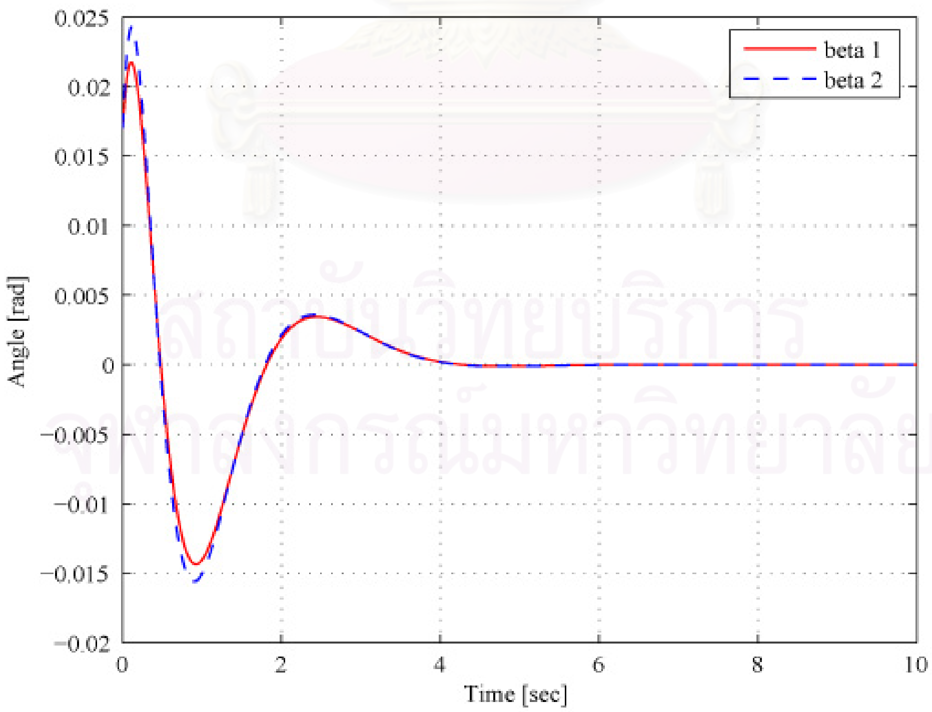

Figure 5.27: The angle of pendulum with initial 1°.

5.3 Summary

In this chapter,the results of the controller designs for the RDIP are shown. The system is stable when the disturbance inputs are injected to the passive joint of the long and short pendulum. The disturbance inputs are always sinusoid therefore the long and short pendulum still oscillate but the magnitude is very small. In addition, we also consider when the initial angle of the long and short pendulum are both 6°. The Fig. 32 shows that the system is stable despite of applied disturbance inputs.

In comparison, it is clear that the performance index in case of the state feedback controller is less than that in case of the output feedback controller. This result can be explained by the angle of long and short pendulum corresponding the disturbance inputs. In short, the state feedback controller gives a better result than the output feedback controller.

CHAPTER VI

CONCLUSIONS

6.1 Summary of Results

This thesis has dealt with a robust controller design problem for a rotary double inverted pendulum. The two controllers which are robust H_∞ state feedback controller robust H_∞ output feedback controller have been designed. The controller design problem were formulated into some standard forms towards the numerical computation of controllers using LMIs. The approach to robust controllers that we have studied is polytopic. With the poly approach, the problem might blow up and the number of solutions increase and consequently LMIs to be solved for becomes 2^n. In other words, that means the increase of the number of uncertain parameter leads to the increase the number of LMIs. The polytopic approach is suited to uncertain systems of which the number of uncertain parameter are not many, otherwise it will cause a computational burden which can ultimately decide the feasibility of the problem. The RDIP has three uncertain parameter which are length of long and short pendulum and inertia of rotating disc. In the process of designing controllers, we have found that the RDIP becomes hard to control when the length of long pendulum is equal to the length of short pendulum. Let us now look at a detailed summary of the results obtained in this thesis.

Chapter 2 is the introduction about a basic knowledge of which some fundamental concepts such as model of uncertain systems were reviewed. Especially, some underlying applications of LMIs were also highlighted. It turns out that LMIs is a very useful tool in the field of control systems. Many problems in control systems can be formulated into the form of a system of LMIs which is able to be solved efficiently using the package YALMIP.

In chapter 3, the nonlinear system RDIP has been modeled and linearized appropriately depending on the control method to be applied. This chapter also stated the controller design problems which are the robustness and the minimization problems.

Chapter 4, which contains the main contents of the thesis, mentioned about the method used to design the two controllers above. In case of the robust H_∞ state feedback controller, we have use a stability region not only to improve the transient responses of the system but also to guarantee the robustness of the system. Thereafter, the state feedback gain were obtained by solving the 32 LMIs constraints. About the robust H_∞ output feedback controller, the problem of design the controller is a minimization problem involving a system of 17 LMIs. The controller parameters are not easy to get after solving the LMI system like the previous case. In order to reconstruct the controller parameters we had also to solve some LMIs more, but it is not a minimization problem, just a mere LMIs problem.

Finally, the simulation results were shown in chapter 5. We have examined the effect of disturbance inputs on the system. In order to calculate the worst value of performance index, the two pendulums are set at the upright unstable position and the uncertain parameters of the system are varied. It can be seen that the robust H_∞ state feedback controller is better than the robust H_∞ output feedback controller. We also compared the proposed controllers with LQR and LPV controllers. The LPV controller seem to be better than the proposed controller whereas the LQR is the worst controller.

6.2 Recommendations

In this research, we have used the polytopic approach to deal with the uncertain system. Increasing of the number of uncertain parameters leads to the increasing the number of LMIs. In addition, this approach seem to be less conservative. That is a drawback of the polytopic approach.

6.2.1 Possible Extensions

A possible extension is LFT approaches. The LFT approach on the other hand tends to be conservative. For instance, the LFT approach is able to cope with a lot more uncertainties. If the number of uncertainties entering the system is n, then LFT deals with $2n$ uncertainties. The LFT approach helps less computational burden which can ultimately decide the feasibility of the problem. It the problem is extended to the complex one with huge number of uncertainties and where the uncertain parameter varies arbitrarily fast, we would recommend the LFT approach. The design of a gain-scheduled controller for the a nonlinear plant can be described as a four-step procedure

1. Compute a linear parameter-varying model the plant. An approach is well know as quasi-LPV scheduling. Some nonlinearities of the plant dynamic are replaced with time-varying parameters used as scheduling variables.

2. This step deals with linear design controller techniques for the LPV plant model. The LPV model will be reduced to LFT form.

3. Design a robust controller such that its coefficients are varied(scheduled) according to the current value of the scheduling variables.

4. Evaluate the controller in order to meet given certain specifications.

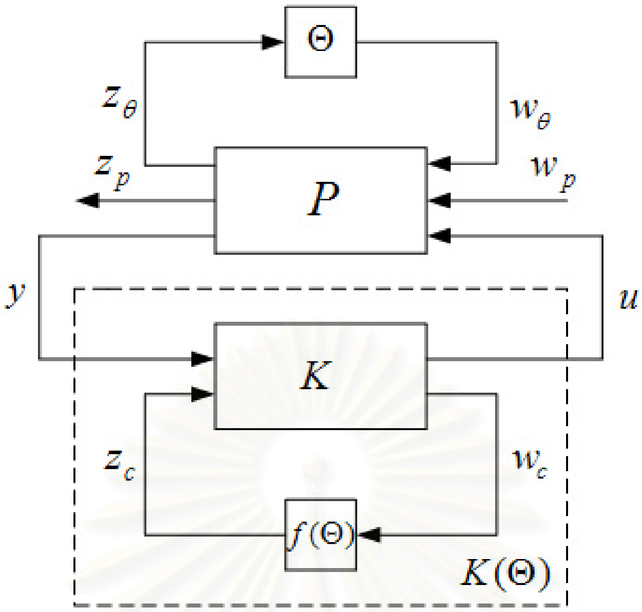

Figure 6.1: The LPV control structure.

LFT approach to gain scheduling using LMIs

LFT description can represent any polynomial or rational matrix function of a scalar variable. The LMIs are able to seek the existence of gain-scheduled controllers. The plant can be represented by using an upper LFT interconnection

$$\begin{pmatrix} z \\ y \end{pmatrix} = F_u(P(s), \Theta) \begin{pmatrix} w \\ u \end{pmatrix}$$

where $P(s)$ is a known LTI plant and Θ is some block diagonal time-varying operator. The block Θ can be defined by

$$\Theta\text{=blockdiag}(\theta_1 I_{r1}, \ldots \theta_k I_{rk})$$

We can use a concept of parameter-dependent H_∞ controllers to solve the problem. The controllers depend on the varying parameters $\theta(t)$ can be presented by

$$\begin{aligned} \dot{x}_c(t) &= A_c(\theta(t))x_c(t) + B_c(\theta(t))y(t) \\ u(t) &= C_c(\theta(t))x_c(t) + D_c(\theta(t))y(t) \end{aligned}$$

where A_c, B_c, C_c, D_c are linear fractional functions of θ. It is noted that in order to apply this approach, the value of $\theta(t)$ must be measured at each time t. As previously presented in Chapter 4, the H_∞ control problem can be considered as finding an internally stabilizing LTI controller K(s) such that

$$\|F(P, K)\|_\infty < \gamma$$

We can attenuate the effect of disturbance inputs by minimizing the value γ. Therefore, the objective is to to minimize closed loop performance $\gamma > 0$ from w to z for all admissible parameter trajectories θ. The minimal realization of the LTI plant $P(s)$ is

$$P(s) = \begin{pmatrix} D_{\theta\theta} & D_{\theta p} & D_{\theta u} \\ D_{p\theta} & D_{pp} & D_{pu} \\ D_{u\theta} & D_{up} & D_{uu} \end{pmatrix} + \begin{pmatrix} C_\theta \\ C_p \\ C_u \end{pmatrix} (sI - A)^{-1} (B_\theta \quad B_p \quad B_u)$$

The LFT of open loop plant can be written as

$$\begin{aligned} \dot{x} &= Ax + B_\theta w_\theta + B_p w_\theta + B_u w_u \\ z_\theta &= C_\theta x + D_{\theta\theta} w_\theta + D_{\theta p} w_p + D_{\theta u} u \\ z_p &= C_p x + D_{p\theta} w_\theta + D_{pp} w_p + D_{pu} u \\ y &= C_y x + D_{y\theta} w_\theta + D_{yp} w_p, \quad w_\theta = \Theta z_\theta \end{aligned}$$

An optimal H_∞ controller can be designed if there exists pairs of symmetric matrices (X, Y) and (R, S) such that

$$\min_{X,Y,R,S} \gamma$$

$$\mathcal{N}_X^T \begin{pmatrix} AX + XA^T & R\widehat{C}_p^T & \widehat{B}_p \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \\ \widehat{C}_p X & -\gamma \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} & \widehat{D}_{pp} \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \widehat{B}_p^T & \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \widehat{D}_{pp}^T & -\gamma \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \mathcal{N}_X < 0 \tag{6.1}$$

$$\mathcal{N}_Y^T \begin{pmatrix} A^T Y + YA & Y\widehat{B}_p & \widehat{C}_p^T \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \\ \widehat{B}_p^T Y & -\gamma \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} & \widehat{D}_p^T \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \\ \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \widehat{C}_p & \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \widehat{D}_{pp} & -\gamma \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \end{pmatrix} \mathcal{N}_Y < 0$$

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \geq 0$$

$$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \geq 0$$

where

$$\widehat{B}_p = (B_\theta, B_p), \quad \widehat{C}_p = \begin{pmatrix} C_\theta \\ C_p \end{pmatrix}, \quad \widehat{D}_{pp} = \begin{pmatrix} D_{\theta\theta} & D_{\theta p} \\ D_{p\theta} & D_{pp} \end{pmatrix}$$

Moreover, there exists an optimal H_∞ controller of order $k < 6$ if the extra rank constraint is satisfied

$$\text{rank}(I - XY) \leq k$$

6.2.2 Future works

In this research, some difficulties have been encountered, hence, these can be considered as avenues for future works:

1. The LFT approach being able to cope with a lot more uncertain parameters gives a conservative result. Future work could consider complex uncertainties and other types of disturbances.

2. Making a real rotary double inverted pendulum will be an interesting work in order to test controllers.

REFERENCES

1. S. Mori, H. Nishihara, and K. Furuta, "Control of unstable mechanical system-control of pendulum," *Int. J. Control*, 23, 5.

2. T. Ishida, N. Shiokawa, T. Nagado, and S. Ganeko, "Learning control of an inverted pendulum using a neural network," *Proc. IEEE Int. Conf. on Industrial Electronics, Control and Instrumentation*.

3. S. Renou and L. Saydy, "Real time control of an inverted pedulum based on approximate linearization," *Proc. IEEE/ Canadian Conference on Electrical and Computer Engineering*.

4. T. Henmi, M. Deng, and A. Inoue, "Swing-up control of a serial double inverted pendulum," *Proc. American Control Conference*.

5. T. Fujinaka, Y. Kishida, M. Yoshioka, and O. S., "Stabilization of double inverted pendulum with self-tuning neuro-pid," *Proc. IEEE-INNS-ENNS Int. Conference on Neural Networks*.

6. K. Furuta, M. Yamakita, and S. Kobayashi, "Swinging up control of inverted pendulum using pseudo-state feedback," *J. Syst. Control Eng*, 206, 14, (1992): 263-269.

7. K. J. Aström and K. Furuta, "Swinging up a pendulum by energy control," *Automatica*, 36, 2, (2000): 287–295.

8. J. Yi, N. Yubazaki, and K. Hirota, "Upswing and stabilization of inverted pendulum system based on sirms dynamically fuzzy inference model," *Fuzzy Sets Syst*, 122, 1, (2001): 139–152.

9. S. Yurkovich and M. Widjaja, "Fuzzy controller synthesis for an inverted pendulum system," *Control Eng. Pract*, 4, 4, (1996): 455–469.

10. H. O. Wang, K. Tanaka, and M. K. Griffin, "An approach to fuzzy control of nonlinear systems: Stability and design issues," *IEEE Trans. Fuzzy Syst.*, 4, 1, (Feb. 1996): 14–23.

11. Q. Wei, W. P. Dayawansa, and W. S. Levine, "Nonlinear controller for an inverted pendulum having restricted travel," *Automatica*, 31, 6, (Jun. 1995): 841–850.

12. C. Chung and J.Hauser, “Nonlinear control of a swinging pendulum,” *Automatica*, 31, 6, (Jun. 1995): 851–862.

13. J. Zhao and M. W. Spong, “Hybrid control for global stabilization of the cart-pendulum systems,” *Automatica*, 37, 12, (Dec. 2001): 1941–1951.

14. P. G. Grossimon, E. Barbieri, and S. Drakunov, “Sliding mode control of an inverted pendulum,” *Proc. Southeastern Sym. on System Theory*.

15. T. Sugie and K. Fujimoto, “Controller design for inverted pendulum based on approximate linearization,” *Int. J. Robust Nonlinear Control*, 8, issue 7.

16. S. Nair and N. E. Leonard, “A normal form for energy shaping: application to the futura pendulum,” *Proc. IEEE Conference on Decision and Control*.

17. P. Pakdeepattarakorn, P. Thamvechvitee, J. Songsiri, M. Wongsaisuwan, and D. Banjerdpongchai, “Dynamic models of a rotary double inverted pendulum system,” *2004 IEEE Region 10 Conference TENCON 2004.*, 4, (2004): 558-561.

18. P. Gahinet and P. Apkarian, “A LMI-based parametrization of all H_∞ controller with applications,” *Proc. CDC*, 2, 3, (1993): 656-661.

19. P. Gahinet, “A convex parameterization of H_∞ suboptimal controllers,” *Proc. CDC*, (1992): 937-942.

20. T. Iwasaki and R. Skelton, “A complete solution to the general H_∞ control problem: LMI existence conditions and state-space formulas,” *Proc. Amer. Contr. Conf*, (1993): 605-609.

21. A. G. Aribowo, Y. Y. Nazaruddin, E. Joelianto, and H. Y. Sutarto, “Stabilization of rotary double inverted pendulum using robust gain-scheduling control,” *SICE Annual Conference*, (2007): .

22. S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishan, *Linear matrix inequalities in system and control theory*. SIAM, 1994.

23. K. Zho and J. C. Doyle, *Essentials of Robust Control*. Pretice Hall, 1998.

24. J. M. Maciejowski, *Multivariable feedback design*. Addison–Wesley Publishing Company, Inc., 1989.

25. P. Gahinet, “Explicit controller formulas for LMI-based H_∞ systhesis,” *Automatica*, 32, 7, (1996): 1007-1014.

26. P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI control toolbox for use with MATLAB.* The mathworks, Inc., Natick, MA, 1996.

27. P. P. Khargonekar, I. R. Petersen, and K. Zhou, “Robust stabilization of uncertain linear systems : Quadratic stabilization and H_∞ control theory,” *IEEE Trans. Automatica. Control*, 35, (1990): 356-361.

28. J. C. Doyle, K. Glover, P. Khargonekar, and B. Francis, “State-space solutions to standard H_2 and H_∞ control problem,” *Automatica*, 34, 8, (1989): 831–847.

29. J. G. Vantwerp and R. D. Braatz, “A tutorial on linear and binear matrix inequalities,” *J. of Process Contrl*, 10, (2000): 363–386.

30. K. Zhou and P. Khargonekar, “An algebraic Riccati equation approach to H_∞ optimization,” *System and Control Letters*, 11, (1988): 85–91.

31. A. Packard, “Gain scheduling via linear fractional transformations,” *System and Control Letters*, 22, (1994): 79–92.

32. D. A. Lawrence and W. J. Rugh, “Gain scheduling dynamic linear controllers for a nonlinear plant,” *Automatica*, 31, (1995): 381–390.

APPENDIX

Appendix

Matlab Source Code for Simulations of Controller Design

---

1 Program for robust H_∞ state feedback controller

```
%==========================================================================
%                            Statefeedback.m
%               Simulation of a rotary double inverted pendulum
%                            Author: Hoang Ha
%                             30 July 2007
%==========================================================================
%==========================================================================
%                         Variable initialization
%==========================================================================
close all; clear all;
J0= 0.06; % Inertia of the rotating disc
J1=0.008; % Inertia of the long pendulum
J2=0.002; % Inertia of the short pendulum
c0=0.004; % Vicous coef. of the rotating disc
c1=0.003; % Vicous coef. of the long pendulum
c2=0.009; % Vicous coef. of the short pendulum
m1=0.25;  % Mass of the long pendulum
m2=0.13;  % Mass of the short pendulum
l1=0.24;  % The displacement from the joint to the c.m of the long pendulum (nominal value)
l2=0.13;  % The displacement from the joint to the c.m of the short pendulum (nominal value)
L=0.172;  % The radius of the rotating disc
g=9.8;    % The gravity constant
Km=0.374; % Torque constant
Kb=0.374; % Back emf. constant
R=8.26;   % Resistant in motor circuit
%==========================================================================
%           Define the maximum and minimum value of the uncertain parameters
%==========================================================================
der_min=0.8; % Variation of the uncertain parameters
der_max=1.2; % Variation of the uncertain parameters
l1_min=l1*der_min;l1_max=l1*der_max; % Length of the long pendulum
l2_min=l2*der_min;l2_max=l2*der_max; % Length of the short pendulum
J0_min=J0*der_min;J0_max=J0*der_max; % Inertia of the rotating disc
%==========================================================================
%           Linearize the nonlinear plant at the upright position
%==========================================================================
%----------- The nominal plant ----------------------------------------------
Y1 = J00+m1*L^2+m2*L^2 ; Y2 = J1+m1*l10^2; Y3= J2+m2*l20^2;
X1 = m1*l10*L; X2= m2*l20*L;
X3 = m1*g*l10; X4 = m2*g*l20;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 X1 X2; X1 Y2  0; X2 0 Y3]; F21=diag([0 -X3 -X4]);
E22= [Y1 -X1 -X2; -X1 Y2  0; -X2 0 Y3]; F21=diag([0 X3 X4]);
% E22= [Y1 -X1 X2; -X1 Y2  0; X2 0 Y3]; F21=diag([0 X3 -X4]);
```

```
E= [eye(3) zeros(3,3); zeros(3,3)  E22];
F = [zeros(3,3) eye(3); F21 F22];
G = [0 0 0 Km/R 0 0]';
A = inv(E)*F;
B = inv(E)*G;
%-------Find linearized model for each vertex------------------------------------------
Y1 = J0_min+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_min^2; Y3= J2+m2*l2_min^2;
X1 = m1*l1_min*L; X2= m2*l2_min*L;
X3 = m1*g*l1_min; X4 = m2*g*l2_min;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 X1 X2; X1 Y2  0; X2 0 Y3]; F21=diag([0 -X3 -X4]);
E22= [Y1 -X1 -X2; -X1 Y2  0; -X2 0 Y3]; F21=diag([0 X3 X4]);
% E22= [Y1 -X1 X2; -X1 Y2  0; X2 0 Y3]; F21=diag([0 X3 -X4]);
E= [eye(3) zeros(3,3); zeros(3,3)  E22];
F = [zeros(3,3) eye(3); F21 F22];
A1 = inv(E)*F;
B1 = inv(E)*G;
%--------------------------------------------------------------------------------------
Y1 = J0_min+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_min^2; Y3= J2+m2*l2_max^2;
X1 = m1*l1_min*L; X2= m2*l2_max*L;
X3 = m1*g*l1_min; X4 = m2*g*l2_max;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2  0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3)  E22];
F = [zeros(3,3) eye(3); F21 F22];
A2 = inv(E)*F;
B2 = inv(E)*G;
%--------------------------------------------------------------------------------------
Y1 = J0_min+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_max^2; Y3= J2+m2*l2_min^2;
X1 = m1*l1_max*L; X2= m2*l2_min*L;
X3 = m1*g*l1_max; X4 = m2*g*l2_min;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2  0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3)  E22];
F = [zeros(3,3) eye(3); F21 F22];
A3 = inv(E)*F;
B3 = inv(E)*G;
%--------------------------------------------------------------------------------------
Y1 = J0_min+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_max^2; Y3= J2+m2*l2_max^2;
X1 = m1*l1_max*L; X2= m2*l2_max*L;
X3 = m1*g*l1_max; X4 = m2*g*l2_max;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2  0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3)  E22];
F = [zeros(3,3) eye(3); F21 F22];
A4 = inv(E)*F;
B4 = inv(E)*G;
%--------------------------------------------------------------------------------------
Y1 = J0_max+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_min^2; Y3= J2+m2*l2_min^2;
X1 = m1*l1_min*L; X2= m2*l2_min*L;
X3 = m1*g*l1_min; X4 = m2*g*l2_min;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2  0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3)  E22];
F = [zeros(3,3) eye(3); F21 F22];
A5 = inv(E)*F;
B5 = inv(E)*G;
%--------------------------------------------------------------------------------------
```

```
Y1 = J0_max+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_min^2; Y3= J2+m2*l2_max^2;
X1 = m1*l1_min*L; X2= m2*l2_max*L;
X3 = m1*g*l1_min; X4 = m2*g*l2_max;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2 0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3) E22];
F = [zeros(3,3) eye(3); F21 F22];
A6 = inv(E)*F;
B6 = inv(E)*G;
%------------------------------------------------------------------------
Y1 = J0_max+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_max^2; Y3= J2+m2*l2_min^2;
X1 = m1*l1_max*L; X2= m2*l2_min*L;
X3 = m1*g*l1_max; X4 = m2*g*l2_min;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2 0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3) E22];
F = [zeros(3,3) eye(3); F21 F22];
A7 = inv(E)*F;
B7 = inv(E)*G;
%------------------------------------------------------------------------
Y1 = J0_max+m1*L^2+m2*L^2 ; Y2 = J1+m1*l1_max^2; Y3= J2+m2*l2_max^2;
X1 = m1*l1_max*L; X2= m2*l2_max*L;
X3 = m1*g*l1_max; X4 = m2*g*l2_max;
F22=-diag([c0+Km*Kb/R c1 c2]);
E22= [Y1 -X1 -X2; -X1 Y2 0; -X2 0 Y3]; F21=diag([0 X3 X4]);
E= [eye(3) zeros(3,3); zeros(3,3) E22];
F = [zeros(3,3) eye(3); F21 F22];
A8 = inv(E)*F;
B8 = inv(E)*G;
%--------Define other matrices for the open-loop system------------------
C2 = [1 0 0 0 0 0;0 1 0 0 0 0; 0 0 1 0 0 0];
D = [0 0 0]';
Q = diag([10 10 10 1 1 1]); % The matrix comes from the benchmark problem
Bw=[0 0;0 0; 0 0;0 0;1 0;0 1];
Bw=inv(E)*B1w; % The disturbance input matrix
Cz=Q; % Define the performance output matrix
%=========================================================================
%            Formulate the problem in terms of LMIs formulation
%=========================================================================
%--------------------------Declare variables------------------------------
Y=sdpvar(6); % A symmetric matrix
W=sdpvar(6); % A symmetric matrix
L=sdpvar(1,6); % A matrix with the dimension of (1,6)
gamma2=sdpvar(1); % sqrt(gamma2) is the H_infinity norm of the system
%---------------The parameters for LMI stability region S(alpha,theta,r)--
theta=pi/3;
r=100000;
alpha=0.00005;
%----------------------------LMIs constraint------------------------------
H=[A1*Y+Y*A1'+B1*L+L'*B1'+B1w*B1w'      Y*Cz'
                Cz                      -gamma2*eye(6)];
%------------------------------------------------------------------------
con12=[A1*Y+Y*A1'+B1*L+(B1*L)'+2*alpha*Y];
con13=[-r*Y              A1*Y+B1*L
       Y*A1'+(B1*L)'    -r*Y    ];
con14=[sin(theta)*(A1*Y+B1*L+Y*A1'+(B1*L)')     cos(theta)*(A1*Y+B1*L-Y*A1'-(B1*L)')
       cos(theta)*(Y*A1'+(B1*L)'-A1*Y-B1*L)     sin(theta)*(A1*Y+B1*L+Y*A1'+(B1*L)')];
```

```
%-----------------------------------------------------------------------
con22=[A2*Y+Y*A2'+B2*L+(B2*L)'+2*alpha*Y];
con23=[-r*Y              A2*Y+B2*L
       Y*A2'+(B2*L)'    -r*Y    ];
con24=[sin(theta)*(A2*Y+B2*L+Y*A2'+(B2*L)')     cos(theta)*(A2*Y+B2*L-Y*A2'-(B2*L)')
       cos(theta)*(Y*A2'+(B2*L)'-A2*Y-B2*L)     sin(theta)*(A2*Y+B2*L+Y*A2'+(B2*L)')];
%-----------------------------------------------------------------------
con32=[A3*Y+Y*A3'+B3*L+(B3*L)'+2*alpha*Y];
con33=[-r*Y              A3*Y+B3*L
       Y*A3'+(B3*L)'    -r*Y    ];
con34=[sin(theta)*(A3*Y+B3*L+Y*A3'+(B3*L)')     cos(theta)*(A3*Y+B3*L-Y*A3'-(B3*L)')
       cos(theta)*(Y*A3'+(B3*L)'-A3*Y-B3*L)     sin(theta)*(A3*Y+B3*L+Y*A3'+(B3*L)')];
%-----------------------------------------------------------------------
con42=[A4*Y+Y*A4'+B4*L+(B4*L)'+2*alpha*Y];
con43=[-r*Y              A4*Y+B4*L
       Y*A4'+(B4*L)'    -r*Y    ];
con44=[sin(theta)*(A4*Y+B4*L+Y*A4'+(B4*L)')     cos(theta)*(A4*Y+B4*L-Y*A4'-(B4*L)')
       cos(theta)*(Y*A4'+(B4*L)'-A4*Y-B4*L)     sin(theta)*(A4*Y+B4*L+Y*A4'+(B4*L)')];
%-----------------------------------------------------------------------
con52=[A5*Y+Y*A5'+B5*L+(B5*L)'+2*alpha*Y];
con53=[-r*Y              A5*Y+B5*L
       Y*A5'+(B5*L)'    -r*Y    ];
con54=[sin(theta)*(A5*Y+B5*L+Y*A5'+(B5*L)')     cos(theta)*(A5*Y+B5*L-Y*A5'-(B5*L)')
       cos(theta)*(Y*A5'+(B5*L)'-A5*Y-B5*L)     sin(theta)*(A5*Y+B5*L+Y*A5'+(B5*L)')];
%-----------------------------------------------------------------------
con62=[A6*Y+Y*A6'+B6*L+(B6*L)'+2*alpha*Y];
con63=[-r*Y              A6*Y+B6*L
       Y*A6'+(B6*L)'    -r*Y    ];
con64=[sin(theta)*(A6*Y+B6*L+Y*A6'+(B6*L)')     cos(theta)*(A6*Y+B6*L-Y*A6'-(B6*L)')
       cos(theta)*(Y*A6'+(B6*L)'-A6*Y-B6*L)     sin(theta)*(A6*Y+B6*L+Y*A6'+(B6*L)')];
%-----------------------------------------------------------------------

con72=[A7*Y+Y*A7'+B7*L+(B7*L)'+2*alpha*Y];
con73=[-r*Y              A7*Y+B7*L
       Y*A7'+(B7*L)'    -r*Y    ];
con74=[sin(theta)*(A7*Y+B7*L+Y*A7'+(B7*L)')     cos(theta)*(A7*Y+B7*L-Y*A7'-(B7*L)')
       cos(theta)*(Y*A7'+(B7*L)'-A7*Y-B7*L)     sin(theta)*(A7*Y+B7*L+Y*A7'+(B7*L)')];
%-----------------------------------------------------------------------

con82=[A8*Y+Y*A8'+B8*L+(B8*L)'+2*alpha*Y];
con83=[-r*Y              A8*Y+B8*L
       Y*A8'+(B8*L)'    -r*Y    ];
con84=[sin(theta)*(A8*Y+B8*L+Y*A8'+(B8*L)')     cos(theta)*(A8*Y+B8*L-Y*A8'-(B8*L)')
       cos(theta)*(Y*A8'+(B8*L)'-A8*Y-B8*L)     sin(theta)*(A8*Y+B8*L+Y*A8'+(B8*L)')];
%---------------Solve the LMIs constraints using Sedumi solver---------------
lmi=set(H>0)+set(con12<0)+set(con13<0)+set(con14<0);
lmi=lmi+set(con22<0)+set(con23<0)+set(con24<0);
lmi=lmi+set(con32<0)+set(con33<0)+set(con34<0);
lmi=lmi+set(con42<0)+set(con43<0)+set(con44<0);
lmi=lmi+set(con52<0)+set(con53<0)+set(con54<0);
lmi=lmi+set(con62<0)+set(con63<0)+set(con64<0);
lmi=lmi+set(con72<0)+set(con73<0)+set(con74<0);
lmi=lmi+set(con82<0)+set(con83<0)+set(con84<0);
opts=sdpsettings;
opts.solver='sedumi';
err=solvesdp(lmi,gamma2);
gamma=sqrt(gamma2);
```

```
%--------------------Get the value of the matrices L, Y----------------------------------
Lsol=double(L);
Ysol=double(Y);
Ksval=Lsol*inv(Ysol);% the state feedback controller is found
%=================================================================================
%                    End program for robust state feedback controller
%=================================================================================
```

2 Program for robust H_∞ output feedback controller

```
%=================================================================================
%                              Outpu tf eedback .m
%                Simulation of a rotary double inverted pendulum
%                              Author: Hoang Ha
%                                30 July 2007
%=================================================================================
Dzu=zeros(6,1); Dyw=zeros(3,2); Dcl=zeros(6,2);
%-------------------------------------------------------------------
C_Dyw=[C2 Dyw];
C2_null=null(C_Dyw); % Compute null space of the matrix [C2 Dyw]
CD=[   C2_null        zeros(8,6)
      zeros(6,5)      eye(6,6)];
B_D12=[B' Dzu'];
B_null=null(B_D12); % Compute null space of the matrix [B' Dzu]
BD=[   B_null         zeros(12,2)
     zeros(2,11)      eye(2,2)]; % (14,13)
%-------------------------------------------------------------------
B1_D12=[B1' Dzu']; B1_null=null(B1_D12); BD1=[   B1_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B2_D12=[B2' Dzu']; B2_null=null(B2_D12); BD2=[   B2_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B3_D12=[B3' Dzu']; B3_null=null(B3_D12); BD3=[   B3_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B4_D12=[B4' Dzu']; B4_null=null(B4_D12); BD4=[   B4_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B5_D12=[B5' Dzu']; B5_null=null(B5_D12); BD5=[   B5_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B6_D12=[B6' Dzu']; B6_null=null(B5_D12); BD6=[   B6_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B7_D12=[B7' Dzu']; B7_null=null(B7_D12); BD7=[   B7_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
%-------------------------------------------------------------------
B8_D12=[B8' Dzu']; B8_null=null(B8_D12); BD8=[   B8_null
zeros(12,2)
      zeros(2,11)     eye(2,2)];% (14,13)
```

```
%---------------------------------------------------------------------------
%                            Declare two variables X,Y
%---------------------------------------------------------------------------
X=sdpvar(6); % A symmetric matrix
Y=sdpvar(6); % A symmetric matrix
gamma=sdpvar(1); XY=[ X      eye(6)
    eye(6)   Y];
%---------------------------------------------------------------------------
%       Formulate the H infinity control problem in terms of LMIs formulation
%---------------------------------------------------------------------------
F11=[A*X+X*A'      X*C1w'                  B1w
     C1w*X      -gamma*eye(6)           zeros(6,2)
      B1w'          zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12=[A'*Y+Y*A      Y*B1w                   C1w'
      B1w'*Y     -gamma*eye(2)           zeros(2,6)
      C1w           zeros(6,2)           -gamma*eye(6)];%(14,14)
con11=BD'*F11*BD; con12=CD'*F12*CD;
%-----------------------------------------------------------
F11=[A1*X+X*A1'      X*C1w'                  B1w
     C1w*X      -gamma*eye(6)            zeros(6,2)
      B1w'          zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12=[A1'*Y+Y*A1      Y*B1w                   C1w'
     B1w'*Y     -gamma*eye(2)            zeros(2,6)
      C1w           zeros(6,2)           -gamma*eye(6)];%(14,14)
con11_1=BD1'*F11*BD1; con12_1=CD'*F12*CD;
%-----------------------------------------------------------
F11_2=[A2*X+X*A2'      X*C1w'                  B1w
     C1w*X      -gamma*eye(6)            zeros(6,2)
      B1w'          zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_2=[A2'*Y+Y*A2      Y*B1w                   C1w'
      B1w'*Y     -gamma*eye(2)            zeros(2,6)
        C1w          zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_2=BD2'*F11_2*BD2; con12_2=CD'*F12_2*CD;
%-----------------------------------------------------------
F11_3=[A3*X+X*A3'      X*C1w'                  B1w
      C1w*X     -gamma*eye(6)            zeros(6,2)
        B1w'         zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_3=[A3'*Y+Y*A3      Y*B1w                   C1w'
      B1w'*Y     -gamma*eye(2)            zeros(2,6)
        C1w          zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_3=BD3'*F11_3*BD3; con12_3=CD'*F12_3*CD;
%-----------------------------------------------------------
F11_4=[A4*X+X*A4'      X*C1w'                  B1w
      C1w*X     -gamma*eye(6)            zeros(6,2)
       B1w'         zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_4=[A4'*Y+Y*A4      Y*B1w                   C1w'
       B1w'*Y    -gamma*eye(2)            zeros(2,6)
         C1w         zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_4=BD4'*F11_4*BD4; con12_4=CD'*F12_4*CD;
%-----------------------------------------------------------
F11_5=[A5*X+X*A5'      X*C1w'                  B1w
      C1w*X     -gamma*eye(6)            zeros(6,2)
       B1w'         zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_5=[A5'*Y+Y*A5      Y*B1w                   C1w'
      B1w'*Y     -gamma*eye(2)            zeros(2,6)
       C1w          zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_5=BD5'*F11_5*BD5; con12_5=CD'*F12_5*CD;
%-----------------------------------------------------------
```

```
F11_6=[A6*X+X*A6'      X*C1w'                B1w
       C1w*X       -gamma*eye(6)          zeros(6,2)
        B1w'        zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_6=[A6'*Y+Y*A6     Y*B1w                 C1w'
     B1w'*Y      -gamma*eye(2)          zeros(2,6)
     C1w            zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_6=BD6'*F11_6*BD6;  con12_6=CD'*F12_6*CD;
%------------------------------------------------------------------------
F11_7=[A7*X+X*A7'      X*C1w'                B1w
       C1w*X       -gamma*eye(6)          zeros(6,2)
        B1w'        zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_7=[A7'*Y+Y*A7      Y*B1w                 C1w'
        B1w'*Y     -gamma*eye(2)          zeros(2,6)
         C1w        zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_7=BD7'*F11_7*BD7;  con12_7=CD'*F12_7*CD;
%------------------------------------------------------------------------
F11_8=[A8*X+X*A8'      X*C1w'                B1w
       C1w*X       -gamma*eye(6)          zeros(6,2)
        B1w'        zeros(2,6)          -gamma*eye(2)]; %(14,14)
F12_8=[A8'*Y+Y*A8     Y*B1w                 C1w'
       B1w'*Y      -gamma*eye(2)          zeros(2,6)
        C1w         zeros(6,2)          -gamma*eye(6)];%(14,14)
con11_8=BD8'*F11_8*BD8;  con12_8=CD'*F12_8*CD;
%------------------------------------------------------------------------
ineq=set(con11<0)+set(con12<0);
ineq=ineq+set(con11_1<0)+set(con12_1<0);
ineq=ineq+set(con11_2<0)+set(con12_2<0);
ineq=ineq+set(con11_3<0)+set(con12_3<0);
ineq=ineq+set(con11_4<0)+set(con12_4<0);
ineq=ineq+set(con11_5<0)+set(con12_5<0);
ineq=ineq+set(con11_6<0)+set(con12_6<0);
ineq=ineq+set(con11_7<0)+set(con12_7<0);
ineq=ineq+set(con11_8<0)+set(con12_8<0);  ineq=ineq+set(XY>0);
opts=sdpsettings;
opts.solver='sedumi'; % use the solver sedumi
err=solvesdp(ineq,gamma); % minimize gamma by solving a system of LMIs
Xsol=double(X); % get the value of variable X
Ysol=double(Y); % get the value of variable Y
gamma_value=double(gamma); % get the value of variable gamma
MN=eye(6)-Xsol*Ysol;  [s,u,v]=svd(MN);  M=(s*sqrt(u)');
N=(v*sqrt(u));%   compute two matrices M and N such that M*N'= I-X*Y
%--------------------compute matrix X_cl----------------------------------------------
bigX=[eye(6)  Ysol  ;zeros(6)  (v*sqrt(u))']*inv([Xsol
eye(6);(s*sqrt(u)')'  zeros(6)]);
%================================================================================
%                      Reconstruct controller parameters
%================================================================================
%-----------------Declare the controller parameters as variables------------------------
 Ak=sdpvar(6);
 Bk=sdpvar(6,3);
 Ck=sdpvar(1,6);
 Dk=sdpvar(1,3);
%--------------------------------------------------------
lmiC1=delta_thetaC1+B1*pi12*phiC+phiC'*pi12*B1';
%--------------------------------------------------------
lmiC2=delta_thetaC2+B2*pi12*phiC+phiC'*pi12*B2';
%---------------------------------------------------------
lmiC3=delta_thetaC3+B3*pi12*phiC+phiC'*pi12*B3';
```

```
%---------------------------------------------------------------
lmiC4=delta_thetaC4+B4*pi12*phiC+phiC'*pi12*B4';
%---------------------------------------------------------------
lmiC5=delta_thetaC5+B5*pi12*phiC+phiC'*pi12*B5';
%---------------------------------------------------------------
lmiC6=delta_thetaC6+B6*pi12*phiC+phiC'*pi12*B6';
%---------------------------------------------------------------
lmiC7=delta_thetaC7+B7*pi12*phiC+phiC'*pi12*B7';
%---------------------------------------------------------------
lmiC8=delta_thetaC8+B8*pi12*phiC+phiC'*pi12*B8';
%---------------------------------------------------------------
 delta21_2=N*Ak*M'+A2'+Ysol*A2*Xsol+Ysol*B2*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w'  ; C1w*Xsol];
 deltaR_2=A2*Xsol+Xsol*A2'+B2*(Ck*M'+Dk*C2*Xsol)+(Ck*M'+Dk*C2*Xsol)'*B2'
         +[B1w'; C1w*Xsol]'*inv(delta_cl)*[B1w'  ; C1w*Xsol];
 deltaS_2=A2'*Ysol+Ysol*A2+C2'*(Bk'*N'+Dk'*B2'*Ysol)+(Bk'*N'+Dk'*B2'*Ysol)'*C2
         +[B1w'*Ysol;C1w]'*inv(delta_cl)*[B1w'*Ysol;C1w];
 lmidelta2=[ deltaR_2    delta21_2'
             delta21_2   deltaS_2];
 %-----------------------------------------------------------------------------------
 delta21_3=N*Ak*M'+A3'+Ysol*A3*Xsol+Ysol*B3*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w'  ; C1w*Xsol];
 deltaR_3=delta_thetaC3+B3*pi12*KC+KC'*pi12*B3';
 deltaS_3=delta_thetaB3+C2'*pi21*KB+KB'*pi21*C2;
 lmidelta3=[ deltaR_3    delta21_3'
             delta21_3   deltaS_3];
 %-----------------------------------------------------------------------------------
 delta21_4=N*Ak*M'+A4'+Ysol*A4*Xsol+Ysol*B4*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w';C1w*Xsol];
 deltaR_4=delta_thetaC4+B4*pi12*KC+KC'*pi12*B4';
 deltaS_4=delta_thetaB4+C2'*pi21*KB+KB'*pi21*C2;
 lmidelta4=[ deltaR_4    delta21_4'
             delta21_4   deltaS_4];
 %-----------------------------------------------------------------------------------
 delta21_5=N*Ak*M'+A5'+Ysol*A5*Xsol+Ysol*B5*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w';C1w*Xsol];
 deltaR_5=delta_thetaC5+B5*pi12*KC+KC'*pi12*B5';
 deltaS_5=delta_thetaB5+C2'*pi21*KB+KB'*pi21*C2;
 lmidelta5=[ deltaR_5    delta21_5'
             delta21_5   deltaS_5];
 %-----------------------------------------------------------------------------------
 delta21_6=N*Ak*M'+A6'+Ysol*A6*Xsol+Ysol*B6*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w'  ; C1w*Xsol];
 deltaR_6=delta_thetaC6+B6*pi12*KC+KC'*pi12*B6';
 deltaS_6=delta_thetaB6+C2'*pi21*KB+KB'*pi21*C2;
 lmidelta6=[ deltaR_6    delta21_6'
             delta21_6   deltaS_6];
 %-----------------------------------------------------------------------------------
 delta21_7=N*Ak*M'+A7'+Ysol*A7*Xsol+Ysol*B7*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w'  ; C1w*Xsol];
 deltaR_7=delta_thetaC7+B7*pi12*KC+KC'*pi12*B7';
 deltaS_7=delta_thetaB7+C2'*pi21*KB+KB'*pi21*C2;
 lmidelta7=[ deltaR_7    delta21_7'
             delta21_7   deltaS_7];
 %-----------------------------------------------------------------------------------
 delta21_8=N*Ak*M'+A8'+Ysol*A8*Xsol+Ysol*B8*Ck*M'+N*Bk*C2*Xsol
          +[Ysol*B1w  C1w]*inv(delta_cl)*[B1w'  ; C1w*Xsol];
 deltaR_8=delta_thetaC8+B8*pi12*KC+KC'*pi12*B8';
```

```
deltaS_8=delta_thetaB8+C2'*pi21*KB+KB'*pi21*C2;
lmidelta8=[ deltaR_8      delta21_8 '
            delta21_8     deltaS_8 ];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% N*Ak*M'
ineqdelta=set(lmidelta2 <0);
% opts=sdpsettings;
% opts.solver='sedumi';
errdelta=solvesdp(ineqdelta);
%=========================================================================================
%                          Compute controller parameters
%=========================================================================================
Ak_sol=double(Ak);
Bk_sol=double(Bk);
Ck_sol=double(Ck);
Dk_sol=double(Dk);
```

Biography

Mr. Ha Hoang was born in Nam dinh, Vietnam in 1982. He received his bachelor's degree in electrical engineering from Hanoi University of Technology, Vietnam in 2005. He has been granted a scholarship by the AUN/SEED-Net (www.seed-net.org) to pursue his master degree in electrical engineering at Chulalongkorn University, Thailand, since 2005. He's now with the Control Systems Research Laboratory, Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University. His research work is in the area of robust control and application of linear matrix inequalities.